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FROM EXPERIENCES IN A DYNAMIC ENVIRONMENT TO WRITTEN NARRATIVES ON FUNCTIONS

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Abstract

This study focusses on high school students' written discourse about their experiences in a dynamic interactive digital environment in which functions were represented in one dimension, as *dynagraphs*, that are digital artefacts in which the independent variable can be acted upon and its movement causes the variation of the dependent variable. After the introduction of the notion of Dynamic Interactive Mediators within the theory of Commognition, we analyze and classify students' written productions describing their experience with the dynagraphs. We present this classification as a tool of analysis that allows us to gain insight into how their writing reflects the temporal and dynamic dimensions of their experience with the dynagraphs. This tool is used to analyze 11 excerpts; finally, epistemological, cognitive and didactic implications of this tool are discussed.

Digital Experiences in Mathematics Education

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1. Introduction

Functions and their graphs have a leading role in mathematical theory, in mathematical practice and in school. Being able to interpret the Cartesian graph of a function, to find out information, and to construct a graph starting from the function's properties are essential processes in mathematical thinking. These processes are based on an understanding of the meaning of *variable* and of the *relation between variations* of the variables (e.g., for small increase of the independent variable how does the dependent variable change?), which are not simple concepts from a cognitive point of view.

A Cartesian graph of a function from real numbers into real numbers consists of the set of points (x, f(x)) with real coordinates on the Cartesian plane, where the independent variable x belongs to the domain of the function and the dependent variable f(x) is its image. This description suggests that a subset of the Cartesian plane representing the graph of a function is made up of points whose coordinates incorporate the functional relation between the two variables. Specifically, these coordinates are two numbers belonging to the same set (the set of real numbers): indeed, *f* takes in and puts out only ordinary numbers which are represented as points belonging to a x-axis and a y-axis. A first hurdle for students approaching functions for the first time is visualizing these two variable numbers as points on lines; the second one is to visualize the graph as a collection of points on the plane that describes a relationship between variables. Colacicco, Lisarelli & Antonini (2017) have described such difficulty for many students in terms of a lack of coordination of the information that each point on the graph carries: a value from the function's domain and a second value that is its image. Indeed, very often students consider the graph to be the function and they identify a point on the graph as "the f(x) value".

Therefore, the Cartesian graph is extremely rich in meaning and powerful but, at the same time, the interpretation and manipulation of a graph requires a deep understanding of the relations between its elements. Several kinds of difficulties that students encounter when grappling with these ideas are widely reported in the literature (Kaput, 1992; Monk & Nemirovsky, 1994; Thompson, 1994; Carlson, 1998). A common finding is that in solving problems about functions, students are at their best when performing numerical evaluation or prescribed algebraic manipulation, but they tend to encounter difficulties in analyzing or even describing a function's behavior by looking at its Cartesian graph. In this respect, Carlson & Oehrtman (2005, p. 2) report:

> Students who think about functions only in terms of symbolic manipulations and procedural techniques are unable to comprehend a more general mapping of a set of input values to a set of output values; they also lack the conceptual structures for modeling function relationships in which the function value (output variable) changes continuously in tandem with continuous changes in the input variable.

Indeed, very often students perceive Cartesian graphs as static pictures representing a physical situation in which the variations, and the relation between variations – *covariation* – related to functional dependency remain hidden. Instead, studies have shown that recognizing variations and the relation between such variations is essential for interpreting the changing nature of a wide array of situations that can be modelled using functions (Carlson, Jacobs, Coe, Larsen & Hsu, 2002).

These studies suggest that it is very important that students have the opportunity to engage in activities on functions that emphasize covariation. The study presented in this paper is part of a larger one exploring the potentials of a sequence of activities for high school students with respect to their appreciation and correct use of Cartesian graphs. To contextualize the study presented in this paper, we now give a short description of this sequence of activities designed to highlight the covariational aspect of functions and its links to their graphs. The sequence was used to introduce functions to high school students in the broader study from which the one in this paper was developed. The sequence starts by introducing dynamic graphs of functions with the variables moving along two parallel lines – a construction previously referred to as *dynagraph* that we will soon return to (Fig. 1a). The sequence continues with a modified construction in which the line containing the dependent variable is rotated 90° with respect to the line with the independent variable, so that the variations of the two variables are now in perpendicular directions (Fig. 1b). Finally, parallel lines to the axes are constructed through the varying points representing the function's variables and their intersections define each point of the Cartesian graph, as they vary (Fig. 1c).

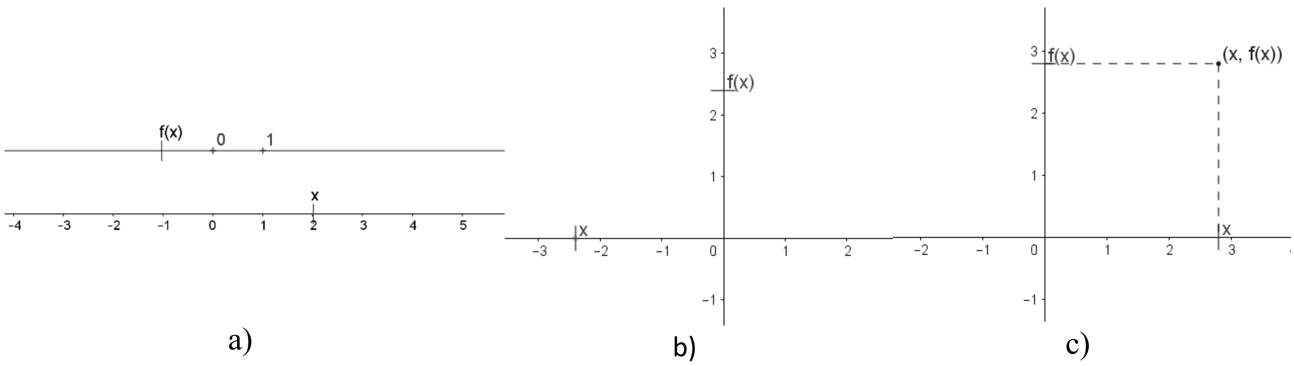


Figure 1: successive representations of functions proposed in the sequence of activities. a) dynagraph with the independent and dependent variables moving on separate axes; b) dynagraph in which the y-axis has been rotated so that it is perpendicular to the x-axis; c) the graph generated point by point as x and f(x) move along perpendicular axes.

We chose to work with dynagraphs because of the known results on how the use of technology can support the teaching of functions (Healy & Sinclair, 2007; Falcade, Laborde & Mariotti, 2007). Indeed, certain technological tools have been found to positively influence students' learning in this domain; among these there is what previously has been referred to as *DynaGraph*[1] (Goldenberg, Lewis & O'Keefe, 1992), a dynamic representation of functions in which movement, identification of invariants and of covariation become central in the exploration. Indeed, DynaGraphs are designed to enable students to experience the dependence relation in terms of direct or indirect movements, to the extent that without moving the independent variable the function itself cannot be identified.

In this paper we focus on a set of exploratory activities representing covariation in functions through artefacts similar to the original DynaGraphs (we will refer to the ones we designed as *dynagraphs*), that were introduced for the first time to a class of Canadian high school students who had not seen this kind of representation before. We describe potentials of these interactive artefacts with respect to the mathematical notion of function by studying how students' written discourse about their interaction with the dynagraphs relates to formal discourse on functions, and in particular how *dynamism* is captured in such discourse.

Assuming that cognitive and communicative processes are closely linked and because of our focus on how students describe their experience in writing, using words and drawings as forms of discourse – we adopt Sfard's communicational framework (Sfard, 2008) to analyze students' productions.

[1] We use "*DynaGraph*" to refer to the original constructions described in the literature and used by Goldenberg, Lewis and O'Keefe, and "*dynagraph*" for the constructions used in this study.

Moreover, we are interested in studying how this lens can be adapted to a context where digital and interactive artifacts, such as dynagraphs, play a central role. Therefore, we will also use this lens to express the dynagraphs' potential to eventually promote formal mathematical discourse. To do this we compare students' written discourse to mathematical discourse on the same functions that might be produced by an expert. We will clarify this approach in the next section, introducing key constructs of the Theory of Commognition that we will be using, and we discuss their specific applications to this study. As we introduce these constructs, we will elaborate on some of them, extending and adapting the theory to our needs. Specifically, we come to a new discursive object, that of *dynamic interactive mediator*, which is central in this study.

