

Hopf-Galois structures on
extensions of degree p^2q
and skew braces of order p^2q :
the cyclic Sylow p -subgroup case

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Abstract

Let p, q be distinct primes, with $p > 2$.

We classify the Hopf-Galois structures on Galois extensions of degree p^2q , such that the Sylow p -subgroups of the Galois group are cyclic.

This we do, according to Greither and Pareigis, and Byott, by classifying the regular subgroups of the holomorphs of the groups (G, \cdot) of order p^2q , in the case when the Sylow p -subgroups of G are cyclic. This is equivalent to classifying the skew braces (G, \cdot, \circ) .

Furthermore, we prove that if G and Γ are groups of order p^2q with non-isomorphic Sylow p -subgroups, then there are no regular subgroups of the holomorph of G which are isomorphic to Γ . Equivalently, a Galois extension with Galois group Γ has no Hopf-Galois structures of type G .

Our method relies on the alternate brace operation \circ on G , which we use mainly indirectly, that is, in terms of the functions $\gamma : G \rightarrow \text{Aut}(G)$ defined by $g \mapsto (x \mapsto (x \circ g) \cdot g^{-1})$. These functions are in one-to-one correspondence with the regular subgroups of the holomorph of G , and are characterised by the functional equation $\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h)$, for $g, h \in G$. We develop methods to deal with these functions, with the aim of making their enumeration easier, and more conceptual.

Keywords: Hopf-Galois extensions, Hopf-Galois structures, holomorph, regular subgroups, braces, skew braces

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1. Introduction

1.1. The general problem, and the classical approach

Let L/K be a finite Galois field extension, and let $\Gamma = \text{Gal}(L/K)$. Then the group algebra $K[\Gamma]$ is a K -Hopf algebra, and its natural action on L endows L/K with a Hopf-Galois structure. In general this is not the only Hopf-Galois structure on L/K , and the study of Galois module structures different from the classical one is important, for example in the context of algebraic number theory. In fact, when L/K is a wildly ramified extension of local fields, there are cases in which the ring of integers \mathcal{O}_L of L is free as a module over a Hopf order in some K -Hopf algebra H , but not in $K[\Gamma]$ (see Child's book [Chi00] for an overview and for the specific results).

Greither and Pareigis [GP87] showed that the Hopf-Galois structures on L/K correspond to the regular subgroups G of the group $S(\Gamma)$ of permutations on the set Γ , which are normalised by the image $\rho(\Gamma)$ of the right

regular representation ρ of Γ (in the relevant literature it is common to use the left regular representation λ instead of the right one ρ we are employing here. We have translated the statements in the literature from left to right).

The groups G and Γ have the same cardinality but they need not be isomorphic. We will say that a Hopf-Galois structure is of *type* G if G is the group associated to it in the Greither-Pareigis correspondence.

As usual we denote by $e(\Gamma, G)$ the number of regular subgroups of $S(\Gamma)$ normalised by $\rho(\Gamma)$, which are isomorphic to G . Equivalently, $e(\Gamma, G)$ is the number of Hopf-Galois structures of type G on a Galois field extension with Galois group isomorphic to Γ .

The direct determination of all regular subgroups of $S(\Gamma)$ normalised by $\rho(\Gamma)$ is in general a difficult task, since the groups $S(\Gamma)$ is large. However, Childs [Chi89] and Byott [Byo96] observed that the condition that $\rho(\Gamma)$ normalises G can be reformulated by saying that Γ is contained in the holomorph $\text{Hol}(G)$ of G , regarded as a subgroup of $S(G)$. This translation turns out to be very useful, since $\text{Hol}(G)$ is usually much smaller than $S(\Gamma)$. One obtains the following result.

Theorem 1.1 ([Byo96, Corollary p. 3320]). *Let L/K be a finite Galois field extension with Galois group Γ . For any group G with $|G| = |\Gamma|$, let $e'(\Gamma, G)$ be the number of regular subgroups of $\text{Hol}(G)$ isomorphic to Γ .*

Then the number $e(\Gamma, G)$ of Hopf-Galois structures on L/K of type G is given by

$$e(\Gamma, G) = \frac{|\text{Aut}(\Gamma)|}{|\text{Aut}(G)|} e'(\Gamma, G). \quad (1.1)$$

Moreover $e(\Gamma)$, the total number of Hopf-Galois structures on L/K , is given by $\sum_G e(\Gamma, G)$ where the sum is over all isomorphism types G of groups of order $|\Gamma|$.

There is a rich literature on Hopf-Galois structures on various classes of field extensions (for a survey up to 2000, see the already mentioned book by Childs [Chi00]). For example, for an odd prime p it is known that there are p^{m-1} Hopf-Galois structures on a cyclic extension of degree p^m , and they are all of cyclic type [Koh98]. An elementary abelian extension of degree p^m , with $p > m$, has at least $p^{m(m-1)-1}(p-1)$ Hopf-Galois structures of abelian type (see [Chi05]). An abelian, non cyclic, extension of degree p^m admits also structures of non-abelian type for $m \geq 3$ [BC12].

The exact number of Hopf-Galois structures is known for some families of Galois extensions, such as those of order the square of a prime, [Byo96], of order the product of two primes [Byo04], of order the cube of a prime [Zen18] and for the cyclic extensions of squarefree order [AB18].

An interesting issue is also to determine which general properties of either Γ or G force those of the other. For certain Galois groups Γ it is known that every Hopf-Galois structure must have type Γ (see [BC12]). For a Galois extension whose Galois group Γ is abelian, the type G of any Hopf-Galois structure must be soluble [Byo15, Theorem 2], although for a soluble, non-abelian Galois group Γ there can be Hopf-Galois structures whose type is not soluble [Byo15, Corollary 3]; see also [TQ19].

The study of Hopf-Galois structures, that is, of regular subgroups of holomorphs, has also a deep connection with the theory of *skew braces*. In fact, if G is a group with respect to the operation “ \cdot ”, classifying the regular subgroups of $\text{Hol}(G)$ is equivalent to determining the operations “ \circ ” on G such that (G, \cdot, \circ) is a (right) skew brace [GV17], that is, (G, \circ) is also a group, and the two group structures on the set G are related by

$$(g \cdot h) \circ k = (g \circ k) \cdot k^{-1} \cdot (h \circ k). \quad (1.2)$$

This connection was first observed by Bachiller in [Bac16, §2] and it is described in detail in the appendix to [SV18].

1.2. The methods

The main goal of this paper is to classify the Hopf-Galois structures on a Galois extension L/K of order p^2q , where p, q are distinct primes and p is odd, such that $\Gamma = \text{Gal}(L/K)$ has a cyclic Sylow p -subgroup. According to the discussion above (but see Subsection 2.7 for a technicality) we do this by following Byott’s approach, that is, by determining the regular subgroups of $\text{Hol}(G)$, for each group $G = (G, \cdot)$ of order p^2q , where p, q are distinct primes and p is odd, such that G has a cyclic Sylow p -subgroup. This is in turn equivalent to determining the right skew braces (G, \cdot, \circ) such that $(G, \circ) \cong \Gamma$ (see [Ven19, Problem 16]).

Our method relies on the use of the alternate brace operation \circ on G , mainly indirectly, that is through the use of the function

$$\begin{aligned} \gamma : G &\rightarrow \text{Aut}(G) \\ g &\mapsto (x \mapsto (x \circ g) \cdot g^{-1}), \end{aligned}$$

which is characterised by the functional equation

$$\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h). \quad (1.3)$$

(See Theorem 2.2 and the ensuing discussion for the details.) The functions γ satisfying (1.3) are in one-to-one correspondence with the regular subgroups of $\text{Hol}(G)$, and occur naturally in the theory of (skew) braces: they are called

μ in [Rum07b], and λ in the literature of skew braces [GV17]. They have been exploited, albeit in a somewhat different context, in [CDVS06, CDV17, CDV18, Car18]. It follows that to determine the number $e'(\Gamma, G)$ defined in Theorem 1.1 we can count the number of functions $\gamma : G \rightarrow \text{Aut}(G)$ verifying (1.3) and such that, for the operation \circ defined on G by

$$g \circ h = g^{\gamma(h)} h,$$

we have $(G, \circ) \cong \Gamma$.

In Section 2 we develop methods to deal with these gamma functions, that will make enumerating them easier. As a side effect, our methods allow us also to give alternative proofs of some results in the literature. In Subsection 2.4 we give a proof of the results of Byott [Byo13, Theorem 1] about the case of finite nilpotent groups. In Subsection 2.9, as a preliminary to our classification, we give a compact treatment of the case of groups of order pq , with p, q distinct primes, dealt with by Byott in [Byo04] (see also [AB19a]). In Subsection 2.10 we present a proof of a generalisation of a result of Kohl [Koh13, Koh16].

1.3. Hopf-Galois structures of order p^2q

Starting with Section 3, we restrict our consideration to the groups of order p^2q , where p and q are distinct primes; those with cyclic Sylow p -subgroups are listed in Table 3.1.

Applying the results of Section 2, we show that for $p > 2$ a Galois field extension L/K with group Γ admits Hopf-Galois structures of type G only for those G such that G and Γ have isomorphic Sylow p -subgroups (see Theorem 3.3 and Corollary 3.4).

In Section 4 we show that if Γ has cyclic Sylow p -subgroups and $p > 2$, each G with cyclic Sylow p -subgroups defines some Hopf-Galois structure on L/K . We then explicitly determine the number of structures for each type.

We accomplish this by calculating, given a group $G = (G, \cdot)$ of order p^2q , $p > 2$, the following equivalent data (see Theorem 2.2 and the ensuing discussion) for each group Γ of order p^2q with cyclic Sylow p -subgroups

1. the total number, and the number and lengths of conjugacy classes within $\text{Hol}(G)$, of regular subgroups isomorphic to Γ ;
2. the total number, and the number of isomorphism classes, of (right) skew braces (G, \cdot, \circ) such that $\Gamma \cong (G, \circ)$.

Remark 1.2. Here and in the following, by the *(total) number of skew braces* we mean, given a group (G, \cdot) , the number of distinct operations \circ on the set G such that (G, \cdot, \circ) is a skew brace.

The following theorems summarise our results, where the notation for the groups is that given in Table 3.1.

Theorem 1.3. *Let L/K be a Galois field extension of order p^2q , where p and q are two distinct primes with $p > 2$, and let $\Gamma = \text{Gal}(L/K)$.*

Let G be a group of order p^2q .

If the Sylow p -subgroups of G and Γ are not isomorphic, then there are no Hopf-Galois structures of type G on L/K .

If the Sylow p -subgroups of Γ and G are cyclic, then the numbers $e(\Gamma, G)$ of Hopf-Galois structures of type G on L/K are given in the following tables.

(i) *For $q \nmid p - 1$:*

$\Gamma \backslash G$	1	2	3
1	p	$2p(p-1)$	$2p(p-1)$
2	pq	$2p(pq-2q+1)$	$2pq(p-1)$
3	pq	$2pq(p-1)$	$2(p^2q-pq-q+1)$

where the upper left sub-tables of sizes 1×1 , 2×2 and 3×3 give respectively the cases $p \nmid q - 1$, $p \parallel q - 1$ and $p^2 \mid q - 1$.

(ii) *For $q \mid p - 1$:*

$\Gamma \backslash G$	1	4
1	p	$2p(q-1)$
4	p^2	$2(p^2q-2p^2+1)$

Theorem 1.4. *Let $G = (G, \cdot)$ be a group of order p^2q , where p, q are distinct primes, with $p > 2$.*

If Γ is a group of order p^2q and the Sylow p -subgroups of G and Γ are not isomorphic, then no regular subgroup of $\text{Hol}(G)$ is isomorphic to Γ .

If G and Γ have both cyclic Sylow p -subgroups, then the following tables give equivalently

1. *the number $e'(\Gamma, G)$ of regular subgroups of $\text{Hol}(G)$ isomorphic to Γ ;*
2. *the number of (right) skew braces (G, \cdot, \circ) such that $\Gamma \cong (G, \circ)$.*

(i) *For $q \nmid p - 1$:*

$\Gamma \backslash G$	1	2	3
1	p	$2pq$	$2q$
2	$p(p-1)$	$2p(pq-2q+1)$	$2q(p-1)$
3	$p^2(p-1)$	$2p^2q(p-1)$	$2(p^2q-pq-q+1)$

where the upper left sub-tables of sizes 1×1 , 2×2 and 3×3 give respectively the cases $p \nmid q - 1$, $p \parallel q - 1$ and $p^2 \mid q - 1$.

(ii) For $q \mid p - 1$:

$\Gamma \backslash G$	1	4
1	p	$2p^3$
4	$q - 1$	$2(p^2q - 2p^2 + 1)$

For each group Γ of order p^2q with cyclic Sylow p -subgroups, the following tables give equivalently

1. the number of conjugacy classes within $\text{Hol}(G)$ of regular subgroups isomorphic to Γ ;
2. the number of isomorphism classes of skew braces (G, \cdot, \circ) such that $\Gamma \cong (G, \circ)$.

(i) For $q \nmid p - 1$:

$\Gamma \backslash G$	1	2	3
1	2	$2p$	2
2	p	$2p(p - 1)$	$2(p - 1)$
3	p	$2p(p - 1)$	$2p(p - 1)$

where the upper left sub-tables of sizes 1×1 , 2×2 and 3×3 give respectively the cases $p \nmid q - 1$, $p \parallel q - 1$ and $p^2 \mid q - 1$.

(ii) For $q \mid p - 1$:

$\Gamma \backslash G$	1	4
1	2	4
4	1	$2(q - 1)$

The lengths of the conjugacy classes are spelled out in Propositions 4.2, 4.4, 4.3, and 4.5.

From Theorem 1.3 we obtain the total number of Hopf-Galois structures.

Corollary 1.5. *Let L/K be a Galois extension of order p^2q , where p and q are two distinct primes with $p > 2$, and let $\Gamma = \text{Gal}(L/K)$.*

Assume that Γ has cyclic Sylow p -subgroups.

The total number of the Hopf-Galois structures on L/K is given by the following table.

<i>Conditions</i>	Γ	<i>Hopf-Galois structures</i>
$q \nmid p-1, p \nmid q-1$	1	p
$q \nmid p-1, p \parallel q-1$	1	$p(2p-1)$
$q \nmid p-1, p^2 \mid q-1$	1	$p(4p-3)$
$q \mid p-1$	1	$p(2q-1)$
$q \nmid p-1, p \parallel q-1$	2	$p(2pq-3q+2)$
$q \nmid p-1, p^2 \mid q-1$	2	$p(4pq-5q+2)$
$q \nmid p-1, p^2 \mid q-1$	3	$4p^2q-3pq-2q+2$
$q \mid p-1$	4	$2p^2q-3p^2+2$

Corollary 1.6. *A cyclic extension L/K of degree p^2q , with $p > 2$, admits exactly p cyclic Hopf-Galois structures.*

Moreover, the cyclic structures are the only Hopf-Galois structures on L/K if and only if there is only one isomorphism type of groups of order p^2q with cyclic Sylow p -subgroup, namely if and only if $p \nmid q-1$ and $q \nmid p-1$.

