Matrix equations in Markov modulated Brownian motion: theoretical properties and numerical solution

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Abstract

A Markov modulated Brownian motion(MMBM) is a substantial generalization of the classical Brownian Motion and is obtained by allowing the Brownian parameters to be modulated by an underlying Markov chain of environments. As in Brownian Motion, the stationary analysis of the MMBM becomes easy once the distributions of the first passage time between levels are determined. Asmussen (Stochastic Models, 1995) proved that such distributions can be obtained by solving a suitable quadratic matrix equation (QME), while, more recently, Ahn and Ramaswami (Stochastic Models, 2017) derived the distributions from the solution of a suitable algebraic Riccati equation (NARE). In this paper we provide an explicit algebraic relation between the QME and the NARE, based on a linearization of a matrix polynomial. Moreover, we discuss the doubling algorithms such as the structure-preserving doubling algorithm (SDA) and alternating-directional doubling algorithm (ADDA), with shifting technique, which are used for finding the sought of the NARE.

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1 Introduction

A Markov modulated Brownian motion (MMBM), denoted by $(F, J) = \{(F(t), J(t), t \geq 0\}$, is a generalization of the classical Brownian motion. The phase process J of the MMBM is an irreducible continuous-time Markov process with finite state space $S = \{1, \cdots, n\}$, infinitesimal generator Q, and stationary probability vector $\boldsymbol{\pi}$. The level process F of the MMBM is defined as the following stochastic integral

$$F(t) = a + \int_0^t \mu_{J(u)} du + \int_0^t \sigma_{J(u)} dB(u), \quad a \geq 0, \tag{1}$$

where, μ_i are real numbers, $\sigma_i \geq 0$ for all $i \in S$, and $\{B(\cdot)\}$ is a standard Brownian motion independent of J. As defined above, the level process behaves like a Brownian motion, but its drift and diffusion parameters change depending on the specific Markovian environmental state of J.

Since when Asmussen [6] and Rogers [22] introduced the MMBM, there have been many subsequent research efforts, and the model has served as the theoretical basis for research in numerous academic fields including queues, finance, and insurance risk theories. The stationary distribution of the one-sided reflection of the MMBM can be represented by certain matrices of the first passage probabilities of the MMBM. But these matrices cannot be obtained in closed forms unlike for the Brownian motion. Hence, it is necessary to develop efficient numerical methods to compute the matrices.

In particular, in [6], the computation of the distribution is ultimately reduced to solving a quadratic matrix equation (QME) of the form $\Delta_{\boldsymbol{\sigma}^2/2} U^2 + \Delta_{\boldsymbol{\mu}} U + Q = \mathbf{0}$, where $\boldsymbol{\mu} = (\mu_1 \ \cdots \ \mu_n)$, $\boldsymbol{\sigma} = (\sigma_1 \ \cdots \ \sigma_n)$, and $\Lambda = \operatorname{diag}\{-[Q]_{ii}, i \in S\}$ and, throughout the paper, for a given vector $\mathbf{v}$, $\Delta_{\mathbf{v}}$ denotes the diagonal matrix with the elements of $\mathbf{v}$ on its diagonal. Several numerical methods have been suggested in the literature, which are based on Cyclic Reduction [19], the eigendecomposition of a linearization [17], or more generally a block diagonal decomposition [1]. More recently, Nguyen and Poloni [20] proposed an algorithm of our special attention. It is an extension of the algorithm developed by Nguyen and Latouche [19] that is based on the cyclic reduction method and designed for the MMBM with $\boldsymbol{\sigma} > \mathbf{0}$. They proved the componentwise accuracy and stability of their algorithm, and also demonstrated its superiority to other algorithms given in [19, 17, 1] with numerical examples.

As opposed to the previous research, Ahn and Ramaswami [5] proposed a new approach based on a nonsymmetric algebraic Riccati equation (NARE) of the form $AZ + ZB + ZCZ + D = \mathbf{0}$, and showed that the first-passage probabilities can be obtained by using the minimal nonnegative solution of the equation. To the best of our knowledge, this is the first approach in the literature to analyze Markov modulated Brownian motion using an NARE. One of the merits of this approach is that one can apply the doubling algorithms such as the structure-preserving doubling algorithm (SDA, [14, 16]) and alternating-directional doubling algorithm (ADDA, [18, 23]). These doubling algorithms quadratically converge except for the so-called null-recurrent case [13, 15, 23]. Furthermore, one can use the so-called shift [13] technique to improve the speed of convergence to be quadratic even for the null-recurrent case.

The contribution of this paper is twofold. From one hand we provide an algebraic connection between the QME and the NARE, more specifically we show that the NARE can be obtained by means of a linearization of a quadratic matrix polynomial associated with the QME. In this way, we provide an explicit relation between the solutions of the two matrix equations, and a characterization of the solutions in terms of location of the eigenvalues in the complex plane. On the other hand, we discuss the doubling algorithms such as the structure-preserving doubling algorithm(SDA, [8]) and alternating-directional doubling algorithm(ADDA, [16]) which are used for finding the minimal nonnegative solution of the NARE. These algorithms are quadratically convergent except for the null-recurrent case of the MMBM. To improve the speed of convergence of the doubling algorithms in the null recurrent case, we introduce a shifted NARE by applying the shift technique, which was investigated by Guo, Iannazzo, and Meini [13]. We observe that the convergence of the doubling algorithms is accelerated and also quadratic even in the null-recurrent case when they are applied to the shifted NARE, as claimed by Guo, Iannazzo, and Meini. Numerical examples show that the algorithm applying ADDA to the shifted NARE is superior to the other doubling algorithms in comparison. This also holds when compared to Nguyen and Poloni's quadratically convergent algorithm [20] that is based on the quadratic matrix equation obtained by Asmussen.

The remainder of this paper is organized as follows. In Section 2 we recall some definitions and properties of nonnegative matrices and matrix polynomials. In sections 3 and 4 we introduce the NARE in the MMBM with $\boldsymbol{\sigma} > \mathbf{0}$ and $\boldsymbol{\sigma} \geq \mathbf{0}$, respectively, investigate the spectral properties of the solutions and show the relationship with the UQME. In section 5, we recall the doubling algorithms and apply the shift technique in the null recurrent case. Numerical examples are given in Section 6. We provide concluding remarks in Section 7.

2 Preliminaries

In this section we recall some definitions and properties on nonnegative matrices and matrix polynomials.

2.1 Nonnegative matrices

We introduce some relevant definitions and notations. For any matrices $A, B \in \mathbb{R}^{m\times n}$, we write $A \geq B(A > B)$ if $[A]_{ij} \geq [A]_{ij}([A]_{ij} > [B]_{ij})$ for all i, j, where $[A]_{ij}$ denotes the (i, j)-th element of A. Given $A \in \mathbb{C}^{m\times n}$, we denote by $|A|$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ the $m \times n$ real matrix matrix whose (i, j)-th entry is $|[A]_{ij}|$, $\mathrm{Re}([A]_{ij})$ and $\mathrm{Im}([A]_{ij})$, respectively.

The comparison matrix of $A \in \mathbb{C}^{m\times m}$ is the matrix $\widehat{A} \in \mathbb{R}^{m\times m}$ such that

$$[\widehat{A}]_{ij} = \begin{cases} \mathrm{Re}([A]_{ii}), & \text{if } i = j \\ -|[A]_{ij}|, & \text{if } i \neq j. \end{cases}$$

A real square matrix A is called a Z-matrix if all its off-diagonal elements are non-positive. Any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius; it is called a singular M-matrix if $s = \rho(B)$ and a non-singular M-matrix if $s > \rho(B)$.

2.2 Matrix polynomials

We refer the reader to the book [11] for a complete treatment on matrix polynomials.

Definition 1. *A $k \times k$* matrix polynomial *of degree ℓ is a polynomial in the form $P(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$, where the A_j are $k \times k$ matrices. The roots of the polynomial $p(\lambda) = \det(P(\lambda))$ are called eigenvalues of $P(\lambda)$. If $p(\lambda)$ has degree $m < \ell k$, we say that $P(\lambda)$ has $\ell k - m$ eigenvalues at infinity.*

Definition 2. *The $k\ell \times k\ell$ matrix polynomial $A - \lambda B$ is a* linerization *of the $k \times k$ matrix polynomial $P(\lambda)$ if*

$$A - \lambda B = E(\lambda) \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{k(\ell-1)} \end{bmatrix} F(\lambda),$$

where $E(\lambda)$ and $F(\lambda)$ are $k\ell \times k\ell$ matrix polynomials such that $\det(E(\lambda))$, $\det(F(\lambda))$ are different from zero and independent of λ.

From the above definition, it follows that the finite eigenvalues of $A - \lambda B$ coincide with the finite eigenvalues of $P(\lambda)$.

Definition 3. *Let* $P(\lambda) = \lambda^\ell I + \sum_{j=0}^{\ell-1} \lambda^j A_j$ *be a* $k \times k$ *monic matrix polynomial. A pair of matrices* (V, T)*, where* V *is* $k \times k\ell$ *and* T *is* $k\ell \times k\ell$*, is called a* standard pair *for* $P(\lambda)$ *if the following properties hold:*

1. *the matrix*

$$W = \begin{bmatrix} V \\ VT \\ \vdots \\ VT^{\ell-1} \end{bmatrix}$$

   is nonsingular;

2. $\sum_{j=0}^{\ell-1} A_j V T^j + V T^\ell = 0.$

With a $k\ell \times k$ *matrix* U *defined as*

$$U = W^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I \end{bmatrix},$$

the triple (V, T, U) *is called a* standard triple *for* $P(\lambda)$.

We say that (T, U) is a left standard pair for $P(\lambda)$ if (U', T') is a standard pair for $P(\lambda)'$. When needed from the context, we will refer to a standard pair as to a right standard pair. Here, $'$ is the transpose operator.

The following result is Theorem 6.2 in [11]:

Theorem 1. *Let* $P(\lambda)$ *be a* $k \times k$ *monic matrix polynomial of degree* ℓ *and assume there exist matrices* V, T, U *of sizes* $k \times k\ell$, $k\ell \times k\ell$, $k\ell \times k$*, respectively, such that*

$$P(\lambda)^{-1} = V(\lambda I - T)^{-1} U$$

for any λ *such that* $\det(\lambda I - T) \neq 0$*. Then* (V, T, U) *is a standard triple for* $P(\lambda)$.

3 The case of positive diffusion parameters

In this section, we assume that $\sigma_i > 0$ for all $i \in S$. First we recall some results relating the Laplace Stilties transform of the first return time with the solution of a suitable algebraic Riccati equation. Secondly, we with complex transform variable s on the closed right half-plane.

3.1 NARE for the first-passage probability matrix

Referring to the MMBM model (F, J), let $\tau = \inf\{t > 0 : F(t) < 0\}$ and define a transform matrix $\hat{\mathbf{f}}(s, a)$ such that for $a \geq 0$ and a complex number s with non-negative real part

$$[\hat{\mathbf{f}}(s,a)]_{i,j} = E[e^{-s\tau}\chi(J(\tau) = j, \tau < \infty) \mid F(0) = a, J(0) = i],\ \ i, j \in S,$$

where $\chi(\cdot)$ denotes the indicator function. Since starting at time 0 at a level a, during any small interval of time, the Brownian motion visits a infinitely often with probability 1, $\hat{\mathbf{f}}(s, 0) = I$.

When $a > 0$, it is well-known in the literature that $\hat{\mathbf{f}}(s, a) = e^{H(s)a}$ for a $n \times n$ square matrix function $H(s)$ of s. Here, the exponential structure originates from the level crossing argument and the spatial homogeneity of the process (F, J) in its levels. However, in general, it is impossible to get exact formula of $H(s)$.

According to the results of [5], the matrix $H(s)$ can be explicitly related to the solution of the NARE

$$A(s)X + XB(s) + XCX + D = \mathbf{0}, \tag{2}$$

where, with $\mathbf{\Delta}(s) = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + \Delta_{\boldsymbol{\sigma}}^{-1}(2sI + 2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^{2})^{1/2}$, $\Lambda = \mathrm{diag}\{-[Q]_{i,i}, i \in S\}$,

$$\begin{aligned}
A(s) &= \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} - \Delta_{\boldsymbol{\sigma}}^{-1}(2sI + 2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^{2})^{1/2},\\
B(s) &= -\mathbf{\Delta}(s), \quad C = \Delta_{\boldsymbol{\sigma}}^{-1}, \quad D = 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q + \Lambda),
\end{aligned}$$

and set $\mathbf{\Delta} = \mathbf{\Delta}(0)$, $A = A(0)$, $B = B(0)$.

More specifically, the following theorem, which can be found in Theorem 5.1 of Ahn and Ramaswami [5], provides a basic result so that $H(s)$ can be obtained from the minimal nonnegative solution of a NARE.

Theorem 2. *Assume $s \in \mathbb{R}$ and $s \geq 0$, and let $X(s) = \Delta_{\boldsymbol{\sigma}}(H(s) + \mathbf{\Delta}(s))$. Then $X(s)$ is the minimal nonnegative solution of the NARE* (2)*.*

Especially when $s = 0$, letting $H = H(0)$, we have $\hat{\mathbf{f}}(0, a) = e^{Ha}$ and it contains first passage probabilities such that $[e^{Ha}]_{ij} = P[\tau < \infty, J(\tau) = j | F(0) = a, J(0) = i]$, $\ \ i, j \in S$. As for the exponent matrix H, we get the following corollary.