2. Dynamic Interactive Mediators (DIMs) from a commognitive perspective

The Theory of *Commognition* (Sfard, 2008) unifies cognitive and communicational processes, conceiving thinking as "an individualized version of interpersonal communication" (Sfard, 2008, p. 81). The word communication includes all communication, not only with others. Indeed, acts of communication can also be with oneself and they can include means that are not only verbal. Communication is considered to be a collectively performed patterned activity in which one action of an individual is followed by a reaction of another individual (or oneself) and the discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with reactions" (Sfard, 2008, p. 297). According to this view, doing mathematics means engaging in the type of communication defined as mathematics and learning means becoming able to access and express this discourse. So, to study students' mathematical learning means to analyze their mathematical discourse, which is characterized by specific *words*, *visual mediators*, *narratives* and *routines*. We now clarify these notions and, at the same time, we introduce some new constructs that are useful for the analysis carried out in this paper.

First of all, mathematical discourse is characterized by the presence of specific <u>words</u>, such as "function", "injective", "injectivity (of a function)", "derivative", "derivability (of a function)", "continuous", "continuity (of a function)"…that are used in experts' discourse, in particular ways. For example, an expert can ask: "Is the function $2x + 3$ injective?" and another expert could answer: "Yes, it is; it is also differentiable, and its derivative is the constant function $f(x) = 2$." The theoretical lens of commognition provides us with the tools to capture, describe and analyze the use of these specific mathematical words in students' discourse. However, for this study, among the interesting words we also consider words and slightly more complex verbal constructs that might not be those of an expert, or that might not be formal, but that are used in "similar" ways. With this we mean that their use is "close enough" to experts' mathematical discourse so that an expert can sensibly (with respect to mathematical discourse) react to them. For example, if a graphical realization of the function 1/x is being discussed, the words: "Around zero y whooshes away!" can be an informal way of saying: "In a neighborhood of zero the function diverges." These informal constructs can be put in relationship with scholarly mathematical discourse, that is experts' discourse in which specific words are used according to the mathematical conventions, within collectively accepted narratives and routines, and possibly accompanied by visual mediators that are endorsed by the community of mathematicians. We believe that these informal constructs can be highly valuable in mathematical learning, because they support a sort of *transitional discourse* that provides important entry points to mathematical discourse for students who are not yet experts, but newcomers to the community of mathematicians. We will return to this idea of transitional discourse in the next section of the paper.

<u>Visual mediators</u> are perceptually accessible objects that are operated upon as a part of the process of communication. While colloquial discourses can usually be mediated by images of material things existing independently from the discourse, mathematical discourses often involve symbolic artifacts

created for the sake of this particular form of communication, for example the writing "$2x + 5$" or a line on the Cartesian plane corresponding to this function. Sfard refers to concrete visual mediation when the mediator is not merely seen, but when it can also be physically manipulated; however, the theory does not go into how such a manipulation might occur or into situations in which such a mediator is also interactive. Indeed, not distinguishing between static and dynamic mediators is a limitation of the theory that has been previously addressed by Ng (2016). For our study, exploring this direction is fundamental and it involves extending the theory in order to explore situations in which the mediators are both *dynamic* – they change over time – and *interactive* – they respond to a person's manipulations. For example, dynagraphs are dynamic and interactive. They are dynamic because the positions of the ticks realizing the variables can move along their axes. They are interactive because they respond to a person's manipulations: the tick realizing the independent variable can be dragged along its axis, and simultaneously the position of the tick realizing the dependent variable varies so that the relationship between the variables is always maintained. Therefore, dynagraphs are an example of *dynamic interactive mediator* (DIM).

Mathematical discourse is special, compared to other discourses, also because it deals with non-accessible objects[2]: a function cannot be accessed directly, unlike, for example, a cow. While a real cow can be seen, touched and interacted with, a function is a purely discursive construct that can only be "seen" through various *realizations* that are perceptually accessible objects that can take the form of spoken or written words, algebraic symbols, drawings, concrete objects and gestures (e.g., a graph, an algebraic expression, a dynagraph…). A realization is not the same as a visual (or dynamic interactive) mediator, because it contains a subjective dimension: a realization of a mathematical object is such for the person carrying out the mathematical discourse, but it may not be a realization of the same mathematical object (or of any mathematical object at all) for another person. This is not the case for a visual mediator or for a DIM, because these can be included as mediators of a discourse even without being realizations of mathematical objects for any of the participants. For example, students who have not yet been introduced to the mathematical object "function" cannot use a Cartesian graph or a dynagraph as a realization of a function; however, their discourse can easily be mediated by one (or both) of these, or it can be a discourse about one (or both) of these. Indeed, a dynagraph is not the realization of a certain function for a student who has not yet learned about functions, even if she is interacting with it, but it is by all means a DIM of her discourse. These DIMs frequently become objects per se of the students' discourse, without, at least initially, realizing any mathematical object.

This is a key idea in our study: an expert can design DIMs that for her are realizations of certain mathematical objects and use them to promote students' discourse that is mediated by these DIMs. We believe that activities that promote discourse mediated by DIMs can be beneficial to mathematical learning, because the emerging discourse mediated by DIMs, a transitional discourse (that we return to below), can be put in relationship with scholarly mathematical discourse. Moreover, such transitional discourse can eventually evolve into mathematical discourse through repeated participation and experts' guidance.

Finally, mathematical discourse is characterized by *narratives* and *routines*, As Sfard writes:

> The overall goal of mathematizing is to produce narratives that can be endorsed, labeled as true, and become known as "mathematical facts." The word *narrative* is used here to

[2] In Sfard's words (2008): "Thus, just as zoology, chemistry, and history can be defined as discourses about animals, chemical substances, and past communities, respectively, so can mathematics be described as a discourse about mathematical objects, such as numbers, functions, sets, and geometrical shapes. The simplicity of this claim is misleading, though, because the notion of mathematical object, unlike that of an animal or chemical substance, is notoriously elusive" (p. 129).

denote any sequence of utterances, spoken or written, framed as a description of objects, of relations between objects, or of activities with or by objects. In colloquial mathematical discourses, narratives are often endorsed on the basis of empirical evidence. Thus, we endorse the equality 2 + 2 = 4 because whenever we put together two pairs of objects and count, the counting ends with the word *four*. At more advanced levels of the colloquial discourse, and at any level of scholarly mathematical discourses, a narrative counts as endorsable if it can be derived according to generally accepted rules from other endorsed narratives. (Sfard 2008, p.223, italics in original)

Therefore *narratives* are descriptions of objects, of relations between objects and of activities with or by objects and include axioms, theorems and definitions. A narrative is endorsable when it "can be endorsed or rejected according to well-defined rules of the given mathematical discourse" (p. 224).

Routines are defined through the notions of task and procedure (Lavie, Steiner & Sfard, 2018). Because of the descriptive and qualitative nature of the requests we made to the participants of this study (they were asked to freely explore and write down their observations about specific DIMs that they were given), we did not use the construct of routine in our analyses.

Discourse with a DIM, about a DIM and mediated by a DIM

Our elaboration of the notion of DIM within the Theory of Commognition leads to an interesting implication: in the original theory discourse can be seen as occurring between two individuals, or between an individual and herself; now, it can also be seen as occurring between an individual and a DIM. We recall that discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with reactions" (Sfard, 2008, p. 297). The reactions of a DIM are those programmed by an expert and, by design, they are in line with those expected of a realization of a certain mathematical object. Moreover, the admissible actions that an individual can carry out are, again by design, a well-defined repertoire. Therefore, it seems meaningful to speak about *discourse with a DIM*, in which the DIM is considered an agent in the discourse. For example, if a student is given a mathematical task such as "Is *ln(x)* ever greater than *5x+3*?", she may turn to a dynamic geometry and algebra system and *ask it* for an answer. The feedback provided by the system may or may not be what the students expects, and it may not even be interpretable to the student. In any case a discourse is taking place.

This is the direction we are exploring in the full-blown study in which DIMs are used to introduce learners to a new (for them) mathematical object (Lisarelli, 2018; Lisarelli et al., in press); what does learners' discourse with a DIM look like? Does the DIM eventually become the realization of a mathematical object for the learner (if so, when and how)? This latter from of discourse can be referred to as *discourse mediated by a DIM*: the DIM is not a participant in the discourse, like a student's "peer" would be, but it is used as an "assistant" to talk about mathematical objects, as a realization of such objects.

Preliminary results have brought our attention to yet another form of discourse that involves DIMs, but in which these are not agents in the discourse, treated as "peers", but objects of the discourse themselves. In this case we speak of *discourse about a DIM*. Both discourse with a DIM and discourse about a DIM seem to be important precursors of scholarly mathematical discourse, characterizing a form of *transitional* discourse. From a teaching perspective, learners' discourse about a well-designed DIM is especially important, because it may provide entry points that the teacher can use to help students access scholarly mathematical discourse. For example, an entry point for speaking about "functions that diverge at a point" could be the "whooshing away" behavior of the dependent tick of a dynagraph, as we described earlier.