Note that the first statement of the above corollary follows also from [Byo13].

Finally, in Section 5 we apply the method proposed in [CRV16] to count the cyclic Hopf-Galois structures on a Galois extension of order p^2q . In fact, in this case one can show that all cyclic structures are induced, and the induced structures are easy to compute, so that we recover our results in Theorem 1.3 for the group G of type 1; note that the only induced structures for extensions of order p^2q are the abelian ones.

The case of groups of order p^2q with $p < q-1$ has been dealt with in [Die18] for braces. Braces have been introduced by Rump in [Rum07a], and correspond to the case when the group (G, \cdot) is abelian.

In a separate paper, we intend to deal with the regular subgroups of holomorphs of groups of order p^2q with elementary abelian Sylow p -subgroups, where p, q are distinct primes, and of order $4q$, where q is an odd prime.

Remark 1.7. The reason why we are not treating here the case $p = 2$ is that in the holomorph of a group G of order $4q$ with cyclic Sylow 2-subgroups there are regular subgroups with elementary abelian Sylow 2-subgroups. This boils down to the fact that in the normaliser of the cyclic subgroup $\langle (1234) \rangle$ in the symmetric group on four letters there is a regular subgroup $\langle (13)(24), (12)(34) \rangle$, which is elementary abelian. In Subsection 2.7 we show that this does not happen when p is odd.

1.4. Another classification

After this manuscript was submitted, Aciri and Bonatto posted on arXiv a manuscript [AB19b], in which they employ the methods of [GV17] to count

the isomorphism classes of skew braces of size p^2q for odd primes p and q . In Subsection 2.2 below, we compare their results and methods, and more generally the methods employed in the theory of skew braces, with ours.

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2. The gamma function

2.1. Describing the regular subgroups of the holomorph

There are several equivalent ways to describe the regular subgroups of the holomorph of a group (G, \cdot) . As explained in the Introduction, we will appeal to the alternate group operation \circ which occurs in the definition of a skew brace [GV17], but then mainly use the equivalent method of gamma functions, employed in [CDVS06, CDV17, CDV18, Car18].

The *abstract holomorph* of a group G is the natural semidirect product $\text{Aut}(G)G$. In this paper we will use a certain concrete realisation of it, namely the *permutational holomorph*, as defined in the following.

Given a group G , denote by $S(G)$ the group of all permutations on the underlying set G . The *right regular representation* of G is the homomorphism

$$\begin{aligned} \rho : G &\rightarrow S(G) \\ g &\mapsto (x \mapsto xg). \end{aligned}$$

Similarly, the *left regular representation* of G is the *antihomomorphism*

$$\begin{aligned} \lambda : G &\rightarrow S(G) \\ g &\mapsto (x \mapsto gx). \end{aligned}$$

Write $\text{inv} : g \mapsto g^{-1}$ for the inversion map on G . Clearly $\text{inv} \in S(G)$.

The following facts are well known, see for instance [PS18, Lemma 3.8].

Proposition 2.1.

1. *The stabiliser of 1 in the normaliser $N_{S(G)}(\rho(G))$ of the image $\rho(G)$ of the right regular representation is $\text{Aut}(G)$.*

2. We have

$$N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G) = \text{Aut}(G)\lambda(G) = N_{S(G)}(\lambda(G)),$$

and this group is isomorphic to the abstract holomorph $\text{Aut}(G)G$ of G .

3. inv centralises $\text{Aut}(G)$, and conjugates $\rho(G)$ to $\lambda(G) \leq \text{Hol}(G)$, that is

$$\rho(G)^{\text{inv}} = \lambda(G).$$

Thus inv normalises $N_{S(G)}(\rho(G))$.

In the following we will refer to $N_{S(G)}(\rho(G))$ as the (*permutational*) *holomorph* of G , and denote it by $\text{Hol}(G)$.

Let G be a finite group, and let $N \leq \text{Hol}(G)$ be a regular subgroup. Since N is regular, the map $N \rightarrow G$ sending $n \in N$ to 1^n is a bijection. Thus for each $g \in G$ there is a unique element $\nu(g) \in N$, such that $1^{\nu(g)} = g$, that is, $\nu : G \rightarrow N$ is the inverse of $n \mapsto 1^n$. Now $1^{\nu(g)\rho(g)^{-1}} = 1$, so that $\nu(g)\rho(g)^{-1} \in \text{Aut}(G)$ by Proposition 2.1(1). Therefore for $g \in G$ we can write $\nu(g)$ uniquely in the form

$$\nu(g) = \gamma(g)\rho(g), \tag{2.1}$$

for a suitable map $\gamma : G \rightarrow \text{Aut}(G)$. We have

$$\nu(g)\nu(h) = \gamma(g)\rho(g)\gamma(h)\rho(h) = \gamma(g)\gamma(h)\rho(g^{\gamma(h)}h). \tag{2.2}$$

Since N is a subgroup of $S(G)$, $\gamma(g)\gamma(h) \in \text{Aut}(G)$, and the expression (2.1) is unique, we have

$$\gamma(g)\gamma(h)\rho(g^{\gamma(h)}h) = \gamma(g^{\gamma(h)}h)\rho(g^{\gamma(h)}h),$$

from which we obtain

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h). \tag{2.3}$$

We obtain

Theorem 2.2. *Let (G, \cdot) be a finite group. The following data are equivalent.*

1. A regular subgroup $N \leq \text{Hol}(G)$.
2. A map $\gamma : G \rightarrow \text{Aut}(G)$ such that

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h). \tag{2.4}$$

3. A group operation \circ on G such that for $g, h, k \in G$

$$(gh) \circ k = (g \circ k)k^{-1}(h \circ k), \tag{2.5}$$

that is, such that (G, \cdot, \circ) is a (*right*) skew brace.

The data of (1)-(3) are related as follows.

- (i) $g \circ h = g^{\gamma(h)}h$ for $g, h \in G$.
- (ii) Each element of N can be written uniquely in the form $\nu(h) = \gamma(h)\rho(h)$, for some $h \in G$.
- (iii) For $g, h \in G$ one has $g^{\nu(h)} = g \circ h$.
- (iv) The map

$$\gamma : (G, \circ) \rightarrow \text{Aut}(G)$$

is a morphism.

- (v) The map

$$\begin{aligned} \nu : (G, \circ) &\rightarrow N \\ h &\mapsto \gamma(h)\rho(h) \end{aligned}$$

is an isomorphism.

This is basically [Car18, Theorem 1.2]. The equivalence of (3) to the other items of Theorem 2.2 follows from [GV17, Theorem 4.2]; for the convenience of the reader, we provide the table below detailing the relations between the properties of γ and the brace axioms. In this table, (G, \cdot) is a group, \circ is an operation on G , and for each $g \in G$ we define a function $\gamma(g) : G \rightarrow G$ by $x \mapsto (x \circ g) \cdot g^{-1}$, so that $x \circ g = x^{\gamma(g)} \cdot g$.

Property of \circ	Property of γ
The brace axiom (2.5) of Theorem 2.2 holds	$\gamma(g)$ is an endomorphism of G , for each $g \in G$
\circ is associative	γ satisfies (2.4) of Theorem 2.2
\circ admits inverses	$\gamma(g)$ is bijective, for each $g \in G$

In the table, the properties on the first line are equivalent. The properties on the second line are equivalent, under the assumption that $\gamma(g)$ is an endomorphism of G , for each $g \in G$. On the third line, the property on the right implies the property on the left, while to prove the left-to-right implication one need to assume (2.4) of Theorem 2.2. The fact that (G, \circ) has an identity follows from the properties in the first line.

Remark 2.3. In the rest of the paper, every time we discuss a regular subgroup as in (1) of Theorem 2.2, or a γ as in (2), we will employ the rest of the notation of Theorem 2.2 without further mention.

Definition 2.4. Let G be a group, $A \leq G$, and $\gamma : A \rightarrow \text{Aut}(G)$ a function. γ is said to satisfy the *gamma functional equation* (or *GFE* for short) if

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h), \quad \text{for all } g, h \in A.$$

γ is said to be a *relative gamma function* (or *RGF* for short) on A if it satisfies the gamma functional equation, and A is $\gamma(A)$ -invariant.

If $A = G$, a relative gamma function is simply called a *gamma function* (or *GF* for short) on G .

A RGF $\gamma : A \rightarrow \text{Aut}(G)$ defines a group operation $g \circ h = g^{\gamma(h)}h$ on A .

2.2. A comparison to other results and methods

As mentioned in the Introduction, Acri and Bonatto have posted on arXiv a manuscript [AB19b] whose results overlap with ours. They enumerate the isomorphism classes of skew braces of size p^2q , for odd primes p and q , including thus the case when the Sylow p -subgroups are elementary abelian, which we do not cover in this paper. Equivalently, they enumerate the conjugacy classes of regular subgroups in the holomorph of groups G of order p^2q , for odd primes p and q . We provide in addition the sizes of the conjugacy classes, and include the case $q = 2$. The common results of [AB19b] agree with ours.

They organise their classification, according to the algorithm of [GV17], which they already employed in [AB19a], on the basis of the size of what in our context is the image $\gamma(G) \leq \text{Aut}(G)$ of the gamma function.

We have already noted that in the literature one usually deals with *left* skew braces, whereas we use the *right* ones; as mentioned in the Introduction, we translate results and methods in the literature into our notation.

In the theory of skew braces (G, \cdot, \circ) , a morphism $\lambda : (G, \circ) \rightarrow \text{Aut}(G, \cdot)$ is defined, which is exactly our γ . (We reserve λ for the left regular representation.)

Several intermediate results of [AB19b] naturally overlap with ours, for instance the fact that $\ker(\gamma) \leq (G, \circ)$. In our context, an ideal of a skew brace (G, \cdot, \circ) is a subgroup $H \leq (G, \cdot)$ which is $\gamma(G)$ -invariant; some results of [AB19b] on $\gamma(G)$ -invariant Sylow subgroups are also similar to ours.

Remark 2.5. Our γ are related to the bijective 1-cocycles of [GV17]. Recall that if the group (H, \bullet) acts (on the right) on the group (G, \cdot) via automorphisms, a map $\pi : H \rightarrow G$ is said to be a 1-cocycle if

$$\pi(a \bullet b) = \pi(a)^b \cdot \pi(b), \quad \text{for } a, b \in H,$$

where g^h denotes the action of $h \in H$ on $g \in G$. It is proved in [GV17, Proposition 1.11] that if (G, \cdot, \circ) is a skew brace, then the identity map

$\pi : (G, \circ) \rightarrow (G, \cdot)$ is a 1-cocycle, where G acts on itself, in our notation, by $g^h = g^{\gamma(h)}$. In fact

$$\pi(a \circ b) = a \circ b = a^{\gamma(b)} \cdot b = \pi(a)^b \cdot \pi(b).$$

Conversely, a bijective 1-cocycle induces a skew brace structure (G, \cdot, \circ) via

$$a \circ b = \pi(\pi^{-1}(a) \bullet \pi^{-1}(b)), \quad \text{for } a, b \in G.$$

In our notation, this means that the function

$$\begin{aligned} \gamma : G &\rightarrow \text{Aut}(G) \\ b &\mapsto (a \mapsto \pi(\pi^{-1}(a) \bullet \pi^{-1}(b)) \cdot b^{-1}) \end{aligned}$$

is a GF.

2.3. Invariant subgroups

The following proposition is a slightly more general version of [TQ19, Proposition 3.3]

Proposition 2.6. *Let G be a finite group, let $H \subseteq G$ and let γ be a GF on G .*

Any two of the following conditions imply the third one:

1. $H \leq G$;
2. $(H, \circ) \leq (G, \circ)$;
3. H is $\gamma(H)$ -invariant.

If these conditions hold, then (H, \circ) is isomorphic to a regular subgroup of $\text{Hol}(H)$.

Proof. The equivalence follows from the fact that for $h_1, h_2 \in H$ we have

$$h_1 \circ h_2 = h_1^{\gamma(h_2)} h_2 \quad \text{and} \quad h_1^{\gamma(h_2)^{-1}} \circ h_2 = h_1 h_2.$$

If the conditions hold, the fact that H is $\gamma(H)$ -invariant implies that we have a map

$$\begin{aligned} \gamma' : H &\rightarrow \text{Aut}(H) \\ h &\mapsto \gamma(h)|_H \end{aligned}$$

which satisfies the GFE because γ does, so that γ' is a GF on H . If \circ' is the group operation on H associated to γ' , the identity map

$$(H, \circ') \rightarrow (G, \circ)$$

is a group morphism, as for $h_1, h_2 \in H$ one has

$$h_1 \circ' h_2 = h_1^{\gamma'(h_2)} h_2 = h_1^{\gamma(h_2)} h_2 = h_1 \circ h_2.$$

It follows that $(H, \circ') = (H, \circ)$ is isomorphic to a regular subgroup of $\text{Hol}(H)$. \square

The subgroups H verifying the conditions of Proposition 2.6 in the language of braces are called *sub-skew braces*.

2.4. An application: nilpotent groups

The following result is due to N. P. Byott.

Theorem 2.7 ([Byo13, Theorem 1]). *Let Γ be a finite nilpotent group of order n .*

Then for each nilpotent group G of order n we have

$$e(\Gamma, G) = \prod_p e(\Gamma_p, G_p),$$

where p ranges over the primes dividing n , and Γ_p resp. G_p denote the Sylow p -subgroups of Γ resp. G .

We now give an alternate proof of Byott's result, using gamma functions.

Proof. Let p_1, \dots, p_k be the distinct primes dividing n . The finite nilpotent group G is the direct product of its distinct Sylow subgroups, each of which is characteristic in G , so we have

$$\text{Aut}(G) = \prod_{i=1}^k \text{Aut}(G_{p_i}). \quad (2.6)$$

and the same holds for Γ . Therefore, in view of (1.1), we can rephrase Byott's result as

$$e'(\Gamma, G) = \prod_{i=1}^k e'((\Gamma_{p_i}, G_{p_i}), \quad (2.7)$$

As we already noted in the Introduction

$$e'(\Gamma, G) = |\{ \gamma \text{ GF on } G : (G, \circ) \cong \Gamma \}|.$$

Let now γ be a gamma function on G such that $(G, \circ) \cong \Gamma$. Let p, q be distinct primes dividing n , and let a be a p -element, and b a q -element of

G and (G, \circ) . Since G and (G, \circ) are nilpotent, a and b commute in both groups. We thus have

$$a^{\gamma(b)}b = a \circ b = b \circ a = b^{\gamma(a)}a = ab^{\gamma(a)},$$

which implies $b^{\gamma(a)} = b$ (and $a^{\gamma(b)} = a$).

This shows that the image $\gamma(G_p)$ of the Sylow p -subgroup of G under γ acts trivially on the Sylow q -subgroups of G , for $q \neq p$. The composition of the restriction of γ to G_p , followed by the projection of $\text{Aut}(G)$ onto $\text{Aut}(G_p)$, is clearly a gamma function on G_p .