Corollary 1. *Let* $X = \Delta_{\boldsymbol{\sigma}}(H + \Delta)$ *with* $\boldsymbol{\Delta} = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + \Delta_{\boldsymbol{\sigma}}^{-1}(2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}$. *Then,* X *is the minimal nonnegative solution of the following Riccati equation*

$$AZ + ZB + ZCZ + D = \mathbf{0}, \tag{3}$$

where $A = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} - \Delta_{\boldsymbol{\sigma}}^{-1}(2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}$, $B = -\boldsymbol{\Delta}$, $C = \Delta_{\boldsymbol{\sigma}}^{-1}$, *and* $D = 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q + \Lambda)$. □

We note that the dual MMBM is an MMBM modulated by the time-reversed process J^d of J and its drift and diffusion vectors are given as $-\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$. Hence, NARE for H^d can be obtained by substituting $\boldsymbol{\mu}$ and Q with $-\boldsymbol{\mu}$ and $Q^d = \Delta_{\boldsymbol{\pi}}^{-1} Q' \Delta_{\boldsymbol{\pi}}$ in the corollary.

3.2 Properties of the NARE

Let $\mathbf{1}$ be the column vector of 1's of appropriate dimension and let $\boldsymbol{\pi}$ the steady state vector of Q, i.e., the vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi} Q = 0$, $\boldsymbol{\pi}\mathbf{1} = 1$.

In association with the NARE (2), we define

$$M(s) = \begin{pmatrix} -B(s) & -C \\ -D & -A(s) \end{pmatrix}, \quad \mathrm{Re}(s) \geq 0,$$

and use M to denote $M(0)$. The comparison matrix for $M(s)$ is

$$\widehat{M}(s) = \begin{pmatrix} -\mathrm{Re}(B(s)) & -C \\ -D & -\mathrm{Re}(A(s)) \end{pmatrix}.$$

Proposition 1. *(a) When* $s \in \mathbb{R}$ *and* $s > 0$, *the matrix* $M(s)$ *is an irreducible non-singular* M*-matrix. (b) When* $s = 0$, *the matrix* M *is an irreducible singular* M*-matrix. In this case, the left and right eigenvectors corresponding to the eigenvalue* 0 *are given as*

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Delta}^{-1}\Lambda\boldsymbol{\pi}' \\ 0.5\Delta_{\boldsymbol{\sigma}}\boldsymbol{\pi}' \end{pmatrix} \quad \text{and} \quad \mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \boldsymbol{\Delta}\Delta_{\boldsymbol{\sigma}}\mathbf{1} \end{pmatrix},$$

which are unique up to a scalar multiple. (c) When $s \in \mathbb{C}$ *with Re*$(s) \geq 0$ *and* $s \neq 0$, *the comparison matrix* $\widehat{M}(s)$ *is an irreducible non-singular* M*-matrix.*

Proof. We note that $A(s)$, $B(s)$, and C are diagonal matrices. Furthermore, if $s \in \mathbb{R}$, it holds that $A(s) < \mathbf{0}$, $B(s) < \mathbf{0}$, $C > \mathbf{0}$, and $D \geq 0$ for all $s \geq 0$. Since Q is assumed to be an irreducible infinitesimal generator, it is clear that $M(s)$ is an irreducible M-matrix.

(a) Assume $s > 0$. The matrix $-B(s) - C[-A(s)]^{-1}D = 2\Delta_{\boldsymbol{\sigma}}^{-2}[-A(s)]^{-1}(sI - Q)$ is invertible and has nonnegative inverse because the diagonal matrix $-A(s)$ and $(sI - Q)^{-1}$ are both nonnegative. Hence, the inverse of $M(s)$ exists and is given as

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (-A(s))^{-1} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} I \\ (-A(s))^{-1}D \end{pmatrix} (sI - Q)^{-1}[-A(s)]\Delta_{\boldsymbol{\sigma}}^2 \begin{pmatrix} I & C(-A(s))^{-1} \end{pmatrix},$$

which is nonnegative. Therefore, $M(s)$ is a non-singular M-matrix [7].

(b) It is simple arithmetic to verify $\mathbf{u}'M = \mathbf{0}$ and $M\mathbf{v} = \mathbf{0}$ and we omit the proofs. These equations imply that M is a singular M-matrix. In this case, 0 is a simple eigenvalue by the Perron-Frobenius theory, hence the corresponding eigenvectors are unique up to a scalar multiple.

(c) Observe that

$$-\mathrm{Re}(B(s)) = \mathrm{Re}(\boldsymbol{\Delta}) = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + \Delta_{\boldsymbol{\sigma}}^{-1}\mathrm{Re}\left((2sI + 2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}\right).$$

On the other hand, if $x \in \mathbb{R}$ is any positive number, then $\mathrm{Re}\left((x+s)^{1/2}\right) > x^{1/2}$, therefore

$$-\mathrm{Re}(B(s)) > \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + \Delta_{\boldsymbol{\sigma}}^{-1}(2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2} = -B(0).$$

Similarly, we may show that $-\mathrm{Re}(A(s)) > -A(0)$. Since $M(0)$ is a singular M-matrix, and since $\widehat{M}(s) \geq M(0)$, with strict inequality on the diagonal entries, and equality on the off diagonal entries, then $\widehat{M}(s)$ is a nonsingular M-matrix [7].

$\square$

In association with the NARE (2), we define

$$L(s) = \begin{pmatrix} -B(s) & -C \\ D & A(s) \end{pmatrix} \tag{4}$$

	$L(s)$			$B(s)+CX(s)$		$A(s)+X(s)C$	
	$\mathbb{C}_+$	$\mathbb{C}_-$	0	$\mathbb{C}_-$	0	$\mathbb{C}_-$	0
$s>0$	n	n	0	n	0	n	0
$s=0,\quad m>0\ (\mu<0)$	$n-1$	n	1	$n-1$	1	n	0
$s=0,\quad m<0\ (\mu>0)$	n	$n-1$	1	n	0	$n-1$	1
$s=0,\quad m=0\ (\mu=0)$	$n-1$	$n-1$	2	$n-1$	1	$n-1$	1

Table 1: Number of eigenvalues in $\mathbb{C}_+$, $\mathbb{C}_-$ and equal to 0 for the matrices $L(s)$, $B(s)+CX(s)$ and $A(s)+X(s)C$.

and use L to denote $L(0)$. From Theorem 2.1 of [8] we get

$$\widetilde{L}(s)=K(s)^{-1}L(s)K(s)=\begin{bmatrix}-(B(s)+CX(s)) & -C\\ 0 & A(s)+X(s)C\end{bmatrix},\quad K(s)=\begin{bmatrix}I & 0\\ X(s) & I\end{bmatrix}. \tag{5}$$

In the case $s=0$, define $m=\mathbf{u}_1'\mathbf{v}_1-\mathbf{u}_2'\mathbf{v}_2$, where $\mathbf{u}_i$, $\mathbf{v}_i$, $i=1,2$, are given in Proposition 1. Through simple arithmetic, we can show that

$$m=\mathbf{u}_1'\mathbf{v}_1-\mathbf{u}_2'\mathbf{v}_2=-\sum_{i\in S}[\boldsymbol{\pi}]_i[\boldsymbol{\mu}]_i.$$

Note that $-m$ is the average drift of the MMBM (F,J). Here, $[\mathbf{a}]_i$ denotes the i-th element of a vector $\mathbf{a}$.

From the results of [8, Section 2.1.2] and from [18] we derive the following result that extends Theorem 2:

Theorem 3. *Let $s\in\mathbb{C}$ have nonnegative real part. (a) When $s\in\mathbb{R}$ the NARE* (2) *has a minimal nonnegative solution $X(s)$. Moreover, $X(s)$ is the unique solution such that $\sigma(B(s)+CX(s))\subset\mathbb{C}_-\cup\{0\}$ and $\sigma(A(s)+X(s)C)\subset\mathbb{C}_-\cup\{0\}$. More specifically, according to the positivity of s and of m, when $s=0$, the eigenvalues of the above matrices are located as according to Table 1.*

(b) When $s\in\mathbb{C}\setminus\mathbb{R}$ the NARE (2) *has a unique solution $X(s)$ such that $|X(s)|\le X(0)$. Moreover, $X(s)$ is the unique solution such that $\sigma(B(s)+CX(s))\subset\mathbb{C}_-$ and $\sigma(A(s)+X(s)C)\subset\mathbb{C}_-$.*

Proof. Part (a) follows from parts (a) and (b) of Proposition 1 and from the results of [8, Section 2.1.2]. Part (b) follows from Proposition 1 and from Theorem 3.1 of [18]. □

In the following, $X(s)$ will denote the solution of the NARE (2) characterized by Theorem 3.

From Theorem 2 we obtain that $H(s)=\Delta_{\boldsymbol{\sigma}}^{-1}(X(s)-\boldsymbol{\Delta}(s))$, i.e., $H(s)=B(s)+CX(s)$. Therefore the eigenvalues of $H(s)$ lie in the (closed) left half plane, according to Table 1.

3.3 NARE and quadratic matrix polynomials

In this section, by using properties of matrix pencils and matrix polynomials, we explicitly relate the solutions of the quadratic matrix equations

$$\Delta_{\boldsymbol{\sigma}^2/2} Z^2 + \Delta_{\boldsymbol{\mu}} Z + Q - sI = 0, \tag{6a}$$

$$Z^2 \Delta_{\boldsymbol{\sigma}^2/2} + Z\Delta_{\boldsymbol{\mu}} + Q - sI = 0, \tag{6b}$$

for $\mathrm{Re}(s) \geq 0$, and the solution $X(s)$ of the Riccati equation (2).

In particular, the solution of interest of (6) is the matrix having eigenvalues in the (closed) left half complex plane, and such solution is used to compute the invariant density of the Markov-modulated Brownian motion.

Define $D_1 = \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}$, $D_2(s) = \Delta_{\boldsymbol{\sigma}}^{-1}(2sI + 2\Lambda + \Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}^2)^{1/2}$, so that $A(s) = D_1 - D_2(s)$, $B(s) = -D_1 - D_2(s)$, and the matrix $L(s)$ in (4) can be written as

$$L(s) = \begin{pmatrix} D_1 + D_2(s) & -\Delta_{\boldsymbol{\sigma}}^{-1} \\ 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q + \Lambda) & D_1 - D_2(s) \end{pmatrix}.$$

Theorem 4. *The matrix pencil $W(\lambda) = \lambda I - L(s)$ can be factored as*

$$W(\lambda) = E(\lambda) \begin{bmatrix} P(\lambda) & 0 \\ 0 & I \end{bmatrix} F(\lambda), \tag{7}$$

with

$$E(\lambda) = \begin{bmatrix} 0 & I \\ -I & (\lambda I - (D_1 - D_2(s))\Delta_{\boldsymbol{\sigma}} \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \\ \lambda I - (D_1 + D_2(s)) & \Delta_{\boldsymbol{\sigma}}^{-1} \end{bmatrix}$$

where $P(\lambda) = \lambda^2 I - 2\lambda\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1}$. Moreover, $W(\lambda)$ is a linearization of the matrix polynomial $P(\lambda)$.

Proof. The factorization (7) of $W(\lambda)$ can be proved by direct inspection: the right hand-side in (7) is equal to

$$\begin{bmatrix} 0 & I \\ -P(\lambda) & (\lambda I - (D_1 - D_2(s))\Delta_{\boldsymbol{\sigma}} \end{bmatrix} \begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \\ \lambda I - (D_1 + D_2(s)) & \Delta_{\boldsymbol{\sigma}}^{-1} \end{bmatrix} = \begin{bmatrix} \lambda I - (D_1 + D_2(s)) & \Delta_{\boldsymbol{\sigma}}^{-1} \\ Q(\lambda) & \lambda I - (D_1 - D_2(s)) \end{bmatrix}$$

where

$$Q(\lambda) = -P(\lambda)\Delta_{\boldsymbol{\sigma}} + (\lambda I - (D_1 - D_2(s)))\Delta_{\boldsymbol{\sigma}}(\lambda I - (D_1 + D_2(s)).$$

Since $D_1^2 - D_2(s)^2 = -2\Delta_{\boldsymbol{\sigma}}^{-2}(sI + \Lambda)$, we may easily conclude that $Q(\lambda) = -2\Delta_{\boldsymbol{\sigma}}^{-1}(Q + \Lambda)$, so that (7) holds. Since $\det(E(\lambda)) = \det(F(\lambda)) = 1$, then $W(\lambda)$ is a linearization of the matrix polynomial $P(\lambda)$. $\square$

The following result gives more insights between the solution $X(s)$ of the NARE (2) and the solutions of the matrix equations (6).

