ESSENTIAL SELF-ADJOINTNESS OF LIOUVILLE OPERATOR FOR 2D EULER POINT VORTICES

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ABSTRACT. We analyse the 2-dimensional Euler point vortices dynamics in the Koopman-Von Neumann approach. Classical results provide well-posedness of this dynamics involving singular interactions for a finite number of vortices, on a full-measure set with respect to the volume measure dx^N on the phase space, which is preserved by the measurable flow thanks to the Hamiltonian nature of the system. We identify a core for the generator of the one-parameter group of Koopman-Von Neumann unitaries on $L^2(dx^N)$ associated to said flow, the core being made of observables smooth outside a suitable set on which singularities can occur.

1. INTRODUCTION

In classical, finite-dimensional Hamiltonian systems whose Hamiltonian function involves singular interaction, there may exist singular trajectories in which, at finite time, the driving vector field diverges. When this happens only for a negligible set of initial conditions with respect to an invariant measure, thus a physically relevant measure on phase space, the motion is said to be *almost complete*. A relevant example is the so called *improbability of collisions* in N-body systems, a problem that has received attention both in classical [1, 32, 31] and more recent [17] works.

In such systems lacking global well-posedness, another natural question is whether the Liouville operator, that is the time evolution generator for the dynamics of observables, is essentially self-adjoint on a class of observables smooth in a dense set obtained by removing singular points from the phase space, [30, Section X.14].

The present work concerns the 2-dimensional Euler point vortices system, a Hamiltonian system of first order differential equations describing the dynamics of point particles (*vortices*) whose interaction potential is singular, perfectly fitting the setting we just outlined.

The point vortices system actually describes the evolution of an incompressible ideal fluid whose vorticity is concentrated in a finite number of points. We consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as space domain: if $G = (-\Delta)^{-1}$ is the Green function of the Laplace operator (with zero space average), the vorticity distribution $\omega_t = \sum_{i=1}^N \xi_i \delta_{x_i(t)}$ with intensities of the vortices $\xi_i \in \mathbb{R}$ and positions $x_i \in \mathbb{T}^2$ satisfying

$$\dot{x}_i(t) = \sum_{j\neq i}^N \xi_j \nabla^\perp G(x_i(t), x_j(t)), \tag{1.1}$$

defines a weak solution to Euler equations in vorticity form,

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ \nabla^\perp \cdot u = \omega \end{cases}, \tag{1.2}$$

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where we can express the velocity field u in terms of ω by the Biot-Savart law, $u = -\nabla^\perp G * \omega$, $\nabla^\perp = (-\partial_2, \partial_1)$.

The point vortices system is a classical model. We refer to the monography [28] for most of the notions we are going to rely on, and to [26] for an overview of the statistical mechanics point of view. Integrable and non integrable behaviours in point vortices systems are also the subject of a considerable literature. We refer to [35, 24] for vortices on $\mathbb{T}^2$, [23] for vortices on $\mathbb{S}^2$ and to [9] for a survey on the topic: complete references can be found in those works, including a large number of studies on vortices on $\mathbb{R}^2$.

Due to the Hamiltonian nature of the system, the dynamics (1.1) preserves the volume measure dx^N on the phase space $(\mathbb{T}^2)^N = \mathbb{T}^{2\times N}$. Notwithstanding the singularity of the interaction $\nabla^\perp G$, according to the result of Dürr and Pulvirenti [12], the dynamics is well-posed for initial data in a full measure set of the phase space, thus defining a measurable flow $T_t : \mathbb{T}^{2\times N} \to \mathbb{T}^{2\times N}$ and giving positive answer to the problem of almost completeness.

Let us consider the one-parameter group of Koopman unitaries U_t associated to such flow,

$$U_t f = f \circ T_t, \quad f \in L^2(\mathbb{T}^{2\times N}).$$

In [5], the authors defined the Liouville operator associated to the evolution problem (1.1) on a set of cylinder functions of Fourier modes, and raised the question of essential self-adjointness. We will discuss the setting of [5] in comparison to ours in subsection 3.4.

The main result of the present paper, Theorem 2.12, is a proof of essential self-adjointness for the Liouville operator on $L^2(\mathbb{T}^{2\times N})$. We will consider a domain of smooth functions on full-measure open sets, vanishing in a neighbourhood of singular points of the driving vector field, on which we are able to explicitly write the generator A of the Koopman group $U_t = e^{\mathrm{i}tA}$, and show that such observables form a core for A.

Even if we will achieve our aim by means of an approximation noticeably differing from the one of [12, 28], much of their understanding of the point vortices system will be crucial to our efforts. Our method also draws ideas from the work [27], which discusses essential self-adjointness of Liouville's operator for an infinite particle system with regular interactions.

The study of the generator of point vortices dynamics might provide some insight in the much more difficult problem of essential self-adjointness for the generator of Euler dynamics with enstrophy-measure (space white noise) fixed time distributions. Euler evolution in such low regularity regimes has been linked to point vortices dynamics by Central Limit Theorems [14, 20, 21]: only existence of solutions (dating back to [7]) is known, and thus essential self-adjointness of the generator is sought as a first uniqueness result [8, 4, 2, 3]. However, as already remarked in [5], the point measures we consider in this paper are singular with respect to white noise. We also mention the recent work [22], concerning identification of a domain for the singular generator of stochastic Burgers equation in an infinite dimensional Gaussian space. More generally, we refer to [13] for an overview of the problem of identifying cores for generators of stochastic dynamics and essential self-adjointness in the context of diffusion operators.

For the sake of simplicity, we will first discuss our result in the case of a fixed number N of vortices on the torus. In section 3 we comment on how one can modify our arguments to cover different geometries and reference measures, and finally discuss the configuration space approach of [5] in subsection 3.4.

2. The Liouville Operator for Point Vortices Systems

In this section, $\underline{x} = (x_1, \dots x_N) \in (\mathbb{T}^2)^N$ are the positions of point vortices on $\mathbb{T}^2$, and $\underline{\xi} = (\xi_1, \dots \xi_N) \in \mathbb{R}^N$ their intensities. We denote by dx^N the uniform (Haar's) measure on $(\mathbb{T}^2)^N$, and by $d(x,y)$ the distance of points $x, y \in \mathbb{T}^2$. Also, we will denote by $|B|$ the measure of measurable subsets $B \subset (\mathbb{T}^2)^N = \mathbb{T}^{2\times N}$. All observables are intended as complex valued, and we will denote $L^2(\mathbb{T}^{2\times N}) = L^2(\mathbb{T}^{2\times N}; \mathbb{C})$. We distinguish the imaginary unit $\mathrm{i} \in \mathbb{C}$ and the index $i \in \mathbb{N}$ (in italics). Time $t \in \mathbb{R}$ ranges the whole real line, but for simplicity we will often consider positive times $t > 0$, the other case being completely analogous.