It follows that every gamma function γ on G is obtained as

$$\gamma(x) = \gamma_1(x_1) \cdots \gamma_k(x_k),$$

where x is uniquely written as $x_1 \cdots x_k$, with $x_i \in G_{p_i}$, and $\gamma_i : G_{p_i} \rightarrow \text{Aut}(G_{p_i}) \leq \text{Aut}(G)$ is a gamma function on G_{p_i} . Proposition 2.6 now implies that the Sylow p -subgroup $(G, \circ)_p$ of (G, \circ) is (G_p, \circ) , which is then isomorphic to Γ_p . This proves equality (2.7). \square

2.5. Isomorphism of Sylow p -subgroups

From Proposition 2.6 we have that if one of the Sylow p -subgroups H of G is invariant under $\gamma(H)$, then the isomorphism type of the Sylow p -subgroups of (G, \circ) is to be found among the isomorphism types of regular subgroups of $\text{Hol}(H)$. If there is no such invariant Sylow p -subgroup, then it is conceivable that the Sylow p -subgroups of regular subgroups of $\text{Hol}(G)$ could take further isomorphism types.

For some classes of p -subgroups the criterion given in Proposition 2.6 can be made more explicit.

Corollary 2.8. *Let G and Γ be finite groups. Suppose $e'(\Gamma, G) \neq 0$. Let $N \leq \text{Hol}(G)$ be a regular subgroup isomorphic to Γ , and γ the GF associated to N .*

If p is an odd prime and H is a $\gamma(H)$ -invariant p -subgroup of G , then

1. *if H is cyclic then $H \cong (H, \circ)$;*
2. *if H is abelian and of rank m , with $m < p - 1$, or $m = 2$ and $p = 3$, and (H, \circ) is abelian too, then $H \cong (H, \circ)$.*

Proof. If H is a cyclic p -group with $p > 2$, by [Rum07b, Corollary, 680] or by [CS19, Proposition 4], every regular subgroup of $\text{Hol}(H)$ is cyclic (see also [Chi00, (8.6) Proposition]).

In the case when H is abelian of rank m , with $m + 1 < p$, or $m = 2$ and $p = 3$, by [FCC12, Theorem 1 and Proposition 4], all the abelian subgroups of $\text{Hol}(H)$ are isomorphic to H .

Both statements now follow from Proposition 2.6. \square

In Theorem 3.3 we will show that in the case of groups of order p^2q with $p > 2$ the conditions of Proposition 2.6 are fulfilled for H a Sylow p -subgroup of G and for each GF. This will simplify our classification since it implies that $e(\Gamma, G) = 0$ whenever the Sylow p -subgroups of Γ and G are not isomorphic (we have already noted in Remark 1.7 that this does not hold for $p = 2$). We will also prove (see Theorem 1.3) that in that case the condition of having isomorphic Sylow p -subgroups is also sufficient for $e(\Gamma, G) \neq 0$.

2.6. Tools of the trade

We now collect several facts related to the gamma functions, which we are going to exploit for our classification.

We begin by rephrasing conjugacy of regular subgroups within $\text{Hol}(G)$ in terms of gamma functions, and recording a simple property that will be useful in Section 4. Conjugacy of regular subgroups of the holomorph of the group (G, \cdot) is equivalent to isomorphism of skew braces (G, \cdot, \circ) , and to a certain equivalence of Hopf-Galois structures.

Note that since N is regular, we have $\text{Hol}(G) = \text{Aut}(G)N$, so that the conjugates of N under $\text{Hol}(G)$ coincide with the conjugates under $\text{Aut}(G)$. Now [SV18, Proposition A.3] (see also the last statement of [CDVS06, Theorem 1]) states that, given a group $G = (G, \cdot)$, there is a bijection between isomorphism classes of skew braces (G, \cdot, \circ) , and classes of regular subgroups of $\text{Hol}(G)$ under conjugation by elements of $\text{Aut}(G)$. We give the simple translation of the latter in terms of gamma functions.

Lemma 2.9. *Let G be a group, N a regular subgroup of $\text{Hol}(G)$, and γ the associated gamma function.*

Let $\beta \in \text{Aut}(G)$.

1. *The gamma function γ^β associated to the regular subgroup N^β is given by*

$$\gamma^\beta(g) = \gamma(g^{\beta^{-1}})^\beta = \beta^{-1}\gamma(g^{\beta^{-1}})\beta, \quad (2.8)$$

for $g \in G$.

2. *If $H \leq G$ is invariant under $\gamma(H)$, then H^β is invariant under $\gamma(H^\beta)$.*

We will refer to the action of $\text{Aut}(G)$ on γ of the Lemma as *conjugation*.

Proof. For $x \in G$ we have $\nu(x)^\beta = \gamma(x)^\beta \rho(x^\beta)$. Since $\nu(x)^\beta$ takes 1 to x^β , we have $\gamma^\beta(x^\beta) = \gamma(x)^\beta$, which yields (2.8) substituting $g = x^\beta$.

If $H \leq G$ is invariant under $\gamma(H)$, and $h_1, h_2 \in H$, then

$$(h_1^\beta)^{\gamma^\beta(h_2^\beta)} = h_1^{\beta\beta^{-1}\gamma(h_2)^\beta} = (h_1^{\gamma(h_2)})^\beta \in H^\beta.$$

□

We now record two simple facts, which we will be using repeatedly, concerning inverses and conjugacy in the group (G, \circ) of Theorem 2.2. We write $a^{\ominus 1}$ for the inverse of $a \in G$ in (G, \circ) .

Lemma 2.10. *In the notation of Theorem 2.2, we have, for $a, b \in G$,*

$$a^{\ominus 1} = a^{-\gamma(a)^{-1}},$$

and

$$a^{\ominus 1} \circ b \circ a = a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)}b^{\gamma(a)}a.$$

Proof. If z is the inverse of a in (G, \circ) , we have $1 = z \circ a = z^{\gamma(a)}a$, whence $z = a^{-\gamma(a)^{-1}}$.

$$\begin{aligned} a^{\ominus 1} \circ b \circ a &= a^{-\gamma(a)^{-1}} \circ b \circ a \\ &= (a^{-\gamma(a)^{-1}\gamma(b)}b) \circ a \\ &= a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)}b^{\gamma(a)}a. \end{aligned}$$

□

Remark 2.11. Note, for later usage, that (2.4) can be rephrased, setting $k = g^{\gamma(h)}$, as

$$\gamma(kh) = \gamma(k^{\gamma(h)^{-1}})\gamma(h). \quad (2.9)$$

Lemma 2.12. *Let G be a finite group, and γ a GF on G . We have*

1. $\ker(\gamma) \trianglelefteq (G, \circ)$, and
2. $\ker(\gamma) \leq G$.

Proof. The first claim is clear as $\gamma : (G, \circ) \rightarrow \text{Aut}(G)$ is a morphism (Theorem 2.2(iv)).

Since $\ker(\gamma)$ is invariant under $\gamma(\ker(\gamma)) = \{1\}$, Proposition 2.6 implies $\ker(\gamma) \leq G$. □

In the statement of the next result we write $[g, \alpha] = g^{-1}g^\alpha$, for $g \in G$, $\alpha \in \text{Aut}(G)$. This is indeed an ordinary commutator in the abstract holomorph of G . We write

$$[A, \gamma(A)] = \{[x, \gamma(y)] : x, y \in A\}.$$

Lemma 2.13. *Let G be a finite group, $A \leq G$, and $\gamma : A \rightarrow \text{Aut}(G)$ a function such that A is invariant under $\gamma(A)$.*

Then any two of the following conditions imply the third one.

1. $\gamma([A, \gamma(A)]) = \{1\}$.

2. $\gamma : A \rightarrow \text{Aut}(G)$ is a morphism of groups.
3. γ satisfies the GFE.

Proof. Suppose $\gamma([A, \gamma(A)]) = \{1\}$. If γ is a morphism, then for $x, y \in A$ we have

$$\gamma(x^{\gamma(y)}y) = \gamma(x[x, \gamma(y)]y) = \gamma(x)\gamma([x, \gamma(y)])\gamma(y) = \gamma(x)\gamma(y),$$

that is, γ satisfies the GFE. Conversely, suppose γ satisfies the GFE, so that $\gamma(A)$ is a subgroup of $\text{Aut}(G)$. Then for $x, y \in A$ we have

$$\begin{aligned} \gamma(xy) &= \gamma(x^{\gamma(y)^{-1}})\gamma(y) \\ &= \gamma(x[x, \gamma(y)^{-1}])\gamma(y) \\ &= \gamma(x^{\gamma([x, \gamma(y)^{-1}])^{-1}})\gamma([x, \gamma(y)^{-1}])\gamma(y) \\ &= \gamma(x)\gamma(y). \end{aligned}$$

Suppose now γ is a morphism and satisfies the GFE. Then for $x, y \in A$ we have

$$\gamma(x)\gamma(y) = \gamma(x^{\gamma(y)}y) = \gamma(x[x, \gamma(y)]y) = \gamma(x)\gamma([x, \gamma(y)])\gamma(y),$$

so that $\gamma([x, \gamma(y)]) = 1$. □

2.7. Lifting and restriction

The next result can be considered as a vestigial form of the First Isomorphism Theorem for gamma functions.

We write

$$\begin{aligned} \iota : G &\rightarrow \text{Aut}(G) \\ g &\mapsto (x \mapsto g^{-1}xg). \end{aligned}$$

Proposition 2.14. *Let G be a finite group and let A, B be subgroups of G such that $G = AB$.*

If γ is a GF on G , and $B \leq \ker(\gamma)$, then

$$\gamma(ab) = \gamma(a), \text{ for } a \in A, b \in B, \tag{2.10}$$

so that $\gamma(G) = \gamma(A)$.

Moreover, if A is $\gamma(A)$ -invariant, then

$$\gamma' = \gamma|_A : A \rightarrow \text{Aut}(G) \tag{2.11}$$

is a RGF on A and $\ker(\gamma)$ is invariant under the subgroup

$$\{ \gamma'(a)\iota(a) : a \in A \}$$

of $\text{Aut}(G)$.

Conversely, let $\gamma' : A \rightarrow \text{Aut}(G)$ be a RGF such that

1. $\gamma'(A \cap B) \equiv 1$,
2. B is invariant under $\{\gamma'(a)\iota(a) : a \in A\}$.

Then the map

$$\gamma(ab) = \gamma'(a), \text{ for } a \in A, b \in B,$$

is a well defined GF on G , and $\ker(\gamma) = \ker(\gamma')B$.

In this situation we will say that γ is a *lifting* of γ' .

Proof. Clearly γ is constant on the cosets of $\ker(\gamma)$, and thus also on the cosets of B , so that (2.10) holds, and thus $\gamma(G) = \gamma(A)$. Assume now that A is $\gamma(A)$ -invariant; by Proposition 2.6, A is a subgroup of (G, \circ) and γ' as in (2.11) satisfies the GFE, so that γ' is a RGF.

For $a \in A$ and $k \in \ker(\gamma)$ we have

$$a^{\ominus 1} \circ k \circ a = a^{-\gamma(a)-1\gamma(k)\gamma(a)} k^{\gamma(a)} a = a^{-1} k^{\gamma(a)} a = k^{\gamma(a)\iota(a)},$$

and since $a^{\ominus 1} \circ k \circ a \in \ker(\gamma)$, we get $k^{\gamma(a)\iota(a)} \in \ker(\gamma)$, namely, $\ker(\gamma)$ is invariant under the action of $\{\gamma'(a)\iota(a) : a \in A\}$.

Note that the latter is a subgroup of $\text{Aut}(G)$, as for $a_1, a_2 \in A$ we have

$$\begin{aligned} \gamma(a_1)\iota(a_1)\gamma(a_2)\iota(a_2) &= \gamma(a_1)\gamma(a_2)\iota(a_1^{\gamma(a_2)})\iota(a_2) \\ &= \gamma(a_1^{\gamma(a_2)}a_2)\iota(a_1^{\gamma(a_2)}a_2), \end{aligned}$$

with $a_1^{\gamma(a_2)}a_2 \in A$, as A is $\gamma(A)$ -invariant.

Conversely, let $\gamma' : A \rightarrow \text{Aut}(G)$ be a RGF such that (1) and (2) hold, and define $\gamma(ab) = \gamma'(a)$ for each $a \in A, b \in B$. The map γ is well-defined; in fact for $i = 1, 2$, let $a_i \in A$ and $b_i \in B$ be such that $a_1b_1 = a_2b_2$; then $a_1 = a_2b_2b_1^{-1}$, so $b = b_2b_1^{-1} \in A \cap B$ and

$$\gamma'(a_1) = \gamma'(a_2b) = \gamma'(a_2^{\gamma'(b)^{-1}})\gamma'(b) = \gamma'(a_2).$$

Moreover

$$\gamma(a_1b_1)\gamma(a_2b_2) = \gamma'(a_1)\gamma'(a_2) = \gamma'(a_1^{\gamma(a_2)}a_2)$$

and

$$\begin{aligned} \gamma((a_1b_1)^{\gamma(a_2b_2)}a_2b_2) &= \gamma(a_1^{\gamma(a_2)}b_1^{\gamma(a_2)}a_2b_2) \\ &= \gamma(a_1^{\gamma(a_2)}a_2b_1^{\gamma(a_2)\iota(a_2)}b_2) \\ &= \gamma'(a_1^{\gamma(a_2)}a_2), \end{aligned}$$

where the last equality holds because of (2); thus γ is a GF on G . Finally, $\gamma(ab) = \gamma'(a) = 1$ if and only if $a \in \ker(\gamma')$, so $\ker(\gamma) = \ker(\gamma')B$. \square

Corollary 2.15. *In the notation of Proposition 2.14, let γ be the lifting of γ' to G . Then γ is a morphism if and only if γ' is a morphism and $\ker(\gamma)$ is a normal subgroup of G .*

Proof. Clearly, if γ is a morphism then so is its restriction γ' , and $\ker(\gamma)$ is a normal subgroup of G .

Conversely, if γ' is a morphism and $\ker(\gamma)$ is a normal subgroup of G then for $a_i \in A$ and $b_i \in B$, $i = 1, 2$, we have

$$\gamma(a_1 b_1 a_2 b_2) = \gamma(a_1 a_2 b_1^{a_2} b_2) = \gamma'(a_1 a_2) = \gamma'(a_1) \gamma'(a_2) = \gamma(a_1 b_1) \gamma(a_2 b_2).$$

□

We now aim at establishing a criterion (Proposition 2.19) that allows us to define a map $\gamma' : A \rightarrow \text{Aut}(G)$ which verifies the GFE, in the case when A is a cyclic p -group.

First, we state separately two elementary arithmetic lemmas which will be useful in the following. The first one is well-known.

Lemma 2.16. *Let $p > 2$ be a prime and let $n > m \geq 0$ be integers. The solutions of the congruence*

$$x^{p^m} \equiv 1 \pmod{p^n} \tag{2.12}$$

are the integers of type $x = 1 + hp^{n-m}$.