Theorem 5. *The matrices $R_1(s) = B(s) + CX(s)$ and $R_2(s) = -\Delta_{\boldsymbol{\sigma}}(A(s) + X(s)C)\Delta_{\boldsymbol{\sigma}}^{-1}$ are solutions of the quadratic matrix equations* (6a) *and* (6b)*, respectively. Moreover, $R_1(s)$ is the solution of* (6a) *having as eigenvalues the n rightmost eigenvalues of $P(\lambda)$, while $R_2(s)$ is the solution of* (6b) *having as eigenvalues the n rightmost eigenvalues of $P(\lambda)$.*

Proof. From (7) we deduce that, for any λ such that $\det P(\lambda) \neq 0$,

$$\begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \\ \lambda I - (D_1 + D_2(s)) & \Delta_{\boldsymbol{\sigma}}^{-1} \end{bmatrix} W(\lambda)^{-1} \begin{bmatrix} 0 & I \\ -I & (\lambda I - (D_1 - D_2(s))\Delta_{\boldsymbol{\sigma}} \end{bmatrix} = \begin{bmatrix} P(\lambda)^{-1} & 0 \\ 0 & I \end{bmatrix},$$

so that

$$\begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \end{bmatrix} W(\lambda)^{-1} \begin{bmatrix} 0 \\ -I \end{bmatrix} = P(\lambda)^{-1}.$$

Hence, for Theorem 1, the triple $(V, L(s), U)$, where $V = \begin{bmatrix} \Delta_{\boldsymbol{\sigma}} & 0 \end{bmatrix}$ and $U = \begin{bmatrix} 0 \\ -I \end{bmatrix}$, is a standard triple for $P(\lambda)$. Therefore, the pair $(V, L(s))$ is left standard pair, while the pair $(L(s), U)$ is a right standard pair. From the definition of standard pair and from the expression of $P(\lambda)$, we obtain

$$VL(s)^2 - 2\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}VL(s) + 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1}V = 0, \tag{8a}$$

$$L(s)^2U - 2L(s)U\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2U\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1} = 0. \tag{8b}$$

Therefore, multiplying (8a) on the right by $K(s)$ and (8b) on the left by $K(s)^{-1}$, where $K(s)$ is defined in (5), yields

$$V\widetilde{L}(s)^2 - 2\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}}V\widetilde{L}(s) + 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1}V = 0, \tag{9a}$$

$$\widetilde{L}(s)^2U - 2\widetilde{L}(s)U\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2U\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1} = 0, \tag{9b}$$

since $VK(s) = V$ and $K(s)^{-1}U = U$. From the structure (5) of $\widetilde{L}(s)$, equations (9a) and (9b) imply that

$$\Delta_{\boldsymbol{\sigma}}(B(s) + CX(s))^2 + 2\Delta_{\boldsymbol{\sigma}}^{-1}\Delta_{\boldsymbol{\mu}}(B(s) + CX(s)) + 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI) = 0,$$

$$(A(s) + X(s)C)^2 - 2(A(s) + X(s)C)\Delta_{\boldsymbol{\sigma}}^{-2}\Delta_{\boldsymbol{\mu}} + 2\Delta_{\boldsymbol{\sigma}}^{-1}(Q - sI)\Delta_{\boldsymbol{\sigma}}^{-1} = 0.$$

From these two equalities we conclude that $R_1(s)$ and $R_2(s)$ solve the matrix equations (6a) and (6b), respectively. The properties of the eigenvalues follow from Table 1 and from the fact that the eigenvalues of $P(\lambda)$ coincide with the eigenvalues of $L(\lambda)$ for Theorem 4. $\square$

4 The case of nonnegative diffusion parameters

In this section, we discuss how to extend the results in Section 3 to the MMBM which include linear states, of which the diffusion parameter σ_i can be zero.

4.1 NARE for the first passage probability matrix

Let $J = \{J(t), t \geq 0\}$ be a continuous time, irreducible Markov process; this Markov process will modulate the environment in which the MMBM operates. We assume that the state space S of J is finite and partitioned into certain subsets as $S = S_b \cup S_u \cup S_d \cup S_0$. Furthermore, we assume that the infinitesimal generator Q of the Markov process is block partitioned correspondingly as

$$Q = [Q_{l,m}, l, m = b, u, d, 0].$$

and has the partitioned stationary probability vector $\boldsymbol{\pi} = (\boldsymbol{\pi}_b\ \boldsymbol{\pi}_u\ \boldsymbol{\pi}_d\ \boldsymbol{\pi}_0)$ satisfying the equations $\boldsymbol{\pi}Q = 0$ and $\boldsymbol{\pi}\mathbf{1} = 1$. Associated with this Markov process, we define a level process $\{F(t) :\ t \geq 0\}$ through the stochastic integral equation (1), where (i) $\sigma_i > 0$ for $i \in S_b$, and $\sigma_i = 0$ for $i \in S_u \cup S_d \cup S_0$; and (ii) $\mu_i > 0$ for $i \in S_u$, $\mu_i < 0$ for $i \in S_d$, and $\mu_i = 0$ for $i \in S_0$.

We let $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_b\ \mathbf{0}_u\ \mathbf{0}_d)$ and $\boldsymbol{\mu} = (\boldsymbol{\mu}_b\ \boldsymbol{\mu}_u\ \boldsymbol{\mu}_d)$, where $\boldsymbol{\sigma}_b = (\sigma_i,\ i \in S_b)$, and the vectors $\boldsymbol{\mu}_b$, $\boldsymbol{\mu}_u$, and $\boldsymbol{\mu}_d$ to denote the row vectors composed of the drift coefficients μ_i of the MMBM provided for S_b, S_u, and S_d respectively. For later use, we define $\Sigma = \Delta_{\boldsymbol{\sigma}_b}$.

Referring to the MMBM models (F, J), let $\tau = \inf\{t > 0 : F(t) < 0\}$ and define a transform matrix $\hat{\mathbf{f}}(s, a)$ such that for $a \geq 0$ and a complex number s with non-negative real part

$$[\hat{\mathbf{f}}(s,a)]_{i,j} = E[e^{-s\tau}\chi(J(\tau) = j, \tau < \infty) \mid F(0) = a, J(0) = i],\ i, j \in S.$$

Then, for $a > 0$, the submatrix $\left([\hat{\mathbf{f}}(s,a)]_{i,j}, i, j \in S_b \cup S_d\right) = e^{H(s)a}$ with $H(s)$ being a $|S_b + S_d|$-dimensional square matrix function of s.

For further description of $H(s)$, we define $D_1 = \Sigma^{-2}\Delta_{\boldsymbol{\mu}_b}$, $D_2(s) = \Sigma^{-1}(2sI + 2\Lambda_b + \Sigma^{-2}\Delta^2_{\boldsymbol{\mu}_b})^{1/2}$, and

$$Q_{l,m}(s) = Q_{l,m} + Q_{l,0}(sI - Q_{0,0})^{-1}Q_{0,m} \text{ for } l, m = b, u, d. \tag{10}$$

Then, the matrix $H(s)$ is related to the minimal nonnegative solution of the NARE

$$A(s)Z + ZB(s) + ZC(s)Z + D(s) = \mathbf{0}, \tag{11}$$

where

$$\begin{aligned} A(s) &= \begin{pmatrix} D_1 - D_2(s) & 2\Sigma^{-1}Q_{b,u}(s) \\ \mathbf{0} & \Delta^{-1}_{\boldsymbol{\mu}_u}[Q_{u,u}(s) - sI] \end{pmatrix}, B(s) = \begin{pmatrix} -D_1 - D_2(s) & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,b}(s) & -\Delta^{-1}_{\boldsymbol{\mu}_d}[Q_{d,d}(s) - sI] \end{pmatrix}, \\ C(s) &= \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,u}(s) \end{pmatrix},\ D(s) = \begin{pmatrix} 2\Sigma^{-1}(Q_{b,b}(s) + \Lambda_b) & 2\Sigma^{-1}Q_{b,d}(s) \\ \Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,b}(s) & \Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,d}(s) \end{pmatrix}. \end{aligned}$$

We use A, B, C, and D to denote $A(0), B(0), C(0)$, and $D(0)$, respectively.

The following theorem, which can be found in Theorem 5.1 of Ahn and Ramaswami [5], provides basic results on probabilistic meanings of the minimal nonnegative solution of the NARE (11), and also its relation with $H(s)$.

Theorem 6. *Assume $s \geq 0$ and let $X(s)$ be the minimal nonnegative solution of the NARE* (11)*. Then,* *(a) $H(s)$ satisfies that $H(s) = B(s) + C(s)X(s)$, and (b) for $i \in S_u$ and $j \in S_b \cup S_d$,*

$$[X(s)]_{i,j} = E[e^{-s\tau}\chi(J(\tau) = j, \tau < \infty) \mid F(0) = 0, J(0) = i].$$

Especially when $s = 0$, letting $H = H(0)$, we have $\left([\hat{\mathbf{f}}(0,a)]_{i,j}, i, j \in S_b \cup S_d\right) = e^{Ha}$, which contains first passage probabilities such that $[e^{Ha}]_{ij} = P[\tau < \infty, J(\tau) = j | F(0) = a, J(0) = i]$ for $i, j \in S_b \cup S_d$. In this case, we get the following corollary.

Corollary 2. *Let $X = X(0)$. Then, (a) H satisfies that $H = B + CX$, and (b) for $i \in S_u$ and $j \in S_b \cup S_d$, $[X]_{i,j} = P[J(\tau) = j, \tau < \infty \mid F(0) = 0, J(0) = i]$.*

4.2 Properties of the NARE

The matrix corresponding to the NARE (11) is

$$M(s) = \begin{pmatrix} -B(s) & -C(s) \\ -D(s) & -A(s) \end{pmatrix}, \; Re(s) \geq 0,$$

and M is used to denote $M(0)$. The comparison matrix for $M(s)$ is

$$\widehat{M}(s) = \begin{pmatrix} -\mathrm{Re}(B(s)) & -C \\ -D & -\mathrm{Re}(A(s)) \end{pmatrix}.$$

The proof of following proposition is similar to that of Proposition 1 and is omitted.

Proposition 2. *(a) When $s > 0$, the matrix $M(s)$ is an irreducible non-singular M-matrix. (b) When $s = 0$, the matrix $M = M(0)$ is an irreducible singular M-matrix. In this case, with*

$$\mathbf{\Delta} = \Sigma^{-2}\Delta_{\boldsymbol{\mu}_b} + \Sigma^{-1}(2\Lambda_b + \Sigma^{-2}\Delta^2_{\boldsymbol{\mu}_b})^{1/2},$$

the left and right eigenvectors $\mathbf{u}$ *and* $\mathbf{v}$ *corresponding to the eigenvalue* 0 *of* M *are given as*

$$\mathbf{u} := \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} \boldsymbol{\Delta}^{-1}\Lambda_b \boldsymbol{\pi}'_b \\ -\Delta_{\boldsymbol{\mu}_d}\boldsymbol{\pi}'_d \end{bmatrix} \\ \\ \begin{bmatrix} 0.5\Sigma\boldsymbol{\pi}'_b \\ \Delta_{\boldsymbol{\mu}_u}\boldsymbol{\pi}'_u \end{bmatrix} \end{pmatrix} \quad \textit{and} \quad \mathbf{v} := \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} \mathbf{1}_b \\ \mathbf{1}_d \end{bmatrix} \\ \\ \begin{bmatrix} \boldsymbol{\Delta}\Sigma\mathbf{1}_b \\ \mathbf{1}_u \end{bmatrix} \end{pmatrix},$$

which are unique up to a scalar multiple. Here, the notation $\mathbf{1}_k$ *with* $k \in \{b, u, d\}$ *is used to denote the* $|S_k|$*-dimensional vector of* 1*'s. (c) When* $s \in \mathbb{C}$ *with* $Re(s) \geq 0$ *and* $s \neq 0$*, the comparison matrix* $\widehat{M}(s)$ *is an irreducible non-singular* M*-matrix.*

In association with the NARE (11), we define

$$L(s) = \begin{pmatrix} -B(s) & -C(s) \\ D(s) & A(s) \end{pmatrix}$$

and use L to denote $L(0)$. We also get from Theorem 2.1 of [8] that

$$\widetilde{L}(s) = K(s)^{-1}L(s)K(s) = \begin{bmatrix} -(B(s) + C(s)X(s)) & -C(s) \\ 0 & A(s) + X(s)C(s) \end{bmatrix}, \quad K(s) = \begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix}. \tag{12}$$

In the case $s = 0$, define $m = \mathbf{u}'_1\mathbf{v}_1 - \mathbf{u}'_2\mathbf{v}_2$, where $\mathbf{u}_i$, $\mathbf{v}_i$, $i = 1, 2$, are given in Proposition 2. Through simple arithmetic, we can also show that

$$m = \mathbf{u}'_1\mathbf{v}_1 - \mathbf{u}'_2\mathbf{v}_2 = -\sum_{i\in S}[\boldsymbol{\pi}]_i[\boldsymbol{\mu}]_i.$$

Note that $-m$ is the average drift of the MMBM (F, J).

From the results of [8, Section 2.1.2] and from [18] we extend Theorem 6 and derive the following result, of which the proof is similar to that of Theorem 3 and is omitted.

Theorem 7. *Let* $s \in \mathbb{C}$ *have nonnegative real part. (a) When* $s \in \mathbb{R}$ *the NARE* (11) *has a minimal nonnegative solution* $X(s)$*. Moreover,* $X(s)$ *is the unique solution such that* $\sigma(B(s) + CX(s)) \subset \mathbb{C}_- \cup \{0\}$ *and* $\sigma(A(s) + X(s)C) \subset \mathbb{C}_- \cup \{0\}$*. More specifically, according to the positivity of* s *and of* m*, when* $s = 0$*,*

	$L(s)$			$B(s)+CX(s)$		$A(s)+X(s)C$	
	$\mathbb{C}_+$	$\mathbb{C}_-$	0	$\mathbb{C}_-$	0	$\mathbb{C}_-$	0
$s>0$	n_b+n_d	n_b+n_u	0	n_b+n_d	0	n_b+n_u	0
$s=0,\quad m>0\ (\mu<0)$	n_b+n_d-1	n_b+n_u	1	n_b+n_d-1	1	n_b+n_u	0
$s=0,\quad m<0\ (\mu>0)$	n_b+n_d	n_b+n_u-1	1	n_b+n_d	0	n_b+n_u-1	1
$s=0,\quad m=0\ (\mu=0)$	n_b+n_d-1	n_b+n_u-1	2	n_b+n_d-1	1	n_b+n_u-1	1

Table 2: Number of eigenvalues in $\mathbb{C}_+$, $\mathbb{C}_-$ and equal to 0 for the matrices $L(s)$, $B(s)+CX(s)$ and $A(s)+X(s)C$. Here, n_b, n_u, and n_d denote the number of states in S_b, S_u, and S_d, respectively.

the eigenvalues of the above matrices are located as according to Table 2.