2.1. Classical Results on Improbability of Collisions.

The system (1.1) is an ordinary differential equation in finite dimension, whose vector field is given by

$$B_i(\underline{x}) = \sum_{j\neq i}^N \xi_j K(x_i, x_j), \quad \underline{x} \in (\mathbb{T}^2)^N, \tag{2.1}$$

where $K = \nabla^\perp G$. The vector field is singular on the generalised diagonal

$$\triangle^N = \left\{ \underline{x} \in (\mathbb{T}^2)^N : x_i = x_j \text{ some } i \neq j \right\},$$

because $K(x,y)$ diverges when $x = y \in \mathbb{T}^2$: indeed, we recall that

$$G(x,y) = -\frac{1}{2\pi} \log d(x,y) + g(x,y), \tag{2.2}$$

with $g \in C^\infty(\mathbb{T}^2 \times \mathbb{T}^2)$ a symmetric, zero averaged function. Since $G(x,y) = G(x-y, 0)$ is translation invariant, the latter representation can be obtained by solving the equation $-\Delta u(x) = \delta_0(x)$ in a ball $B \subset \mathbb{T}^2$ centred in 0 with Dirichlet boundary conditions, and considering the difference $G - u$.

Although B is smooth outside $\triangle^N$, classical well-posedness theorems can only provide existence and uniqueness of solutions only locally in time. Indeed, if some vortices collapse, that is if a solution reaches $\triangle^N$, the vector field diverges. However, B is divergence free (in the sense of distributions) and it thus formally preserves the measure dx^N. In fact, the point vortices system is Hamiltonian with respect to conjugate coordinates $(x_{i,1}, \xi_i x_{i,2})$, and Hamiltonian function

$$H(\underline{x}) = \sum_{i<j}^N \xi_i \xi_j G(x_i, x_j)$$

(the interaction energy of vortices). By exploiting this peculiar structure of B, it is possible to prove that in fact, for any fixed –but arbitrary– choice of intensities ξ, the system (1.1) has a global (in time), smooth solution for almost every initial condition with respect to dx^N.

The case in which all intensities ξ_i have the same sign is easier, since the minimum distance between vortices along a trajectory in phase space can be controlled by means of the Hamiltonian H. Indeed, from the definition of H and (2.2) (using in particular the fact that g is uniformly bounded), we have

$$\begin{aligned} H(\underline{x}) &\geq \left(\min_i |\xi_i|\right)^2 \sum_{i<j}^N \left(-\frac{1}{2\pi} \log d(x_i, x_j) + \min_{\mathbb{T}^2} g \right) \\ &\geq -C \log \min_{i\neq j} d(x_i, x_j) - C', \end{aligned}$$

where $C, C' > 0$ are constants depending on $\underline{\xi}, N$, and thus

$$\min_{i\neq j} d(x_i, x_j) \geq e^{-CH(\underline{x}) - C'}. \tag{2.3}$$

Since the right-hand side is a first integral of the motion, we can extend local-in-time solutions of (1.1) starting from $\underline{x} \in \mathbb{T}^{2\times N} \setminus \triangle^N$ to global solutions which are also smooth in time.

When vortices intensities $\underline{\xi} \in \mathbb{R}^N$ take both positive and negative values, there might exist initial conditions leading to collapse, see [28, Section 4.2] and the references above on integrable motion of vortices. Indeed, the energy $H(\underline{x})$ gives us no control whatsoever on the vortices distances along the trajectory of $\underline{x}$, since H include now both positive and negative terms which can cancel out large contributions of close couples of vortices.

Almost completeness in the general case of intensities with positive and negative signs is a classical result due to Dürr and Pulvirenti, [12]; we also refer to [28] for the case of vortices on the whole plane (see section 3 below). The following result summarises Theorems 2.1 and 2.2 of [12]

Theorem 2.1 (Dürr-Pulvirenti)**.** *Let $\underline{\xi} \in \mathbb{R}^N$ be fixed. There exists a full-measure set $M \subset (\mathbb{T}^2)^N$ and a one-parameter group of maps $T_t : M \to M$ such that $\underline{x}(t) = T_t(\underline{x})$ is the unique, smooth solution of (1.1) with initial datum $\underline{x}(0) = \underline{x} \in M$. For all $t \in \mathbb{R}$, T_t defines a measurable, measure preserving, dx^N-almost everywhere invertible transformation of $(\mathbb{T}^2)^N$.*

Define moreover, for $t > 0$ and $\underline{x} \in (\mathbb{T}^2)^N$,

$$d_t(\underline{x}) = \inf_{s\in[0,t]} \min_{i\neq j} |(T_s\underline{x})_i - (T_s\underline{x})_j|.$$

Then there exists a constant $C > 0$ independent of $c \in (0, 1)$ such that

$$|\{d_t(\underline{x}) < c\}| \leq \frac{C(t+1)}{-\log c}. \tag{2.4}$$

The core idea in the proof of Theorem 2.1 is to consider a smooth vector field on $\mathbb{T}^{2\times N}$ given by

$$B_i^\varepsilon(\underline{x}) = \sum_{j\neq i}^{N} \xi_j K_\varepsilon(x_i, x_j), \quad K_\varepsilon = \nabla^\perp G_\varepsilon, \tag{2.5}$$

that is the point vortices vector field (2.1) with a smoothed interaction obtained by $G_\varepsilon \in C^\infty(\mathbb{T}^2)$ such that:

$$G_\varepsilon|_{B(0,\varepsilon)^c} = G|_{B(0,\varepsilon)^c}, \quad |\nabla G^\varepsilon(x)| \leq |\nabla G(x)| \leq \frac{C}{|x|} \quad \forall x \in \mathbb{T}^2, \tag{2.6}$$

where $B(0, \varepsilon) \subset \mathbb{T}^2$ is the ball of radius ε centred in 0 and $C > 0$ is a universal constant. We denote by $T_t^\varepsilon \underline{x} = T^\varepsilon(t, \underline{x})$ the flow of the ordinary differential equation

$$\begin{cases} \dot{\underline{x}}(t) = B^\varepsilon(\underline{x}(t)) \\ \underline{x}(0) = \underline{x} \end{cases},$$

which is globally well-posed since its driving vector field is smooth. Moreover, we define for $\varepsilon > 0, t > 0$ and $\underline{x} \in \mathbb{T}^{2\times N}$,

$$d_t^\varepsilon(\underline{x}) = \inf_{s\in[0,t]} \min_{i\neq j} |(T_s^\varepsilon\underline{x})_i - (T_s^\varepsilon\underline{x})_j|. \tag{2.7}$$