Lemma 2.17. *Let $p > 2$ be a prime and let $s \in \mathbb{Z}$, $s \equiv 1 \pmod{p}$. Define $e_s(0) = 0$ and for each $k > 0$*

$$e_s(k) = \sum_{i=0}^{k-1} s^i.$$

Then, for each $n \in \mathbb{N}$, the set $\{e_s(0), \dots, e_s(p^n - 1)\}$ is a set of representatives of the classes modulo p^n .

Proof. We will show that for $k, h \in \mathbb{Z}$

$$e_s(k) \equiv e_s(h) \pmod{p^n} \iff k \equiv h \pmod{p^n}$$

and this implies the lemma. Let

$$s = 1 + p^l h \quad \text{with } \gcd(h, p) = 1; \tag{2.13}$$

by hypothesis $l > 0$. If $k > h \geq 0$, taking into account equality (2.13), we have

$$\begin{aligned}
e_s(k) \equiv e_s(h) \pmod{p^n} &\iff s^h \sum_{i=0}^{k-h-1} s^i \equiv 0 \pmod{p^n} \\
&\iff \sum_{i=0}^{k-h-1} s^i \equiv 0 \pmod{p^n} \\
&\iff (s-1) \sum_{i=0}^{k-h-1} s^i \equiv 0 \pmod{p^{n+l}} \\
&\iff s^{k-h} \equiv 1 \pmod{p^{n+l}}.
\end{aligned}$$

By Lemma 2.16, $s = 1 + p^l h$ ($\gcd(h, p) = 1$) is a solution of the last equation if and only if $k - h \equiv 0 \pmod{p^n}$. \square

Corollary 2.18. *Let G be a finite group, and let $A = \langle a \rangle$ be a cyclic subgroup of G of order p^n , where p is an odd prime. Let*

$$\gamma : A \rightarrow \text{Aut}(G)$$

be a RGF and let $a^{\gamma(a)} = a^s$.

Then, for each k ,

$$a^{\circ k} = a^{e_s(k)}, \tag{2.14}$$

so $\text{ord}_{(A, \circ)}(a) = \text{ord}_A(a)$ and (A, \circ) is generated by a .

Proof. The equality in (2.14) can be easily shown by induction; the corollary follows then from Lemma 2.17 since $a^{\circ k} \in A$ for each k . \square

Proposition 2.19. *Let G be a finite group, and let $A = \langle a \rangle$ be a cyclic subgroup of G of order p^n , where p is an odd prime.*

Let $\eta \in \text{Aut}(G)$.

The following are equivalent.

1. *There is a RGF*

$$\gamma : A \rightarrow \text{Aut}(G)$$

such that $\gamma(a) = \eta$.

2. (a) *A is η -invariant, and*
(b) *$\text{ord}(\eta) \mid p^n$.*

When these conditions hold, γ is uniquely defined.

Proof. Assume first that the map $\gamma: A \rightarrow \text{Aut}(G)$ is a RGF, and let $\gamma(a) = \eta$. Then, $\gamma: (A, \circ) \rightarrow \text{Aut}(G)$ is a morphism, so

$$\text{ord}(\eta) \mid \text{ord}_{(A, \circ)}(a) = \text{ord}_A(a) = p^n$$

where the first equality follows from Corollary 2.18.

As to the converse, assume (2) holds. Then $\eta|_A \in \text{Aut}(A) \cong (\mathbb{Z}/p^n\mathbb{Z})^*$ and, if $\eta(a) = a^s$, then $s \in (\mathbb{Z}/p^n\mathbb{Z})^*$ and

$$\text{ord}(s) = \text{ord}(\eta|_A) \mid \text{gcd}(p^n, \phi(p^n)) = p^{n-1},$$

so that, by Lemma 2.16,

$$s \equiv 1 \pmod{p}. \quad (2.15)$$

In the notation of Lemma 2.17 we have that $\{e_s(0), \dots, e_s(p^n - 1)\}$ is a set of representatives of the classes modulo p^n , hence $A = \{a^{e_s(k)}\}_{k=0}^{p^n-1}$. Therefore we can define γ on A letting, for all k ,

$$\gamma(a^{e_s(k)}) = \eta^k,$$

and we have only to check that it satisfies the GFE. Now $a_1 = a^{e_s(k_1)}$, $a_2 = a^{e_s(k_2)}$, for some k_1, k_2 , so that

$$\begin{aligned} \gamma(a_1^{\gamma(a_2)} a_2) &= \gamma((a^{e_s(k_1)})^{\gamma(a^{e_s(k_2)})} a^{e_s(k_2)}) \\ &= \gamma((a^{e_s(k_1)})^{\eta^{k_2}} a^{e_s(k_2)}) \\ &= \gamma(a^{s^{k_2} \sum_{i=0}^{k_1-1} s^i} a^{\sum_{i=0}^{k_2-1} s^i}) \\ &= \gamma(a^{\sum_{i=0}^{k_1+k_2-1} s^i}) \\ &= \gamma(a^{e_s(k_1+k_2)}) \\ &= \eta^{k_1+k_2} \\ &= \gamma(a^{e_s(k_1)}) \gamma(a^{e_s(k_2)}) \\ &= \gamma(a_1) \gamma(a_2). \end{aligned}$$

Finally, if $\gamma': A \rightarrow \text{Aut}(G)$ is a RGF such that $\gamma'(a) = \eta$, denoting by \circ' the operation $a_1 \circ' a_2 = a_1^{\gamma'(a_2)} a_2$, we have that $\gamma'(a^{\circ' k}) = \eta^k$, for each k . On the other hand,

$$a^{\circ' k} = a^{\sum_{i=0}^{k-1} \eta^i} = a^{e_s(k)},$$

so that

$$\gamma'(a^{e_s(k)}) = \gamma'(a^{\circ' k}) = \eta^k = \gamma(a^{e_s(k)}),$$

that is, $\gamma' = \gamma$. □

Corollary 2.20. *Let G be a finite group, and let $A = \langle a \rangle$ be a cyclic subgroup of G of order p^n where p is an odd prime.*

Let $\gamma: A \rightarrow \text{Aut}(G)$ be a RGF.

Then the following are equivalent:

1. γ is a morphism, and
2. $a^{\gamma(a)} = a^s$, with $s \equiv 1 \pmod{\text{ord}(\gamma(a))}$.

Proof. Let $\gamma(a) = \eta$. Then γ is a morphism if and only if, for all k ,

$$\gamma(a^{e_s(k)}) = \gamma(a)^{e_s(k)},$$

or equivalently

$$\eta^k = \eta^{e_s(k)},$$

namely, $e_s(k) \equiv k \pmod{\text{ord}(\eta)}$. The last condition is easily seen to be equivalent to $s \equiv 1 \pmod{\text{ord}(\eta)}$.

In fact, if $s \equiv 1 \pmod{\text{ord}(\eta)}$ then clearly $e_s(k) \equiv k \pmod{\text{ord}(\eta)}$ for all k .

On the the hand, if $e_s(k) \equiv k \pmod{\text{ord}(\eta)}$ for all k , then, in particular, $e_s(2) = s^0 + s = 2 + (s - 1) \equiv 2 \pmod{\text{ord}(\eta)}$ namely $s \equiv 1 \pmod{\text{ord}(\eta)}$. \square

Corollary 2.21. *In the notation of Corollary 2.20, assume*

$$\text{ord}(\gamma(a)) = p.$$

Then γ is a morphism.

Proof. This follows from Corollary 2.20 and equation (2.15). \square

2.8. Duality

The GF associated to the image $\rho(G)$ of the right regular representation is $\gamma(G) = \{1\}$, and the associated circle operation on G is the defining operation on the group G .

The GF associated to the image $\lambda(G) = \rho(G)^{\text{inv}}$ of the left regular representation (see Proposition 2.1(3)) is $\iota(y^{-1})$, and the associated circle operation is the opposite operation, $x \circ y = yx$, as $x^{\iota(y^{-1})}y = yx = x^{\lambda(y)}$. In particular,

$$\begin{aligned} \text{inv} : G &\rightarrow (G, \circ) \\ x &\mapsto x^{-1} \end{aligned}$$

is an isomorphism in this case.

Our next result is an extension of the above pairing between the images of the right and the left regular representations to all regular subgroups of $\text{Hol}(G)$. This will be useful, as it allows us to halve the number of GF we have to consider, when G is non-abelian, and also because it allows us in some circumstances to choose a GF with a kernel that is more suitable for calculations (see Proposition 2.24 below).

Proposition 2.22. *Let G be a finite group, $\gamma : G \rightarrow \text{Aut}(G)$ a GF, N the associated regular subgroup of $\text{Hol}(G)$, and \circ the associated operation.*

Then

$$\begin{aligned}\tilde{\gamma} : G &\rightarrow \text{Aut}(G) \\ x &\mapsto \gamma(x^{-1})\iota(x^{-1})\end{aligned}$$

is also a GF, which corresponds to the regular subgroup N^{inv} , that is, the conjugate of N under $\text{inv} \in S(G)$. If $\tilde{\circ}$ is the operation associated to $\tilde{\gamma}$, then

$$\text{inv} : (G, \circ) \rightarrow (G, \tilde{\circ})$$

is an isomorphism.

Proof. From Proposition 2.1(3) we have that inv normalises $\text{Hol}(G)$.

Consider the conjugate of $N = \{ \gamma(y)\rho(y) : y \in G \}$ under inv , which will be another regular subgroup of $\text{Hol}(G)$. We have

$$\begin{aligned}x^{(\gamma(y)\rho(y))^{\text{inv}}} &= (x^{-\gamma(y)}y)^{-1} \\ &= y^{-1}x^{\gamma(y)} \\ &= x^{\gamma(y)\iota(y)\rho(y^{-1})}.\end{aligned}$$

In particular, $1^{(\gamma(y)\rho(y))^{\text{inv}}} = y^{-1}$, that is, $(\gamma(y)\rho(y))^{\text{inv}}$ is the element of N^{inv} taking 1 to y^{-1} . Therefore the GF associated to N^{inv} is

$$\begin{aligned}\tilde{\gamma} : G &\rightarrow \text{Aut}(G) \\ y &\mapsto \gamma(y^{-1})\iota(y^{-1}).\end{aligned}$$

The operation associated to $\tilde{\gamma}$ is

$$\begin{aligned}x \tilde{\circ} y &= x^{\tilde{\gamma}(y)}y = x^{\gamma(y^{-1})\iota(y^{-1})}y = \\ &= yx^{\gamma(y^{-1})} = (x^{-\gamma(y^{-1})}y^{-1})^{-1} = (x^{-1} \circ y^{-1})^{-1},\end{aligned}$$

so that, as in the case of G and its opposite group, $\text{inv} : (G, \circ) \rightarrow (G, \tilde{\circ})$ is an isomorphism. (See also [CDV18, Lemma 1.4].) \square

Lemma 2.23. *Let G be a finite non-abelian group. Let C be a non-trivial subgroup of G such that:*

1. C is abelian;
2. C is characteristic in G ;
3. $C \cap Z(G) = \{1\}$.

Let $\gamma: G \rightarrow \text{Aut}(G)$ be a GF, and suppose that for every $c \in C$ we have $\gamma(c) = \iota(c^{-\sigma})$ for some function $\sigma: C \rightarrow C$.

Then $\sigma \in \text{End}(C)$, and for every $a \in G$ the following relation holds in $\text{End}(C)$:

$$\sigma \gamma(a)_{|C} (\sigma - 1) = (\sigma - 1) \gamma(a)_{|C} \iota(a)_{|C} \sigma \quad (2.16)$$

Proof. For $c_1, c_2 \in C$ we have

$$\begin{aligned} \iota((c_1 c_2)^{-\sigma}) &= \gamma(c_1 c_2) \\ &= \gamma(c_1^{\gamma(c_2)^{-1}}) \gamma(c_2) \\ &= \gamma(c_1^{\iota(c_2^\sigma)}) \gamma(c_2) \\ &= \iota(c_1^{-\sigma}) \iota(c_2^{-\sigma}) \\ &= \iota(c_1^{-\sigma} c_2^{-\sigma}). \end{aligned}$$

Since C is abelian, and $C \cap Z(G) = \{1\}$, we obtain that σ is an endomorphism of C .

For $a \in G$ and $c \in C$ we have

$$\begin{aligned} a^{\ominus 1} \circ c \circ a &= a^{-\gamma(a)^{-1} \gamma(c) \gamma(a)} c^{\gamma(a)} a \\ &= a^{\iota(c^{-\sigma \gamma(a)})} a c^{\gamma(a) \iota(a)} \\ &= c^{\sigma \gamma(a)} a^{-1} c^{-\sigma \gamma(a)} a c^{\gamma(a) \iota(a)} \\ &= c^{\sigma \gamma(a) - \sigma \gamma(a) \iota(a) + \gamma(a) \iota(a)}. \end{aligned} \quad (2.17)$$

On the other hand

$$\gamma(a^{\ominus 1} \circ c \circ a) = \gamma(a)^{-1} \gamma(c) \gamma(a) = \iota(c^{-\sigma \gamma(a)}),$$

so that, writing τ for the exponent $\sigma \gamma(a) - \sigma \gamma(a) \iota(a) + \gamma(a) \iota(a)$ of c in the last term of (2.17), we get

$$\gamma(c^\tau) = \iota(c^{-\tau \sigma})$$

Since $C \cap Z(G) = \{1\}$, we obtain $\sigma \gamma(a) = \tau \sigma$, and thus (2.16) □

Proposition 2.24. *Let G be a finite non-abelian group. Let C be a subgroup of G such that:*

1. $C = \langle c \rangle$ is cyclic, of order a power of the prime r ,
2. C is characteristic in G ,
3. $C \cap Z(G) = \{1\}$, and
4. there is $a \in G$ which induces by conjugation on C an automorphism whose order is not a power of r .

Let $\gamma: G \rightarrow \text{Aut}(G)$ be a GF, and suppose that for every $c \in C$ we have $\gamma(c) = \iota(c^{-\sigma})$, for some function $\sigma: C \rightarrow C$.

Then

1. either $\sigma = 0$, that is, $C \leq \ker(\gamma)$,
2. or $\sigma = 1$, that is, $\gamma(c) = \iota(c^{-1})$, so that $C \leq \ker(\tilde{\gamma})$.

Note that the hypotheses of Proposition 2.24 contain those of Lemma 2.23.

Proof. It is immediate that $\sigma \in \text{End}(C)$, so that we can identify σ with an integer modulo the order of c .

Since $\text{End}(C)$ is abelian, and $\gamma(a)|_C \in \text{Aut}(C)$, we obtain from (2.16) the equality

$$\sigma(\sigma - 1)(\iota(a)|_C - 1) = 0$$

in $\text{End}(C)$, for all $a \in G$. Choose now $a \in G$ which induces on C an automorphism $\iota(a)$ whose order is not a power of r . Then $\iota(a)|_C - 1$ is not a zero divisor in $\text{End}(C)$, so that

$$\sigma(\sigma - 1) = 0.$$

Since $\text{End}(C)$ is a local ring, we obtain that either $\sigma = 0$, or $\sigma = 1$.