(b) When $s\in\mathbb{C}\setminus\mathbb{R}$ the NARE (11) *has a unique solution $X(s)$ such that $|X(s)|\le X(0)$. Moreover, $X(s)$ is the unique solution such that $\sigma(B(s)+CX(s))\subset\mathbb{C}_-$ and $\sigma(A(s)+X(s)C)\subset\mathbb{C}_-$.*

In the following, $X(s)$ will denote the solution of the NARE (11) characterized by Theorem 7.

From Theorem 6 we obtain that $H(s)=B(s)+CX(s)$, hence the eigenvalues of $H(s)$ lie in the (closed) left half plane, according to Table 2.

4.3 NARE and quadratic matrix polynomials

In this section, by using properties of matrix pencils and matrix polynomials, we explicitly relate the solutions of the quadratic matrix equations given in Nguyen and Poloni [20] and the solution $X(s)$ of the NARE (11).

Let $Q^r(s)$ be the block-partitioned matrix composed of the submatrices $Q_{l,m}(s)$ in (10) such that $Q^r(s)=[Q_{l,m}(s),l,m=b,d,u]$. We define $\boldsymbol{\kappa}_s(\lambda)=\lambda^2\Delta^r_{\boldsymbol{\sigma}^2/2}-\lambda\Delta^r_{\boldsymbol{\mu}}+Q^r(s)-sI$ with $\Delta^r_{\boldsymbol{\sigma}^2/2}=\mathrm{diag}\{\Delta^2_{\boldsymbol{\sigma}_b}/2,\mathbf{0},\mathbf{0}\}$ and $\Delta^r_{\boldsymbol{\mu}}=\mathrm{diag}\{\Delta_{\boldsymbol{\mu}_b},\Delta_{\boldsymbol{\mu}_d},\Delta_{\boldsymbol{\mu}_u}\}$, and denote $\boldsymbol{\kappa}_0(\lambda)$ simply by $\boldsymbol{\kappa}(\lambda)$. We note that the (i,j)-th element of the matrix exponential $e^{t\boldsymbol{\kappa}(\lambda)}$ has the following probabilistic meaning(see Proposition 5.1 of Asmussen[6]):

$$\left[e^{t\boldsymbol{\kappa}(\lambda)}\right]_{ij}=E\left[e^{-\lambda F_0(t)}\chi(J_0(t)=j)|F_0(0)=0,J_0(0)=i\right],$$

where (F_0,J_0) is the process to be obtained from (F,J) by cutting off the period where J stays in S_0.

We let matrix polynomial $P(\lambda)=\mathrm{diag}\{2\Sigma^{-2},-\Delta^{-1}_{\boldsymbol{\mu}_d},-\Delta^{-1}_{\boldsymbol{\mu}_u}\}\boldsymbol{\kappa}_s(\lambda)$. Then $P(\lambda)$ can be represented as $P(z)=\lambda^2A_2+\lambda A_1+A_0$, where

$$A_2=\mathrm{diag}\{I_b,\mathbf{0}_d,\mathbf{0}_u\},A_1=\mathrm{diag}\{-2\Sigma^{-2}\Delta_{\boldsymbol{\mu}_b},I_d,I_u\}\text{ and }A_0=\mathrm{diag}\{2\Sigma^{-2},-\Delta^{-1}_{\boldsymbol{\mu}_d},-\Delta^{-1}_{\boldsymbol{\mu}_u}\}\left(Q^r(s)-sI\right).\quad(13)$$

The following theorem shows the relation between the solutions of our NARE and the quadratic matrix equation in Nguyen and Poloni[20].

Theorem 8. *Let $X(s)$ be the minimal nonnegative solution of the NARE (11) and define*

$$V_1 = \begin{bmatrix} I_b & \mathbf{0} \\ \mathbf{0} & I_d \\ X_{ub}(s) & X_{ud}(s) \end{bmatrix} \text{ and } U_2 = \begin{bmatrix} 2\Sigma^{-1} & \mathbf{0} & -X_{bd}(s)\Delta_{\boldsymbol{\mu}_d}^{-1} \\ \mathbf{0} & \Delta_{\boldsymbol{\mu}_u}^{-1} & -X_{ud}(s)\Delta_{\boldsymbol{\mu}_d}^{-1} \end{bmatrix} \tag{14}$$

Then, matrix $B(s) + C(s)X(s)$ is a solution of the following quadratic equation

$$\Delta_{\boldsymbol{\sigma}^2/2}^r V_1 Z^2 + \Delta_{\boldsymbol{\mu}}^r V_1 Z + (Q^r(s) - sI)V_1 = \mathbf{0}.$$

Furthermore, the matrix $-(A(s) + X(s)C(s))$ is a solution of the following quadratic equation

$$Z^2 U_2 \Delta_{\boldsymbol{\sigma}^2/2} + Z U_2 \Delta_{\boldsymbol{\mu}} + U_2(Q(s) - sI) = \mathbf{0}.$$

For the proof of the theorem, we introduce the following two lemmas, of which the proofs are deffered to Appendix. Recall the matrix $L(s)$ in (12) and define a corresponding matrix pencil $W(\lambda) = \lambda I - L(s)$. The following lemma shows a result on the linearization of $P(\lambda)$.

Lemma 1. *The matrix pencil $W(\lambda)$ can be factored as*

$$W(\lambda) = \boldsymbol{\eta}(\lambda) \begin{bmatrix} P(\lambda) & \mathbf{0} \\ \mathbf{0} & I_b \end{bmatrix} \boldsymbol{\zeta}(\lambda), \tag{15}$$

where

$$\boldsymbol{\eta}(\lambda) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & I_b \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ -\Sigma & \mathbf{0} & \mathbf{0} & \Sigma[\lambda I - (D_1 - D_2(s))] \\ \mathbf{0} & \mathbf{0} & I_u & \mathbf{0} \end{pmatrix} \text{ and } \boldsymbol{\zeta}(\lambda) = \begin{pmatrix} I_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_u \\ \lambda I - (D_1 + D_2(s)) & \mathbf{0} & \Sigma^{-1} & \mathbf{0} \end{pmatrix}$$

Moreover, $W(\lambda)$ is a linearization of the matrix polynomial $\boldsymbol{\kappa}_s(\lambda)$.

Lemma 2. *(a) Define matrices U and V such that*

$$V = \begin{bmatrix} I_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_u \end{bmatrix} \text{ and } U = \begin{bmatrix} \mathbf{0}_b & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} \\ -\Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_u \end{bmatrix}.$$

Then, it holds that $V(\lambda I - L(s))^{-1}U = P(\lambda)^{-1}$.

(b) Matrix $\left[\begin{smallmatrix} V \\ VL(s) \end{smallmatrix}\right]$ is a full-column-rank matrix and $\left[\begin{smallmatrix} U & L(s)U \end{smallmatrix}\right]$ is a full-row-rank matrix.

(c) The following quadratic equations hold:

$$\mathbf{0} = A_0 V + A_1 V L(s) + A_2 V L(s)^2 \quad \text{and} \quad \mathbf{0} = U A_0 + L(s) U A_1 + L(s)^2 U A_2. \tag{16}$$

Proof. **(Proof of Theorem 8)** The equations in (16) can be rewritten as

$$\Delta^r_{\boldsymbol{\sigma}^2/2} V L(s)^2 - \Delta^r_{\boldsymbol{\mu}} V L(s) + (Q^r(s) - sI)V = 0, \tag{17a}$$

$$L(s)^2 U^* \Delta^r_{\boldsymbol{\sigma}^2/2} - L(s) U^* \Delta^r_{\boldsymbol{\mu}} + U^*(Q^r(s) - sI) = 0. \tag{17b}$$

with $U^* = U \ \mathrm{diag}\{2\Sigma^{-2}, -\Delta^{-1}_{\boldsymbol{\mu}_d}, -\Delta^{-1}_{\boldsymbol{\mu}_u}\}$. We recall that

$$\widetilde{L}(s) = K(s)^{-1} L(s) K(s) = \begin{bmatrix} -(B(s) + C(s)X(s)) & -C(s) \\ 0 & A(s) + X(s)C(s) \end{bmatrix}, \quad K(s) = \begin{bmatrix} I & 0 \\ X(s) & I \end{bmatrix},$$

where $X(s)$ is the minimal nonnegative solution of the NARE (11). Therefore, multiplying (17a) on the right by $K(s)$ and (17b) on the left by $K(s)^{-1}$ yields

$$\Delta^r_{\boldsymbol{\sigma}^2/2} \tilde{V} \tilde{L}(s)^2 - \Delta^r_{\boldsymbol{\mu}} \tilde{V} \tilde{L}(s) + (Q^r(s) - sI)\tilde{V} = 0, \tag{18a}$$

$$\tilde{L}(s)^2 \tilde{U} \Delta^r_{\boldsymbol{\sigma}^2/2} - \tilde{L}(s) \tilde{U} \Delta^r_{\boldsymbol{\mu}} + \tilde{U}(Q^r(s) - sI) = 0, \tag{18b}$$

where

$$\tilde{V} = VK(s) = \begin{bmatrix} I_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ X_{ub}(s) & X_{ud}(s) & \mathbf{0} & I_u \end{bmatrix}, \tilde{U} = K(s)^{-1}U^* = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\Delta_{\boldsymbol{\mu}_d}^{-1} & \mathbf{0} \\ -2\Sigma^{-1} & X_{bd}(s)\Delta_{\boldsymbol{\mu}_d}^{-1} & \mathbf{0} \\ \mathbf{0} & X_{ud}(s)\Delta_{\boldsymbol{\mu}_d}^{-1} & -\Delta_{\boldsymbol{\mu}_u}^{-1} \end{bmatrix}.$$

Letting $R_1(s) = B(s)+C(s)X(s)$ and $R_2(s) = -(A(s)+X(s)C(s))$, from the structure of $\widetilde{L}(s)$, equations (18a) and (18b) imply that

$$\Delta^r_{\boldsymbol{\sigma}^2/2}V_1R_1(s)^2 + \Delta^r_{\boldsymbol{\mu}}V_1R_1(s) + (Q^r(s) - sI)V_1 = \mathbf{0},$$

$$R_2(s)^2U_2\Delta_{\boldsymbol{\sigma}^2/2} + R_2(s)U_2\Delta_{\boldsymbol{\mu}} + U_2(Q(s) - sI) = \mathbf{0},$$

which completes the proof. □

5 Doubling algorithms

5.1 Structure-preserving doubling algorithm

We introduce the dual NARE of (2) given as

$$B(s)Z + ZA(s) + ZDZ + C = \mathbf{0}, \quad s \geq 0, \tag{19}$$

and denote by $Y(s)$ its minimal nonnegative solution.

The SDA presented in [16] is given in Table 3.

SDA for an NARE $AZ + ZB + ZCZ + D = \mathbf{0}$	
1.	Choose $\gamma \geq \max\{-[A]_{ii}, -[B]_{ii}, i \in S\}$ and set $\begin{pmatrix} E_0 & G_0 \\ H_0 & F_0 \end{pmatrix} = \begin{pmatrix} \gamma I - B & -C \\ -D & \gamma I - A \end{pmatrix}^{-1} \begin{pmatrix} \gamma I + B & C \\ D & \gamma I - A \end{pmatrix}$
2.	$E_{k+1} = E_k(I - G_kH_k)^{-1}E_k\,;\;\; F_{k+1} = F_k(I - H_kG_k)^{-1}F_k\,;$ $G_{k+1} = G_k + E_k(I - G_kH_k)^{-1}G_kF_k\,;\;\; H_{k+1} = H_k + F_k(I - H_kG_k)^{-1}H_kE_k\,;$
3.	$Z = H_\infty\,;$

Table 3: Structure-preserving doubling algorithm

With these matrices, we define Cayley transforms $R_\gamma(s) = (R(s) + \gamma I)^{-1}(R(s) - \gamma I)$ and $S_\gamma(s) =$

$(S(s)+\gamma I)^{-1}(S(s)-\gamma I)$ of $R(s)=-B(s)-CX(s)$ and $S(s)=-A(s)-DY(s)$. Since $M(s)$ is a non-singular M-matrix by Proposition 1, using the results in Section 4 of [13], it follows that $\rho(R_\gamma(s))<1$, $\rho(S_\gamma(s))<1$, and

$$\limsup_{k\to\infty} \sqrt[2^k]{||H_k - X(s)||} \le \rho(R_\gamma(s))\rho(S_\gamma(s)) < 1.$$

Hence the SDA for the NARE (2) has quadratic convergence and it is efficient enough for computation of $H(s)$ for all $s>0$.

However when $s=0$, which is of our main interest, quadratic convergence of the SDA is not always guaranteed because M is an irreducible singular matrix. Hereafter, we only consider the case with $s=0$.

For the non-negative solutions X of the NARE (3) and Y of the NARE (19) with $s=0$, we let R_γ and S_γ denote the Cayley transform of $R=-B-CX$ and $S=-A-DY$, that is,

$$R_\gamma = (R+\gamma I)^{-1}(R-\gamma I) \text{ and } S_\gamma = (S+\gamma I)^{-1}(S-\gamma I).$$

The following result can be found in Theorem 4.1 of [13].

Theorem 9. *Note that $\{H_k\}$, $\{E_k\}$, and $\{F_k\}$ denote the matrices in the k-th iteration of the SDA (Table 3).*

(a) If $m>0$ (positive recurrent case), then $\rho(R_\gamma)=1$ and $\rho(S_\gamma)<1$. Furthermore, $\{H_k\}$ converges to X quadratically with

$$\limsup_{k\to\infty} \sqrt[2^k]{||H_k - X||} \le \rho(S_\gamma),$$

$\{F_k\}$ converges to $\mathbf{0}$ quadratically with $\limsup_{k\to\infty} \sqrt[2^k]{||F_k||} \le \rho(S_\gamma)$, and $\{E_k\}$ is bounded. The notation $||A||$ denotes the maximum of the absolute values of the elements in a matrix A.