In order to control the minimum distance between vortices, instead of the Hamiltonian one can resort to the Lyapunov function

$$\mathcal{L}^\varepsilon(\underline{x}) = \sum_{i\neq j} G_\varepsilon(x_i, x_j) = \sum_{i\neq j} G_\varepsilon(x_i - x_j),$$

for which the following analogue of (2.3) holds:

$$d_t^\varepsilon(\underline{x}) \geq \exp\left(-C \sup_{s\in[0,t]} |\mathcal{L}^\varepsilon(T_s^\varepsilon \underline{x})| - C'\right), \tag{2.8}$$

with $C, C' > 0$ only depending on N. By differentiating in time $\mathcal{L}^\varepsilon(T_t^\varepsilon \underline{x})$, and exploiting the Hamiltonian structure of the vector field B^ε –in particular volume conservation on phase space– [12] established a uniform bound:

$$\int_{\mathbb{T}^{2\times N}} \sup_{s\in[0,t]} |\mathcal{L}^\varepsilon(T_s^\varepsilon \underline{x})| dx^N \leq C(t+1)$$

with $C > 0$ independent of $\varepsilon > 0$. This, in combination with Markov inequality and (2.8) produces the crucial estimate, for $C > 0$ independent of $c \in (0,1)$,

$$|\{d_t^\varepsilon(\underline{x}) < c\}| \leq \frac{C(t+1)}{-\log c}, \tag{2.9}$$

from which (2.4) follows, since $\{d_t < c\}$ is the almost sure limit of $\{d_t^\varepsilon < c\}$.

In fact, on the set $\{d_t^\varepsilon(\underline{x}) > \varepsilon\}$ the flow $T_s^\varepsilon(\underline{x})$ of B^ε and $T_t(\underline{x})$ of B coincide: for all $t, \varepsilon > 0$,

$$T_s^\varepsilon(\underline{x}) = T_s(\underline{x}) \quad \forall s \in [0,t], \underline{x} \in \{d_t > \varepsilon\}\,. \tag{2.10}$$

Sending $\varepsilon \to 0$ we obtain the full-measure set $\{d_t(\underline{x}) > 0\}$ on which the flow $T_t(\underline{x})$ of B is well-defined, and intersecting the sets $\{d_{t_n}(\underline{x}) > 0\}$ over a sequence of times $t_n \uparrow \infty$ (and one $t_n \downarrow -\infty$) Theorem 2.1 is completed.

2.2. **Functional Analytic Setting.** This paragraph collects abstract definitions and results we are going to apply to point vortices systems. We assume knowledge of basic notions in the theory of groups of unitary operators on Hilbert spaces, for which we refer the reader to [30, Chapter VIII].

Let $(X, \mathcal{F}, \mu)$ a standard Borel probability space, *i.e.* X is a Polish space and $\mathcal{F}$ the associated Borel σ-algebra. The following results establishes a relation between groups of maps on X and groups of operators on $L^2(\mu) = L^2(X, \mathcal{F}, \mu)$. Its first part, the easier one, is well known as Koopman's Lemma, whereas the second part, a converse implication, is a relevant result in Ergodic Theory, for the proof of which we refer to [19].

Theorem 2.2. *Let the mapping*

$$\mathbb{R} \times X \ni (t,x) \mapsto T_t(x) \in X$$

be such that: for μ-almost every $x \in X$, $t \mapsto T_t(x)$ is a continuous map; for all $t \in \mathbb{R}$, $x \mapsto T_t(x)$ is a μ-almost surely invertible, measurable and measure preserving map and for all $t, s \in \mathbb{R}$

$$T_t \circ T_s(x) = T_{t+s}(x)$$

(that is, $(T_t)_{t\in\mathbb{R}}$ is a group). Then

$$L^2(X, \mathcal{F}, \mu) \ni f \mapsto U_t f = f \circ T_t \in L^2(X, \mathcal{F}, \mu)$$

*defines a strongly continuous group of unitary, positive and unit-preserving operators on $L^2(X, \mathcal{F}, \mu)$ (*Koopman group*).*

Conversely, let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous group of of unitary, positive and unit-preserving operators on $L^2(X, \mathcal{F}, \mu)$ with generator A; then there exists a group of μ-almost surely invertible, measurable and measure preserving maps $T_t : X \to X$, $t \in \mathbb{R}$, such that $U_t f = f \circ T_t$ for all $f \in L^2(X, \mathcal{F}, \mu)$; moreover, $t \mapsto T_t(x)$ is weakly measurable for all $t \in \mathbb{R}$.

Remark 2.3. It is worth mentioning that the characterisation of Koopman groups is an important problem in Ergodic Theory. We refer to [25] for a review on the topic, and to the works [18, 36] for a characterisation of Koopman groups in terms of properties of their generators.

Our aim is to identify a core for the generator of vortex dynamics. This problem is intimately linked to the one of uniquely extending densely defined symmetric operators and essential self-adjointness. We recall the following terminology.

Proposition 2.4. *Consider a symmetric linear operator* (L, D) *on* $L^2(\mu)$*; each of the following statements implies the next one:*

- Essential self-adjointness*: the closure of* (L, D) *is self-adjoint;*
- $L^2(\mu)$ uniqueness*: there exists a unique one-parameter strongly continuous group of unitaries whose generator extends* (L, D)*.*
- Markov uniqueness*: there exists a unique one-parameter strongly continuous group of unitaries preserving positivity and unit whose generator extends* (L, D)*.*

While the second implication is trivial, the first one is due to Stone's theorem: any one-parameter strongly continuous groups of unitaries on a Hilbert space H is generated by a self-adjoint operator. We also recall that the first two definitions coincide if (L, D) is semi-bounded; however this will not be the case in our discussion.

The basic self-adjointness criterion is the following (see [30, Theorem VIII.1]).

Proposition 2.5. *Let* H *be a complex Hilbert space,* $U_t = e^{\mathrm{i}tA}$ *a strongly continuous unitary group on* H *and* A *its generator. If* $D \subset D(A)$ *is a dense linear subset such that* $U_t(D) \subseteq D$*, then* $(A|_D, D)$ *is essentially self-adjoint and* D *is a core for* A*,* $\overline{A|_D} = A$*.*

We will in fact use a modified version of this criterion: the proof is a standard argument, but we report it for the sake of completeness.

