A phenomenological model for chain transmissions efficiency

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Abstract. This work deals with the development of a simplified and time-efficient phenomenological model for chain transmissions efficiency. Firstly, a complete multibody model for a final transmission of a motorcycle is developed in MSC Adams View environment, and is properly validated in terms of efficiency with experimental tests on a real chain system. Results from multibody simulations are used to develop a new analytical model, that relates all the main operating and geometry parameters of the transmission system to the transmission efficiency with simple relations. In particular, the influence of the polygonal effect on system efficiency is investigated, finding a relationship with the angle of the sprockets pitch polygon.

A new parameter, called chain tension efficiency, is introduced to model the distribution of losses within the system. A linear relationship between this parameter and the number of sprockets teeth and with the system efficiency was assumed and validated, with good results. In addition, the dependence of slack span tension from several parameters, like speed, torque and number of teeth of sprockets, is investigated. In particular, it is highly dependent on the chain peripheral speed, both for a centrifugal tension and for a further linear component of tension, which might be originated by friction between links.

The presented simple model can describe the chain system dynamics with low computational effort, allowing the designer to use a smart tool to select the proper transmission parameters. Due to its computational efficiency, the model is also useful for real-time and hardware-in-the-loop simulations.

1. Introduction

Roller chain is the most commonly used type of chain drive for transmission of mechanical power, and one of its main application fields is in vehicle power transmission, especially for motorcycles. A deep understanding of chain transmission behavior is important in order to maximize motorcycle performance and to reduce fuel consumption. Despite that, chain mechanics is highly complex and they are still not fully understood, especially for what concerns dynamic effects on chain spans tension and on transmission efficiency.

The very first steps to the study of chain links tension distribution were made by Binder in 1956 [1], who presented a geometric progressive load distribution (GPLD) model under several assumptions, like negligible friction, constant sprockets angular speed and tooth reaction angle equal to the pressure angle. Binder's model was revised and deepened in the 1980s by Naji and Marshek [2], including the effects of friction and sprockets' teeth elasticity. This model turned out to be accurate enough for low-speed and high preload working conditions. A similar force model was used by Hollingworth and Hills [3] in 1986 to calculate the losses due to sliding friction in a chain drive during articulation [4]. Conwell and Johnson in 1995 [5] gave an important contribution by designing a machine to measure chain link tension and link-sprocket impact intensity. A further model to describe load distribution and links tension in a two-sprocket chain drive was presented by Troedsson and Vedmar between 1999 [6] and 2001 [7], which introduced a more detailed ANSI tooth profile, links elasticity, a slack span tension calculation and the effect of some dynamic loads. In 2002, Lodge and Burgess [8] extended the previous researches presenting an improved chain link tension model, that was then used to formulate an efficiency model based on friction losses to predict chain drive efficiency. However, the only dynamic load considered is the centrifugal one, while further relevant dynamic loads are neglected, as well as the energy losses due to impacts and vibrations. The model shows a good precision at relatively low torque and speeds, where the neglected dynamic effects are less relevant.

All presented models showed some drawbacks, thus being inadequate to describe both chain tension and system efficiency in all chain drive operating conditions, especially at the highest speeds and torques, which are the most common operating conditions for motorcycle applications. A great effort towards a better description of high-speed chains dynamic effects was made by the various multibody models presented in the very last decades. A first step in this direction was made by Wang and Liu [9], that in 1991, by analysing previous researches, stated that new integrated models describing the full dynamics of roller chain systems were needed, thus opening the way to multibody analysis on chain systems. In particular, Pedersen developed in 2004 [10] a multibody dynamic model for the analysis of generic chain transmissions, with some insights on naval diesel engine applications. These multibody models allow precise and in-depth analysis of chain systems behavior, but with very long calculation times due to their high complexity.

This work aims to combine the precision of multibody models with the simplicity of analytical models. It starts from the creation of a complete multibody model, developed in MSC Adams View environment, for a final transmission of a motorcycle, that is properly validated in terms of efficiency with experimental tests on real chain system and it is used to create a database of test performed for steady state conditions and for different chain transmission geometry and operating conditions. The database is used to develop a new analytical model, that relates all the main operating and geometry parameters of the transmission system to the transmission efficiency with simple relationships. This simple model can describe the chain system efficiency and chain tension, for different geometries and working conditions and with low computational effort, allowing the designer to use a smart tool to select the proper transmission parameters.