(b) If $m<0$ (transient case), then $\rho(R_\gamma)<1$ and $\rho(S_\gamma)=1$. Furthermore, $\{H_k\}$ converges to X quadratically with

$$\limsup_{k\to\infty} \sqrt[2^k]{||H_k - X||} \le \rho(R_\gamma),$$

$\{E_k\}$ converges to $\mathbf{0}$ quadratically with $\limsup_{k\to\infty} \sqrt[2^k]{||E_k||} \le \rho(R_\gamma)$, and $\{F_k\}$ is bounded.

(c) If $m=0$ (null recurrent case), then $\rho(R_\gamma)=1$ and $\rho(S_\gamma)=1$. In this case, $\{H_k\}$ converges to X and

$\{E_k\}$, $\{F_k\}$ *are bounded.*

5.1.1 Alternating-directional doubling algorithm

The ADDA algorithm for NARE, which was developed by Wang, Wang, and Li [23], can be considered to be an extension of the SDA. It differs from the SDA only in its initial setup that build E_0, F_0, G_0, and H_0. We use $\hat{E}_k, \hat{G}_k, \hat{H}_k$, and $\hat{F}_k$ with $k \geq 0$ to denote the matrices in the k-th iteration of the ADDA. For the setup, we adopt initialization suggested by Poloni and Reis [21] that unifies initializations for doubling algorithms. We let

$$0 \leq \alpha \leq \alpha_{opt} := [\max\{-[A]_{ii}\}]^{-1} \text{ and } 0 \leq \beta \leq \beta_{opt} := [\max\{-[B]_{ii}\}]^{-1},\ \ \max\{\alpha, \beta\} \neq 0,$$

then the matrix $[\hat{E}_0\ \hat{G}_0; \hat{H}_0\ \hat{F}_0]$ is determined as

$$\begin{pmatrix} \hat{E}_0 & \hat{G}_0 \\ \hat{H}_0 & \hat{F}_0 \end{pmatrix} = \begin{pmatrix} I - \alpha B & -\beta C \\ -\alpha D & I - \beta A \end{pmatrix}^{-1} \begin{pmatrix} I + \beta B & \alpha C \\ \beta D & I - \alpha A \end{pmatrix}. \tag{20}$$

The following is given in Theorem 3.1 of [24].

Theorem 10. *(a) For all $k \geq 0$, $\hat{E}_k \geq 0$, $\hat{F}_k \geq 0$, and they are uniformly bounded with respect to k.*
(b) For all $k \geq 0$, $I - \hat{H}_k\hat{G}_k$ and $I - \hat{G}_k\hat{H}_k$ are non-singular M-matrices.
(c) Let X denote the minimal nonnegative solution of the NARE (3), then $0 \leq \hat{H}_k \leq \hat{H}_{k+1} \leq X$ and

$$\limsup_{k\to\infty} \sqrt[2^k]{||\hat{H}_k - X||} \leq \rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}), \tag{21}$$

where $R_{\beta,\alpha} = (\beta R - I)(\alpha R + I)^{-1}$ and $S_{\alpha,\beta} = (\alpha S - I)(\beta S + I)^{-1}$. The optimal α and β that minimize the right-hand sid of (21) *are $\alpha = \alpha_{opt}$ and $\beta = \beta_{opt}$.*

Remark 1. *(a) The SDA is a particular case of the ADDA. That is, if we let $\alpha = \beta = \gamma^{-1}$, then the ADDA is equivalent to the SDA.*
(b) In [23], it is shown that $\rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}) < 1$ if the original NARE is not in the null-recurrent case, otherwise $\rho(R_{\alpha,\beta})\rho(S_{\alpha,\beta}) = 1$. Furthermore, the upper-bound is less than that of the SDA, that is, $\rho(R_{\beta,\alpha})\rho(S_{\alpha,\beta}) \leq \rho(R_\gamma)\rho(S_\gamma)$ with $\gamma = \max\{\alpha^{-1}, \beta^{-1}\}$. Hence the ADDA converges faster than the SDA (Section 5 of [23] and Section 3 of [24]).

5.2 Shifted NARE for the MMBM with $\boldsymbol{\sigma} > 0$

As for the null-recurrent case, it is known that the SDA and ADDA can show a linear convergence of rate 1/2 [13, 15]. In this section, we introduce a shift technique for improving the convergence rate, which is proposed by Guo, Iannazzo, and Meini [13]. We note that the shift consists in performing a rank-one correction which moves one zero eigenvalue to a suitable nonzero real number (see also [10]). For more details of this section, we refer to [13] and [23].

We first assume $m \geq 0$ which includes positive and null recurrent cases. We recall $\mathbf{v}' = (\mathbf{v}_1' \;\; \mathbf{v}_2') = [\mathbf{1}' \;\; \mathbf{1}'\boldsymbol{\Delta}\Delta_{\boldsymbol{\sigma}}] > \mathbf{0}$ and define $\mathbf{p} = (\mathbf{p}_1' \;\; \mathbf{p}_2')' = (\mathbf{v}'\mathbf{1})^{-1}\mathbf{1}$ so that $\mathbf{p} > \mathbf{0}$ and $\mathbf{p}'\mathbf{v} = 1$. We note that $\mathbf{p}_1 > \mathbf{0}$ is a sufficient condition for the results in the following theorems [13]. We define the new NARE

$$\widehat{A}Z + Z\widehat{B} + Z\widehat{C}Z + \widehat{D} = \mathbf{0} \tag{22}$$

where, with a scalar $\eta > 0$,

$$\widehat{A} = A + \eta\mathbf{v}_2\mathbf{p}_2', \quad \widehat{B} = B - \eta\mathbf{v}_1\mathbf{p}_1', \quad \widehat{C} = C - \eta\mathbf{v}_1\mathbf{p}_2', \quad \widehat{D} = D + \eta\mathbf{v}_2\mathbf{p}_1'.$$

The following result can be found in Section 6.1 of [13].

Theorem 11. *Assume $m \geq 0$ and let $\widehat{H}_k$ denote the H_k-matrix in the k-th iteration of the SDA when it is applied to the shifted NARE (22). Then $\widehat{H}_k$ approximates X which is the minimal nonnegative solution of the NARE (3) and its convergence is quadratic with*

$$\limsup_{k\to\infty} \sqrt[2^k]{||\hat{H}_k - X||} \leq \rho(\widehat{R}_\gamma)\rho(\widehat{S}_\gamma) < \rho(R_\gamma)\rho(S_\gamma) \leq 1,$$

where $\widehat{R}_\gamma$ and $\widehat{R}_\gamma$ are the Cayley transform of $\widehat{R} = -\widehat{B} - \widehat{C}X$ and $\widehat{S} = -\widehat{A} - \widehat{D}\widehat{Y}$ with $\widehat{Y}$ being the minimal solution of the dual NARE of (22).

Remark 2. *(a) When $m \geq 0$, $\rho(\widehat{R}_\gamma) < \rho(R_\gamma) = 1$. See Section 6.1 of [13].*
(b) When the ADDA is applied to the shifted NARE, the upper bound of the limit is $\rho(\widehat{R}_{\beta,\alpha})\rho(\widehat{S}_{\alpha,\beta})$ where $\widehat{R}_{\beta,\alpha} = (\beta\widehat{R} - I)(\alpha\widehat{R} + I)^{-1}$ and $\widehat{S}_{\alpha,\beta} = (\alpha\widehat{S} - I)(\beta\widehat{S} + I)^{-1}$. It holds that $\rho(\widehat{R}_{\beta,\alpha})\rho(\widehat{S}_{\alpha,\beta}) \leq \rho(\widehat{R}_\gamma)\rho(\widehat{S}_\gamma) < \rho(R_\gamma)\rho(S_\gamma) \leq 1$ with $\gamma = \max\{\alpha^{-1}, \beta^{-1}\}$.

The transient case ($m < 0$) is easily reduced to the case of $m > 0$. It is shown in Lemma 5.1 of [13] that the matrix X is the minimal nonnegative solution of the NARE (3) if and only if $Z = X'$ is the minimal nonnegative solution of the equation $B'Z + ZA' + ZC'Z + D' = \mathbf{0}$. Furthermore, this NARE is positive recurrent if and only if the NARE (3) is transient. In this case, the corresponding shifted NARE is given as $\widehat{A}_t Z + Z\widehat{B}_t + Z\widehat{C}_t Z + \widehat{D}_t = \mathbf{0}$, where

$$\widehat{A}_t = B' + \eta\upsilon\mathbf{u}_1\mathbf{1}', \quad \widehat{B}_t = A' - \eta\upsilon\mathbf{u}_2\mathbf{1}', \quad \widehat{C}_t = C' - \eta\upsilon\mathbf{u}_2\mathbf{1}', \quad \widehat{D}_t = D' + \eta\upsilon\mathbf{u}_1\mathbf{1}' \text{ with } \upsilon = (\mathbf{u}'\mathbf{1})^{-1}.$$

For more details, we refer to Section 5 of [13].

5.3 Shifted NARE for the MMBM with $\boldsymbol{\sigma} \geq 0$

Let τ be the first passage time of the MMBM to level 0. As for the first passage probabilities, if we consider a matrix of which the (i,j)-th element is $P[\tau < \infty, J(\tau) = j | F(0) = a, J(0) = i]$ with $i,j \in S_b \cup S_d$, then the matrix is also represented as a matrix exponential form e^{Ha}. The exponent matrix H is $(|S_b| + |S_d|)$-dimensional square matrix which forms a sub-stochastic generator, that is, its off-diagonal elements are nonnegative, diagonal elements are negative, and its row sums are less than 0. The exponent matrix H is also an important quantity for describing the stationary distribution of the MMBM as it is for the MMBM with only positive diffusion coefficients. (See [6].)

The following result concerns how to compute H using NARE's and is a corollary of Theorem 6.

Corollary 3. *Define $Q^o_{l.m} = Q_{l,m}(0)$ for $l, m = b, u, d$, and $D_2 = D_2(s)$. Let X denote the minimal nonnegative solution of the NARE*

$$AZ + ZB + ZCZ + D = \mathbf{0} \tag{23}$$

where

$$\begin{aligned} A &= \begin{pmatrix} D_1 - D_2 & 2\Sigma^{-1}Q^o_{b,u} \\ \mathbf{0} & \Delta^{-1}_{\boldsymbol{\mu}_u} Q^o_{u,u} \end{pmatrix}, B = \begin{pmatrix} -D_1 - D_2 & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_d} Q^o_{d,b} & -\Delta^{-1}_{\boldsymbol{\mu}_d} Q^o_{d,d} \end{pmatrix}, \\ C &= \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d} Q^o_{d,u} \end{pmatrix}, \ D = \begin{pmatrix} 2\Sigma^{-1}(Q^o_{b,b} + \Lambda_b) & 2\Sigma^{-1}Q^o_{b,d} \\ \Delta^{-1}_{\boldsymbol{\mu}_u} Q^o_{u,b} & \Delta^{-1}_{\boldsymbol{\mu}_u} Q^o_{u,d} \end{pmatrix}. \end{aligned}$$

The matrix H satisfies $H = B + CX$. □

The M-matrix corresponding to the NARE (23) is

$$M = \begin{pmatrix} -B & -C \\ -D & -A \end{pmatrix}.$$

We can also show that M is an irreducible singular M-matrix in a similar way as in the proof of Proposition 1, which is omitted here. For construction of the shifted NARE of (23), we define

$$\mathbf{p} := \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} \nu\ \mathbf{1}_{bd} \\ \nu\ \mathbf{1}_{bu} \end{pmatrix} \quad \text{with } \nu = \mathbf{1}/(\mathbf{v}'\mathbf{1}), \tag{24}$$

where $\mathbf{v}$ is the eigenvector given in Proposition 2, $\mathbf{1}'_{bd} = (\mathbf{1}'_b\ \mathbf{1}'_d)$, and $\mathbf{1}'_{bu} = (\mathbf{1}'_b\ \mathbf{1}'_u)$.

To obtained the quadratic convergence of the doubling algorithms, we consider different NARE's depending on the categories of the original NARE (23). When $m \leq 0$, we use the shifted NARE of (23)

$$\widehat{A}Z + Z\widehat{B} + Z\widehat{C}Z + \widehat{D} = \mathbf{0} \tag{25}$$

where

$$\widehat{A} = A + \gamma\mathbf{v}_2\mathbf{p}'_2, \quad \widehat{B} = B - \gamma\mathbf{v}_1\mathbf{p}'_1, \quad \widehat{C} = C - \gamma\mathbf{v}_1\mathbf{p}'_2, \quad \widehat{D} = D + \gamma\mathbf{v}_2\mathbf{p}'_1.$$

As for the case $m > 0$, we consider the shifted NARE of the transposed NARE of (23), which is

$$\widehat{A}_t Z + Z\widehat{B}_t + Z\widehat{C}_t Z + \widehat{D}_t = \mathbf{0}, \tag{26}$$

where, with $\upsilon = (\mathbf{u}'\mathbf{1})^{-1}$,

$$\widehat{A}_t = B' + \gamma\upsilon\mathbf{u}_1\mathbf{1}'_{bd}, \quad \widehat{B}_t = A' - \gamma\upsilon\mathbf{u}_2\mathbf{1}'_{bu}, \quad \widehat{C}_t = C' - \gamma\upsilon\mathbf{u}_2\mathbf{1}'_{bd}, \quad \widehat{D}_t = D' + \gamma\upsilon\mathbf{u}_1\mathbf{1}'_{bu}.$$

We note that, when X is the minimal nonnegative solution of the original NARE (23), the minimal nonnegative solutions of the NARE (25) and (26) are equal to X and X', respectively. For more details, refer to Section 5 of [13].