Proposition 2.6. *Let* H *be a complex Hilbert space,* $U_t = e^{\mathrm{i}tA}$ *a strongly continuous unitary group on* H *and* A *its generator. If* $D \subset D(A)$ *is a dense subset,* $L = A|_D$ *and*

$$\forall t \in \mathbb{R}, \forall f \in D, \quad U_t f \in D\left(\overline{L}\right), \tag{2.11}$$

then (L, D) *is essentially self-adjoint and* D *is a core for* A*,* $\overline{L} = A$*.*

Proof. By [30, Theorem VIII.3], if $\ker(L^* \pm \mathrm{i}) = \{0\}$, then $\overline{L}$ is self-adjoint. Assume by contradiction that there exists $f \in D(L^*)$ such that $L^* f = \mathrm{i} f$ (the case of $L^* f = -\mathrm{i} f$ is analogous). Then, for all $g \in D = D(L)$ it holds

$$\frac{d}{dt} \langle U_t g, f \rangle_H = \langle \mathrm{i} A U_t g, f \rangle_H = \left\langle \mathrm{i} \overline{L} U_t g, f \right\rangle_H = \langle U_t g, f \rangle_H , \tag{2.12}$$

where the second passage makes use of the hypothesis $U_t g \in D(\overline{L})$, and the last one of $L^* = (\overline{L})^*$. The operator U_t is unitary, so the only solution to the above differential equation for $\langle U_t g, f \rangle$ in t is the constant 0, and thus, since g varies on the dense set D, $f = 0$.

We are left to show that $\overline{L} = A$: this follows easily by differentiating in time $e^{\mathrm{i}t\overline{L}}$ on D and noting that the result coincides by definition with the derivative of U_t, so that $U_t = e^{\mathrm{i}t\overline{L}}$. □

Let us note that condition (2.11) can be rephrased as: for all $t \in \mathbb{R}$ and $f \in D$, there exists a sequence $g_n \in D$ such that

$$g_n \xrightarrow{n\to\infty} U_t f, \quad L g_n \xrightarrow{n\to\infty} \overline{L} U_t f \tag{2.13}$$

in the strong topology of H.

2.3. **The Koopman Group for Point Vortices Systems.** We denote by T_t the group of transformations of $\mathbb{T}^{2\times N}$ defined in Theorem 2.1 –that is the point vortices flow– and U_t its associated Koopman group for the remainder of this section. We now define a first set of observables on which we are able to write explicitly the generator of U_t, and which will turn out to be a core for the generator in the simple case where vortices all have positive (or negative) intensity.

Proposition 2.7. *The linear subspace*

$$D = \left\{ f \in C^\infty(\mathbb{T}^{2\times N}) : \operatorname{supp} f \cap \triangle^N = \emptyset \right\}.$$

is dense in $L^2(\mathbb{T}^{2\times N})$.

Fix $\underline{\xi} \in \mathbb{R}^N$. *For any* $f \in D$ *the following expression is well defined as a function in* $L^\infty(\mathbb{T}^{2\times N})$*:*

$$Lf(\underline{x}) = -\mathrm{i} \sum_{i=1}^{N} \sum_{j\neq i} \nabla_i f(\underline{x}) \cdot \xi_j K(x_i - x_j), \tag{2.14}$$

where $\nabla_i f$ *denotes the gradient in the* i*-th coordinate of* $\mathbb{T}^{2\times N} = (\mathbb{T}^2)^N$.

The operator (L, D) *is symmetric; moreover, if* A *is the generator of* U_t*, then* $D \subset D(A)$ *and* $L = A|_D$.

For the sake of clarity, we recall that $\operatorname{supp} f$, the *support* of f, is the closure of the set of points on which $f \neq 0$. Let us also introduce the useful notation

$$\triangle^N_\varepsilon = \left\{ \underline{x} \in \mathbb{T}^{2\times N} : d(x_i, x_j) \leq \varepsilon \text{ for some } i \neq j \right\}, \tag{2.15}$$

and notice that the support of any $f \in D$ and $\triangle^N_\varepsilon$ are disjoint for any small enough $\varepsilon > 0$.

Proof. The density statement is straightforward: smooth functions $C^\infty(\mathbb{T}^{2\times N})$ are dense in $L^2(\mathbb{T}^{2\times N})$, so we only need to show that we can approximate in L^2-norm the elements of $C^\infty(\mathbb{T}^{2\times N})$ with the ones of D. This is readily done by means of Urysohn's lemma —or rather its C^∞ version on smooth manifolds, see [10, Theorem 3.5.1])— which ensures existence of smooth functions g_ε vanishing on $\triangle^N_\varepsilon$ and coinciding with a given $g \in C^\infty(\mathbb{T}^{2\times N})$ outside $\triangle^N_{\varepsilon'}$ for $0 < \varepsilon < \varepsilon'$.

The expression (2.14) is well-defined for $f \in D$ since ∇f vanishes in a neighbourhood of $\triangle^N_\varepsilon$, and moreover

$$\|Lf\|_\infty \leq C_{\xi,N} \|f\|_{C^1(\mathbb{T}^{2\times N})} \left(\min_{\underline{x}\in \operatorname{supp} f} \min_{i\neq j} |x_i - x_j| \right)^{-1} < \infty,$$

because $K(x,y) \sim |x-y|^{-1}$ for $x \to y$ in $\mathbb{T}^2$.

As for the symmetry part: first one replaces $K = \nabla^\perp G$ with the cut off kernel $K_\varepsilon = \nabla^\perp G_\varepsilon$ as in (2.5). Integration by parts and the fact that K_ε is divergence free readily show that

$$\int_{\mathbb{T}^{2\times N}} \nabla_i f(\underline{x}) \cdot K_\varepsilon(x_i - x_j) g(\underline{x}) dx^N = - \int_{\mathbb{T}^{2\times N}} \nabla_i g(\underline{x}) \cdot K_\varepsilon(x_i - x_j) f(\underline{x}) dx^N,$$

in which we can send $\varepsilon \to 0$ by bounded convergence. Summing up all contributions, multiplying by i and taking into account complex conjugation in the scalar product of $L^2(\mathbb{T}^{2\times N})$ we conclude that (L, D) is symmetric.

It remains to show the following limit in $L^2(\mathbb{T}^{2\times N})$,

$$\lim_{t\to 0} \frac{U_t f - f}{t} = Lf, \quad \forall f \in D.$$

But thanks to Theorem 2.1, for almost every $\underline{x} \in \mathbb{T}^{2\times N}$ we have that $U_t f(\underline{x}) = f(T_t\underline{x})$ is a smooth function of t and

$$\frac{d}{dt} f(T_t\underline{x})\bigg|_{t=0} = \sum_{i=1}^{N} \nabla_i f(\underline{x}) \cdot \dot{x}_i(0) = Lf(\underline{x}), \tag{2.16}$$

so that we can conclude by bounded convergence. □

Uniqueness of the flow in the almost-everywhere sense of Theorem 2.1 already gives us, by means of Theorem 2.2, the following uniqueness result.