To gain further insight into students' discourse about dynagraphs, seen as particular DIMs that for experts can be realizations of functions, we set up the smaller study presented in this paper.

Summarizing, we have, on one hand, scholarly mathematical discourse of functions, and, on the other hand, transitional discourse with and about DIMs, which does not have the same characteristics as the scholarly mathematical discourse (for example it does not share the same set of specific words) but it is by all means a form of discourse. This discourse can be put in relationship with scholarly mathematical discourse, and we expect that the more careful the design of the DIM is the more easily relationships with mathematical discourse can be identified. Moreover, characterizing features of transitional discourse, and in particular discourse about DIMs, can be captured even after it has occurred, for example, in a written narrative. We set out to analyze these written narratives exploring if and how they can be put in relationship with scholarly mathematical discourse. Specifically, in this study, we want to focus on how the accounts of the dynamism which is intrinsic in the discourse with our DIMs allows us to capture key mathematical aspects of the functions realized through the DIMs. In particular, we expect that the dynamism and interactivity of our DIMs will play a key role in identifying discursive constructs that constitute potential entry points into scholarly mathematical discourse, that a teacher could use.

3. RESEARCH QUESTION

We are interested in studying students' emerging discourse in their experiences with DIMs and its relationships with scholarly mathematical discourse. In this paper we focus, specifically, on students' discourse in the context of dynagraphs, seen as DIMs that for experts are realizations of certain functions. Since dynamism is key in unlocking mathematically relevant properties of the functions realized, our analyses aim at characterizing its appearance in students' written narratives. Specifically, we address the following research questions:

How is dynamism captured in students' written narratives about their interaction with the dynagraphs? What are the relationships between the students' written narratives and scholarly mathematical discourse?

4. METHODOLOGY

The data used for this study were collected by the third author, who was interested in seeing how the use of dynagraphs would work in a Canadian setting, since this was the topic of her dissertation, and since she had been introduced to teachers in 10th grade Canadian classrooms who were also graduate students at Simon Fraser University. The researcher, together with these teachers, planned one lesson about dynagraph-activities and implemented it in four 10th grade classrooms. During the class, that lasted 1 hour, students were taught by their regular teacher and the researcher observed and took notes.

The class was organized as follows. Students worked in small groups, composed by two or three students, with one iPad per group and pre-designed interactive Sketchpad files. Each of these files consisted of a dynagraph and a task. They were given a set of 7 dynagraphs and they were asked to: "Experiment with the construction. Describe what you notice. Write down your observations." For some dynagraphs they were also asked the question "How does this compare to the previous dynagraph?". Students explored as many dynagraphs as they had time to, during one lesson period, and then they produced written descriptions that were collected. We collected 154 excerpts, some of which we are going to present and analyze in the next section, and each excerpt contains the whole answer to one dynagraph-activity given by a group of students.

Most of these students had never used Sketchpad before and none of them had ever worked with dynagraphs. Moreover, they had not spent much time studying functions during that school year, although most of them had spoken informally about functions in previous grades.

4.1 Dynagraphs and the DIMs designed for this study

The researchers Goldenberg, Lewis & O'Keefe (1992) realized R → R functions using DynaGraphs where the input and output were represented separately, each on its own (horizontal) visualization of a segment of the real line. These horizontal segments were originally referred to as "the x Line" and "the f(x) Line". The independent variable was draggable along the x Line and the dependent variable would move accordingly on the f(x) Line.

A DynaGraph cannot be constructed in a static environment; objects need to be moved on the screen. In particular, the student can obtain two possible types of movement: indirect and direct. Direct motion occurs when a basic element is dragged by acting directly on it; while indirect motion occurs when there is a construction of elements that depend on the basic elements.

In the case of DynaGraphs the independence of the x variable is realized by the possibility of freely dragging a point, bound to a line (the x Line), and the resulting movement visually mediates the variation of the point within a specific domain. The dependence of the f(x) variable is realized by an indirect motion: dragging the independent variable along its axis causes the motion of a point, bound to another line (the f(x) Line), that cannot be directly dragged. The dependency between two different types of motion, direct and indirect motion, has been investigated in the context of open problems in geometry (Baccaglini-Frank & Mariotti, 2010).We are interested in working with DynaGraphs and in studying how the dependency between such different types of motion, resulting from their manipulation, can be a possible realization of functional dependency. The authors built on the original idea of DynaGraphs to realize functions within the dynamic geometry software Sketchpad. The underlying mathematical relations in the idea of DynaGraph were not changed, we only made some changes in the layout, obtaining what we indicate with the word *dynagraph* and can be seen in Figure 2.

We asked students to write their observations during the explorations of the DIMs, "forcing" them to describe in writing an experience that had a strong dynamic component. Each file contained two horizontal lines with two markers moving on them according to the two types of motion described above. In the first dynagraph these markers had no labels, because we wanted to allow students to decide which words, symbols or gestures to use. This process is part of the act of distinguishing the two variables, which contributes to the development of a discourse on functions. In the following dynagraphs they are labeled A and B respectively. There is also a red segment linking the two markers together, which changes its length and moves according to the movements of the markers along the lines. This red segment was used to highlight possible changes or invariants in the relative positions of the two variables. Moreover, the two points 0 and 1 were marked on the lines to highlight that the lines realize two copies of the real number line.

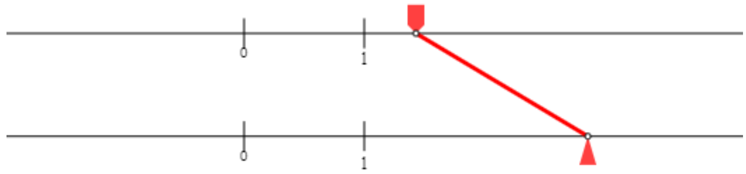


Figure 2. The screenshot of a dynagraph

This dynagraph is a DIM. For an expert it can be seen as a realization of a function; we expect that students, at least initially, refer to it as the main object of their discourse and as a mediator in communication with other students or with the teacher.

Most of the elementary properties of a function can be realized within the dynagraph, as is the case for the functional dependency between the two variables. In this study we asked students to explore and describe 7 dynagraphs that we designed choosing 7 different functions (see Appendix for a screenshot of each file). This allowed us to investigate whether students' written discourse could be put in relationship with scholarly mathematical discourse. For example, when exploring a dynagraph, it is quite easy to recognize whether the movements of the two variables follow the same direction or opposite directions. For example, we used an always-decreasing function that could elicit informal discourse specifically related to the dynagraph such as "the variables always move in opposite directions"; this could be put in relationship with scholarly mathematical discourse like "the function is decreasing". In a similar way, a change in direction of the dependent variable, with the independent one always following the same direction, reveals the presence of a point of maximum or minimum. Students' discourse used to describe such behavior of a dynagraph could be "the dependent marker stops and goes back again", possibly paired with visual mediators like a gesture reproducing the same movement in the air or a drawing on the sheet of paper. This could be put in relationship with scholarly mathematical discourse about a maximum point.

In addition to changes in direction, the dynamic graphs also provide information about the rate of change. For example, moving the independent variable at constant speed along the x-axis can result in constant growth, which the students can feel while actually observing that the dependent variable always has the same increment. Similarly, moving x at a constant speed can result in accelerated growth and the students will see f(x) whisking off the screen. Students' written discourse could be about changes in speed of the markers; and this could be put in relationship with scholarly mathematical discourse on the slope of the function, that is its derivative.

A vertical asymptote is realized in the dynagraphs by disappearing of the marker of the dependent variable as soon as the other marker is dragged over the value where the function is not defined. For example, if the left and right limits have different signs, this marker quickly shoots back from the other side of the screen. This sudden disappearing and reappearing of the marker can be surprising for someone who does not know which function has been defined; very often students produce original narratives to describe such behavior (Sinclair, Healy & Reis Sales, 2009). For example, when students investigate a dynamic realization of the function $f(x)=1/(x-3)$, by moving the x-marker from the far left to the far right, they see the f(x) marker starts very close to zero, moving slowly, and then it begins to race quickly to the left as the x marker approaches 3. When $x = 3$ the f(x) marker zooms back in from the right side to catch up with x and then it drastically slows down and approaches zero. This description is rich in references to space, speed, time and movement, and it can be put in relationship with experts' discourse about several properties of the function, including the presence of both vertical and horizontal asymptotes.