2. Multibody model

2.1. Modeling

A multibody model (shown in figure 1) was built using the *ADAMS/View* software and is composed by two sprockets, connected to the *ground* by revolute joints, and a chain made of a large number of links. Even if the bodies are tridimensional, they are assumed to belong to the same mid-plane. All bodies are modeled as rigid, as the effects of deformability are demanded to interface items, such as contacts between sprockets and rollers and bushings between chain links.

Real tooth profile has been taken into account to model sprockets: two-arches of circumference construction has been used to make the model faster and more stable, but the real tooth shape, based on a four-arches construction conforming to ANSI B29.1, was approximated using the method of least squares.

The chain is modeled using the *multilink* mode, consisting in using an alternation of two different link types, whose inertial properties are calculated through a simple model. Links are connected with elastic bushings, that allow to model chain elasticity and damping with a series of concentrated spring-and-damper systems. The chain stiffness is obtained from an experimental

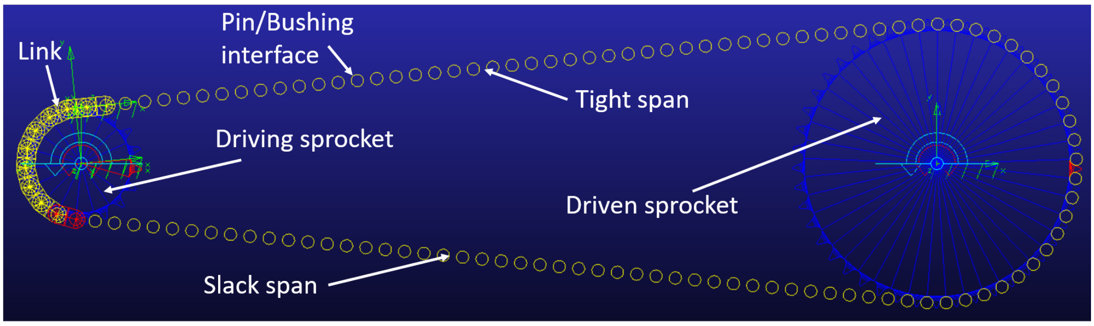

Figure 1. Multibody model.

tensile test performed on a real chain, while the damping is obtained as a percentage of the critical damping.

The energy dissipated due to relative rotation between two links has been modeled in two different ways: using a friction model or introducing viscous rotational damping. The two models have been compared and the latter was chosen in order to favor the simplicity of the model, while the former has been used to properly tune the rotational damping parameter.

Particular attention was paid to model the contact between the sprockets' teeth and the rollers. The "IMPACT" function, available in the multibody code, was used to model the contact force as a non-linear elastic contact with the addition of a damping term, according to the relationship

$$F_N = k_c \delta^n + c_c \dot{\delta} \tag{1}$$

where δ is the interpenetration between rollers and sprockets [11]. Contact parameters (k_c, c_c and n) have been chosen as a trade-off between the approximation of empirical formulas for cylindrical contact and the need to stabilize the simulation with corrections for small mass bodies (link mass is much lower than 1 kg). The effect of friction between sprockets and rollers has also been considered. Numerical data for model parameters are not provided for confidentiality reasons.

The result of the modeling phase is an easily editable model that is able to run quite fast simulations (calculations time of about 2 minutes for a single second of simulation time) providing stable and interpretable results.

2.2. Validation of the model

In order to validate the model and to ensure results reliability, a comparison with data from experimental tests is done. Experimental measurements are taken from efficiency tests made by a chain supplier on the real chain. Tests were made on a test bench, using a 16 teeth driving sprocket and a 40 teeth driven sprocket. The chain is made of 120 links and was mounted with a chain-transversal clearance (which is the maximum movement allowed to the chain at the center of the slack span in a direction orthogonal to the span itself) of 20 mm, which corresponds to a center distance of 727.6 mm between the sprockets. The efficiency in steady state condition was obtained by measuring the output speed and the torque applied by a brake at the driven sprocket, thus calculating the output power and then dividing it by the input power. Tests were made for several values of driving sprocket speed and torque, thus obtaining an efficiency map for a large part of the chain range of use. Multibody results have been obtained by simulating the dynamic model in the same conditions. Results are shown in figures 2 and 3.