Proposition 2.8. *For any fixed $\underline{\xi} \in \mathbb{R}^N$, (L, D) is Markov unique, and it extends to the self-adjoint generator A of $U_t = e^{itA}$, the Koopman group of T_t.*

Before we move on, in the next section, to identify a core for the generator of the Koopman group in the general case $\underline{\xi}$, let us analyse the simpler case of vortices with positive (equivalently, negative) intensities, $\underline{\xi} \in (\mathbb{R}^+)^N$. This indeed is a simpler case because the energy $H(\underline{x})$ controls the minimum distance of vortices as noted above in (2.3)

Theorem 2.9. *Let $\underline{\xi} \in (\mathbb{R}^+)^N$. Then the operator (L, D) is essentially self-adjoint, and its closure coincides with the generator of U_t.*

Proof. We apply the classical criterion of Proposition 2.5 by showing that D is left invariant by U_t, that is for any $f \in D$ and $t \in \mathbb{R}$ it holds $f \circ T_t \in D$. By (2.3) and invariance of H, it holds

$$\forall t \in \mathbb{R}, \quad \min_{\underline{x}\in \operatorname{supp} f} \min_{i\neq j} d\left((T_t\underline{x})_i, (T_t\underline{x})_j\right) \geq \exp\left(-C \max_{\underline{x}\in \operatorname{supp} f} H(\underline{x}) - C'\right) > 0,$$

with $C, C' > 0$ constants depending only on $\underline{\xi}$ and N, so that for any $\varepsilon > 0$ smaller than the right-hand side of the latter inequality, $f \circ T_t = f \circ T_t^\varepsilon$, with T^ε being the flow of (2.5) as above. This implies that $f \circ T_t$ is still a smooth function and that its support is disjoint from $\triangle^N$, which concludes the proof. □

2.4. A Core for the Liouville Operator: The General Case.

As we mentioned above, when vortices intensities $\underline{\xi} \in \mathbb{R}^N$ take both positive and negative values, there might exist initial conditions leading to collapse. More generally, the minimum distance of vortices along a globally defined trajectory of the flow might be 0, that is the configuration might pass arbitrarily close to $\triangle^N$.

As a consequence, even if for $f \in D$ the support of f has a positive distance from the diagonal $\triangle^N$, trajectories starting from $\operatorname{supp} f$ can travel arbitrarily close to $\triangle^N$ in finite time, and D is thus not invariant for U_t. Moreover, there is no clue that U_t should preserve C^∞ regularity.

Instead of Proposition 2.5, we rely in this case on Proposition 2.6, which allows us to check a sort of "approximate invariance" of the candidate core. The key is in choosing the correct approximation of U_t, and a natural choice might be to consider the Koopman group of the flow T^ε of the smoothed vector field B^ε: unfortunately this choice is inadequate to our purposes, see subsection 2.5 below.

We now define a new set of observables, which we will prove to be a core for A, and a truncated flow that will serve us to check conditions of Proposition 2.6.

Definition 2.10. *We denote by $\tilde{D}$ the linear space of functions $f \in L^\infty(\mathbb{T}^{2\times N})$ such that:*

- *there exists a version of f and a full-measure open set $M \subset \mathbb{T}^{2\times N}$ on which $f|_M \in C^\infty(M)$, and moreover $\nabla f|_M \in L^\infty(M)$;*
- *f vanishes in a neighbourhood of $\triangle^N$.*

Proposition 2.11. *The linear subspace $\tilde D$ is dense in $L^2(\mathbb{T}^{2\times N})$, and for any $\underline{\xi} \in \mathbb{R}^N$, $f \in \tilde D$ the following expression is well defined as a function in $L^\infty(\mathbb{T}^{2\times N})$:*

$$Lf(\underline{x}) = -\mathrm{i}\sum_{i=1}^{N}\sum_{j\neq i}\nabla_i f(\underline{x})\cdot \xi_j K(x_i - x_j). \tag{2.17}$$

Moreover, $(L, \tilde D)$ is a symmetric operator and if A is the generator of U_t, then $\tilde D \subset D(A)$ and $L = A|_{\tilde D}$.

The proof of the latter Proposition is completely analogous to the one for D above. The following is the main result of the present paper.

Theorem 2.12. *Let $\underline{\xi} \in \mathbb{R}^N$. Then the operator $(L, \tilde D)$ is essentially self-adjoint, and its closure coincides with the generator A of U_t.*

Instead of smoothing the driving vector field, we simply stop trajectories of the flow drawing too close to $\triangle^N$. Since $T^\varepsilon : [0,t] \times \mathbb{T}^{2\times N} \to \mathbb{T}^{2\times N}$ is a smooth function on a compact set, by definition $d_t^\varepsilon : \mathbb{T}^{2\times N} \to \mathbb{R}$, defined in (2.7), is a continuous function. In particular, the sets $\{d_t^\varepsilon < c\}$, $\{d_t^\varepsilon > c\}$ are open subsets of $\mathbb{T}^{2\times N}$. Moreover, since

$$\left|\bigcup_{c\geq 0}\{\underline{x} : d_t^\varepsilon(\underline{x}) = c\}\right| = 1,$$

the closed sets $\{d_t^\varepsilon = c\}$ are negligible for almost all $c \geq 0$. Let us stress that

$$\forall \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\}, \quad \forall s \in [0,t], \quad T_s^\varepsilon \underline{x} = T_s \underline{x}.$$

We define the (lower semicontinuous) function

$$\tau_{t,\varepsilon}(\underline{x}) = \begin{cases} t & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\} \\ 0 & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \end{cases}, \tag{2.18}$$

and the arrested flow

$$R_t^\varepsilon \underline{x} = T_{\tau_{t,\varepsilon}(\underline{x})}\underline{x} = \begin{cases} T_t\underline{x} & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} = \{d_t > \varepsilon\} \\ \underline{x} & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \end{cases}. \tag{2.19}$$

We can assume without loss of generality that $|\{d_t^\varepsilon = \varepsilon\}| = 0$, so that (2.18) and (2.19) define $\tau_{t,\varepsilon}, R_t^\varepsilon$ on a full-measure open set. Indeed, if $\{d_t^\varepsilon = \varepsilon\}$ has positive measure, we can redefine $R_t^\varepsilon = T_t = T^\varepsilon = T^{\varepsilon'}$ on $\{d_t^\varepsilon > \varepsilon'\}$ with a slightly larger $\varepsilon' > \varepsilon$ such that $\{d_t^\varepsilon = \varepsilon'\}$ is negligible, and the identity outside $\{d_t^\varepsilon > \varepsilon'\}$: this does not influence any of the forthcoming arguments. That being said, we see that R_t^ε has the following properties: for any $\varepsilon > 0, t \in \mathbb{R}$,

- it is a diffeomorphism on the full-measure open set $\{d_t^\varepsilon \neq \varepsilon\}$,
- it is a discontinuous but measurable transformation of the whole $\mathbb{T}^{2\times N}$,
- it is a measure preserving map.

Finally, we define the approximating Koopman operators

$$V_t^\varepsilon f(\underline{x}) = f(R_t^\varepsilon \underline{x}) \quad f \in L^2(\mathbb{T}^{2\times N}), \tag{2.20}$$

which are positivity and unit preserving maps taking values in $L^2(\mathbb{T}^{2\times N})$.