If the latter holds we have then

$$\tilde{\gamma}(c) = \gamma(c^{-1})\iota(c^{-1}) = \iota(c)\iota(c^{-1}) = 1.$$

□

Corollary 2.25. *Let G be a finite non-abelian group, r a prime dividing the order of G , and B a Sylow r -subgroup of G .*

Suppose that

- B is cyclic, and
- B contains a subgroup C of order r , which satisfies the hypotheses of Proposition 2.24.

Let γ be a GF on G . For each group \mathcal{G} of the same order as G , let

$$k_r(\mathcal{G}) = |\{ \gamma \text{ GF on } G : r \mid |\ker(\gamma)| \text{ and } (G, \circ) \cong \mathcal{G} \}|.$$

Then

$$e'(\mathcal{G}, G) = |\{ \gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \}| = 2k_r(\mathcal{G}).$$

Proof. Write

$$\begin{aligned} X &= \{ \gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \}, \\ X_1 &= \{ \gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \text{ and } r \mid |\ker(\gamma)| \}, \\ X_2 &= \{ \gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \text{ and } r \nmid |\ker(\gamma)| \}. \end{aligned}$$

We have

$$e'(\mathcal{G}, G) = |X| = |X_1| + |X_2| = k_r(\mathcal{G}) + |X_2|.$$

If we show that there is a bijection between X_1 and X_2 , it will follow that $e'(\mathcal{G}, G) = 2k_r(\mathcal{G})$. Consider

$$\begin{aligned} \psi : X &\rightarrow X \\ \gamma &\mapsto \tilde{\gamma}, \end{aligned}$$

where $\tilde{\gamma}$ is as in Proposition 2.22. The map ψ is well defined, indeed $\tilde{\gamma}$ is a GF on G and $(G, \tilde{\circ}) \cong (G, \circ) \cong \mathcal{G}$ (see the proof of Proposition 2.22); moreover

$$\psi^2(\gamma) = \psi(\tilde{\gamma}) = \tilde{\tilde{\gamma}}.$$

By Theorem 2.2 each GF on G corresponds to a unique regular subgroup of $\text{Hol}(G)$, so let N be the regular subgroup corresponding to γ ; then, by Proposition 2.22, the regular subgroup corresponding to $\tilde{\gamma}$ is $(N^{\text{inv}})^{\text{inv}} = N$, so that we get that $\psi^2 = 1$, and ψ is bijective. Now, using Proposition 2.24, we get that $\psi(X_2) = X_1$, so that $|X_2| = |X_1|$. \square

Note that this duality is equivalent to the notion of an *opposite skew brace* as introduced by Koch and Truman in [KT19]. In fact, given a brace (G, \cdot, \circ) , Koch and Truman define the opposite brace to be (G, \cdot', \circ) , where $x \cdot' y = yx$ gives the opposite group (G, \cdot') of (G, \cdot) . With our construction, the circle operation associated to the regular subgroup N^{inv} is given by $x \tilde{\circ} y = x^{\tilde{\gamma}(y)} \cdot y = x^{\gamma(y^{-1})\iota(y^{-1})} \cdot y = y \cdot x^{\gamma(y^{-1})}$.

Now $\text{inv} : (G, \cdot', \circ) \rightarrow (G, \cdot, \tilde{\circ})$ is an isomorphism of skew braces, as for $x, y \in G$ we have $(x \cdot' y)^{\text{inv}} = (y \cdot x)^{-1} = x^{-1} \cdot y^{-1} = x^{\text{inv}} \cdot y^{\text{inv}}$, and, $(x \circ y)^{\text{inv}} = (x^{\gamma(y)} \cdot y)^{-1} = y^{-1} \cdot x^{-\gamma(y)} = x^{-\gamma(y)\iota(y)} \cdot y^{-1} = x^{\text{inv}} \tilde{\circ} y^{\text{inv}}$.

2.9. An application: groups of order pq

To exemplify our methods, we first apply them to the case, dealt with by Byott in [Byo04], of groups of order pq , where p and q are distinct primes. We also recover the classification of skew braces of order pq of Aciri and Bonatto [AB19a].

So let $p > q$ be two primes. We will write \mathcal{C}_{pq} for the cyclic group of order pq , and $\mathcal{C}_p \rtimes \mathcal{C}_q$ for the non-abelian one, which occurs when $q \mid p - 1$.

Byott has proved the following

Theorem 2.26 ([Byo04, Section 6]). *Let L/K be a Galois field extension of order pq , and let $\Gamma = \text{Gal}(L/K)$.*

Then the following table gives the numbers $e(\Gamma, G)$ of Hopf-Galois structures on L/K of type G for each group G of order pq .

	G	\mathcal{C}_{pq}	$\mathcal{C}_p \rtimes \mathcal{C}_q$
Γ			
	\mathcal{C}_{pq}	1	$2(q - 1)$
	$\mathcal{C}_p \rtimes \mathcal{C}_q$	p	$2(pq - 2p + 1)$

We now compute with our methods the number $e'(\Gamma, G)$ of the regular subgroups of $\text{Hol}(G)$ which are isomorphic to Γ , in the form

	G	\mathcal{C}_{pq}	$\mathcal{C}_p \rtimes \mathcal{C}_q$
Γ			
	\mathcal{C}_{pq}	1	$2p$
	$\mathcal{C}_p \rtimes \mathcal{C}_q$	$q - 1$	$2(pq - 2p + 1)$

from which the previous theorem can be obtained using formula (1.1) of Theorem 1.1.

In terms of conjugacy classes of regular subgroups, we have

Theorem 2.27. *Let $G = (G, \cdot)$ be a group of order pq , where p, q are primes, with $p > q$.*

For each group Γ of order pq , the following table gives equivalently

1. *the number (and lengths) of conjugacy classes within $\text{Hol}(G)$ of regular subgroups isomorphic to Γ ;*
2. *the number of isomorphism classes of braces (G, \cdot, \circ) such that $\Gamma \cong (G, \circ)$.*

	G	\mathcal{C}_{pq}	$\mathcal{C}_p \rtimes \mathcal{C}_q$
Γ			
	\mathcal{C}_{pq}	(1, 1)	(2, p)
	$\mathcal{C}_p \rtimes \mathcal{C}_q$	(1, $q - 1$)	(2, 1), (2($q - 2$), p)

Here (c, l) denotes c conjugacy classes of length l ; the full table refers to the case when $q \mid p - 1$, and the 1×1 upper left sub-table refers to the case $q \nmid p - 1$.

We obtain that when $q \mid p - 1$ there are $2q + 2$ isomorphism classes of braces of order pq , which coincides with the results of [AB19a].

Let γ be a GF on G , let B be the Sylow p -subgroup of G and A a Sylow q -subgroup.

If $G = \mathcal{C}_{pq}$, then $B \leq \ker(\gamma)$, since in $\text{Aut}(G) \cong \mathcal{C}_{p-1} \times \mathcal{C}_{q-1}$ there are no elements of order p .

If $G = \mathcal{C}_p \rtimes \mathcal{C}_q$, we may take $r = p$ and $C = B$ in the hypotheses of Corollary 2.25 here, so that we need only to consider the case $B \leq \ker(\gamma)$.

If $\ker(\gamma) = G$, we get the right regular representation, whose image forms a conjugacy class in itself.

Suppose thus $\ker(\gamma) = B$, so that $|\gamma(G)| = q$. We claim that there is a unique $\gamma(G)$ -invariant Sylow q -subgroup A of G . This is clearly true for $G = \mathcal{C}_{pq}$. When $G = \mathcal{C}_p \rtimes \mathcal{C}_q$, the q -elements of $\text{Aut}(G) \cong \mathcal{C}_p \rtimes \mathcal{C}_{p-1}$ are inner automorphisms, so that $\gamma(G) = \langle \iota(a) \rangle$ for some $a \in G$ of order q . It follows that $A = \langle a \rangle$ is the unique $\gamma(G)$ -invariant Sylow q -subgroup.

Now $A\gamma(G)$ is a subgroup of order q^2 of $\text{Hol}(G)$, so that $[A, \gamma(G)] = 1$, and thus

$$[G, \gamma(G)] = [AB, \gamma(G)] = [B, \gamma(G)] = [B, \langle \iota(a) \rangle] = B = \ker(\gamma).$$

By Lemma 2.13, all such GF's are precisely the morphisms $\gamma : G \rightarrow \text{Aut}(G)$ of groups with kernel B .

If $G = \mathcal{C}_{pq}$, and $q \nmid p - 1$, there are no such morphisms, as in this case $q \nmid |\text{Aut}(G)|$. If $q \mid p - 1$, there are exactly $q - 1$ such morphisms with kernel of order p , as they correspond to sending a fixed element of order q of G to one of the $q - 1$ elements of order q of $\text{Aut}(G)$. The corresponding regular subgroups are non-abelian by Lemma 2.10, as for $b \in B$ we have

$$a^{\ominus 1} \circ b \circ a = b^{\gamma(a)} \neq b.$$

Let $\beta \in \text{Aut}(G)$ send a to a^t , for some t . Then, according to Lemma 2.9, and since $\text{Aut}(G)$ is abelian, we have $\gamma^\beta(a) = \gamma(a^{\beta^{-1}}) = \gamma(a)^{t^{-1}}$. It follows that all these $q - 1$ GF's are conjugate.

If $G = \mathcal{C}_p \rtimes \mathcal{C}_q$, one has first to choose a $\langle \iota(a) \rangle$ among the p Sylow q -subgroups of $\text{Aut}(G)$. Since for $b \in B$ Lemma 2.10 yields

$$a^{\ominus 1} \circ b \circ a = a^{-1} b^{\gamma(a)} a = b^{\gamma(a)\iota(a)},$$

the choice $\gamma(a) = \iota(a)^{-1}$ yields p instances of $(G, \circ) = \mathcal{C}_{pq}$. According to Lemma 2.9.(2), all these γ are conjugate under $\iota(B)$, which conjugates transitively the Sylow q -subgroups.

The other non-trivial choices of $\gamma(a) = \iota(a)^s$, with $s \neq 0, -1$ yield $p(q-2)$ instances of $(G, \circ) = \mathcal{C}_p \rtimes \mathcal{C}_q$. Once more, the action of $\iota(B)$, which conjugates transitively the Sylow q -subgroups, shows that the conjugacy classes are of length at least p . The cyclic complement of order $p-1$ of $\iota(B)$ in $\text{Aut}(G)$ which contains $\gamma(a) = \iota(a)^s$ then centralises a and $\gamma(a)$, so that it centralises γ by Lemma 2.9, and the conjugacy classes have length precisely p .

2.10. An application: a result of Kohl

In [Koh13, Koh16], Kohl gives a method to determine the Hopf-Galois structures on a family of Galois extensions of degree pm , where p is a prime, $\gcd(m, p) = 1$, and some additional hypotheses hold.

With our methods we obtain the following slight generalisation of the main result [Koh16, Theorem 1.3], which we have reformulated in terms of Byott's translation.

Theorem 2.28. *Let G be a group of order mp , with $p \nmid m$. Assume G has a unique Sylow p -subgroup P .*

Let $M \leq G$ be a subgroup of order m . Assume p does not divide $|\text{Aut}(M)|$.

Let N be a regular subgroup of $\text{Hol}(G)$.

Then

1. $\nu(P) \leq N$, so that $\nu(P)$ is a Sylow p -subgroup of N , which need not be unique.
2. $\nu(P) \in \{\rho(P), \lambda(P)\}$.
3. $[\rho(G), \nu(P)] \leq \nu(P)$.

Note that the subgroup M as in the statement of the Theorem exists by the theorem of Schur-Zassenhaus.

Proof. Since $P = \langle x \rangle$ is characteristic in G , it is also a subgroup of (G, \circ) , so that $\nu(P) \leq N$, as $\nu : (G, \circ) \rightarrow N$ is a morphism.

Proposition 2.19 yields that $\gamma(x)$ is an element of $\text{Aut}(G)$ of order dividing p .

Suppose first $[P, M] = 1$. Then there are no elements of order p in $\text{Aut}(G) = \text{Aut}(P) \times \text{Aut}(M)$, so that $\gamma(x) = 1$. It follows that $\nu(P) = \rho(P)$ in this case, so that $[\rho(G), \nu(P)] = [\rho(G), \rho(P)] = \rho([G, P]) = 1$.

Let thus $[P, M] \neq 1$, so that $\iota(P)$, the group of inner automorphisms of G induced by conjugation by the elements of P , has order p .

We now go through the arguments of [Koh13, Lemma 1.1] and [Koh16, Lemma 1.2]. According to [Cur08, Theorem 1] since p does not divide

$|\text{Aut}(P)|$ and $|\text{Aut}(M)|$, the p -elements of $\text{Aut}(G)$ are in one-to-one correspondence with the maps $\beta : M \rightarrow P$ such that

$$\beta(yz) = \beta(z)\beta(y)^z. \quad (2.18)$$

(Among these maps, one finds the maps corresponding to the inner automorphisms $\iota(P)$, in the form $\beta(y) = [x^i, y]$, for $i = 0, 1, \dots, p-1$.) According to (2.18), the restriction of β to the centraliser $C_M(P)$ of P in M yields a morphism $C_M(P) \rightarrow P$; since $\gcd(|M|, |P|) = 1$, we obtain that β is trivial on $C_M(P)$. Since $M/C_M(P)$ is isomorphic to a subgroup of the cyclic group $\text{Aut}(P)$, (2.18) shows that β is completely determined by its value $\beta(g)$ on an element g , whose image generates $M/C_M(P)$. There are at most p choices for $\beta(g) \in P$. It follows that the automorphisms of G of order p are inner, induced by conjugation by elements of P .

Proposition 2.24 then yields that $\gamma(x) = \iota(x^{-\sigma})$, where

- either $\sigma = 0$, so that $\nu(P) = \rho(P)$, and thus

$$[\rho(G), \nu(P)] = [\rho(G), \rho(P)] = \rho([G, P]) = \rho(P) = \nu(P),$$

- or $\sigma = 1$, and thus $\nu(P) = \lambda(P)$ is centralised by $\rho(G)$.

□

3. The groups of order p^2q and their automorphism groups

In this section we prepare the field for the proof of Theorem 1.3, which will take place in the next section. Although in this paper we deal with those groups of order p^2q , where p and q are distinct primes, which have cyclic Sylow p -subgroups, some of our results will be stated for the general case.

We will first recall the classification of the groups of order p^2q with cyclic Sylow p -subgroups, where p and q are distinct primes, and of their automorphism groups. We will then show how to apply the duality results of Subsection 2.8 to the non-abelian groups of order p^2q with a cyclic Sylow p -subgroup. Finally we will show that for odd p , if G is a group of order p^2q , a Sylow p -subgroup of a regular subgroup of $\text{Hol}(G)$ is isomorphic to a Sylow p -subgroup of G .

The classification of groups of order p^2q , where p, q are distinct primes, goes back to O. Hölder [Höl93]. In particular, Hölder showed that in such a group there is always a normal Sylow subgroup. As a handy reference, we have recorded the classification of these groups and of their automorphism

groups in [CCDC19]. We list in the table below those groups of order p^2q with a cyclic Sylow p -subgroup, referring to [CCDC19] for the details. Here each type corresponds to an isomorphism class. We use the notation \mathcal{C}_n for a cyclic group of order n .