6 Numerical study

In this section, we compare the following algorithms, applied to (2) with $s = 0$:

$A1$: SDA applied to the original NARE ;

$A2$: ADDA applied to the original NARE;

$A3$: SDA applied to the shifted NARE;

$A4$: ADDA applied to the shifted NARE;

$NP2$: Nguyen and Poloni's algorithm using GTH-like algorithm (Algorithm 3 in [20]).

For the numerical study, we used MATLAB(R2016a) running on a Windows 7 64 bit in DELL PowerEdge R730 Server with Processor Intel Xeon E5-2620 v3 @ 2.40GHz and 64GB of main memory.

We consider two different examples for the cases of $\boldsymbol{\sigma} > \mathbf{0}$ and $\boldsymbol{\sigma} \geq \mathbf{0}$, respectively, in which we can obtain exact values of first-passage probabilities. To compare the performance of the algorithms, we consider the cputime and iteration number necessary for the convergence of the algorithms when maximum matrix norm and 10^{-12} are used for their stopping criterion. We also take into account the error, difference between the computed and exact values of certain first-passage probability to be considered in each example.

6.1 Example for the case of $\boldsymbol{\sigma} > \mathbf{0}$

For $n = 10, 100, 1000$, we let $\boldsymbol{\mu} = \mu \mathbf{1}_n$ and $\boldsymbol{\sigma} = \sigma \mathbf{1}_n$ with $\mu = 0, 1, 10$ and $\sigma = 1, 10$. We determine the values of the off-diagonal elements of Q using ceiling number of the uniform random numbers in $(0, 100)$, then diagonal elements are given so that the row sums of Q are to be 0. With this choice, for any Q, the MMBM is simply an ordinary Brownian motion with drift parameter μ and diffusion parameter σ, whose first passage probability is explicitly given as $P(\tau < \infty | B(0) = a) = \exp(-a(\mu + |\mu|)/\sigma^2)$. (See [9].) This observation allows us to construct a set of problems for use in comparison of algorithms both with respect to speed and accuracy. Note that we let $a = 3$ in this example.

Table 4 contains the absolute error values, that is, the differences between the exact value($e^{-3(\mu+|\mu|)/\sigma^2}$) of the first passage probability and its numerical values computed by the algorithms. To compare the speed of the algorithms, we investigated the total number of iterations (Table 6) and also cpu-times (Table 5) necessary for the algorithms to produce their values of the first passage probability.

n	$-m$	σ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	1	2.7E-06	2.7E-06	3.0E-14	3.0E-14	5.1E-13
10	0	10	3.2E-07	3.2E-07	2.4E-14	2.4E-14	5.3E-14
10	1	1	4.3E-15	6.2E-15	4.8E-17	1.6E-15	2.2E-18
10	1	10	3.4E-12	2.1E-12	1.3E-14	6.1E-15	5.6E-16
10	10	1	5.5E-40	3.6E-39	5.3E-40	1.8E-40	5.5E-41
10	10	10	2.5E-13	1.1E-13	2.4E-14	7.5E-15	6.7E-16
100	0	1	NaN	NaN	2.9E-14	2.9E-14	3.2E-12
100	0	10	7.6E-07	7.6E-07	1.3E-14	1.3E-14	1.6E-13
100	1	1	1.1E-15	3.7E-14	4.2E-16	1.6E-15	1.6E-16
100	1	10	4.1E-13	6.4E-13	3.0E-15	2.6E-15	4.4E-15
100	10	1	9.6E-41	2.1E-40	1.5E-40	6.2E-38	1.7E-40
100	10	10	5.4E-14	4.6E-14	3.3E-16	5.0E-15	1.6E-15
1000	0	1	2.6E-06	2.6E-06	3.1E-14	3.1E-14	9.9E-12
1000	0	10	NaN	NaN	2.0E-15	2.0E-15	9.8E-13
1000	1	1	5.5E-15	4.0E-15	1.1E-16	6.9E-17	2.8E-16
1000	1	10	5.1E-12	1.1E-11	1.7E-15	3.2E-15	6.1E-15
1000	10	1	6.8E-40	8.7E-40	2.2E-40	3.4E-40	1.9E-40
1000	10	10	2.2E-14	6.6E-14	2.2E-15	1.1E-15	1.1E-15

Table 4: Comparison of error values.

n	$-m$	σ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	1	10	0.0E+00	0.0E+00	0.0E+00	4.7E-02	0.0E+00
10	10	1	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
10	10	10	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
100	0	1	1.3E+00	1.1E+00	1.3E-01	1.3E-01	3.0E+00
100	0	10	6.3E-01	6.9E-01	1.9E-01	4.7E-02	2.4E+00
100	1	1	1.1E-01	1.7E-01	1.7E-01	9.4E-02	4.7E-01
100	1	10	2.7E-01	2.7E-01	1.7E-01	1.6E-02	5.5E-01
100	10	1	9.4E-02	9.4E-02	9.4E-02	1.1E-01	4.2E-01
100	10	10	1.4E-01	1.7E-01	9.4E-02	9.4E-02	5.0E-01
1000	0	1	9.8E+01	9.2E+01	1.9E+01	1.9E+01	1.6E+03
1000	0	10	2.2E+02	2.2E+02	2.0E+01	1.9E+01	1.6E+03
1000	1	1	4.0E+01	4.0E+01	1.9E+01	2.0E+01	5.8E+02
1000	1	10	4.9E+01	4.7E+01	1.9E+01	1.9E+01	7.0E+02
1000	10	1	3.0E+01	2.9E+01	2.2E+01	2.3E+01	4.6E+02
1000	10	10	3.8E+01	3.9E+01	1.9E+01	2.0E+01	5.8E+02

Table 5: Comparison of cpu times.

n	-m	σ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	1	28	28	6	6	48
10	0	10	28	28	5	5	48
10	1	1	9	9	5	5	11
10	1	10	12	12	5	5	14
10	10	1	6	6	5	5	8
10	10	10	9	9	5	5	11
100	0	1	71	71	5	5	47
100	0	10	28	28	5	5	48
100	1	1	11	11	5	5	13
100	1	10	14	14	5	5	16
100	10	1	8	8	5	5	9
100	10	10	11	11	5	5	13
1000	0	1	31	31	4	4	47
1000	0	10	74	74	4	4	47
1000	1	1	12	12	4	4	14
1000	1	10	15	15	4	4	17
1000	10	1	9	9	5	5	11
1000	10	10	12	12	4	4	14

Table 6: Comparison of iteration number.

6.2 Example for the case of $\boldsymbol{\sigma} \geq 0$

6.2.1 Asmussen's example

In this section, we take an example, referring to Example 6.1 of Asmussen [6], where it is assumed for (F, J) that $S = \{1, 2, 3\}$ with $S_b = \{1\}$, $S_u = \{2\}$, $S_d = \{3\}$,

$$\sigma_1 = \sqrt{\frac{7}{4}}, \mu_1 = \frac{3}{2}, \mu_2 = \frac{1}{2}, \mu_3 = -\frac{1}{2} \text{ and } Q = \begin{pmatrix} -\frac{15}{8} & \frac{15}{16} & \frac{15}{16} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

The corresponding stationary probability vector is given as $\boldsymbol{\pi} = (8/23 \;\; 15/46 \;\; 15/46)$ and the first passage probability $f(x) = P[\tau < \infty | J(0) = 1, F(0) = x]$ satisfies that $f(x) = (3/4)e^{-x} + (1/4)e^{-3x}$, which is to be used as a target function to check the accuracy of the algorithms. (For $f(x)$, refer to the matrix U in Example 6.1 of Asmussen [6].)

To check our computations even more, we consider the MMBM (F, J) for which $S = \{1, \cdots, n\}$ with

$n = 3k,\ k = 10, 20, 50, 100, 500, 1000,\ \sigma_b = \sqrt{\frac{7}{4}}\mathbf{1}'_k,\ \boldsymbol{\mu}_u = \frac{3}{2}\mathbf{1}'_k,\ \boldsymbol{\mu}_u = \frac{1}{2}\mathbf{1}'_k,\ \boldsymbol{\mu}_d = -\frac{1}{2}\mathbf{1}'_k$, and

$$Q = \begin{pmatrix} -\frac{15}{8}I_k & \frac{15}{16k}\mathbf{J}_k & \frac{15}{16k}\mathbf{J}_k \\ \frac{1}{k}\mathbf{J}_k & -I_k & \mathbf{0}_{k\times k} \\ \frac{1}{k}\mathbf{J}_k & \mathbf{0}_{k\times k} & -I_k \end{pmatrix},$$

where $\mathbf{J}_k$ denotes the k-dimensional square matrix of 1's. For example, when $n(=3k) = 6$, Q is given as

$$\begin{pmatrix} -\frac{15}{8} & 0 & \frac{15}{32} & \frac{15}{32} & \frac{15}{32} & \frac{15}{32} \\ 0 & -\frac{15}{8} & \frac{15}{32} & \frac{15}{32} & \frac{15}{32} & \frac{15}{32} \\ 0.5 & 0.5 & -1 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & -1 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & -1 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that the level process in this model is stochastically equivalent to that of the original. All we have done is to mask the exponential duration of the phase in states 1, 2, 3 into a equal mixture of k exponential distributions with the same rate by adding $k-1$ more phase to the model. Hence, both models have the same first-passage time distributions. This gives us yet another computational check on the first passage time distribution of the algorithms.

We also note that the value of $-m$, the average drift value of (F, J), is given as $12/23$ for all k. To investigate the effects of m, we consider another parameter $d_\ell = (23/12) \times \ell$ with $\ell > 0$, which makes the $(-m)$-value of $(d_\ell \times F, J)$ be ℓ. For the MMBM $(d_\ell \times F, J)$ with τ_ℓ and H_ℓ being its first passage time to 0 and the H-matrix, it holds that for all ℓ and x

$$\begin{aligned} P[\tau < \infty | J(0) = 1, F(0) = x] &= P[\tau_\ell < \infty | J(0) = 1, d_\ell F(0) = d_\ell x] \\ &= \mathbf{e}_1 e^{(d_\ell x)H_\ell}\mathbf{1} = (3/4)e^{-x} + (1/4)e^{-3x}, \end{aligned}$$

where $\mathbf{e}_1$ is a unit vector with appropriate dimension, in which its first element is 1 and the others are all 0. In this study, we consider the MMBM's $(d_\ell \times F, J)$ with $\ell = 0.0001, 0.001, 0.1, 5, 10, 20$, and compute $P[\tau_\ell < \infty | J(0) = 1, d_\ell F(0) = 3d_\ell]$. Note that the diffusion and drift vector of $(d_\ell \times F, J)$ are given as $d_\ell \boldsymbol{\sigma}_b$ and $d_\ell \boldsymbol{\mu}$.

$k(n=3k)$	$-m$	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	2.40E-08	2.40E-08	4.40E-16	4.40E-16	1.11E-15
10	0.0001	2.70E-16	2.40E-16	2.20E-16	2.10E-16	3.47E-17
10	0.001	2.70E-16	2.60E-16	2.40E-16	2.10E-16	2.78E-17
10	0.1	2.50E-16	2.70E-16	2.40E-16	2.20E-16	2.78E-17
10	5	2.80E-16	2.80E-16	2.60E-16	2.40E-16	9.71E-17
10	10	2.80E-16	2.80E-16	2.60E-16	2.40E-16	9.71E-17
10	20	2.80E-16	2.80E-16	2.60E-16	2.40E-16	9.71E-17
20	0	NaN	NaN	0.00E+00	0.00E+00	2.00E-15
20	0.0001	3.30E-16	2.80E-17	6.90E-18	2.10E-17	3.19E-16
20	0.001	3.00E-16	3.50E-17	2.80E-17	6.90E-18	3.47E-16
20	0.1	3.30E-16	2.80E-17	6.90E-18	1.40E-17	3.61E-16
20	5	3.40E-16	2.80E-17	1.40E-17	6.90E-18	3.40E-16
20	10	3.40E-16	2.80E-17	1.40E-17	6.90E-18	3.40E-16
20	20	3.40E-16	2.80E-17	1.40E-17	6.90E-18	3.40E-16
50	0	NaN	NaN	1.30E-15	1.30E-15	4.66E-15
50	0.0001	2.10E-16	2.80E-17	1.20E-16	2.80E-17	4.09E-16
50	0.001	2.40E-16	6.90E-17	1.50E-16	6.90E-17	4.16E-16
50	0.1	1.70E-16	2.80E-17	9.70E-17	1.40E-17	4.37E-16
50	5	2.10E-16	1.40E-17	1.10E-16	4.90E-17	4.44E-16
50	10	2.10E-16	1.40E-17	1.10E-16	4.90E-17	4.44E-16
50	20	2.10E-16	1.40E-17	1.10E-16	4.90E-17	4.44E-16
100	0	NaN	NaN	3.70E-14	3.70E-14	2.64E-14
100	0.0001	1.20E-15	5.20E-16	1.00E-15	9.70E-16	2.37E-15
100	0.001	1.20E-15	5.10E-16	1.00E-15	9.20E-16	2.43E-15
100	0.1	1.10E-15	2.80E-16	1.00E-15	9.40E-16	2.47E-15
100	5	1.20E-15	4.60E-16	1.00E-15	9.90E-16	2.43E-15
100	10	1.20E-15	4.60E-16	1.00E-15	9.90E-16	2.43E-15
100	20	1.20E-15	4.60E-16	1.00E-15	9.90E-16	2.43E-15
500	0	NaN	NaN	3.60E-14	3.60E-14	1.47E-14
500	0.0001	3.70E-16	4.60E-16	9.70E-17	5.60E-17	7.42E-16
500	0.001	4.00E-16	4.90E-16	9.70E-17	1.40E-17	1.11E-16
500	0.1	3.20E-16	4.20E-16	8.30E-17	2.80E-17	6.94E-16
500	5	3.60E-16	4.60E-16	4.20E-17	1.40E-17	4.72E-16
500	10	3.60E-16	4.60E-16	4.20E-17	1.40E-17	4.72E-16
500	20	3.60E-16	4.60E-16	4.20E-17	1.40E-17	4.72E-16
1000	0	4.10E-08	4.10E-08	5.40E-15	5.40E-15	3.62E-14
1000	0.0001	2.60E-15	2.50E-15	1.00E-15	8.00E-16	9.02E-16
1000	0.001	2.70E-15	2.60E-15	1.00E-15	9.80E-16	8.33E-16
1000	0.1	2.60E-15	2.50E-15	1.10E-15	1.00E-15	1.07E-15
1000	5	2.70E-15	2.50E-15	7.40E-16	1.00E-15	1.53E-15
1000	10	2.70E-15	2.50E-15	7.40E-16	1.00E-15	1.53E-15
1000	20	2.70E-15	2.50E-15	7.40E-16	1.00E-15	1.53E-15

Table 7: Comparison of error values.