Figure 4 shows the difference between the two efficiency maps. This difference is always lower than 1%. It is possible to notice also that the multibody model overestimates efficiency at low

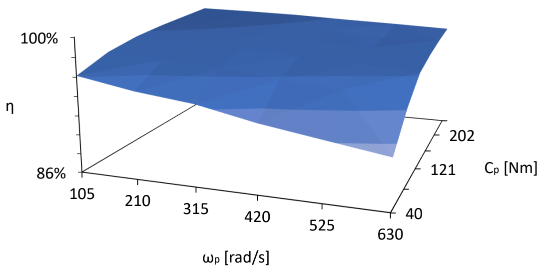

Figure 2. Efficiency map from experimental tests.

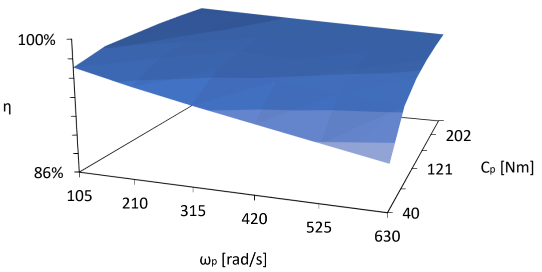

Figure 3. Efficiency map from multibody simulations.

speeds and slightly underestimates it at high speeds. Anyway, at high speeds, which are the most common working conditions, the error is lower than 0,5%.

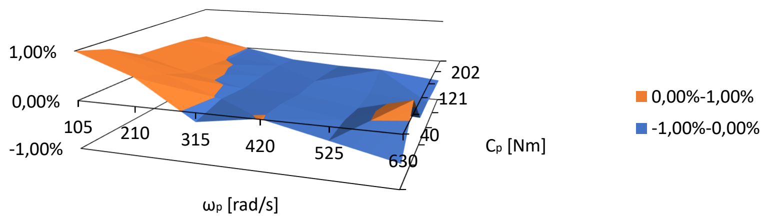

Figure 4. Differences between efficiency maps in figures 2 and 3.

The comparison between simulation and experimental results confirmed the reliability of the model and its parameters.

3. Analytical model

The new analytical model presented is shown in figure 5. Sprockets are modeled as cylindrical wheels, whose diameter is the Pitch diameter, and chain spans are modeled as two inextensible ropes, whose tensions are T_t and T_s for the tight and the slack span, respectively. The efficiency of the system is considered by reducing the output torque C_c, that is

$$C_c = \frac{\eta}{\tau} C_p \tag{2}$$

where C_p is the input torque, τ is the velocity ratio $\omega_c/\omega_p \simeq R_p/R_c$ and η is the system efficiency.

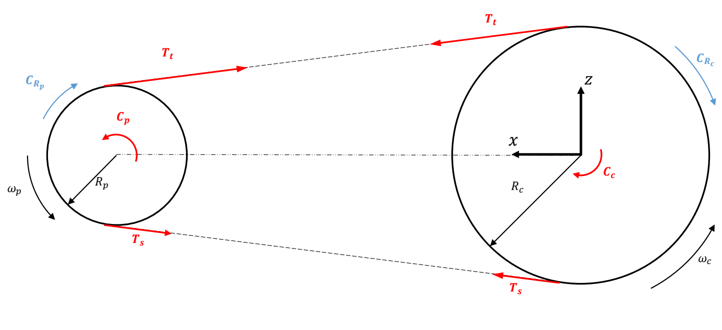

Figure 5. Analytical model.

A parameter η_T, called *chain tension efficiency*, is used to model the distribution of losses within the system. It is defined as the ratio between the difference in tension of the two spans and the tight span ideal tension T_{ideal}:

$$\eta_T = \frac{T_t - T_s}{T_{ideal}} \tag{3}$$

where the ideal tension is calculated as the ratio between the input torque and the radius of the driving sprocket

$$T_{ideal} = \frac{C_p}{R_p} \tag{4}$$

According to this definition, it is possible to calculate power losses on each sprocket through two (fictitious) friction torques C_{Rp} and C_{Rc} on the driving and driven sprockets, respectively:

$$C_{Rp} = C_p - (T_t - T_s)R_p = C_p - \frac{T_t - T_s}{T_{ideal}} C_p = C_p(1 - \eta_T) \tag{5}$$

$$C_{Rp} = (T_t - T_s)R_c - C_c = \left(\frac{T_t - T_s}{T_{ideal}}\right)\frac{C_p}{\tau} - \eta\frac{C_p}{\tau} = \frac{C_p}{\tau}(\eta_T - \eta) \tag{6}$$

Therefore, power losses on the driving sprocket are $1 - \eta_T$, while losses on the driven sprocket are $\eta_T - \eta$.