The most defining feature of our DIM is the possibility of dragging. Dragging allows students to see (and even interact with) the behavior of the mathematical objects over time. Moreover, the realization of a function (as a relationship between two variables) through a dynagraph can only be perceived in time, since the covariation can only be seen dynamically, in time, as the ticks realizing the variables move together along their axes. By analyzing students' discourse about this realization of the function we can identify possible connections with scholarly mathematical discourse but also, we can observe how their discourse captures the dynamism characterizing the interaction with the dynagraphs. Indeed, for describing their exploration, which happened over a time interval, in writing, students have to choose what to communicate about that time interval and how to do this in a paper-and-pencil environment.

4.2 Tools of analysis

In this section we present a characterization of students' narratives according to the way they capture the dynamism in writing (section 4.2.1) and to features that capture the specific-generic dialectic (Mason & Pimm, 1984; see section 4.2.2). The result will be a tool of analysis that allows us to gain insight into how students' writing reflects the temporal and dynamic dimensions of their experience with the DIMs. Then we use this tool to analyze 11 excerpts that have been chosen to exemplify recurring combinations of characteristics of students' narratives.

4.2.1 Snapshots, live-photos, and scenes

From analyses of the students' written productions we reached a characterization that was later refined through further rounds of analysis of the excerpts. Consistently with our interest in understanding how movement (the variation of position in time) enters students' written discourse, we introduced a taxonomy of written narratives of the experiences with the DIMs. In Table 1 we define the main types of accounts and describe features of the discourse it characterizes, both in the case in which it is about the dynagraph (which may or may not be related to the notion of function for the students) and when it is about the function.

Table 1: Characterization of students' written narratives

Type of narratives about the experience with DIMs		**Discourse about a DIM**	**Scholarly mathematical discourse about function**
snapshot	<u>snapshot</u>: photo at a certain position showing one or more properties	expresses properties in one selected position in which movement is stopped	properties at a point (point-wise properties)
	<u>snapshot-album</u>: set of snapshots, each showing something in particular	expresses properties in selected positions; these constitute a finite discrete set; each property is shown in one snapshot	pointwise properties at a particular set of points
	<u>snapshot-cluster</u>: set of snapshots showing	expresses one or more properties through a comparison of snapshots of the dynagraph; the focus is on	properties that refer to variation and/or to invariants in particular

	relationships between them	the relationships between snapshots. The cluster may or may not be in a sequence consistent with the order on the line.	intervals identified by analyzing and comparing different points
live-photo	live-photo: animated photo showing what happens a bit before and a bit after a certain instant	expresses one or more properties that depend on the nearby positions and they are described at a certain position; certain positions of the dynagraph are shown close to a specific position	local properties of functions, at a specific point and the variations in any neighborhood of the point.
	live-photo-album: set of live-photos, each showing something in particular	expresses properties in selected positions; these constitute a finite discrete set; each property is shown in one live-photo	local properties in neighborhoods of a particular set of points
	live-photo-cluster: set of live-photos showing relationships between them	expresses one or more properties through a comparison of live-photos; the focus is on the relationships between live-photos.	properties that refer to variation and/or to invariants in particular intervals identified by analyzing and comparing different neighborhoods
scene	scene: video showing movement over a period of time; it can be realized through snapshots, or live-photos, in an album or cluster.	expresses one or more properties of the movement over an arc of time, in an interval of real numbers, or in the space covered by the point moving along the line; it includes various positions. The intervals can be limited or unlimited; the scene can be described statically or dynamically	properties of functions that are global, general, or present in a whole interval

Table 1 shows the taxonomy used as a tool of analysis (snapshot, live-photo and scene) and a further characterization with respect to ways in which more than one could be used (as albums or clusters). The table above includes more types of accounts than we actually identified in the current study. However, theoretically, we do not see why we shouldn't be able to find examples for all 9 types in a larger sample.

4.2.2 The specific-generic dimension

Another characteristic that we analyzed is the reference to *specific* or *generic* objects, in line with the distinction used by Mason and Pimm (1984). The discourse may be about points in space, numbers on the real line, temporal instants that are specific (e.g., "x=1") or generic (e.g., "any point"). Although it is not always possible to characterize students' discourse as about specific or generic objects, frequently there appear to be words or visual mediators that support this distinction. Sometimes, for example, students' sketches of dynagraphs include specific labels (numbers, letters like “x” or “y”, ticks, etc.); or they use words like “no matter how far”, which in formal mathematical discourse could become “for every” or “in every”.

For example, in Figure 3 the student writes: “no matter how far you drag ‘A’” and the markers A and B are not at labeled positions. The excerpt of discourse is constituted of written text and a cluster of snapshots, a visual mediator realizing the property “d(AB)=constant (for every A)”. We recognize the students’ production as representing *generic* objects by noting the absence of markers other than those for the extremes of the interval [0,1]. In this sense, the snapshots are as good as any others with A more to the left or right of the interval, for realizing the property.

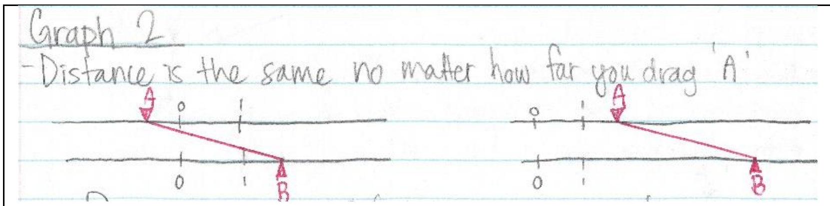


Figure 3. example of a students’ written discourse about a generic property

“Distance is the same no matter how far you drag ‘A’”

This dimension of characterization of students’ written discourse is important because of its relationship to formal mathematical discourse. Being able to identify this characteristic helps gain insight into potentials of using the DIMs we designed to initiate mathematical discourse.

Finally, we observe that a same property of a function can be realized in different ways. For example, the property of “being constant” can be realized as a cluster of snapshots using a sentence like “at every point the function takes on the same value” or as a scene using a sentence like “however I move A, B stays still”; etc..

5. ANALYSIS OF STUDENTS’ WRITTEN PRODUCTIONS

The following excerpts exemplify recurring combinations of characteristics of students’ discourses stimulated by experiences with dynagraphs.

Excerpt 1

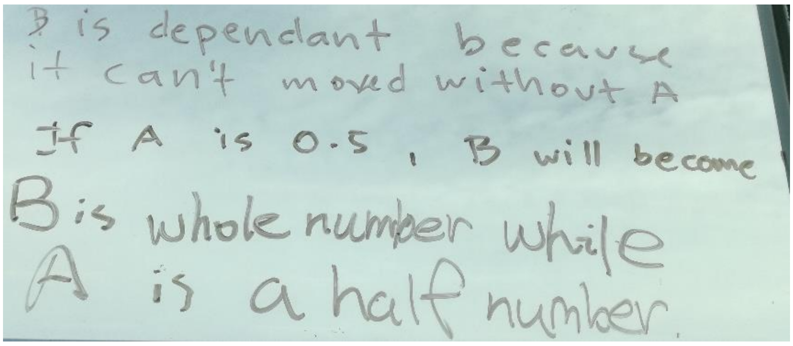
<table>
<tr>
<td>Students’ discourse:


Figure 4. two scenes and a live photo</td>
<td>“B is dependent because it can’t moved without A
If A is 0.5, B will become 1
B is whole number while A is a half number”</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

B = f(A) (B is a function of A)
The limit for x approaching 0.5 equals 1
If n is an odd integer, then f(n/2) is an integer</td>
</tr>
</table>

The students who wrote excerpt 1 (in Figure 4) used only words to describe a dynagraph realizing the nearest integer function. The students depict: 1) a scene, “B is dependent because it can’t [be] moved without A”; 2) a live photo, “if A is 0.5, B will become 1”[3]; 3) a scene, “B is whole number while A is a half number”. Both the scenes are described statically and depict general properties of the movement of the markers in the space covered by each point as it moves along its number line.

The sentence “if A is 0.5, B will become 1” is articulated in two parts: a static antecedent (“if A is 0.5”) and a dynamic consequent (“B will become 1”). The co-presence of the static and dynamic components reveals that the object of the discourse is both the dynagraph at a point (0.5) and the movement (“will become”) in a neighborhood. Indeed, the property described at a certain position depends on the nearby positions. This is why we consider this sentence as a live-photo and not a snapshot.