Proposition 2.13. *Fix $f \in \tilde D$ and $t \in \mathbb{R}$. Then:*

(i) *$V_t^\varepsilon f \in \tilde D$;*
(ii) *$V_t^\varepsilon f$ converges to $U_t f$ in $L^2(\mathbb{T}^{2\times N})$ as $\varepsilon \to 0$;*
(iii) *$AV_t^\varepsilon f = LV_t^\varepsilon f$ is well-defined since $V_t^\varepsilon f \in \tilde D$, and it converges to $AU_t f$ in $L^2(\mathbb{T}^{2\times N})$ as $\varepsilon \to 0$.*

Proof. Starting from item (i), first of all we notice that $V_t^\varepsilon f = f \circ R_t^\varepsilon \in L^\infty(\mathbb{T}^{2\times N})$ because $f \in L^\infty(\mathbb{T}^{2\times N})$. Let M be, as above, the open set on which (a version of) f is smooth, then

$$f \circ R_t^\varepsilon(\underline{x}) = \begin{cases} f(T_t\underline{x}) & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M \\ f(\underline{x}) & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \cap M \end{cases}. \tag{2.21}$$

The sets on the right-hand side are disjoint since $\{d_t^\varepsilon > \varepsilon\} \cap \{d_t^\varepsilon < \varepsilon\} = \emptyset$, and open because intersection of open sets. Moreover, since T_t^ε is measure-preserving,

$$\left|(T_t^\varepsilon)^{-1}M\right| = |M| = 1 \quad \Rightarrow \left|\{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M\right| = |\{d_t^\varepsilon > \varepsilon\}|$$

and also $|\{d_t^\varepsilon < \varepsilon\} \cap M| = |\{d_t^\varepsilon < \varepsilon\}|$. This shows that $f \circ R_t^\varepsilon$ coincides with a smooth function on a full-measure open set. As for its gradient,

$$\nabla\left(f \circ R_t^\varepsilon\right)(\underline{x}) = \begin{cases} \nabla f(T_t^\varepsilon \underline{x}) DT_t^\varepsilon(\underline{x}) & \underline{x} \in \{d_t^\varepsilon > \varepsilon\} \cap (T_t^\varepsilon)^{-1}M \\ \nabla f(\underline{x}) & \underline{x} \in \{d_t^\varepsilon < \varepsilon\} \cap M \end{cases},$$

where $\|DT_t^\varepsilon(\underline{x})\|_\infty < \infty$, and thus $\nabla(V_t^\varepsilon f) \in L^\infty(\mathbb{T}^{2\times N})$ since $\nabla f \in L^\infty(M)$. By definition, $R_t^\varepsilon(\underline{x}) = \underline{x}$ on $\{d_t^\varepsilon < \varepsilon\}$, which is a neighbourhood of $\triangle^N$, since it contains all $\triangle^N_{\varepsilon'}$ for $\varepsilon' < \varepsilon$; thus on the intersection of $\{d_t^\varepsilon < \varepsilon\}$ and the neighbourhood of $\triangle^N$ on which f vanishes, so must vanish also $V_t^\varepsilon f$, concluding item (i).

Item (ii) follows directly from (2.9) and the fact that U_t is unit preserving:

$$\begin{aligned} \|U_t f - V_t^\varepsilon f\|_{L^2}^2 &= \int_{\{d_t^\varepsilon < \varepsilon\}} |U_t f(\underline{x}) - f(\underline{x})|\, dx^N \\ &\leq 2\,\|f\|_\infty^2\, |\{d_t^\varepsilon < \varepsilon\}| \leq \frac{Ct\,\|f\|_\infty^2}{-\log\varepsilon}. \end{aligned}$$

Let us now consider how the generator A acts on $V_t^\varepsilon f$. By definition,

$$AV_t^\varepsilon f(\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} U_s V_t^\varepsilon f(\underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s \underline{x}).$$

For a fixed $\underline{x}$ in the open set $\{d_t^\varepsilon > \varepsilon\}$, $T_s\underline{x}$ is well-defined for s in a neighbourhood of 0, and it is a smooth function in such time interval. Thus, for small enough s depending on the $\underline{x}$ we are fixing, $T_s\underline{x} \in \{d_t^\varepsilon > \varepsilon\}$, and the same is true if $\underline{x} \in \{d_t^\varepsilon < \varepsilon\} \setminus \triangle^N$ (we are removing the closed negligible diagonal $\triangle^N$). As a consequence, for all $\underline{x} \in \{d_t^\varepsilon > \varepsilon\}$,

$$\left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s \underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_t T_s \underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_s T_t \underline{x}) = Lf(T_t\underline{x}) = U_t Lf(\underline{x}),$$

and analogously for $\underline{x} \in \{d_t^\varepsilon < \varepsilon\} \setminus \triangle^N$,

$$\left.\frac{d}{ds}\right|_{s=0} f(R_t^\varepsilon T_s \underline{x}) = \left.\frac{d}{ds}\right|_{s=0} f(T_s \underline{x}) = Lf(\underline{x}).$$

We thus see that, for $\underline{x}$ in a full-measure set,

$$LV_t^\varepsilon f(\underline{x}) = V_t^\varepsilon Lf(\underline{x}).$$

Since a strongly continuous unitary group always commutes with its generator on the domain of the latter, and since $Af = Lf$ for $f \in \tilde{D}$,

$$\begin{aligned} \|AU_t f - AV_t^\varepsilon f\|_{L^2}^2 = \|U_t Lf - V_t^\varepsilon Lf\|_{L^2}^2 &= \int_{\{d_t^\varepsilon < \varepsilon\}} |U_t Lf(\underline{x}) - Lf(\underline{x})|\, dx^N \\ &\leq 2\,\|Lf\|_\infty^2\, |\{d_t^\varepsilon < \varepsilon\}| \leq \frac{Ct\,\|f\|_{C^1}^2}{-\log\varepsilon}, \end{aligned}$$

where C is a constant depending on N and $\underline{\xi}$. This concludes the proof of (iii). □

Theorem 2.12 is a direct consequence of Proposition 2.6 and Proposition 2.13. Indeed, for fixed $f \in \tilde{D}$ and $t \in \mathbb{R}$, the we have shown that $V_t^\varepsilon f$ satisfies condition (2.13), and thus $\tilde{D}$ is a core for A.

2.5. **Considerations on unsuccessful approaches.** In the proof of Theorem 2.12 we use in an essential way the peculiar structure of our approximating flow R_t^ε in items (i) and (iii), while (ii) still holds true if we replace $U_t f$ with $U_t^\varepsilon f = f \circ T_t^\varepsilon$, the approximating flow of [12], for any smooth $f \in D$. There are two reasons why we are not able to treat the latter setting.