Type 1 Cyclic group.

Type 2 This is the non-abelian group with centre of order p for $p \mid q - 1$, which we denote by $\mathcal{C}_{p^2} \rtimes_p \mathcal{C}_q$.

Type 3 This is the non-abelian group with trivial centre for $p^2 \mid q - 1$, which we denote by $\mathcal{C}_{p^2} \rtimes_1 \mathcal{C}_q$.

Type 4 This is the non-abelian group for $q \mid p - 1$, which we denote by $\mathcal{C}_{p^2} \rtimes \mathcal{C}_q$.

Table 3.1: Groups of order p^2q with cyclic Sylow p -subgroups and their automorphisms

Type	Conditions	G	$\text{Aut}(G)$
1		$\mathcal{C}_{p^2} \times \mathcal{C}_q$	$\mathcal{C}_{p(p-1)} \times \mathcal{C}_{q-1}$
2	$p \mid q - 1$	$\mathcal{C}_{p^2} \rtimes_p \mathcal{C}_q$	$\mathcal{C}_p \times \text{Hol}(\mathcal{C}_q)$
3	$p^2 \mid q - 1$	$\mathcal{C}_{p^2} \rtimes_1 \mathcal{C}_q$	$\text{Hol}(\mathcal{C}_q)$
4	$q \mid p - 1$	$\mathcal{C}_{p^2} \rtimes \mathcal{C}_q$	$\text{Hol}(\mathcal{C}_{p^2})$

The next Propositions apply to the groups in Table 3.1, and also to the other groups of order p^2q . For the types of such groups not dealt with in this paper, we refer to the notation of [CCDC19].

Proposition 3.1. *Let G be a non-abelian group of order p^2q , and let γ be a GF on G .*

1. *Let B be the normal Sylow r -subgroup of G ($r \in \{p, q\}$);*
2. *if $r = p$, assume B cyclic;*
3. *denote by C the unique subgroup of B of order r .*

Then

$$C \leq \ker(\gamma) \text{ if and only if } C \not\leq \ker(\tilde{\gamma}).$$

Moreover, for each group \mathcal{G} of order p^2q , let

$$k_r(\mathcal{G}) = |\{\gamma \text{ GF on } G : C \leq \ker(\gamma) \text{ and } (G, \circ) \cong \mathcal{G}\}|.$$

Then

$$e'(\mathcal{G}, G) = |\{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G}\}| = 2k_r(\mathcal{G}).$$

Proof. We show that G, B, C fulfil the the assumptions of Proposition 2.24 and Corollary 2.25, from which the result will follow.

The subgroup B is cyclic and characteristic in G , so C is the only subgroup of order r of G , and (1) and (2) of Proposition 2.24 hold. Let now $r = q$; in this case $C = B$ and G can be of type 2, 3 or 11 and we always have $B \cap Z(G) = \{1\}$. Moreover, $\text{ord}(\gamma(b)) \mid \text{ord}_{(G, \circ)}(b) \mid q$ since $(B, \circ) \leq (G, \circ)$ by Proposition 2.6 and $\gamma: (G, \circ) \rightarrow \text{Aut}(G)$ is a morphism. For G of type 2, 3 or 11, $|\text{Aut}(G)/\text{Inn}(G)|$ is coprime to q , thus all the elements of order q in $\text{Aut}(G)$ belong to $\text{Inn}(G)$ and so $\gamma(b) = \iota(b^{-\sigma})$ for some σ .

On the other hand, for $r = p$ we have $B = \langle b \rangle$, and G is of type 4, so that $p > 2$. Here $Z(G) = \{1\}$, so that we have only to show that for all $c \in C = \langle b^p \rangle$ we have $\gamma(c) = \iota(c^{-\sigma})$. According to Theorem 3.4 and Remark 3.5 of [CCDC19], the Sylow p -subgroup of $\text{Aut}(G) = \text{Hol}(\mathcal{C}_{p^2})$ is a non-abelian group $X = \mathcal{C}_{p^2} \rtimes \mathcal{C}_p$ of order p^3 , spanned by $\iota(b)$, of order p^2 , and another element ψ of order p which maps $b \mapsto b^{1+p}$, and fixes elementwise a Sylow q -subgroup A of G of one's choice. The derived subgroup $\langle \iota(b^p) \rangle$ of X is central, of order p . We now quote the elementary

Remark 3.2. For $x, y \in X$ we have

$$(xy)^p = x^p y^p [y, x]^{\binom{p}{2}} = x^p y^p,$$

as p is odd.

We have once more $(B, \circ) \leq (G, \circ)$ and since $p > 2$, by Corollary 2.18, $\text{ord}_{(B, \circ)}(b^{\circ k}) = \text{ord}_B(b^k)$ for all k , thus $\text{ord}(\gamma(b^p)) \mid p$. If $\gamma(b^p) = 1$ we are done, otherwise $\text{ord}(\gamma(b^p)) = p$. Let $\gamma(b) = \iota(b^{-\sigma})\psi^t$; then

$$b^{\circ p} = b^{\sum_{i=0}^{p-1} \gamma(b)^i} = b^{\sum_{i=0}^{p-1} (1+ipt)} = b^p,$$

so that

$$\gamma(b^p) = \gamma(b^{\circ p}) = \gamma(b)^p = (\iota(b^{-\sigma})\psi^t)^p = \iota(b^{-\sigma p}),$$

where we have used Remark 3.2.

Finally, since G is not abelian, then $\text{Inn}(G)$ contains elements of order both p and q , so there exists $a \in G$ whose order is not a power of r and Proposition 2.24 and Corollary 2.25 can be applied. \square

Theorem 3.3. *Let G be a group of order p^2q and γ a GF on G .*

Then there exists a Sylow p -subgroup A of G which is $\gamma(A)$ -invariant.

In particular, for $p > 2$, G and (G, \circ) have isomorphic Sylow p -subgroups.

Proof. We show that there is always a Sylow p -subgroup A of G which is $\gamma(A)$ -invariant; then the result will follow from Corollary 2.8, since the groups

of order p^2 are abelian. Clearly, this is always the case when A is characteristic; otherwise the set \mathcal{P} of the Sylow p -subgroups of G has q elements. For each γ , the group $\gamma(G)$ acts on \mathcal{P} , partitioning it into orbits whose length divides $|\gamma(G)|$. So, denoting by N_l the number of orbits of length l , we have

$$\sum_{l \mid |\gamma(G)|} N_l l = |\mathcal{P}| = q \equiv 1 \pmod{p}.$$

If $q \mid |\ker(\gamma)|$, then $|\gamma(G)| = 1, p$ or p^2 and necessarily $N_1 \geq 1$, namely there exist $A \in \mathcal{P}$ which is $\gamma(G)$ -invariant.

This argument covers the cases when G is of type 1, 4, 5, 6, 7, 8, 9, 10 (that is, the Sylow p -subgroup is characteristic) and when G is of type 2, 3 or 11, and the Sylow q -subgroup B is contained in $\ker(\gamma)$. So suppose G of type 2, 3 or 11 and $B \not\leq \ker(\gamma)$; here B is characteristic and by Proposition 3.1 we get that $B \leq \ker(\tilde{\gamma})$. The previous argument ensures that there exists a Sylow p -subgroup A of G which is $\tilde{\gamma}(G)$ -invariant and, by Proposition 2.6, it is also a Sylow p -subgroup of $(G, \tilde{\circ})$.

Now, $(G, \tilde{\circ})$ is isomorphic to (G, \circ) via the map $\text{inv}: x \mapsto x^{-1}$ (see the proof of Proposition 2.22), thus $A^{\text{inv}} = A$ is also a Sylow p -subgroup of (G, \circ) . Using Proposition 2.6 again we get that A is $\gamma(A)$ -invariant.

We can conclude that, for each G and for each GF γ , there exists a Sylow p -subgroup of G which is also a Sylow p -subgroup of (G, \circ) . Corollary 2.8 allows us to conclude that A and (A, \circ) are isomorphic. \square

We immediately get

Corollary 3.4. *Let $p > 2$ and q be distinct primes. Let Γ and G be groups of order p^2q with non isomorphic Sylow p -subgroups. Then $e'(\Gamma, G) = e(\Gamma, G) = 0$.*

Remark 3.5. If G is a group of order p^2q with cyclic Sylow p -subgroups, then either G has a unique Sylow q -subgroup or it is of type 4. In the latter case it follows from Subsection 4.4 that there exists a unique Sylow q -subgroup Q invariant under $\gamma(Q)$. So, a posteriori, Theorem 3.3 is also true replacing Sylow p -subgroups with Sylow q -subgroups, but we do not have a general argument to prove it.

4. Proof of Theorems 1.3 and 1.4

We are now ready to prove our main Theorem 1.3 We will mostly rely on the general results established in Sections 2 and 3, appealing occasionally to ad hoc arguments.

In enumerating the GF's γ on G , we will usually tacitly ignore the case $\gamma(G) = \{1\}$, that is, $\ker(\gamma) = G$, as it corresponds to the (trivial) case of the right regular representation.

According to Theorem 3.3, if G is a group of order p^2q , with p, q distinct primes and $p > 2$, then G and (G, \circ) have isomorphic Sylow p -subgroups. Thus to prove Theorem 1.3 we only need to consider groups G with cyclic Sylow p -subgroups, that is, those in Table 3.1. For these groups each type corresponds to an isomorphism class. We will proceed by analysing the types/isomorphism classes under consideration one by one.

We fix the following notation: q and $p > 2$ are distinct primes, G is a group of order p^2q with cyclic Sylow p -subgroups, and $\gamma: G \rightarrow \text{Aut}(G)$ is a GF on G .

Remark 4.1. We discuss here a recurring pattern which occurs in the application of Proposition 2.14. Let $\{r, s\} = \{p, q\}$ and let A be a Sylow r -subgroup, and B be a Sylow s -subgroup of G . Clearly $G = AB$ and we know that at least one of A and B is characteristic. In the following we restrict our attention to the GF's $\gamma: G \rightarrow \text{Aut}(G)$ such that $B \leq \ker(\gamma)$.

Suppose first A is characteristic. Then γ is the lifting of $\gamma' = \gamma|_A: A \rightarrow \text{Aut}(G)$. On the other hand, by Proposition 2.14, in this case the RGF's $\gamma': A \rightarrow \text{Aut}(G)$ which can be lifted to G are exactly those for which B is invariant under $\{\gamma'(a)\iota(a): a \in A\}$.

If A is not characteristic in G , and thus B is, the situation is slightly more involved.

Consider the action of $\gamma(G)$ on the set \mathcal{R} of the Sylow r -subgroups of G . Since by assumption $\gamma(G)$ has order a power of r , Sylow's theorems imply that $\gamma(G)$ has $N_1 > 0$ fixed points in this action. Let $\bar{A} \in \mathcal{R}$ be one of these Sylow r -subgroups of G invariant under $\gamma(G)$. Then $\gamma|_{\bar{A}}: \bar{A} \rightarrow \text{Aut}(G)$ is a RGF. On the other hand, since B is characteristic, it is invariant under $\{\gamma'(a)\iota(a): a \in A\} \leq \text{Aut}(G)$, and thus each RGF $\gamma': \bar{A} \rightarrow \text{Aut}(G)$ can be lifted to a GF on G . It follows that when A is not characteristic, each γ with $B \leq \ker(\gamma)$ can be obtained as a lifting of a γ' defined on a Sylow r -subgroup in N_1 ways, one for each Sylow r -subgroup \bar{A} which is invariant under $\gamma(G)$.

4.1. G of type 1

In this case $G = \mathcal{C}_{p^2} \times \mathcal{C}_q$; the Sylow p -subgroup $A = \langle a \rangle$ and the Sylow q -subgroup $B = \langle b \rangle$ are both cyclic and characteristic. By Proposition 2.6 A and B are also subgroups of (G, \circ) , for each operation \circ induced on G by γ . Moreover, $\text{Aut}(G) = \text{Aut}(A) \times \text{Aut}(B) \cong \mathcal{C}_{p(p-1)} \times \mathcal{C}_{q-1}$ is abelian.

4.1.1. The case $B \leq \ker(\gamma)$

This occurs in particular when $q \nmid p-1$, that is, $q \nmid |\text{Aut}(G)|$.

Since A and B are both characteristic in G , Remark 4.1 ensures that the GF's on G are in one-to-one correspondence with the RGF's

$$\gamma': A \rightarrow \text{Aut}(G).$$

On the other hand, since A is cyclic, by Proposition 2.19 each γ' is uniquely defined by assigning the image of the generator a as $\gamma'(a) = \eta \in \text{Aut}(G)$ where $\text{ord}(\eta) \mid p^2$ and $\eta = (\eta|_A, \eta|_B)$.

For such a γ' , the unique γ induced on G defines an operation \circ for which, according to Lemma 2.10,

$$a^{\ominus 1} \circ b \circ a = a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)} b^{\gamma(a)} a = b^\eta. \quad (4.1)$$

Thus if $\eta|_B = 1$ (which is the only possibility when $p \nmid q-1$), then (G, \circ) is abelian, that is, of type 1: there are p choices of $\eta \in \text{Aut}(G)$, that is, of $\eta|_A \in \text{Aut}(A)$ of order dividing p , with this property.

As to the conjugacy classes, let

$$\gamma(a) : \begin{cases} a \mapsto a^{1+ph} \\ b \mapsto b \end{cases}$$

and consider an automorphism of G , defined for $\text{gcd}(u, p) = \text{gcd}(v, q) = 1$,

$$\beta : \begin{cases} a \mapsto a^u \\ b \mapsto b^v. \end{cases} \quad (4.2)$$

Then

$$\begin{aligned} \gamma^\beta(a) &= \gamma(a^{\beta^{-1}}) = \gamma(a^{u^{-1}}) = \gamma(a^{\circ f_s(u^{-1})}) = \\ &= \gamma(a)^{f_s(u^{-1})} : \begin{cases} a \mapsto a^{1+phf_s(u^{-1})} \\ b \mapsto b \end{cases} \end{aligned}$$

Here f_s is the inverse of the function e_s of Lemma 2.17. So if $h = 0$ we have the conjugacy class of length 1 of $\rho(G)$, whereas if $h \neq 0$, the stabiliser is given by $f_s(u^{-1}) \equiv 1 \pmod{p}$ and any v , so the stabiliser has order $p(q-1)$, and there is a conjugacy class of length $p-1$.

If $p \mid q-1$, we can also choose $\text{ord}(\eta|_B) = p$. There are $p-1$ choices for such an $\eta|_B$, which paired with the p choices for $\eta|_A \in \text{Aut}(A)$ of order dividing p yield $p(p-1)$ choices for $\eta \in \text{Aut}(G)$. In this case (4.1) shows that (G, \circ) is of type 2.