Moreover, we believe that the two components (the static one and the dynamic one) play a role in the transition towards a reification of the markers into numbers. Indeed, in the first scene, A and B are points on the screen and they can be acted on in different ways. In the last scene, A and B are referred to as “numbers”; they have become subjects in the sentences, and neither time nor dynamism nor interactions with them appear any longer in the discourse (“B is whole number”… “A is a half number”). From a cognitive point of view the two scenes are very different and the transition between them could be an important process in which the live-photo, with its two components, plays a bridging role.

[3] There is a “1” written on the far right of the screen.

Excerpt 2

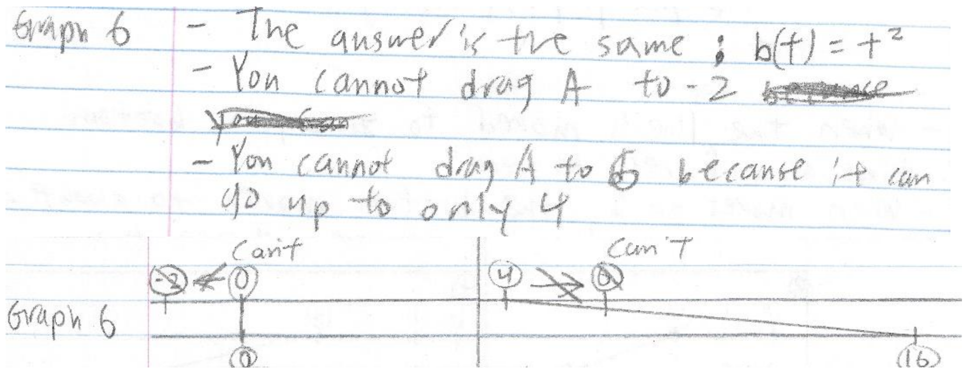
<table>
<tr>
<td>Students' discourse:


Figure 5a,b: live photo presented with written words (a) and visual mediators (b)</td>
<td>"The answer is the same: $b(t) = t^2$
You cannot drag A to -2
You cannot drag A to 6 because it can go up to only 4"</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

If $x < 0$, then x does not belong to the domain of the function.
If $x > 4$, then x does not belong to the domain of the function.</td>
</tr>
</table>

The dynagraph described by the students in Figure 5a, b realizes the function $f(x) = x^2$ with domain restricted to the interval [0,4] (i.e. the point realizing the independent variable is constrained to the segment with endpoints 0 and 4). The students describe what they can or cannot do while manipulating the dynagraph. The description includes two limiting configurations (A at 0 and A at 4) and properties that depend on nearby positions (attempt to drag A to the left of 0 and then to the right of 4), so we classify this written narrative as a live photo. This interpretation is strengthened by the presence of arrows and prohibition signs that indicate two impossible movements for the independent variable in the neighborhood of the point 0 and in the neighborhood of the point 4. Overall the live photo depicts the negation of movements at certain places on the top line; the students chose to also use two numbers, -2 and 6, that we see as generic elements (-2 is used as *any* number less than 0 and 6 as *any* number greater than 4)

Excerpt 3

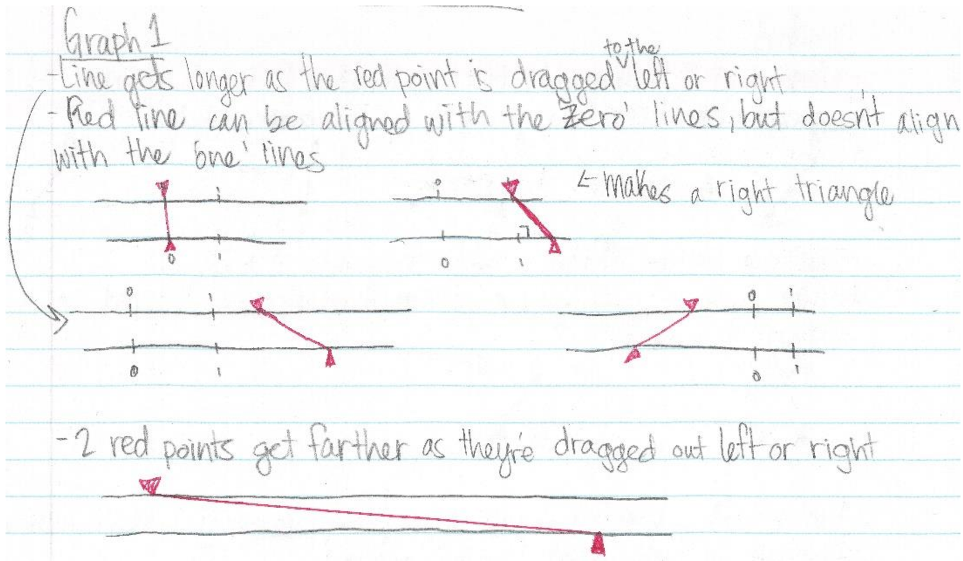
<table>
<tr><td>Students’ discourse:


Figure 6. two albums of snapshots and a scene</td><td>“Line gets longer as the red point is dragged to the left or right.
Red line can be aligned with the zero’ lines, but doesn’t align with the one’ lines.
2 red points get farther as they’re dragged out left or right.”</td></tr>
<tr><td colspan="2">Related scholarly mathematical discourse:

The function $|f(x)-x|$ is decreasing
$f(0) = 0$ and $f(1) \neq 1$</td></tr>
</table>

Excerpt 3 (Figure 6) describes a dynagraph of the function f(x)=2x. The students use words and visual mediators for each of three features that they describe. The words “line gets longer as the red point is dragged to the left or right” depict a scene dynamically, which is accompanied (see the arrow) by the sketches of the dynagraph associated to extreme positions (left and right). The sketches constitute an album of snapshots, because they express properties in selected positions, which constitute a finite discrete set, and each property is shown in one snapshot: each snapshot is realized to show a property, one on the right and one on the left of the [0,1] interval. These snapshots have generic connotations (since no specific ticks or numbers are indicated).

The first feature noticed by the students is described through the words “red line can be aligned with the ‘zero’ lines” and accompanied by the first sketch of the dynagraph at the specific position in which the alignment happens. The second property is depicted through an album of snapshots and is described through the words “red line […] doesn’t align with the ‘one’ lines” and the second sketch shows a generic position other than at zero. Again, the words and visual mediators expressing properties in selected positions, which constitute a finite discrete set, and each property is shown in one snapshot.

Finally, the students consider the “2 red points” that “get farther as they’re dragged left or right”, realizing the scene together with a snapshot that has a generic connotation (there are no explicit markers or numbers on the sketch of the dynagraph). We also notice that the “red line” no longer appears in the written words, but it is drawn in the sketch used as visual mediator.

Excerpt 4

<table>
<tr>
<td>Students’ discourse:

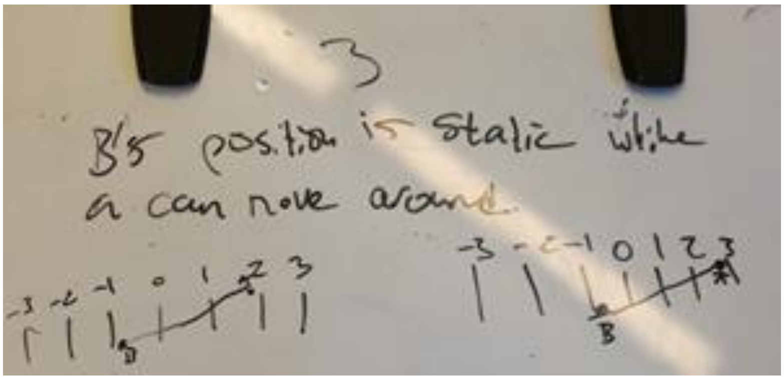



Figure 7. scene described through a cluster of snapshots</td>
<td>“B’s position is static while a can move around”</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

f is a constant function</td>
</tr>
</table>

Figure 7 describes a dynagraph of the constant function f(x) = -1. The students compare the static position of B with A’s possibility of moving from 2 to 3. The sentence “B’s position is static while a [A] can move around” describes in words a general property of the dynagraph; we classified it as a scene. Indeed we find a description of two properties of the movements of A and B in an interval of real numbers. The sketches depict a possible movement of A from 2 to 3 (that could be generic points) and the fact that B is stopped at -1.

The two visual mediators are snapshots that realize properties together: it is the relationship between the two snapshots that realizes the possibility of movement of A and the static position of B. This is why we classify this system of snapshots as a cluster.