Using the fact that T_t^ε is smooth one can show with some care that U_t^ε preserves D. This and estimate (2.9) would show that D is a core for A provided that we can also show that $AU_t^\varepsilon f$ strongly converges to $AU_t f$ for fixed $f \in D, t \in \mathbb{R}$ (*cf.* Proposition 2.13). Since $U_t f$ and $U_t^\varepsilon f$ coincide on $\{d_t^\varepsilon > \varepsilon\}$, we only need to evaluate their difference on $\{d_t^\varepsilon \le \varepsilon\}$. The set over which we integrate has small measure $t \log\left(\frac{1}{\varepsilon}\right)$, but if we try to bound $LU_t^\varepsilon f$ uniformly in $\underline{x}$ ($LU_t f = U_t L f$ is uniformly bounded since Lf is), we are led to control terms including $\|DT_t^\varepsilon\|_\infty$: since the vector field $\|B^\varepsilon\|_\infty \simeq \varepsilon^{-2}$, we get $\|DT_t^\varepsilon\|_\infty \simeq e^{C\varepsilon^{-2}}$, which is way too large to be compared with the measure of the integration set. Considering estimates in L^2 or L^p norms does not seem to solve the issue, either.

We have seen above how an abrupt truncation of the flow allows us to show that $\tilde{D}$ is a core for A, and it is clear that allowing functions of lower regularity was necessary to employ this kind of approximation. We further mention only one more smooth approach. One might define the vector field

$$B^\delta(\underline{x}) = M^\delta(\underline{x}) B(\underline{x}), \tag{2.22}$$

with $M^\delta \in C^\infty(\mathbb{T}^{2\times N})$ vanishing on a δ-neighbourhood of $\triangle^N$ and taking value 1 far from it. The Koopman operators V_t^δ of its associated flow would preserve D and strongly converge to U_t; however, L and V_t^δ would not commute unless M^δ is a first integral of the motion. As there can not be invariants of the vortex motions controlling the minimum distance of vortices (as M^δ would do) in the case of coexisting positive and negative vortices, we would not be able to continue the proof as we did for Theorem 2.12, and thus have to face explicit computations, in which the difficulties of the same kind of the ones outlined above appear.

3. Generalisations

3.1. **Point Vortices on the Sphere.** All the arguments above still work with almost no modifications when the torus $\mathbb{T}^2$ is replaced with a smooth compact surface with no boundary, such as the sphere $\mathbb{S}^2 = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$ (to be regarded as an embedded surface). On $\mathbb{S}^2$, $d\sigma$ is the Riemannian volume, so that $\int_{\mathbb{S}^2} d\sigma = 1$, and $x \cdot y, x \times y$ respectively denote scalar and vector products in $\mathbb{R}^3$. Euler equations on $\mathbb{S}^2$ are given by, for $x \in \mathbb{S}^2$,

$$\begin{cases} \partial_t \omega(x,t) = x \cdot (\nabla \psi(x,t) \times \nabla \omega(x,t)), \\ -\Delta \psi(x,t) = \omega(x,t). \end{cases}$$

Here Δ is the Laplace-Beltrami operator, and we have to supplement the Poisson equation for the *stream function* ψ with the zero average condition. The Green function of $-\Delta$ is simply given by

$$-\Delta G(x,y) = \delta_y(x) - 1, \quad G(x,y) = -\frac{1}{2\pi} \log|x-y| + c,$$

$c \in \mathbb{R}$ a universal constant making G zero-averaged. To satisfy in weak sense Euler equations, the point vortices vorticity distribution $\omega = \sum_1^N \xi_i \delta_{x_i}$ must evolve according to

$$\dot{x}_i = \frac{1}{2\pi}\sum_{i\neq j}^{N} \xi_j \frac{x_j \times x_i}{|x_i - x_j|^2}, \tag{3.1}$$

which is still a Hamiltonian system with

$$H(x_1, \dots x_N) = \sum_{i<j}^{N} \xi_i \xi_j G(x_i, x_j).$$

In fact, setting $K(x,y) = \frac{1}{2\pi}\frac{x\times y}{|x-y|^2}$, (3.1) takes the same form of (1.1), and K is still a skew-symmetric, divergence free function on $\mathbb{S}^2$ (divergence being the adjoint of the gradient operator on functions of $\mathbb{S}^2$). We refer to [29] for a more complete discussion of this setting. It is easy to see that all the features we relied on in section 2 are still present:

(i) the flow is locally well posed when positions of vortices do not coincide, and it is measure-preserving because of the Hamiltonian nature of the equations;

(ii) the crucial cancellation leading to integrability of the Lyapunov function $\mathcal{L}(x_1, \dots x_N) = \sum_{i\neq j} G(x_i, x_j)$ and thus required for the proof of Theorem 2.1 to work still takes place, in this case because $(x \times y) \perp (x - y)$ for any $x, y \in \mathbb{S}^2$;

(iii) as a consequence, the almost-surely well defined point vortices flow T_t coincide with a smooth one on open sets of large measures, so that we can implement again the strategy of subsection 2.4.

3.2. **Point Vortices on Bounded Domains.** Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and Lebesgue measure $|\mathcal{D}| = 1$, $G(x,y)$ the Green function of $-\Delta$ on $\mathcal{D}$ with Dirichlet boundary conditions, which can be represented as the sum of its free version $G_{\mathbb{R}^2}(x,y) = -\frac{1}{2\pi}\log|x-y|$ and the harmonic extension in $\mathcal{D}$ of the values of $G_{\mathbb{R}^2}$ on $\partial\mathcal{D}$,

$$G(x,y) = -\frac{1}{2\pi}\log|x-y| + g(x,y), \qquad \begin{cases} \Delta g(x,y) = 0 & x \in \mathcal{D} \\ g(x,y) = \frac{1}{2\pi}\log|x-y| & x \in \partial D \end{cases} \tag{3.2}$$

for all $y \in \mathcal{D}$. Both G and g are symmetric, and maximum principle implies that

$$-\frac{1}{2\pi}\log(d(x) \vee d(y)) \le g(x,y) \le \frac{1}{2\pi}\log \operatorname{diam}(\mathcal{D}), \tag{3.3}$$

with $d(x)$ the distance of $x \in \mathcal{D}$ from the boundary $\partial\mathcal{D}$.

The motion of a system of N vortices with intensities $\xi_1, \dots, \xi_N \in \mathbb{R}$ and positions $x_1, \dots, x_N \in \mathcal{D}$ is governed by the Hamiltonian function

$$H(x_1, \dots, x_n) = \sum_{i<j}^{N} \xi_i \xi_j G(x_i, x_j) + \sum_{i=1}^{N} \xi_i^2 g(x_i, x_i),$$

leading to the system of equations

$$\dot{x}_i(t) = \sum_{j\neq i}^{N} \xi_j \nabla^\perp G(x_i(t), x_j(t)) + \xi_i^2 \nabla^\perp g(x_i, x_i).$$

The additional (with respect to the cases with no boundary) self-interaction terms involving g are due to the presence of an impermeable boundary: it is thanks to these terms that the system satisfies (in weak sense) Euler's equations. We refer to [28, Section 4.1] for a thorough motivation of this fact.