With these definitions, the whole dynamic behavior of the system can be described as a function of three main quantities: system efficiency η, chain tension efficiency η_T and slack span tension T_s.

In the next section, these quantities are investigated under two main assumptions: constant chain pitch (equal to 15.9mm for the chain studied) and constant mounting transversal clearance, equal to 20mm at the centre of the slack span. All other parameters are variable, including the number of teeth of sprockets, angular speed and torque applied on the driving sprocket. The dependencies of the three quantities are studied, also from a phenomenological point of view: three simple equations are introduced to fit the multibody simulations, performed for driving sprockets having 16 to 18 teeth and for driven sprockets having 38 to 45 teeth.

4. Efficiency and tension determination

4.1. Efficiency η

Aside from working speed and torque, efficiency η also shows a dependency from the number of teeth of sprockets. Figures 6 and 7 show off the efficiency variation in function of sprockets number of teeth for several torque and speed working conditions. It can be observed that:

- Efficiency increases almost linearly with the number of teeth of both sprockets.
- The shape of efficiency curves is the same for every working condition.
- The efficiency variation due to driving sprocket number of teeth is higher than the one due to the number of teeth of the driven sprocket.

The main sources of energy loss in a roller chain system are: the frictional forces between pins and bushings in the links as they articulate onto and off the sprockets, impacts between chain rollers and sprockets teeth during engagement and vibrations resulting from those impacts. These factors depend on the discrete nature of the chain, which leads to the so-called polygonal effect, related to the wrapping of the chain around the sprockets forming a polygon, instead of a circle. Consequently, the magnitude of the polygonal effect can be related to the angle between

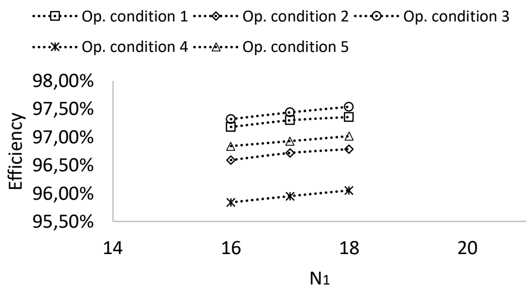

Figure 6. System efficiency *Vs* Number of teeth of driving sprocket for different operating conditions (speed-torque pairs).

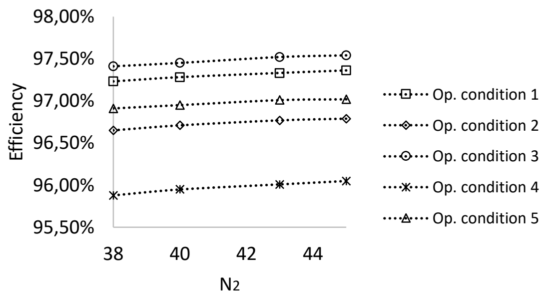

Figure 7. System efficiency *Vs* Number of teeth of driven sprocket for different operating conditions (speed-torque pairs).

each link and the following while they are engaged. This angle can be expressed as the interior angle of the pitch polygon and depends on the number of teeth of the sprocket:

$$\alpha = \pi - \theta = \pi - \frac{2\pi}{N} \tag{7}$$

where N is the sprocket number of teeth and θ is the exterior angle between two adjacent links.

The angle α is not globally linear with the number of teeth, but locally, between 16 and 18 teeth and between 38 and 45 teeth, the trend is almost linear and it is similar to the efficiency trend, with a higher slope for the driving sprocket than for the driven sprocket. It is therefore possible to assume a linear dependency between the system efficiency and the interior angle α of the pitch polygon.

The following expression was assumed to describe the system efficiency η:

$$\eta = c + m_1 C_p^{m_2}\,\omega_p + a_1\Big(\frac{2\pi}{N_1^0} - \frac{2\pi}{N_1}\Big) + a_2\Big(\frac{2\pi}{N_2^0} - \frac{2\pi}{N_2}\Big) \tag{8}$$

where an exponential dependency (with $m_2 < 1$) has been assumed for input torque C_p and a linear dependency has been assumed for angular speed ω_p of driving sprocket. The apex 0 refers to the sprockets configuration that has been used to determine the starting map of efficiency (respectively 16 and 40 teeth).