As to the conjugacy classes, if

$$\gamma(a) : \begin{cases} a \mapsto a^{1+ph} \\ b \mapsto b^r \end{cases}, \quad (4.3)$$

with r of order p , then

$$\gamma^\beta(a) = \begin{cases} a \mapsto a^{1+phf_s(u^{-1})} \\ b \mapsto b^{r \cdot f_s(u^{-1})} \end{cases},$$

so the stabiliser is the same as in the previous case, and we get p classes of length $p - 1$.

If $p^2 \mid q - 1$, we can also choose $\text{ord}(\eta|_B) = p^2$. As above, in this case there are $p^2(p - 1)$ choices of $\eta \in \text{Aut}(G)$ with this property, and (4.1) show that (G, \circ) is of type 3.

As to the conjugacy classes, this time r in (4.3) has period p^2 , so for the stabiliser we need $f_s(u^{-1}) \equiv 1 \pmod{p^2}$, that is, $u = 1$. Therefore the stabiliser has order $q - 1$, and we get p classes of length $p(p - 1)$.

4.1.2. The case $B \not\leq \ker(\gamma)$

Here $q \mid p - 1$, so that (G, \circ) can only be of type 1 or 4. Moreover, $p \parallel |\text{Aut}(G)|$, so that $p \mid |\ker(\gamma)|$.

If $A \leq \ker(\gamma)$, since B is the unique Sylow q -subgroup of G , Remark 4.1 yields that the GF's on G are in one-to-one correspondence with the RGF's $\gamma' : B \rightarrow \text{Aut}(G)$. In turn, the latter are uniquely determined by the assignment $b \mapsto \gamma'(b)$, where $\text{ord}(\gamma'(b)) \mid q$. Note that all such γ' are morphisms: this follows either from Corollary 2.21, or from Lemma 2.13, as $\gamma(G)$, of order q , acts trivially on the group G/A of order q , so that $[G, \gamma(G)] \leq A \leq \ker(\gamma)$. For each such γ' , the unique γ induced on G defines an operation \circ such that:

$$b^{\ominus 1} \circ a \circ b = b^{-1} a^{\gamma(b)} b = a^{\gamma(b)}.$$

Since $B \not\leq \ker(\gamma)$, then $\text{ord}(\gamma'(b)) = q$. There are $q - 1$ choices for such a $\gamma'(b) \in \text{Aut}(G)$. Here $a^{\gamma(b)} \neq a$, and thus (G, \circ) is of type 4.

As to the conjugacy classes, here

$$\gamma(b) : \begin{cases} a \mapsto a^t \\ b \mapsto b \end{cases}$$

with t of order q modulo p^2 . With β as in (4.2), we have

$$\gamma^\beta(b) = \gamma(b^{\beta^{-1}}) = \gamma(b^{v^{-1}}) = \gamma(b)^{v^{-1}} : \begin{cases} a \mapsto a^{t^{v^{-1}}} \\ b \mapsto b \end{cases},$$

which coincides with γ if $v = 1$. Therefore the stabiliser has order $p(p - 1)$, and we get a single conjugacy class of length $q - 1$.

If $A \not\leq \ker(\gamma)$, we show that $\gamma(b) = 1$, contradicting $B \not\leq \ker(\gamma)$. The group B is characteristic, hence by Proposition 2.6 it is also a subgroup of (G, \circ) , and by Corollary 2.18 $\text{ord}_{(G, \circ)}(b) = q$, so that $\text{ord}(\gamma(b)) \mid q$. In particular, $b^{\gamma(b)} = b$. Since $|\ker(\gamma)| = pq$ or q in this case, and a group of type 4 has no normal subgroup of this order (G, \circ) is abelian. So we have

$$b = a^{\ominus 1} \circ b \circ a = a^{-\gamma(b)} b^{\gamma(a)} a = a^{-\gamma(b)} a b^{\gamma(a)},$$

which gives $a^{\gamma(b)} = a$, so that $\gamma(b) = 1$ as claimed.

We summarise, including the right regular representations.

Proposition 4.2. *Let G be of type 1, that is cyclic.*

Then in $\text{Hol}(G)$ there are:

1. p regular subgroups of type 1, which split in one conjugacy class of length 1, and one conjugacy class of length $p - 1$.
2. if $p \mid q - 1$,
 - (a) $p(p - 1)$ regular subgroups of type 2, which split in p conjugacy classes of length $p - 1$;
 - (b) $p^2(p - 1)$ further regular subgroups of type 3, if $p^2 \mid q - 1$, which split in p conjugacy classes of length $p(p - 1)$.
3. if $q \mid p - 1$,
 - (a) $q - 1$ regular subgroups of type 4, which form a single conjugacy class.

4.2. G of type 2

In this case $p \mid q - 1$, and $G = \mathcal{C}_q \rtimes_p \mathcal{C}_{p^2}$. The Sylow q -subgroup $B = \langle b \rangle$ is characteristic in G . Here an element of order p^2 of G induces an automorphism of order p of B .

We have

$$\text{Aut}(G) \cong \text{Hol}(\mathcal{C}_q) \times \mathcal{C}_p.$$

According to Subsection 4.5 of [CCDC19], the second direct factor is generated by the automorphism ψ of G which fixes b , and maps every element of order p^2 to its $(1 + p)$ -th power. It follows that ψ fixes every element of the unique subgroup of G of order pq .

Since G is non-abelian, in counting the GF's Proposition 3.1 allows us to consider only the case $B \leq \ker(\gamma)$, and then double the number of regular subgroups we find.

Setting aside as always the case of the right regular representation, $\gamma(G)$ will thus be a subgroup of $\text{Aut}(G)$ of order p or p^2 .

A Sylow p -subgroup of $\text{Aut}(G)$ is abelian, isomorphic to the direct product of one of the Sylow p -subgroups of $\text{Hol}(\mathcal{C}_q)$ (which is cyclic of order p^e where $p^e \parallel q - 1$), and the group $\langle \psi \rangle$. Moreover, the elements of $\text{Hol}(\mathcal{C}_q)$ of order dividing p are of the form $\iota(x)$ where x is a p -element of G .

4.2.1. The case $|\gamma(G)| = p^2$

This case can only occur when $p^2 \mid q - 1$, as by Theorem 2.2(iv) we have $\gamma(G) \cong (G, \circ) / \ker(\gamma)$, and this is a cyclic group by Theorem 3.3. Therefore $\gamma(G)$ is generated by an element (η, ψ^t) , where $\eta \in \text{Hol}(\mathcal{C}_q)$ has order p^2 and $0 \leq t < p$. Since η^p is an element of order p of $\text{Hol}(\mathcal{C}_q)$, we have $\eta^p = \iota(a)$ for an element of $a \in G$ of order p^2 . Since every Sylow p -subgroup of G is self-normalising, $A = \langle a \rangle$ is the only $\gamma(G)$ -invariant Sylow p -subgroup.

Once one of the q Sylow p -subgroups A has been chosen, Remark 4.1 tells us that to count the GF's on G we can count the RGF's $\gamma': A \rightarrow \text{Aut}(G)$. By Proposition 2.19, these are as many as the possible images

$$\gamma'(a) = (\eta, \psi^t), \quad (4.4)$$

with $\text{ord}(\eta) = p^2$, $0 \leq t < p$, such that A is invariant under $\gamma'(a)$. Since $\langle \psi \rangle$ fixes all the Sylow p -subgroups of G , it follows as above that $A = \langle a \rangle$, where $\eta^p = \iota(a)$.

Therefore, once A is chosen, we have $p(p - 1)$ choices for η , and p choices for t . So we have $qp^2(p - 1)$ GF's on G with $\gamma(G)$ of order p^2 .

In this case (G, \circ) is always of type 3. In fact, if $b^\eta = b^j$, then j has order p^2 modulo q and

$$a^{\ominus 1} \circ b \circ a = a^{-1} b^{\gamma(a)} a = a^{-1} b^\eta a = a^{-1} b^j a = b^{j\iota(a)}.$$

Since $\iota(a)$ is an automorphism of B of order p , conjugation by a in (G, \circ) is an automorphism of B of order p^2 .

As to the conjugacy classes, $\iota(B)$ conjugates transitively the Sylow p -subgroups of G , so that, by Lemma 2.9(2), all classes have order a multiple of q . Since the Sylow p -subgroups of $\text{Aut}(G)$ are abelian, we then have for the action of ψ on one of our γ

$$\gamma^\psi(a) = \psi^{-1} \gamma(a^{\psi^{-1}}) \psi = \gamma(a^{1-p}),$$

so that all classes have also order a multiple of p . Finally, if ϑ is an element of order $q - 1$ of $\text{Aut}(G)$ which fixes a , then $\langle \vartheta \rangle$ is in the stabiliser of each γ . It follows that we have $p(p - 1)$ classes of length qp here.

4.2.2. *The case $|\gamma(G)| = p$*

Here $\ker(\gamma)$ is the unique subgroup of G of index p . Since $\gamma(G)$ acts trivially on $G/\ker(\gamma)$, we have $[G, \gamma(G)] \leq \ker(\gamma)$, and thus by Lemma 2.13 all the GF's are morphisms $G \rightarrow \text{Aut}(G)$ here.

In the case when $\gamma(G) = \langle \psi \rangle$, each Sylow p -subgroup of G is $\gamma(G)$ -invariant, thus every RGF on such a Sylow subgroup lifts to the same GF on G . If $\langle a \rangle$ is any of the Sylow p -subgroups, there are $p - 1$ choices for $\gamma(a)$, and such a choice determine γ uniquely. It is immediate to check that $a^{\ominus 1} \circ b \circ a = a^{-1}ba$, so that the corresponding groups (G, \circ) are all of type 2.

As to the conjugacy classes, ψ is central in $\text{Aut}(G)$ and [CCDC19, Remark 3.3] implies that $\text{Aut}(G)$ acts trivially on $G/\ker(\gamma)$ so that for $\beta \in \text{Aut}(G)$ we have $\gamma^\beta(a) = \gamma(a^{\beta^{-1}}) = \gamma(a)$, and we end up with $p - 1$ conjugacy classes of length 1.

If $\gamma(G) \neq \langle \psi \rangle$, as above there is a unique Sylow p -subgroup $A = \langle a \rangle$ fixed by $\gamma(G)$, and

$$\gamma(a) = (\iota(a)^{-s}, \psi^t), \quad \text{for some } 0 < s < p, 0 \leq t < p.$$

Since there are q choices for the Sylow p -subgroup A , this gives a total of $q(p - 1)p$ GF's. For the operation \circ we have

$$a^{\ominus 1} \circ b \circ a = a^{-1}b^{\gamma(a)}a = a^{-1}b^{\iota(a)^{-s}}a = b^{a^{-s+1}}.$$

If $s \equiv 1 \pmod{p}$, then (G, \circ) is of type 1 for each of the qp choices of A and t .

If $s \not\equiv 1 \pmod{p}$, (G, \circ) is of type 2; here there are q choices for A , $p - 2$ choices for s , and p choices for t .

As to the conjugacy classes, with the above arguments we see that they have all length a multiple of q , and that the subgroup $\langle \psi, \vartheta \rangle$ of order $p(q - 1)$ is in the stabiliser, so that each class has indeed length q .

We summarise, including the right and left regular representations.

Proposition 4.3. *Let G be a group of type 2, that is, $G = C_q \rtimes_p C_{p^2}$.*

Then in $\text{Hol}(G)$ there are:

1. $2pq$ regular subgroups of type 1, which split into $2p$ conjugacy classes of length q ;
2. $2qp(p - 2) + 2p$ of type 2, which split into $2p(p - 2)$ conjugacy classes of length q , and $2p$ conjugacy classes of length 1;
3. $2qp^2(p - 1)$ further regular subgroups of type 3, if $p^2 \mid q - 1$, which split into $2p(p - 1)$ conjugacy classes of length qp .

4.3. G of type 3

In this case $G = \mathcal{C}_q \rtimes_1 \mathcal{C}_{p^2}$, with $p^2 \mid q - 1$. The Sylow q -subgroup $B = \langle b \rangle$ is characteristic in G , and an element of order p^2 of G induces an automorphism of order p^2 on B . Here $\text{Aut}(G) \cong \text{Hol}(\mathcal{C}_q)$, and since G has trivial centre we have $\text{Inn}(G) \cong G$.

Since

$$\left| \frac{\text{Aut}(G)}{\text{Inn}(G)} \right| = \frac{q-1}{p^2}$$

is coprime to q , and the Sylow p -subgroups of $\text{Aut}(G)$ are cyclic, we get $\gamma(G) \leq \text{Inn}(G)$.

By Proposition 2.6, B is also a Sylow q -subgroup of (G, \circ) , so that $|\gamma(B)|$ divides q , and

$$\gamma(B) \leq \iota(B) = \{ \iota(b^x) : 0 \leq x < q \}.$$

Proposition 3.1 now allows us to consider only the case $B \leq \ker(\gamma)$, and then double the number of regular subgroups we find.

Now Theorem 1 of [Cur08] (as recorded in Theorem 3.2 and Remark 3.3 of [CCDC19]) yields that $\text{Aut}(G)$ acts trivially on G/B , so that $[G, \gamma(G)] \leq [G, \text{Aut}(G)] \leq B \leq \ker(\gamma)$, and then by Lemma 2.13 all the GF's are morphisms $\gamma : G \rightarrow \text{Aut}(G)$ in this case.

If $\gamma(G) \neq \{1\}$, we claim that there is exactly one Sylow p -subgroup of G which is $\gamma(G)$ -invariant. In fact, $|\gamma(G)| = p$ or p^2 , and $\gamma(G)$ is a cyclic subgroup of $\langle \iota(a) \rangle$ for some $a \in G$ with $\text{ord}(a) = p^2$. Since every Sylow p -subgroup of G is self-normalising, $A = \langle a \rangle$ is the only $\gamma(G)$ -invariant Sylow p -subgroup.

Let $\gamma(a) = \iota(a^{-s})$, for some $0 < s < p^2$.

In G we have $a^{-1}ba = b^t$, for some t of order p^2 modulo q . We get

$$a^{\ominus 1} \circ b \circ a = a^{-1}b^{\gamma(a)}a = a^{-1}b^{\iota(a)^{-s}}a = a^{-(1-s)}ba^{1-s} = b^{t^{1-s}}. \quad (4.5)$$

Equation (4.5) yields the following.

- If $s \equiv 1 \pmod{p^2}$, then (G, \circ) is of type 1, and for each of the q Sylow p -subgroups A of G there is exactly one GF with this property.
- If $s \equiv 1 \pmod{p}$ but $s \not\equiv 1 \pmod{p^2}$, then conjugation by a in (G, \circ) has order p , so (G, \circ) is of type 2. For each of the q Sylow p -subgroups A there are $p-1$ such GF's, so that we get $q(p-1)$ GF's in this case.
- If $s \not\equiv 1 \pmod{p}$, conjugation by a in (G, \circ) has order p^2 , hence (G, \circ) is of type 3. For each of the q Sylow p -subgroups A of G there are p^2-p-1 such choices of s with $0 < s < p^2$, so that we get $q(p^2-p-1)$ groups in this case.