Excerpt 5

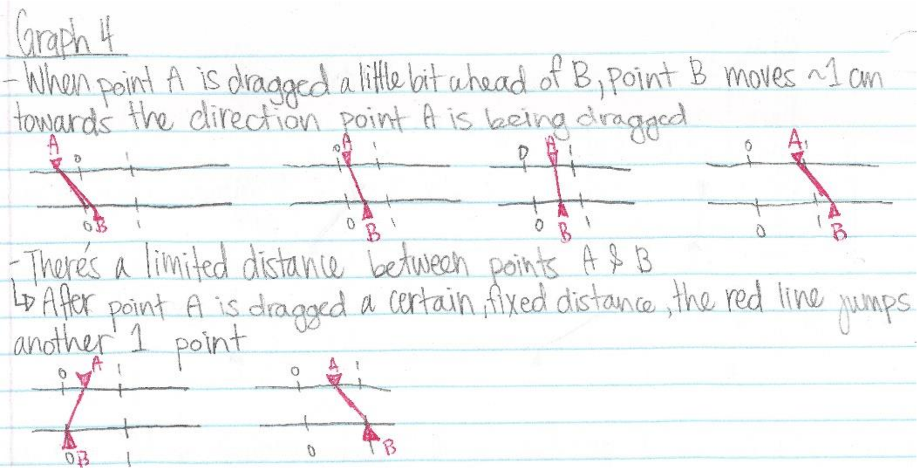
<table>
<tr>
<td>Students' discourse:


Figure 8. scene described through album of snapshots in sequence, and live photo depicted through two snapshots showing 'initial' and 'final' states.</td>
<td>"When point A is dragged a little bit ahead of B, point B moves ~1cm towards the direction point A is being dragged.
There's a limited distance between points A & B. After point A is dragged a certain fixed distance, the red line jumps another 1 point."</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

For every value of B there exists x_0 such that $f(x_0) = B$ and $f(x_0) - \lim_{x \to x_0^-} f(x) = 1$.
There exists M such that |f(x)-x| < M. Moreover, the set of images is discrete.</td>
</tr>
</table>

Figure 8 describes the dynagraph of the nearest integer function. The first part of the students' discourse contains four different visual mediators, following a temporal evolution for A moving in a neighborhood of 0. The words and the visual mediators depict a single scene through an album of snapshots; indeed they express properties in selected positions, that constitute a finite discrete set, and each property is shown in one snapshot. The snapshots show what happens "when point A is dragged a little bit ahead of B".

In the second part of the excerpt in Figure 8, the students describe the jumping of the red line as A is dragged across the midpoint of the interval [0,1]. This part of their discourse appears to be a live photo depicted through two visual mediators that are particular snapshots showing a neighborhood of 0.5.

Excerpt 6

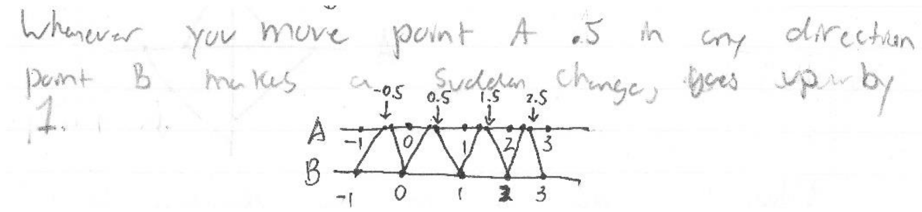
<table>
<tr>
<td>Students' discourse:


Figure 9. album of live photos</td>
<td>"Whenever you move point A .5 in any direction point B makes a sudden change, goes up by 1."</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

If n is an integer, then $\left|\lim_{x\to n+\frac{1}{2}^{+}} f(x) - \lim_{x\to n+\frac{1}{2}^{-}} f(x)\right| = 1$</td>
</tr>
</table>

Excerpt 6 is also about a dynagraph of the nearest integer function. The students draw a single visual mediator depicting the dynagraph's interval [-1,3] on both axes (Figure 9), and its behavior written in words: "whenever you move point A .5 in any direction point B makes sudden changes, goes up by 1." We classify this narrative as live photo album because each one depicts a property that depends on the nearby positions, shown close to each specific position. In this case each live photo shows A near the midpoint of a unit interval. Moreover, each 'sudden change' is shown by a position of the segment AB immediately before and immediately after the transition of A across the midpoints (the ones realized are -0.5, 0.5, 1.5, 2.5) of the unit intervals. The word "whenever" gives this album of live photos a generic connotation.

Excerpt 7

<table>
<tr>
<td>Students’ discourse:

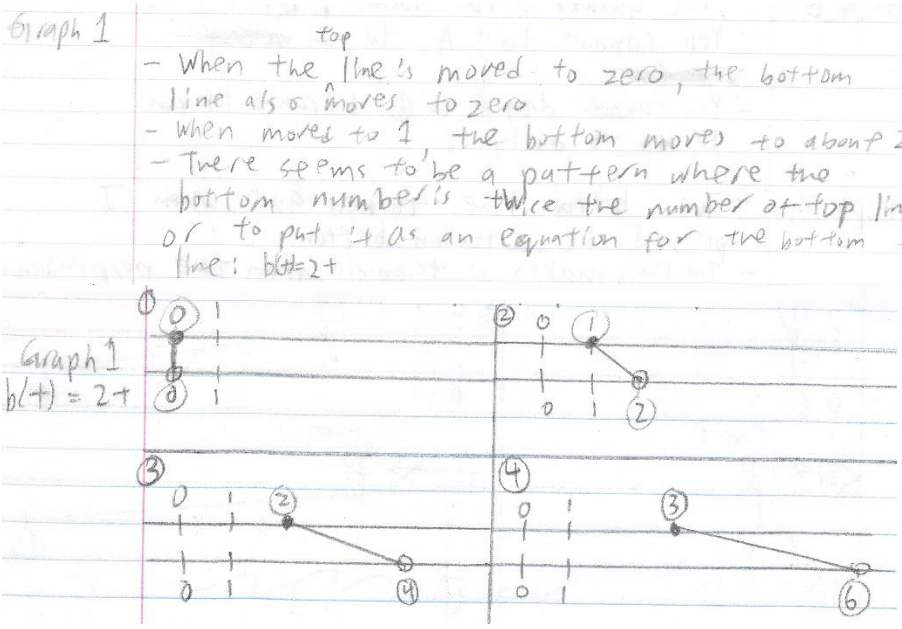



Figure 10 a,b: live-photos and scene realized through snapshots</td>
<td>“When the top line is moved to zero, the bottom line also moves to zero.
When moves to 1, the bottom moves to about 2
There seems to be a pattern where the bottom number is twice the number at top line, or to put it as an equation for the bottom line: b(t)=2t”</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

$\lim_{x \to 0} f(x) = 0$ and f(0) = 0
$\lim_{x \to 1} f(x) = 2$ and f(1) = 1
f(x) = 2x</td>
</tr>
</table>

Excerpt 7 contains discourse, with both text and visual mediators, about the dynagraph of the function f(x) =2x. Figure 10a and 10b depict written text and visual mediators. The first property in **Error! Reference source not found.** is described through a live-photo taken at ‘A close to 0’ (“when the top line is moved to zero, the bottom line also moves to zero”); the first visual mediator in Figure 10b presents this specific instance through a snapshot in 0. The second property is described analogously as a live photo taken at “A close to 1”; indeed, the second visual mediator in Figure 8b presents this specific instance through a snapshot in 1. In other words, the written text refers to the neighborhoods; it consists in live-photos, and it is realized through visual mediators that can be classified as static snapshots.

The third property is a scene described through reified discourse (“b(t)=2t”), with a generic connotation. This scene is presented through two snapshots at specific instants (A=2 and A=3 in Figure 8b) forming a cluster of snapshots. Together the two specific snapshots and the written text convey a generic connotation to this property.