Table 1. Constants from fitting for efficiency.

c	m_1 ((Nm)$^{-m_2}$ s/rad)	m_2	a_1	a_2
98.687	−0.37372	−0.90772	4.7222	5.4665

Fitting results are shown in figures 8 and 9, where green points refer to multibody simulation results. Values obtained for the constants are reported in table 1 and provide accurate results, with a maximum percentage error of about 0.28%. Note that efficiency values obtained are in percentage points.

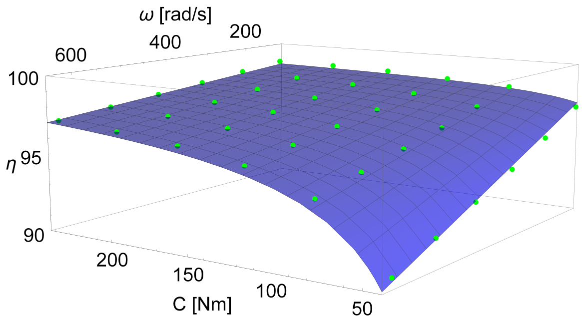

Figure 8. Fitting results: efficiency map.

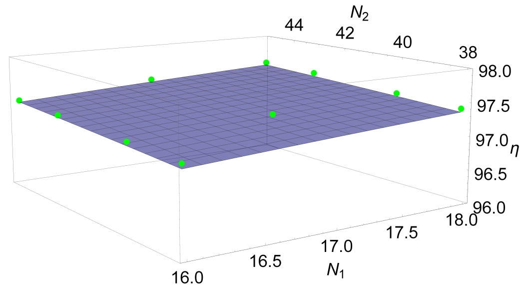

Figure 9. Fitting results: efficiency *Vs* Number of teeth of sprockets.

4.2. Chain tension efficiency η_T

By the analysis of multibody simulation, a linear relationship has been used to relate the efficiency of the chain to the system efficiency.

$$\eta_T = k_1 + k_2\eta \tag{9}$$

The parameters k1 and k2 are variable with the number of teeth. Figure 10 shows the variation of these two parameters with the number of teeth of the driving sprocket, while the different shape of the curves refers to two different numbers of teeth of the driven sprocket. Similarly, figure 11 shows the variation of these two parameters with the number of teeth of the driven sprocket, while the different shape of the curves refers to two different numbers of teeth of the driving sprocket. Anyway, k_1 increases with driving sprocket number of teeth and decreases for bigger numbers of teeth of the driven sprocket, while k_2 makes the opposite (note that $k_1+k_2 \simeq 1$ to grant that $\eta_T > \eta$).

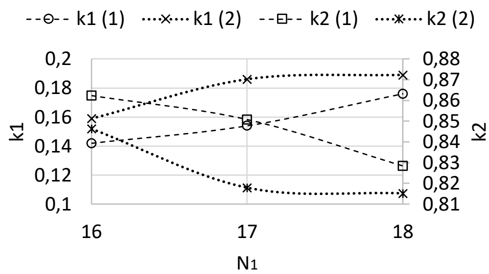

Figure 10. k_1 and k_2 *Vs* Number of teeth of driving sprocket.

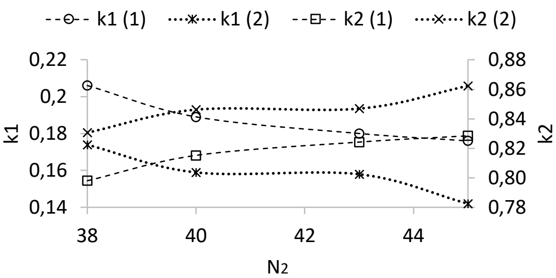

Figure 11. k_1 and k_2 *Vs* Number of teeth of driven sprocket.

In first approximation, dependency of k_1 and k_2 from number of teeth can be expressed with a linear function:

$$k_1 = c_{01} + c_1N_1 + c_2N_2 \qquad\qquad k_2 = c_{02} + c_3N_1 + c_4N_2 \tag{10}$$

Equation 9 can then be rewritten, considering equation 10, as follows:

$$\eta_T = (c_{01} + c_1N_1 + c_2N_2) + (c_{02} + c_3N_1 + c_4N_2)\eta \tag{11}$$

with $c_1, c_4 > 0$ and $c_2, c_3 < 0$ and where $c_{01}, c_{02}, c_1, c_2, c_3$ and c_4 are fitting parameters whose values are reported in table 2. Results, shown in figures 12 and 13, are very accurate, since the maximum percentage error between analytical and multibody results is less than 0.1%.