$k(n=3k)$	$-m$	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	0.00E+00	3.10E-02	0.00E+00	0.00E+00	5.00E-01
10	0.0001	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.72E-01
10	0.001	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.88E-01
10	0.1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.88E-01
10	5	0.00E+00	0.00E+00	0.00E+00	0.00E+00	9.38E-02
10	10	0.00E+00	0.00E+00	0.00E+00	0.00E+00	4.69E-02
10	20	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.09E-01
20	0	1.90E-01	1.90E-01	9.40E-02	0.00E+00	6.56E-01
20	0.0001	0.00E+00	0.00E+00	0.00E+00	0.00E+00	4.69E-02
20	0.001	0.00E+00	0.00E+00	0.00E+00	0.00E+00	2.03E-01
20	0.1	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.56E-01
20	5	0.00E+00	0.00E+00	9.40E-02	0.00E+00	2.19E-01
20	10	1.10E-01	0.00E+00	0.00E+00	0.00E+00	7.81E-02
20	20	0.00E+00	0.00E+00	0.00E+00	0.00E+00	1.41E-01
50	0	9.40E-01	8.10E-01	1.60E-02	1.60E-02	5.41E+00
50	0.0001	1.10E-01	9.40E-02	1.60E-02	9.40E-02	1.06E+00
50	0.001	0.00E+00	9.40E-02	9.40E-02	1.60E-02	1.45E+00
50	0.1	0.00E+00	9.40E-02	1.70E-01	9.40E-02	1.16E+00
50	5	1.10E-01	1.70E-01	9.40E-02	1.60E-02	2.02E+00
50	10	9.40E-02	9.40E-02	1.60E-02	1.60E-02	1.06E+00
50	20	9.40E-02	9.40E-02	1.60E-02	1.30E-01	1.38E+00
100	0	3.20E+00	3.30E+00	3.60E-01	3.60E-01	3.70E+01
100	0.0001	3.80E-01	3.80E-01	2.70E-01	4.10E-01	8.13E+00
100	0.001	3.00E-01	5.20E-01	3.80E-01	3.90E-01	8.84E+00
100	0.1	3.80E-01	3.00E-01	3.30E-01	3.90E-01	9.22E+00
100	5	3.90E-01	3.80E-01	4.20E-01	3.60E-01	7.92E+00
100	10	4.10E-01	3.80E-01	3.40E-01	4.40E-01	8.23E+00
100	20	3.60E-01	3.80E-01	3.80E-01	4.20E-01	8.45E+00
500	0	1.90E+02	1.90E+02	3.70E+01	3.70E+01	2.47E+03
500	0.0001	2.10E+01	2.10E+01	3.40E+01	3.40E+01	5.69E+02
500	0.001	2.00E+01	2.00E+01	3.50E+01	3.40E+01	5.70E+02
500	0.1	2.10E+01	2.10E+01	3.40E+01	3.40E+01	5.73E+02
500	5	2.10E+01	2.10E+01	3.50E+01	3.50E+01	5.77E+02
500	10	2.10E+01	2.10E+01	3.40E+01	3.40E+01	5.77E+02
500	20	2.10E+01	2.00E+01	3.40E+01	3.40E+01	5.75E+02
1000	0	1.20E+03	1.20E+03	3.30E+02	3.20E+02	1.64E+04
1000	0.0001	1.40E+02	1.30E+02	3.10E+02	3.10E+02	3.81E+03
1000	0.001	1.40E+02	1.40E+02	3.10E+02	3.10E+02	3.82E+03
1000	0.1	1.40E+02	1.40E+02	3.10E+02	3.10E+02	3.82E+03
1000	5	1.40E+02	1.30E+02	3.10E+02	3.10E+02	3.81E+03
1000	10	1.40E+02	1.30E+02	3.10E+02	3.00E+02	3.81E+03
1000	20	1.40E+02	1.40E+02	3.10E+02	3.10E+02	3.81E+03

Table 8: Comparison of cpu times.

$k(n=3k)$	$-m$	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	30	30	5	5	66
10	0.0001	6	6	4	4	11
10	0.001	6	6	4	4	11
10	0.1	6	6	4	4	11
10	5	6	6	4	4	11
10	10	6	6	4	4	11
10	20	6	6	4	4	11
20	0	64	64	5	5	63
20	0.0001	6	6	4	4	11
20	0.001	6	6	4	4	11
20	0.1	6	6	4	4	11
20	5	6	6	4	4	11
20	10	6	6	4	4	11
20	20	6	6	4	4	11
50	0	61	61	5	5	59
50	0.0001	6	6	4	4	11
50	0.001	6	6	4	4	11
50	0.1	6	6	4	4	11
50	5	6	6	4	4	11
50	10	6	6	4	4	11
50	20	6	6	4	4	11
100	0	61	61	5	5	59
100	0.0001	6	6	4	4	11
100	0.001	6	6	4	4	11
100	0.1	6	6	4	4	11
100	5	6	6	4	4	11
100	10	6	6	4	4	11
100	20	6	6	4	4	11
500	0	61	61	5	5	61
500	0.0001	6	6	4	4	11
500	0.001	6	6	4	4	11
500	0.1	6	6	4	4	11
500	5	6	6	4	4	11
500	10	6	6	4	4	11
500	20	6	6	4	4	11
1000	0	60	60	5	5	57
1000	0.0001	6	6	4	4	11
1000	0.001	6	6	4	4	11
1000	0.1	6	6	4	4	11
1000	5	6	6	4	4	11
1000	10	6	6	4	4	11
1000	20	6	6	4	4	11

Table 9: Comparison of iteration numbers.

For the case of $m = 0$, we used the MMBM (F_0, J), of which all the parameters are same as (F, J) except $\boldsymbol{\mu}_b$ being given as $\mathbf{0}_{k\times 1}$. We compute $P[\tau_0 < \infty | J(0) = 1, F_0(0) = 3] = \mathbf{e}_1 e^{3H_0}\mathbf{1}$, which is equal to 1. Here, τ_0 and H_0 denotes the first passage time to 0 and the H-matrix of (F_0, J).

The numerical results are presented in Tables 7, 8, and 9.

6.2.2 Random example

In this example, we intend to check the effect of the variation of the drift and diffusion parameters on the performance of the algorithms. We consider an MMBM of which the drift vector $\boldsymbol{\mu}$, diffusion vector $\boldsymbol{\sigma}$, and infinitesimal generator Q are given as $\boldsymbol{\mu} = [\boldsymbol{\nu} + \mu\mathbf{1}' \;\; -\boldsymbol{\nu} + \mu\mathbf{1}' \;\; \boldsymbol{\nu} \;\; -\boldsymbol{\nu}]$, $\boldsymbol{\sigma} = [\boldsymbol{\zeta} \;\; \boldsymbol{\zeta} \;\; \mathbf{0} \;\; \mathbf{0}]$ and

$$Q = \begin{pmatrix} \Xi & \Theta & \Theta & \Theta \\ \Theta & \Xi & \Theta & \Theta \\ \Theta & \Theta & \Xi & \Theta \\ \Theta & \Theta & \Theta & \Xi \end{pmatrix}.$$

We fix the sizes of $\boldsymbol{\nu}$ and $\boldsymbol{\zeta}$ at 50 so that Q is an 200×200 matrix. We determine the values of the off-diagonal elements of Ξ and the elements of Θ using ceiling number of the uniform random numbers in $(0, 100)$, then diagonal elements of Ξ are given so that the row sums of Q are to be 0. We also use ceiling number of the uniform random numbers in $(0, K)$ and $(0, \sqrt{K})$ with $K = 10, 50, 100, 200$ to determine the values of $\boldsymbol{\nu}$ and $\boldsymbol{\zeta}$, respectively. Note that the average drift of this example is given as μ and we consider $0, 5, 10, 20$ for its values. For this example, it is impossible to get the exact value of the first passage probability. So we consider the normalized residual(N_{Res}), which is defined as

$$N_{Res} = \frac{||\Delta^r_{\boldsymbol{\sigma}^2/2} V_1 H^2 + \Delta^r_{\boldsymbol{\mu}} V_1 H + Q^r V_1||_1}{\left(||\Delta^r_{\boldsymbol{\sigma}^2/2}||_1 ||V_1||_1 ||H||_1 + ||\Delta^r_{\boldsymbol{\mu}}||_1 ||V_1||_1\right) ||H||_1 + ||Q^r||_1 ||V_1||_1}$$

The numerical results are presented in Tables 10, 11, and 12.

7 Concluding Remarks

In this paper we have shown, by using linear algebra tools, an explicit algebraic relation between the QME and the NARE that characterize MMBM. We have compared the performances of several existing algorithms for computation of first-passage probabilities of the MMBM, among which the component-wise stable and

K	μ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	NaN	NaN	2.53E-18	2.53E-18	3.50E-18
10	5	7.69E-19	7.69E-19	2.17E-18	2.17E-18	3.32E-18
10	10	6.81E-19	6.81E-19	1.43E-18	1.43E-18	3.18E-18
10	20	6.83E-19	6.83E-19	1.80E-18	1.80E-18	4.96E-18
50	0	1.37E-18	1.37E-18	1.04E-18	1.04E-18	6.98E-18
50	5	4.75E-19	4.75E-19	8.84E-19	8.84E-19	4.15E-18
50	10	9.01E-19	9.01E-19	1.74E-18	1.74E-18	5.90E-18
50	20	4.26E-19	4.26E-19	6.99E-19	6.99E-19	3.17E-18
100	0	1.79E-18	1.79E-18	2.21E-18	2.21E-18	1.46E-17
100	5	3.25E-19	3.25E-19	5.79E-19	5.79E-19	1.46E-18
100	10	2.25E-19	2.25E-19	8.22E-19	8.22E-19	2.37E-18
100	20	3.01E-19	3.01E-19	6.77E-19	6.77E-19	2.47E-18
200	0	NaN	NaN	1.88E-18	1.88E-18	5.85E-18
200	5	1.80E-18	1.80E-18	3.07E-18	3.07E-18	1.47E-17
200	10	5.92E-19	5.92E-19	1.17E-18	1.17E-18	3.95E-18
200	20	2.61E-18	2.61E-18	3.37E-18	3.37E-18	2.84E-17

Table 10: Comparison of error values.

K	μ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	2.22E+00	1.83E+00	5.31E-01	4.22E-01	1.38E+01
10	5	5.47E-01	4.53E-01	4.38E-01	4.53E-01	5.19E+00
10	10	5.78E-01	6.41E-01	5.78E-01	4.84E-01	5.31E+00
10	20	5.78E-01	5.31E-01	4.69E-01	4.06E-01	5.25E+00
50	0	1.00E+00	9.69E-01	5.47E-01	6.41E-01	1.30E+01
50	5	4.69E-01	5.47E-01	3.91E-01	4.84E-01	5.39E+00
50	10	4.69E-01	3.75E-01	3.75E-01	4.69E-01	5.03E+00
50	20	5.47E-01	3.75E-01	4.06E-01	3.91E-01	4.73E+00
100	0	1.02E+00	9.06E-01	3.44E-01	4.06E-01	1.41E+01
100	5	5.31E-01	5.94E-01	5.31E-01	5.16E-01	5.63E+00
100	10	5.47E-01	6.09E-01	4.69E-01	6.41E-01	5.84E+00
100	20	5.63E-01	5.94E-01	5.47E-01	5.94E-01	5.11E+00
200	0	1.95E+00	1.89E+00	3.13E-01	3.91E-01	1.35E+01
200	5	5.47E-01	5.16E-01	3.91E-01	2.97E-01	5.73E+00
200	10	6.09E-01	6.56E-01	5.31E-01	4.69E-01	5.45E+00
200	20	3.75E-01	5.63E-01	4.22E-01	3.44E-01	4.48E+00

Table 11: Comparison of cpu times.

K	μ	$A1$	$A2$	$A3$	$A4$	$NP2$
10	0	66	66	13	13	62
10	5	17	17	13	13	22
10	10	16	16	13	13	21
10	20	15	15	13	13	20
50	0	33	33	13	13	63
50	5	18	18	14	14	23
50	10	17	17	13	13	21
50	20	17	17	14	14	22
100	0	33	33	12	12	63
100	5	19	19	15	15	24
100	10	19	19	15	15	24
100	20	18	18	14	14	22
200	0	65	65	13	13	63
200	5	17	17	12	12	22
200	10	18	18	14	14	23
200	20	15	15	11	11	19

Table 12: Comparison of iteration numbers.

quadratically convergent algorithm by Nguyen and Poloni [20] for solving the QME. In the null recurrent case, SDA and ADDA applied to the original NARE are not satisfactory, since they don't provide an accurate approximation of the solution. Their performance is improved when a shift technique is applied. The Nguyen-Poloni algorithm provides accurate results also in the null recurrent case, but the convergence is slower than that of SDA and ADDA applied to the shifted equation.