Excerpt 8

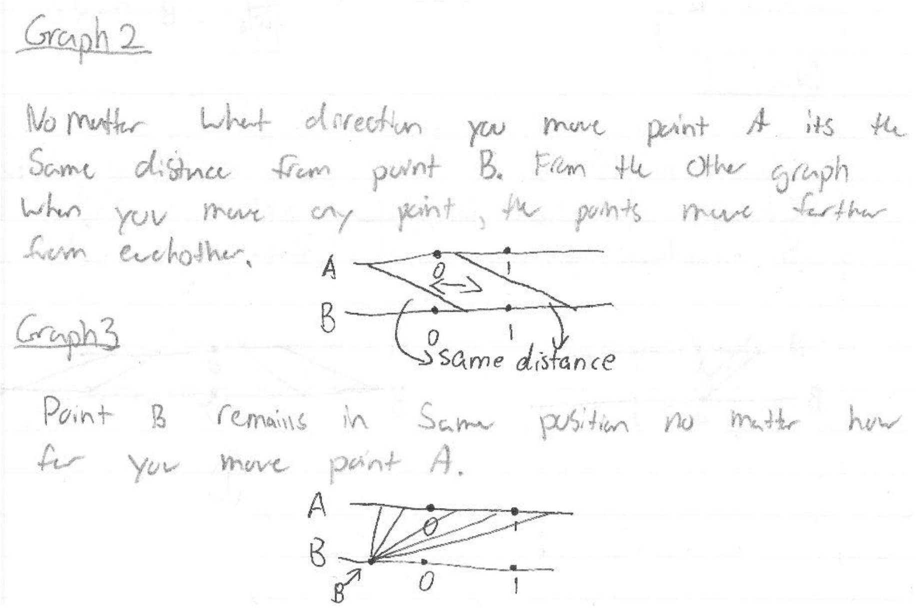
<table>
<tr>
<td>Students’ discourse:


Figure 11. two scenes depicted through clusters of snapshots</td>
<td>“Graph 2: No matter what direction you move point A it’s the same distance from point B. From the other graph when you move any point, the points move further from each other.
Graph 3: Point B remains in same position no matter how far you move point A.”</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

Graph 2: There exists a constant c such that f(x) – x = c.
In other graphs the function f(x)-x is not constant
Graph 3: f is a constant function</td>
</tr>
</table>

The students who author the discourse in excerpt 8 (Figure 11) emphasize the invariance of the distance between A and B (the function is f(x)=x+5), which has a generic connotation, conveyed through the words “no matter what direction” and the two congruent parallel segments in the visual mediator with in between a double-headed arrow and two arrows pointing to the words “same distance”. We classify this visual mediator as a cluster of snapshots because it shows a property described through a comparison of snapshots of the dynagraph, condensed to focus on the relationship between them, which is the invariant property identified.

Analogously, the discourse in the second part of the excerpt 8 (identified as “graph 3” by the students) appears to describe a scene through a cluster of snapshots, this time depicting the generic (notice the words “no matter how”) invariance of the position of B as A varies (“you move point A”). The visual mediator seems to realize a generic set of points.

Excerpt 9

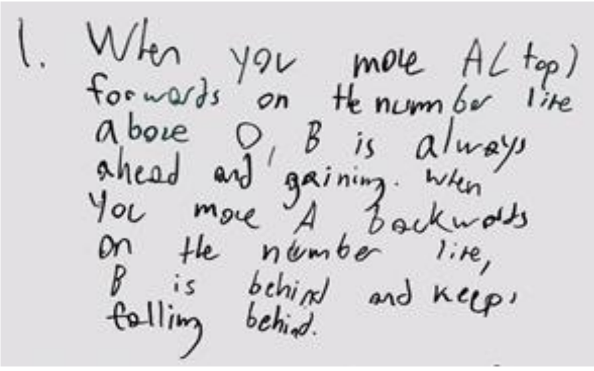
<table>
<tr>
<td>Students' discourse:


Figure 1. scene album</td>
<td>"When you move A (top) forwards on the number line above 0, B is always ahead and gaining. When you move A backwards on the number line, B is behind and keeps falling behind."</td>
</tr>
<tr>
<td colspan="2">Related scholarly mathematical discourse:

If $x > 0$, then $f(x) > x$ and the function $F(x) = f(x) – x$ is increasing.
If $x < 0$, then $f(x) < x$ and the function $F(x) = f(x) – x$ is increasing.</td>
</tr>
</table>

In excerpt 9 (Figure 12) the students use only written words to outline the behavior of a dynagraph of the function $f(x) = 2x$. They consider a whole interval of time (and space) articulated in an album of two scenes. Indeed, the students describe two properties of the movement of the dynagraph as A is dragged in different directions ("A forwards" and "A backwards"). The intervals within which A is dragged are not limited in the description, and the scene appears to be dynamic. The students make several references to possible movements and to the reciprocal position of the two variables ("forwards", "ahead", "backwards", "behind") underlining invariants with the adverb "always". Moreover, the students start with the present tense of the verb "to be" ("B is always ahead […] B is behind"), which gives their statements the form of an absolute observable truth, but then they add details on the quality of movement, passing to the "–ing form" of other verbs, indicating spatial dynamic features ("always […] gaining", "keeps falling behind"). This could suggest that the students treat the movement as not ending even when it cannot be seen anymore within a large interval.

Excerpt 10

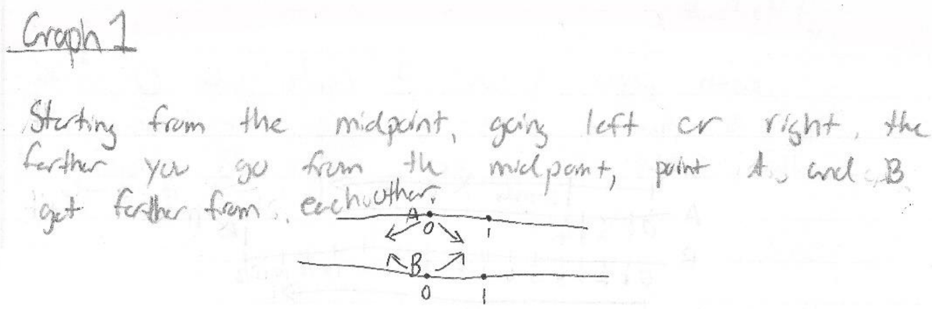
<table>
<tr><td>Students' discourse:


Figure 2. scene album in text and visual mediators</td><td>"Starting from the midpoint, going left or right, the further you go from the midpoint, point A and B get further from each other."</td></tr>
<tr><td colspan="2">Related scholarly mathematical discourse:

If $|x_1|<|x_2|$, then $|f(x_1) - x_1| < |f(x_2) - x_2|$</td></tr>
</table>

On the same dynagraph (f(x) = 2x) as in excerpt 9 (Figure 13) this pair of students speaks about the property: "A and B get farther from each other". The words describe an album of two scenes because they contain the description of two properties of the movement in the dynagraph as A is dragged to the left and to the right from the point 0; the descriptions includes various positions. The description specifically depicts the directions of movement of the two variables "starting from the midpoint [which appears to be 0 from the visual mediator they draw]". The visual mediator drawn conveys a global description of the behavior of the dynagraph, evolving over time, and it shows as a live-photo in a neighborhood of the "midpoint" with A and B at 0 and four arrows beginning at these (two at A and two at B), pointing in opposite directions.

Excerpt 11

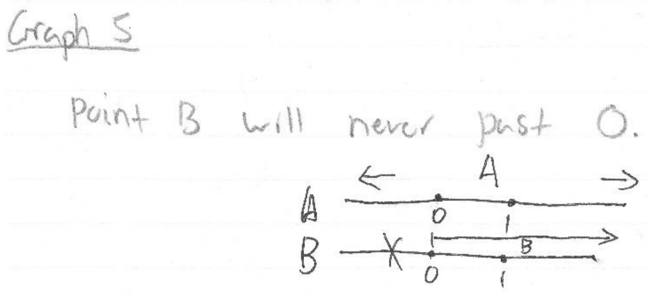
<table>
<tr><td>Students' discourse:


Figure 3. scene album in a visual mediator</td><td>"Point B will never past 0."</td></tr>
<tr><td colspan="2">Related scholarly mathematical discourse:

$f(x) \geq 0$ for every x</td></tr>
</table>

In the excerpt in Figure 14the function is $f(x)= x^2$ (defined for every real number). This is the scene album depicting the generic property "point B will never pass 0". The visual mediator contains arrows next to A, suggesting that this variable moves in both directions along its axis. For describing the path followed by B the students use an arrow and a cross, indicating possible and impossible movements along the axes. A single visual mediator, therefore, contains different scenes collected in an album, respectively identified with the two arrows next to A.

6. CONCLUSION

The analysis of students' written narratives shows that temporality and movement do very frequently characterize students' emergent discourse on functions; and this seems to be supported by their experience with the dynamic realization proposed. In particular, students' use of words, visual mediators and their ways of identifying invariants showed different forms of emerging written discourse. Students use different approaches to realize the dynamism of their interaction with the DIMs, or to bypass it. The analyses show how students' discourses vary quite a bit from one to the other, along the two dimensions identified: the use of snapshots, live photos or scenes; and the generic or specific connotation of the written discourse produced. Our classification in snapshots, live photos and scenes seems to capture well the forms of written discourse collected. Some students fix a particular time section and focus on it; others describe one or more instants together with what immediately precedes and follows them; others try to capture a behavior over a longer period of time (and space). The written discourse can include words alone, visual mediators alone (less common), or a combination of the two. In some cases, the discourse describes not only the possible behaviors of the variables but also impossible ones, using additional visual mediators in the form of arrows or prohibition signs.