Table 2. Constants from fitting for chain tension efficiency.

c_{01}	c_1	c_2	c_{02}	c_3	c_4
0.1623	0.01772	−0.003692	0.8429	−0.01835	0.003707

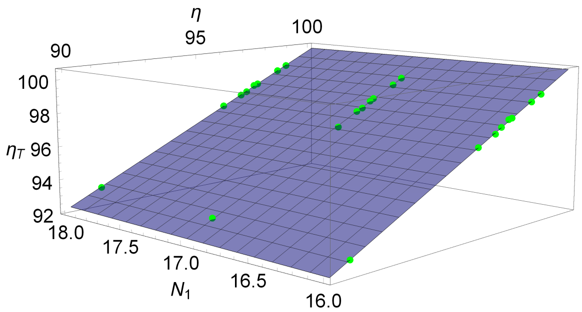

Figure 12. Chain tension efficiency *Vs* Number of teeth of driving sprocket.

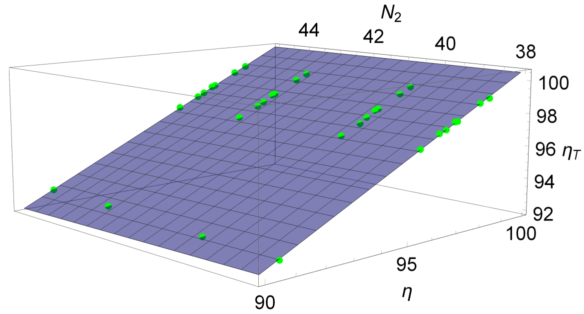

Figure 13. Chain tension efficiency *Vs* Number of teeth of driven sprocket.

As it could be expected from graphics and from the condition $k_1 + k_2 \simeq 1$, it can be observed that $c_1 \simeq -c_3$, $c_2 \simeq -c_4$ and $c_{01} + c_{02} \simeq 1$. Consequently, constants can be reduced from 6 to 3 with no loss of accuracy.

4.3. Slack span tension T_s

Slack span tension turned to be relevantly influenced by sprockets dimensions, working speed and driving sprocket input torque.

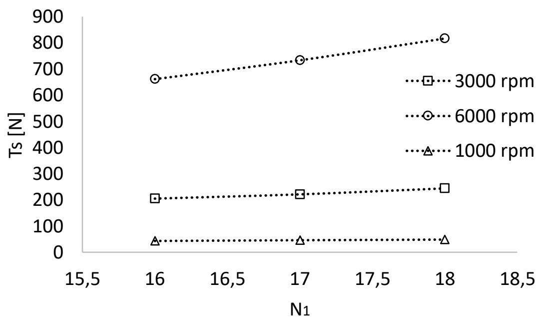

Figure 14. Slack span tension *Vs* Number of teeth of driving sprocket.

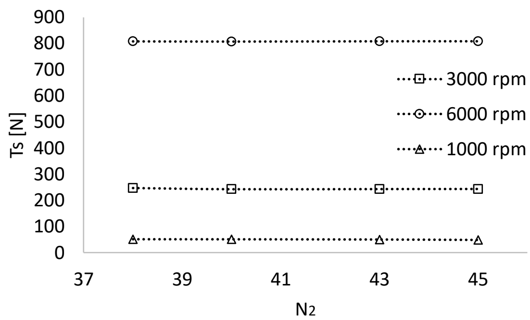

Figure 15. Slack span tension *Vs* Number of teeth of driven sprocket.

Figure 14 shows that slack span tension increases with the number of teeth of the driving sprocket. This could be due to the fact that, for a larger driving sprocket, link disengagement occurs with a higher horizontal speed, thus causing a larger deviation of link trajectory from the "ideal tangent" between the two sprockets and emphasizing effects of damping and friction between links on chain tension. This hypothesis is endorsed by the fact that slope of the curves increases with rotational speed, confirming the influence of speed in the phenomenon. In figure 15 the dependency of slack span tension from driven sprocket number of teeth is shown to be irrelevant, and therefore this dependency can be neglected.