In certain applications of MMBM, the corresponding NARE has large scale matrix coefficients, but the most common structure of these matrices is sparsity, which refers to a matrix having a relatively large number of zero coefficients [4]. It is reported in Chapter 6 of [8] that this kind of large-scale and sparsity problem can be handled by applying Newton's method. We will investigate this subject in our further studies.

8 Appendix: Proofs

8.1 Proof of Lemma 1

Let E_{34}^r and E_{34}^c denote permutation matrices to be obtained by exchanging the 3rd and 4th rows, and the 3rd and 4th columns of the block-diagonal matrix diag$\{I_b, I_d, I_b, I_u\}$, respectively. Note that $E_{34}^r E_{34}^c = I =$

$E^c_{34}E^r_{34}$. Define $P^*(\lambda) = E^r_{34}\ \mathrm{diag}\{P(\lambda), I_b\}E^c_{34}$. Then $P^*(\lambda)$ is equal to

$$\begin{pmatrix} \lambda^2 I_b - \lambda\left(2\Sigma^{-2}\Delta_{\boldsymbol{\mu}_b}\right) + 2\Sigma^{-2}[Q_{bb}(s) - sI] & 2\Sigma^{-2}Q_{bd}(s) & \mathbf{0} & 2\Sigma^{-2}Q_{bu}(s) \\ -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{db}(s) & \lambda I_d - \Delta^{-1}_{\boldsymbol{\mu}_d}[Q_{d,d}(s) - sI] & \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{du}(s) \\ \mathbf{0} & \mathbf{0} & I_b & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,b}(s) & -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{ud}(s) & \mathbf{0} & \lambda I_u - \Delta^{-1}_{\boldsymbol{\mu}_u}[Q_{u,u}(s) - sI] \end{pmatrix}.$$

Note that the matrix $W(\lambda) = \lambda I - L(s)$ equals to

$$\begin{pmatrix} \begin{pmatrix} \lambda I-(D_1+D_2(s)) & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,b}(s) & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_d}[Q_{d,d}(s)-sI] \end{pmatrix} & \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,u}(s) \end{pmatrix} \\ \\ \begin{pmatrix} -2\Sigma^{-1}(Q_{b,b}(s)+\Lambda_b) & -2\Sigma^{-1}Q_{b,d}(s) \\ -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,b}(s) & -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,d}(s) \end{pmatrix} & \begin{pmatrix} \lambda I-(D_1-D_2(s)) & -2\Sigma^{-1}Q_{b,u}(s) \\ \mathbf{0} & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_u}[Q_{u,u}(s)-sI] \end{pmatrix} \end{pmatrix}. \tag{27}$$

Define

$$\boldsymbol{\eta}^*(\lambda) = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d \end{pmatrix} & \begin{pmatrix} I_b & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \\ \begin{pmatrix} -\Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \Sigma[\lambda I-(D_1-D_2(s))] & \mathbf{0} \\ \mathbf{0} & I_u \end{pmatrix} \end{pmatrix}, \boldsymbol{\zeta}^*(\lambda) = \begin{pmatrix} \begin{pmatrix} I_b & \mathbf{0} \\ \mathbf{0} & I_d \end{pmatrix} & \mathbf{0} \\ \begin{pmatrix} \lambda I-(D_1+D_2(s)) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} & \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & I_u \end{pmatrix}. \end{pmatrix}.$$

Then, with simple arithmetic, we can show that

$$\boldsymbol{\eta}^*(\lambda)P^*(\lambda) = \begin{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,b} & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_d}[Q_{d,d}(s)-sI] \end{pmatrix} & \begin{pmatrix} I_b & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,u} \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{\omega}_1 & -2\Sigma^{-1}Q_{b,d}(s) \\ -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,b}(s) & -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,d}(s) \end{pmatrix} & \begin{pmatrix} \Sigma[\lambda I-(D_1-D_2(s))] & -2\Sigma^{-1}Q_{b,u}(s) \\ \mathbf{0} & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_u}[Q_{u,u}(s)-sI] \end{pmatrix} \end{pmatrix}.$$

with $\boldsymbol{\omega}_1 = -2\Sigma^{-1}\left[\lambda^2\Sigma^2/2 - \lambda\Delta_{\boldsymbol{\mu}_b} + Q_{bb}(s) - sI\right]$. Using this, it is easy to check

$$\boldsymbol{\eta}^*(\lambda)P^*(\lambda)\boldsymbol{\zeta}^*(\lambda) = \begin{pmatrix} \begin{pmatrix} \lambda I-(D_1+D_2(s)) & \mathbf{0} \\ -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,b} & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_d}[Q_{d,d}(s)-sI] \end{pmatrix} & \begin{pmatrix} \Sigma^{-1} & \mathbf{0} \\ \mathbf{0} & -\Delta^{-1}_{\boldsymbol{\mu}_d}Q_{d,u} \end{pmatrix} \\ \begin{pmatrix} \boldsymbol{\omega}_2 & -2\Sigma^{-1}Q_{b,d}(s) \\ -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,b}(s) & -\Delta^{-1}_{\boldsymbol{\mu}_u}Q_{u,d}(s) \end{pmatrix} & \begin{pmatrix} \lambda I-(D_1-D_2(s)) & -2\Sigma^{-1}Q_{b,u}(s) \\ \mathbf{0} & \lambda I-\Delta^{-1}_{\boldsymbol{\mu}_u}[Q_{u,u}(s)-sI] \end{pmatrix} \end{pmatrix}.$$

with $\boldsymbol{\omega}_2 = -2\Sigma^{-1}\left[\lambda^2\Sigma^2/2 - \lambda\Delta_{\boldsymbol{\mu}_b} + Q_{bb}(s) - sI\right] + \Sigma[\lambda I - (D_1 - D_2(s))][\lambda I - (D_1 + D_2(s))]$. Noting

$$\begin{aligned} \boldsymbol{\omega}_2 &= -\lambda^2\Sigma + 2\lambda\Sigma^{-1}\Delta_{\boldsymbol{\mu}_b} - 2\Sigma^{-1}Q_{b,b}(s) + 2s\Sigma^{-1} + \lambda^2\Sigma - 2\lambda\Sigma^{-1}\Delta_{\boldsymbol{\mu}_b} - 2\Sigma^{-1}(sI + \Lambda_b) \\ &= -2\Sigma^{-1}(Q_{b,b}(s) + \Lambda_b) \end{aligned}$$

and then comparing with (27), we can observe that $W(\lambda) = \boldsymbol{\eta}^*(\lambda)P^*(\lambda)\boldsymbol{\zeta}^*(\lambda)$. Furthermore, Using the permutation matrices, we can show that

$$W(\lambda) = \boldsymbol{\eta}^*(\lambda) E_{34}^c E_{34}^r P^*(\lambda) E_{34}^c E_{34}^r \boldsymbol{\zeta}^*(\lambda) = \boldsymbol{\eta}(\lambda)\ \mathrm{diag}\{P(\lambda), I_b\}\ \boldsymbol{\zeta}(\lambda).$$

Since $det(\boldsymbol{\eta}(\lambda))$ and $det(\boldsymbol{\zeta}(\lambda))$ are non-zero constants, that is, they do not depend on λ, $W(\lambda)$ is a linearization of the matrix polynomial $\boldsymbol{\kappa}(\lambda)$.

8.2 Proof of Lemma 2

(a) From (15), we deduce that, for any λ such that $\det P(\lambda) \neq 0$, $\boldsymbol{\zeta}(\lambda)(\lambda I - L(s))^{-1}\boldsymbol{\eta}(\lambda) = \mathrm{diag}\{P(\lambda)^{-1}, I_b\}$, that is,

$$\begin{bmatrix} I_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I_u \\ \lambda I - (D_1 + D_2(s)) & \mathbf{0} & \Sigma^{-1} & \mathbf{0} \end{bmatrix} (\lambda I - L(s))^{-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & I_b \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} \\ -\Sigma & \mathbf{0} & \mathbf{0} & \Sigma[\lambda I - (D_1 - D_2(s))] \\ \mathbf{0} & \mathbf{0} & I_u & \mathbf{0} \end{bmatrix} = \begin{bmatrix} P(\lambda)^{-1} & \mathbf{0} \\ \mathbf{0} & I_b \end{bmatrix}.$$

Therefore, it holds that $V(\lambda I - L(s))^{-1}U = P(\lambda)^{-1}$.

(b) Define

$$Q_{d+u}^r = \begin{pmatrix} Q_{d,d} & Q_{d,u} \\ Q_{u,d} & Q_{u,u} \end{pmatrix}, \quad \Lambda_{d+u}^r = \mathrm{diag}\left\{-[Q_{d+u}^r]_{ii}, i \in S_d \cup S_u\right\},$$

and $Q_{d+u}^* = \left[\mathrm{diag}\{\Delta_{\boldsymbol{\mu}_d}^{-1}, -\Delta_{\boldsymbol{\mu}_u}^{-1}\}(Q_{d+u}^r + \Lambda_{d+u}^r)\right]'$. Noting that $P(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$ with A_0, A_1, A_2 being defined in (13), we can observe that $det\,[P(\lambda)]$ is a polynomial of degree $(2|S_b| + |S_d| + |S_u|)$ and that $P^{-1}(\lambda)$ can be represented as

$$P^{-1}(\lambda) = \lambda^{-1}\mathrm{diag}\{\mathbf{0}_b, I_d, I_u\} + \lambda^{-2}\mathrm{diag}\{I_b, -Q_{d+u}^*\} + \lambda^{-3}Z_1 + \cdots,$$

for some matrices $Z_1, Z_2, \cdots$ and for $|\lambda|$ sufficiently large. Using complex integral for a circle Γ in the complex plane having zeros of $P(\lambda)$ in its interior,

$$\frac{1}{2\pi i}\oint_\Gamma P^{-1}(\lambda)d\lambda = \mathrm{diag}\{\mathbf{0}_b, I_d, I_u\} \text{ and } \frac{1}{2\pi i}\oint_\Gamma \lambda P^{-1}(\lambda)d\lambda = \mathrm{diag}\{I_b, -Q_{d+u}^*\}. \tag{28}$$

But we may also choose Γ large enough so that

$$\frac{1}{2\pi i}\oint_\Gamma \lambda^i(\lambda I - L(s))^{-1}d\lambda = L(s)^i, i = 0, 1, 2, \cdots . \tag{29}$$

Noting that, for $i, j = 0, 1, 2, \cdots$,

$$VL(s)^i L(s)^j U = VL(s)^{i+j}U = V\frac{1}{2\pi i}\oint_\Gamma \lambda^{i+j}(\lambda I - L(s))^{-1}d\lambda U = \frac{1}{2\pi i}\oint_\Gamma \lambda^{i+j}P(\lambda)^{-1}d\lambda$$

and Equation (28), we have

$$\begin{bmatrix} V \\ VL(s) \end{bmatrix}\begin{bmatrix} U & L(s)U \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi i}\oint_\Gamma P(\lambda)^{-1}d\lambda & \frac{1}{2\pi i}\oint_\Gamma \lambda P(\lambda)^{-1}d\lambda \\ \frac{1}{2\pi i}\oint_\Gamma \lambda P(\lambda)^{-1}d\lambda & \frac{1}{2\pi i}\oint_\Gamma \lambda^2 P(\lambda)^{-1}d\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0}_b & \mathbf{0} & \mathbf{0} & I_b & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_d & \mathbf{0} & \mathbf{0} & * & * \\ \mathbf{0} & \mathbf{0} & I_u & \mathbf{0} & * & * \\ I_b & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & * & * & & & \\ \mathbf{0} & * & * & & & \end{bmatrix}.$$

From the structure of the last matrix, we can observe that its rank is greater than or equal to $2|S_b|+|S_d|+|S_u|$. But the number of columns of $\left[\begin{smallmatrix} V \\ VL(s) \end{smallmatrix}\right]$ and the number of the rows of $\left[\begin{smallmatrix} U & L(s)U \end{smallmatrix}\right]$ equal to $2|S_b| + |S_d| + |S_u|$. Hence, the ranks of $\left[\begin{smallmatrix} V \\ VL(s) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} U & L(s)U \end{smallmatrix}\right]$ are equal to $2|S_b| + |S_d| + |S_u|$.

(c) Using (29), we can also observe that

$$\begin{aligned} \mathbf{0} &= \left[\frac{1}{2\pi i}\oint_\Gamma P(\lambda)P(\lambda)^{-1}d\lambda \quad \frac{1}{2\pi i}\oint_\Gamma \lambda P(\lambda)P(\lambda)^{-1}d\lambda\right] \\ &= \left[A_0V + A_1VL(s) + A_2VL(s)^2\right]\left[U \quad L(s)U\right], \quad \text{and} \\ \mathbf{0} &= \begin{bmatrix} \frac{1}{2\pi i}\oint_\Gamma P(\lambda)^{-1}P(\lambda)d\lambda \\ \frac{1}{2\pi i}\oint_\Gamma P(\lambda)^{-1}P(\lambda)d\lambda \end{bmatrix} = \begin{bmatrix} V \\ VL(s) \end{bmatrix}\left[UA_0 + L(s)UA_1 + L(s)^2UA_2\right]. \end{aligned}$$

Since $\left[\begin{smallmatrix} V \\ VL(s) \end{smallmatrix}\right]$ is a full-column-rank matrix and $\left[\begin{smallmatrix} U & L(s)U \end{smallmatrix}\right]$ is a full-row-rank matrix, we have that

$$\mathbf{0} = A_0V + A_1VL(s) + A_2VL(s)^2 \quad \text{and} \quad \mathbf{0} = UA_0 + L(s)UA_1 + L(s)^2UA_2,$$

which completes the proofs.

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