As to the conjugacy classes, $\iota(B)$ conjugates transitively the Sylow p -subgroups of G , so that by Lemma 2.9.(2) each conjugacy class for $s \neq 0$ has length a multiple of q . The cyclic complement of order $q - 1$ of $\iota(B)$ in $\text{Aut}(G)$ which contains $\gamma(a) = \iota(a^{-s})$ centralises a and $\gamma(a)$, so that, by Lemma 2.9, it centralises γ . It follows that the conjugacy classes have length precisely q .

We summarise, including the right and left regular representations.

Proposition 4.4. *Let G be a group of type 3, namely $G = \mathcal{C}_q \rtimes_1 \mathcal{C}_{p^2}$. Then in $\text{Hol}(G)$ there are:*

1. $2q$ regular subgroups of type 1, which split into 2 conjugacy classes of length q ;
2. $2q(p-1)$ regular subgroups of type 2, which split into $2(p-1)$ conjugacy classes of length q ;
3. $2(1 + q(p^2 - p - 1))$ regular subgroups of type 3, which split into 2 conjugacy classes of length 1, and $2(p^2 - p - 1)$ conjugacy classes of length q .

4.4. G of type 4

Here $q \mid p - 1$, and $G = \mathcal{C}_{p^2} \rtimes \mathcal{C}_q$, where an element of order q acts non-trivially on the unique Sylow p -subgroup $B = \langle b \rangle$. We have $\text{Aut}(G) \cong \text{Hol}(\mathcal{C}_{p^2})$, and $\text{Inn}(G) \cong G$, as $Z(G) = 1$. The groups (G, \circ) can be of type 1 or 4.

As discussed in the proof of Proposition 3.1, $\text{Aut}(G)$ has a unique Sylow p -subgroup, which is isomorphic to $\mathcal{C}_{p^2} \rtimes \mathcal{C}_p$ where the normal factor is $\langle \iota(b) \rangle$. For the second factor we have p choices. In fact, according to Theorem 3.4 and Remark 3.5 of [CCDC19], for each of the p^2 Sylow q -subgroups A we can choose a generator ψ of the second factor such that $\psi: b \mapsto b^{1+p}$, and ψ restricts to the identity on A : we will make a convenient choice of A , and thus ψ , later. Note that if ψ is the identity on the Sylow q -subgroup $A = \langle a \rangle$, then it is also the identity on the p Sylow q -subgroups $\langle ab^{pi} \rangle$, for $0 \leq i < p$.

Since the Sylow q -subgroups of $\text{Aut}(G)$ are cyclic, the elements of $\text{Aut}(G)$ of order q are inner automorphisms, given by conjugation by an element of G of order q .

By Proposition 3.1, we can restrict ourselves to counting the GF's such that p divides $|\ker(\gamma)|$.

4.4.1. The case $|\ker(\gamma)| = p^2$

Here $\ker(\gamma) = B$ and $|\gamma(G)| = q$. Hence $\gamma(G) = \langle \iota(a) \rangle$ for some $a \in G$ of order q , so that $A = \langle a \rangle$ is the unique $\gamma(G)$ -invariant subgroup. Since

$[G, \gamma(G)] \leq B = \ker(\gamma)$, each such γ is a morphism. For the operation \circ we have

$$a^{\ominus 1} \circ b \circ a = a^{-1} b^{\gamma(a)} a = b^{\gamma(a)\iota(a)}.$$

Once we have made one of the p^2 choices for A , the value $\gamma(a) = \iota(a)^{-1}$ will give an abelian group (G, \circ) of type 1, while all other $q - 2$ choices for $\gamma(a)$ will give groups of type 4.

As to the conjugacy classes, $\iota(B)$ conjugates transitively the Sylow q -subgroups, so that all classes have length a multiple of p^2 . The cyclic complement of order $p(p - 1)$ of $\iota(B)$ in $\text{Aut}(G)$ which contains $\gamma(G) = \langle \iota(a) \rangle$ centralises a and $\gamma(a)$ so that, by Lemma 2.9, it fixes γ . Hence the conjugacy classes have length precisely p^2 .

4.4.2. The case $|\ker(\gamma)| = pq$

This case does not occur. In fact, since $\ker(\gamma) \trianglelefteq (G, \circ)$, we have that (G, \circ) is abelian here. Thus if $a \in \ker(\gamma)$ is an element of order q , we have

$$a = b^{\ominus 1} \circ a \circ b = b^{-\gamma(b)^{-1}\gamma(a)\gamma(b)} a^{\gamma(b)} b = a^{\gamma(b)\iota(b)}.$$

Now $\gamma(b)$ is an element of order p in $\text{Aut}(G)$. Therefore, by Remark 3.2, we have

$$(\gamma(b)\iota(b))^p = \iota(b^p).$$

This implies that a and b^p commute, a contradiction.

4.4.3. The case $|\ker(\gamma)| = p$

Here $\ker(\gamma) = B^p$, and $|\gamma(G)| = pq$. Clearly $\gamma(B)$ has order p , as $B \leq (G, \circ)$. An element of order q of $\gamma(G)$ will thus be of the form $\gamma(a)$, for some $a \in G$, which will be of order q , as all elements of G outside of B have order q .

Therefore

$$\gamma(a) = \iota(a^{-l}b^m)$$

for some $0 < l < q$ and m , so that $a^{-l}b^m$ has also order q . Now choose a ψ as above that fixes $a^{-l}b^m$.

$\gamma(b)$ will be an element of order p of $\text{Aut}(G)$, that is, an element of $\langle \iota(b^p), \psi \rangle$, an elementary abelian group of order p^2 . Since $\gamma(G)$ is a subgroup of $\text{Aut}(G)$ of order pq , and $p > q$, we will have that $\langle \gamma(b) \rangle$ is normalised by $\iota(a^{-l}b^m)$, an element of order q . Now $\iota(a^{-l}b^m)$ centralises ψ by the choice of the latter, and normalises but does not centralise $\langle \iota(b^p) \rangle$. In other words, $\iota(a^{-l}b^m)$ has two distinct eigenvalues in its action on $\langle \iota(b^p), \psi \rangle$, so that there are two possibilities for $\gamma(b)$ to be normalised by $\iota(a^{-l}b^m)$.

If $\gamma(b) = \iota(b^{ps}) \neq 1$, for some s , Proposition 2.24 yields that $ps \equiv -1 \pmod{p^2}$, a contradiction.

Therefore $\gamma(b) = \psi^t$ for some $t \neq 0$. It follows that $\gamma(G) = \langle \psi, \iota(a^{-l}b^m) \rangle$ is abelian, and thus (G, \circ) is of type 1, as a group of type 4 does not have an abelian quotient of order pq .

Comparing $a \circ b^p = ab^p$ with

$$b^p \circ a = b^{p\gamma(a)}a = b^{p\iota(a^{-l})}a = ab^{p\iota(a^{-l+1})}$$

in the abelian group (G, \circ) , we get $l = 1$.

Now

$$(a^{-1})^\psi = (a^{-1}b^mb^{-m})^\psi = a^{-1}b^m(b^{-m})^\psi = a^{-1}b^mb^{-m(1+p)} = a^{-1}b^{-pm},$$

so that, taking the inverse, we get

$$a^\psi = b^{pm}a.$$

Comparing

$$a \circ b = a^{\gamma(b)}b = b^{pmt}ab = b^{pmt}b^{\iota(a^{-1})}a$$

with

$$b \circ a = b^{\gamma(a)}a = b^{\iota(a^{-1})}a,$$

we obtain $p \mid m$. Write $m = pn$ for some n . We have thus

$$\gamma(b^{-pn}a) = \gamma(a) = \iota(a^{-1}b^{pn}) = \iota(b^{-pn}a)^{-1},$$

so that all the GF's with kernel of order p can be constructed as follows.

Choose first one of the p^2 Sylow q -subgroups $A = \langle a \rangle$. Then define ψ as the automorphism of G that is the power $1 + p$ on B , and fixes a . Finally define γ as

$$\begin{cases} \gamma(b) = \psi^t \\ \gamma(a) = \iota(a^{-1}). \end{cases} \quad (4.6)$$

It is immediate to see that $\gamma(a^ib^j) = \iota(a^{-i})\psi^{tj}$ defines indeed a GF satisfying (4.2). Note that (4.2) determines $\langle a \rangle$ uniquely as the only Sylow q -subgroup A of G which is $\gamma(A)$ -invariant. In fact, if $A = \langle ab^k \rangle$ is $\gamma(A)$ -invariant, we have, writing $b^{\iota(a^{-1})} = b^\lambda$,

$$A \ni (ab^k)^{\gamma(ab^k)} = ab^{k\lambda(1+ptk)},$$

so that the latter equals ab^k , and we have

$$k(\lambda(1 + ptk) - 1) \equiv 0 \pmod{p^2}.$$

Since λ has order q modulo p^2 , we have that $\lambda \not\equiv 1 \pmod{p}$, so that $k \equiv 0 \pmod{p^2}$ and $A = \langle a \rangle$.

Therefore the p^2 choices for A and the $p - 1$ choices for t yield $p^2(p - 1)$ choices for γ .

As to the conjugacy classes, take γ defined as in (4.6). By Lemma 2.9(2), $\langle \iota(b) \rangle$ acts regularly on the γ 's, so that all conjugacy classes have order a multiple of p^2 .

Consider now the group R of automorphisms of G , of order $p(p - 1)$, of the form

$$\beta : \begin{cases} a \mapsto a \\ b \mapsto b^r \end{cases} .$$

We claim that the stabiliser in R of any γ is $\langle \psi \rangle$, of order p . It follows that all conjugacy classes have order a multiple of $p - 1$, and thus all conjugacy classes have order $p^2(p - 1)$.

In fact one sees immediately, using the fact that $\psi \in R$, and that the latter is cyclic, that

$$\begin{cases} \gamma^\beta(b) = \psi^{tr^{-1}} \\ \gamma^\beta(a) = \iota(a^{-1}) \end{cases}$$

Thus $\gamma^\beta = \gamma$ if and only if $r \equiv 1 \pmod{p}$, as claimed.

We summarise, including the right and left regular representations.

Proposition 4.5. *Let G be a group of type 4, namely $G = \mathcal{C}_{p^2} \rtimes \mathcal{C}_q$. Then in $\text{Hol}(G)$ there are:*

1. $2p^3$ regular groups of type 1, which split into 2 conjugacy classes of length p^2 , and 2 conjugacy classes of length $p^2(p - 1)$;
2. $2(1 + p^2(q - 2))$ regular groups of type 4, which split into 2 conjugacy classes of length 1, and $2(q - 2)$ conjugacy classes of length p^2 .

4.5. Proof of Theorem 1.3 and 1.4

From Theorem 1.1 we have that, for each pair of finite groups Γ, G with $|\Gamma| = |G|$,

$$e(\Gamma, G) = \frac{|\text{Aut}(\Gamma)|}{|\text{Aut}(G)|} e'(\Gamma, G).$$

By assumption the Sylow p -subgroups of the group Γ are cyclic. If the Sylow p -subgroups of the group G are also cyclic, then the values of $e'(\Gamma, G)$ computed in Propositions 4.2, 4.3, 4.4, and 4.5, and the cardinalities of the automorphism groups given in Table 3.1 yield the values of $e(\Gamma, G)$.

Propositions 4.2, 4.3, 4.4, and 4.5 also yield, for $G = (G, \cdot)$, the numbers of conjugacy classes of regular subgroups of $\text{Hol}(G)$, that is, the numbers of isomorphism classes of skew braces (G, \cdot, \circ) .

If the Sylow p -subgroups of the group G are not cyclic, then Corollary 3.4 yields $e'(\Gamma, G) = e(\Gamma, G) = 0$.

5. Induced structures

The goal of this section is to record an alternative proof of Proposition 4.2, using the methods of [CRV16].

Let L/K be a finite Galois field extension and let $\Gamma = \text{Gal}(L/K)$. A Hopf-Galois structure on L/K is called *split* if it is of type G with $G = G_1 \times G_2$ in a non-trivial way.

On the other hand, let F be a field with $K \subseteq F \subseteq L$ such that $\Gamma' = \text{Gal}(L/F)$ has a normal complement in Γ . Assume that F/K and L/F have Hopf-Galois structures of type G_1 and G_2 , respectively. Then $G = G_1 \times G_2$ gives a Hopf-Galois structure on L/K [CRV16, Theorem 3]. A Hopf-Galois structure on L/K is called *induced* if it is obtained as above for some field F with $K \subsetneq F \subsetneq L$.

Remark 5.1. It is immediate to see that each induced Hopf-Galois structure is split, but the converse it is not true in general (see [CRV16, Section 3.2]). However, in the case of extensions of degree p^2q all *abelian* (split) structures are induced. In fact, in this case the Galois group Γ is always a semidirect product of a Sylow p -subgroup and a Sylow q -subgroup. On the other hand, if N is an abelian group inducing a Hopf-Galois structure on L/K , then its Sylow p and q -subgroups are characteristic, so they are stable under the action of $\rho(\Gamma)$ and [CRV16, Theorem 9] guarantees that each Hopf-Galois structure of type N is induced.

In the case when Γ has cyclic Sylow p -subgroups the only split structure is the cyclic one, so all split structures are induced in this case.

We give here a short sketch of how the number of cyclic structures can be obtained by computing induced structures in the case of Galois extension of degree p^2q , $p > 2$ and cyclic Sylow p -subgroups.

So, let L/K be a Galois extension of degree p^2q and assume that Γ has cyclic Sylow p -subgroups. From Theorem 1.3 we have that the number of cyclic structures on L/K is p , pq , pq or p^2 for Γ of type 1, 2, 3, or 4, respectively.

If Γ is of type 1, 2, or 3, then it has a normal Sylow q -subgroup B . Denote by A a Sylow p -subgroup of Γ (there is a unique choice for A if G is of type 1 and there are q choices if G is of type 2 or 3) and let F be

the fixed field by A . Then L/F is a Galois extension of degree p^2 , so it admits p Hopf-Galois structures, all of cyclic type (see [Byo96, Corollary p. 3226]). Moreover, B is a normal complement of A in Γ , so the degree q extension F/K is almost classically Galois and it always admits a unique Hopf-Galois structure (see [Par90, Theorem 5.2]). By [CRV16, Theorem 3] each of these structures induces a split structure on L/K , so we obtain the p or pq structures, depending on Γ being of type 1, or of type 2 or 3, already found in Theorem 1.3. If Γ is of type 2 or 3 those just described are the only possible splittings of L/F which give induced structures.

If Γ is of type 1, in principle, we should also consider the tower $K \subseteq M \subseteq L$ where M is the subfield of L fixed by the Sylow q -subgroup B . However, in this case, since the group Γ splits, the Hopf-Galois structures on L/M are the same as those on F/K and those on M/K are the same as those on L/F , hence all of them have already been considered.

If Γ is of type 4, the Sylow p -subgroup is normal, and we have p^2 possible intermediate extensions F with L/F cyclic of order q (one Hopf-Galois structure for each of them), whereas by [CS18, Theorem 9] F/K has a unique Hopf-Galois structure, so we recover the p^2 cyclic structures as in Theorem 1.3.

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