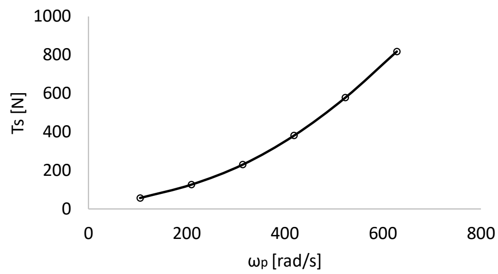

Figure 16. Slack span tension *Vs* Speed.

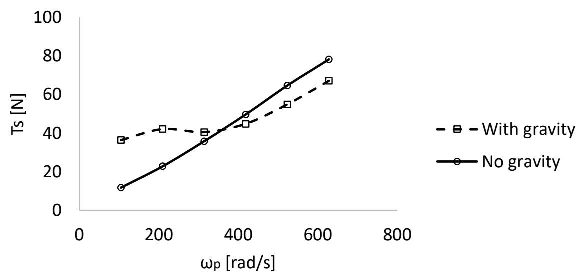

Figure 17. Slack span tension *Vs* Speed (no centrifugal tension).

Several multibody simulations made with constant input torque (figure 16) show that slack span tension has an almost quadratic dependence on driving sprocket angular speed. This trend was predictable since centrifugal effects become more relevant as speed increases. Some interesting information may also come from considering slack span tension obtained deactivating the effect of the lineic mass of the chain and, consequently, neglecting centrifugal effect (figure 17), in order to investigate speed effects on the residual tension. In particular, this trend is evaluated with and without the effect of gravity. Two observations can be done:

- The tension computed without the gravity force is proportional to rotational speed. A dependence from rotational damping between links can then be hypothesized. Opposing to chain straightening at disengagement, the rotational damping causes the disengaging links to deviate higher from their ideal trajectory. Links must then be brought back down to engage the driven sprocket, and this generates a tension in the chain span.
- The influence of gravity increases slack span tension at low speeds, while reduces it at high speeds. This behavior can be justified with slack span orientation: as speed and resulting dynamic effects increase, the slack span suffers more the influence of rotational damping, taking higher trajectories. Gravity at high speeds tends to bring links down, thus reducing both the deviation from the ideal tangent and the slack span tension. Anyway, especially for higher speed working conditions, the gravity contribution to slack span tension is much less than the others and can be neglected.

Finally, torque effect on slack span tension has been investigated. Figure 18 shows that, as input torque increases, T_s decreases with an almost hyperbolic trend, that can be described with a C^n function, with $n < 0$

For what previously told, the chosen function to describe slack span tension is

$$T_s = C_p^n \Big(a \frac{\omega_p^2}{\sin^2 \frac{\pi}{N_1}} + b\omega_p + c \Big)(N_1 - d) \tag{12}$$

where C_p is the input torque, ω_p is the angular speed of the driving sprocket, N_1 is the number of teeth of the driving sprocket and n, a, b, c and d are fitting parameters.

Fitting results, reported in table 3 and in figures 19 and 20, are accurate enough, giving an average precision of over 95%.

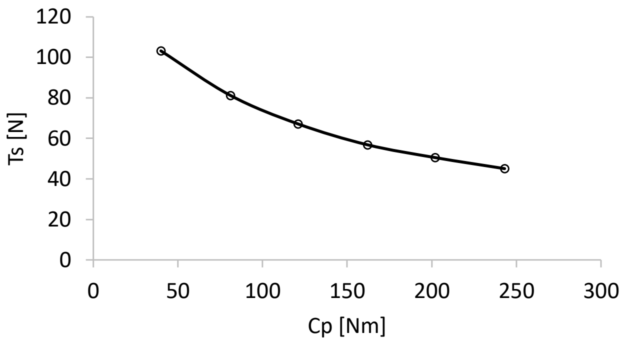

Figure 18. Slack span tension *Vs* Input torque.

Table 3. Constants from fitting for slack span tension.

n	a ($\mathrm{N}^{1-n}\,\mathrm{m}^{-n}/(\mathrm{rad/s})^2$)	b ($\mathrm{N}^{1-n}\,\mathrm{m}^{-n}/(\mathrm{rad/s})$)	c ($\mathrm{N}^{1-n}\,\mathrm{m}^{-n}$)	d
-0.05334	$-6.304\ 10^{-7}$	$7.267\ 10^{-5}$	-0.3734	144.1

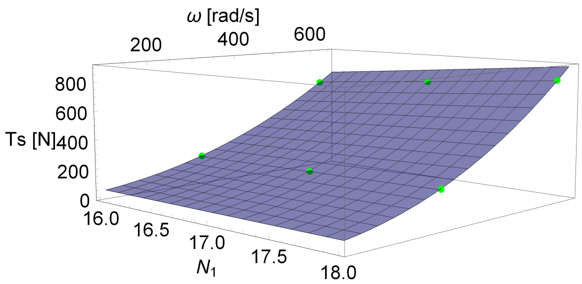

Figure 19. Fitting result: Effect of driving sprocket dimension and speed on T_s.