We did not expect students' discourses to contain formal mathematical terms – and indeed they mostly did not – but, rather, we expected that by creating a challenge for the students, they would produce a discourse, somehow consistent with that of experts, and find efficient ways to communicate in the specific context of the activity. Similarly, expert mathematicians operate a continuous dialectic between the temporal dimension and the products of objectification during their mathematical discourse. As discussed in Lisarelli (2017), much time and effort in mathematics teaching is employed to build fluency in the use of mathematical terms and in developing a formal mathematical discourse that eliminates the temporal dimension in favor of the objectification. Our claim is that students can engage in reasoning and conjecturing without necessarily working with this type of fluency. This does not mean the formal mathematical vocabulary is less important, but that more time and effort should be devoted to allowing students to discuss and explore using transitional discourse, even though this may not involve formal terms. In other words, in order to develop the dynamic aspects of mathematical discourse in the study of functions and calculus, we believe that introducing students to the mathematical objects through explorations that involve DIMs and then providing situations for them to talk about them can be beneficial.

According to the commognitive framework a narrative can be recognized as a mathematical discourse if it has some specific properties. For example, experts communicate about mathematics by using de-temporalized sentences and specific technical words. Therefore, this lens provided us with the tools to analyze and describe students' discourse (and so their learning) as it was characterized by such properties. However, in this study we have seen many examples of transitional discourse, with and about DIMs, which does not have the same characteristics as mathematical discourse but it is by all means an interesting (from a didactical point of view) form of discourse. In order to express the potential of this discourse, from a mathematical point of view, we spoke about relationships with scholarly mathematical discourse. Identifying such possible relationships allowed us to highlight in the episodes possible entry points into mathematical discourse on functions. More specifically, we expect that the more careful the design of the DIM is the more easily relationships with scholarly mathematical discourse can be identified.

The analytical tool we have introduced allows to characterize features of transitional discourse, and in particular discourse about DIMs, that can be captured in students' written narratives. In the paper we showed examples of how we used this tool to analyze students' discourse about dynagraphs (written narratives about their experience with the DIMs). We believe that the characterization has value at three levels: the cognitive, the didactical and the epistemological levels.

From a cognitive point of view, the tool is relevant because the classification proposed was identified a posteriori, through an empirical analysis of the data collected. Epistemologically, the types of narratives can be put in relation with different mathematical properties of functions in scholarly mathematical discourse, as shown in Table 1. Thinking about properties that are pointwise, local or global, the discourse – even that of experts – that can emerge may have characteristics such as those we have found in students' discourses, focusing on "instants" that capture the behavior of isolated points, systems of points, sequences, neighborhoods of points, points as limits, intervals… Moreover, scholarly mathematical discourse can capture properties at certain points, variations and invariants. For example, a description of what happens in a neighborhood of a point can be conveyed through a live photo, while the behavior at infinity requires a scene, since a neighborhood of infinity can be seen as a scene "from a certain point on", that is, as a live photo at infinity.

From a didactical point of view the classification is significant, assuming that we value promoting various forms of students' discourse, and therefore students' flexibility in constructing such discourse. In this respect, the teacher plays a very important role. Firstly, she can design DIMs that for her are realizations of certain functions and, secondly, she can use them to promote students' discourse that is mediated by these DIMs. We believe that activities that promote discourse mediated by DIMs can be beneficial to mathematical learning, because transitional discourse can be put in relationship with scholarly mathematical discourse. The teacher can interpret such transitional discourse from a mathematical point of view, by establishing relationships with scholarly mathematical discourse about pointwise properties, local or global properties, variations, and invariants. Doing this, the teacher can then promote these forms of discourse through appropriate tasks (e.g., by using the behavior of dynagraphs in certain positions, intervals, etc.), knowing what to expect and how to gradually foster the transition to more formal mathematical discourse.

We note that students' narratives within transitional discourse can be much more varied than the corresponding scholarly mathematical discourse: when thinking of the possible related scholarly mathematical discourse to each of the excerpts analyzed, we found ourselves returning to very similar if not identical mathematical sentences even when the students' narratives were different. We also observe that a specific scholarly mathematical discourse can be put in relationship with different students' discourses about the DIMs. Moreover, interestingly, all students had something (insightful) to write about every dynagraph – this is definitely not the case when students are invited to talk about functions using scholarly mathematical discourse. This phenomenon supports our hypothesis that DIMs can offer excellent entry points into mathematical discourse for most (if not all) students, including those with learning styles that usually lead them to struggle and fail in learning mathematics.

Finally, we believe that one particularly fruitful type of discourse in the context of dynagraphs seems to be that involving transitions from pointwise to local properties, and thus discourse involving live photos. These seem to be interesting directions worth pursuing in future research, some of which is currently under development.

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Declaration of Conflicting Interests

On behalf of all authors, the corresponding author states that there is no conflicts of interest.

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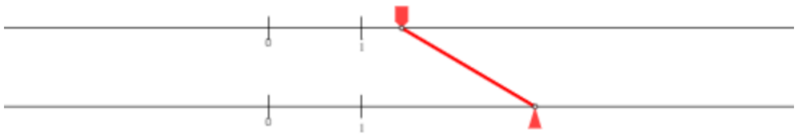
Appendix

Below we provide screenshots of the 7 dynagraphs that we referred to in this study.

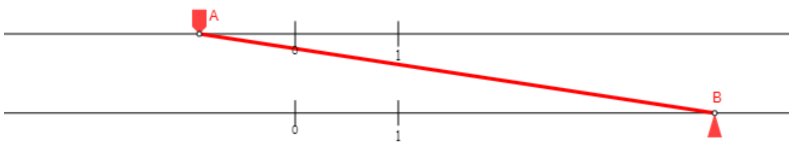
Dynagraph 1: f(x)=2x

Experiment with the construction. Describe what you notice. Write down your observations.



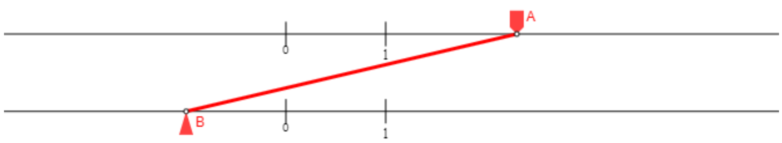
Dynagraph 2: f(x)=x+5

Drag A. Describe what you notice. Write down your observations. How does this compare to the previous dynagraph?



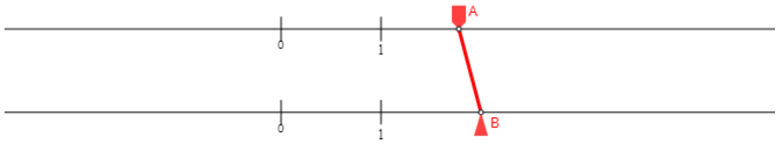
Dynagraph 3: f(x)= -1

Drag A. Describe what you notice. Write down your observations.



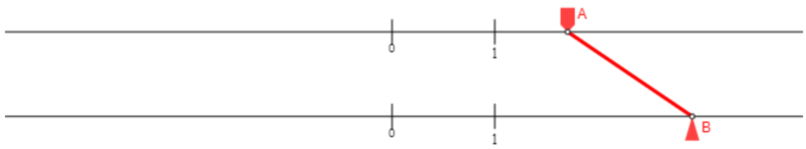
Dynagraph 4: f(x)=round(x)

Drag A. Describe what you notice. Write down your observations.
Can you get B to be 0.5?



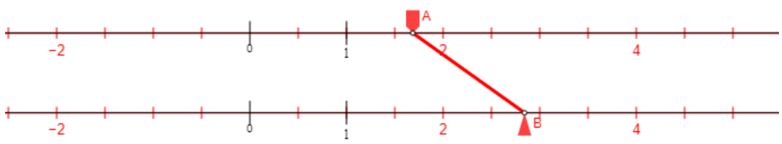
Dynagraph 5: $f(x)=x^2$

Drag A. Describe what you notice. Write down your observations.
How does the behaviour compare to Dynagraph 1?



Dynagraph 6: $f(x)=x^2$ defined in [0, 4]

Drag A. Describe what you notice. Write down your observations.
Can you drag A to -2? Can you drag A to 6?



Dynagraph 7: $f(x)=2x$

Press Make Cartesian to turn the dynagraph into a Cartesian graph.
Drag A. What do you notice?
Press Show Perpendiculars and then Show Intersection then drag A again.
What shape does the trace make when you drag A?