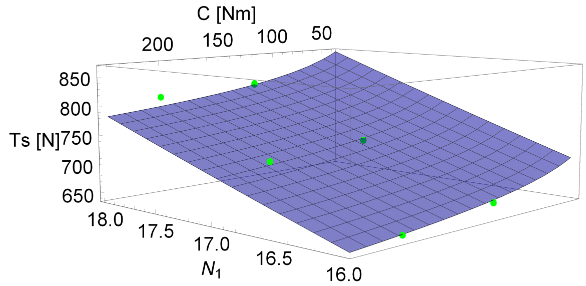

Figure 20. Fitting result: Effect of driving sprocket dimension and torque on T_s.

5. Analytical model validation

In order to test the model accuracy, two more simulations have been performed. The first one simulates a straight run of the motorbike at a really high speed (about 300km/h), using a pair of sprockets not tested before (15-teeth driving sprocket and 36-teeth driven sprocket) and with a torque of 150Nm applyed to the driving sprocket. The comparison between multibody results and previsions from the analytical model are reported in table 4 and shows a very accurate match.

Table 4. Straight run at 300km/h: results comparison.

	ω_p [rad/s]	C_p [Nm]	η	T_s [N]	T_t [N]	η_T
Multibody	666	150	95.78%	679.27	4492.35	97.04%
Analytical	666	150	95.83%	681.81	4494.77	97.04%

The second simulation is made for a very different pair of sprockets, with 20 teeth each, in order to test the precision of the analytical model equations far outside the range of parameters used to build the model. Results, listed in table 5, show that the error is a bit increased, but analytical model still provides very precise previsions of the system behavior.

Table 5. 20-teeth sprockets simulation: results comparison.

	ω_p [rad/s]	C_p [Nm]	η	T_s [N]	T_t [N]	η_T
Multibody	308	121	96.32%	294.79	2637.73	98.25%
Analytical	308	121	96.71%	272.78	2608.51	97.95%

6. Conclusions

In this paper, a new analytical model to determine the behavior of a chain system for a motorbike has been presented, basing it on results from a complete multibody model validated in terms of efficiency with experimental tests. The analytical model, starting from a map of the transmission system efficiency and estimating a limited number of coefficients, allows to foresee the chain system behavior for several geometries and for the whole working range, in terms of torque and speed, of interest for motorcycle applications.

Regarding coefficients estimation for the model, some considerations can be done:

- parameters m_1 and m_2 for the dependence of efficiency from torque and speed could not be necessary if an efficiency map of the whole working range is provided, since the effect of the number of teeth of sprockets is just to translate that map to higher or lower efficiencies;
- parameters to express the influence of angle α of sprockets on efficiency can be easily estimated with just three experimental tests: the first two, for different driving sprocket sizes and with all other parameters constants, in order to determine a_1, and the third, for a different number of teeth of the driven sprocket, to determine a_2; moreover, values of a_1 and a_2 are quite similar, therefore they could be reduced to a single parameter without a significant loss of accuracy;
- as previously said, the 6 parameters to determine chain tension efficiency can be reduced to 3, with no repercussions on model accuracy; Also, these parameters can be estimated taking into account that $k_1 \sim 0.1 \div 0.3$ and $k_2 \sim 0.7 \div 0.9$;
- parameters to determine slack span tension can be estimated with a limited number of experimental tests, if slack span tension T_s can be measured (for example using a single modified link in the chain); moreover, some of them are related to particular quantities of the system: for example, a depends on the mass of links, since it is an inertial constant, b is related to friction and damping parameters between links and c depends on the "static" load on the slack span due to self weight; also, it is $-1 < n < 0$.

In conclusion, the simple model presented can describe the chain system dynamics with low computational effort, allowing the designer to use a smart tool to select the proper transmission parameters. The model is also useful for real-time simulation, being able to reproduce the transmission characteristics with low computational effort.

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