In this setting, the relevant features (i)–(iii) we individuated in the last paragraph are still present, but the boundary enters as an additional singular set of the vector field, and thus our arguments must take it into account. Without going into details, we just mention the relevant required modifications:

- the smoothed vector field B^ε and its associated flow T^ε of subsection 2.1 must be defined by smoothing both $\log|\cdot|$ and g in (3.2): B^ε will coincide with the original vortices vector field B whenever vortices are at least $\varepsilon > 0$ apart from each other and from the boundary;
- functions of D must now satisfy $\operatorname{supp} f \subset \mathcal{D}^N \setminus \triangle^N$, where $\triangle^N$ is the diagonal set of $\mathcal{D}^N$ (the definition being the same as in the torus case), whereas function of $\tilde{D}$ must vanish not only around $\triangle^N$, but also in a neighbourhood of $\left\{\underline{x} \in \overline{\mathcal{D}}^N : \exists i : x_i \in \partial\mathcal{D}\right\}$.

3.3. **Gibbsian Ensembles and Vortices on the Whole Plane.** We now return to point vortices on $\mathbb{T}^2$. Besides the uniform measure dx^N on $\mathbb{T}^{2\times N}$, the point vortices flow T_t also preserves all *(Canonical) Gibbs measures* defined by

$$d\mu_{\beta,N}(\underline{x}) = \frac{1}{Z_{\beta,N}} e^{-\beta H(\underline{x})} dx^N, \quad Z_{\beta,N} = \int_{\mathbb{T}^{2\times N}} e^{-\beta H(\underline{x})} dx^N. \tag{3.4}$$

In fact, the above density is integrable as soon as $|\beta| \leq C(N,\underline{\xi})$, $C > 0$ being a constant depending only on N and the intensities, and the *partition function* $Z_{\beta,N} > 0$ is there to make $\mu_{\beta,N}$ a probability measure, see [21, 26].

When $\beta > 0$, $\mu_{\beta,N}$ gives more weight to configurations where vortices of the same sign are far from each other, but positive and negative vortices are close. Vice-versa, if $\beta < 0$, vortices of the same sign tend to cluster. Invariance of $\mu_{\beta,N}$ is an easy consequence of the one of $H(\underline{x})$ and dx^N, and it can be achieved by considering the smoothed vortices interaction B^ε with Hamiltonian H^ε and sending $\varepsilon \to 0$.

Whatever β is, since $\mu_{\beta,N}$ is absolutely continuous with respect to dx^N, the flow T_t is still globally well-defined on a full-measure set. However, the density of $\mu_{\beta,N}$ is singular in $\triangle^N$ (save for trivial cases), so uniform integrability of the Lyapunov functions $\mathcal{L}^\varepsilon$ in Theorem 2.1 is spoiled. As a consequence, the arguments in subsection 2.4 also fail.

Let us now spend a few words on point vortices on $\mathbb{R}^2$. The system is given by (1.1) with $G(x) = -\frac{1}{2\pi}\log|x|$, and it is well-posed for almost all initial conditions with respect to the product Lebesgue measure provided that no subset of the intensities $\{\xi_1, \dots \xi_N\}$ sums to zero, see [28]. The latter condition ensures that vortices can not travel to infinity in finite time.

The product Lebesgue measure on $\mathbb{R}^{2\times N}$ is not a probability measure, so we are led to look for an integrable density on $\mathbb{R}^2$ left invariant by the dynamics. To the best of our knowledge, this is only achieved by the Gaussian measure

$$d\mu_{\alpha,\eta,N}(\underline{x}) = \frac{1}{Z_{\alpha,\eta,N}} e^{-\eta\cdot M(\underline{x}) - \alpha I(\underline{x})} dx^N, \quad \eta \in \mathbb{R}^2, \alpha \in \mathbb{R}^+,$$

$$M(\underline{x}) = \sum_{i=1}^N \xi_i x_i, \qquad I(\underline{x}) = \sum_{i=1}^N \xi_i |x_i|^2,$$

when *all vortices are positive*, I and M being first integrals of vortices motion, the *moment of inertia* and *centre of vorticity* (see [26, Section 5.3]). The interaction energy H can be also added to the Gibbs exponential, but this is not a substantial modification. As we have seen above, the case of positive vortices can be dealt with by exploiting conservation of energy, so we shall not discuss it further. Unfortunately, the more interesting case of arbitrary signs seems to be impossible to include in our discussion.

3.4. **The Configuration Space and Non-Uniqueness.** In the point vortices time evolution, the number and intensities of vortices are constant — at least when no vortices collide, as we will see. As a consequence, everything we said still applies if instead of fixing $N, \underline{\xi}$ we choose them at random, provided that all objects are well defined. In order to discuss an arbitrary number of vortices, one can consider the phase space

$$\bigcup_{N\geq 0} (\mathbb{T}^2 \times \mathbb{R})^N,$$

on which, conditioned to the random choice of N, to be made for instance with a sample of a Poisson distribution, we consider the product measures $dx^N \otimes \nu^{\otimes N}$, with ν the probability law of a single intensity $\xi_i \in \mathbb{R}$.

An equivalent (up to symmetrisation of products) point of view is the *configuration space* setting, in which one looks at the law of the vorticity distribution $\omega = \sum_{i=1}^N \xi_i \delta_{x_i}$ (the empirical measure of vortices) under the law of the aforementioned ensemble of vortices. This is the approach of [5]. Let us define

$$\Gamma = \bigcup_{N\geq 0} \Gamma_N, \quad \Gamma_N = \left\{ \gamma = \sum_{i=1}^N \xi_i \delta_{x_i} : \xi_i \in \mathbb{R}, x_i \in \mathbb{T}^2, x_i \neq x_j \text{ if } i \neq j \right\},$$

to be regarded as a subset of finite signed measures $\mathcal{M}(\mathbb{T}^2)$. There is a one-to-one correspondence between elements of Γ_N and classes of equivalence of $(\mathbb{T}^2 \times \mathbb{R})^N$ up to permutations. Let ν be a probability measure on $\mathbb{R}$ with finite second moment and $\lambda > 0$; we define the measure μ_N on Γ_N as the image of $dx^N \otimes \nu^{\otimes N}$ on $(\mathbb{T}^2 \times \mathbb{R})^N$ under the aforementioned correspondence, and then we define μ on Γ as

$$\mu = e^{-\lambda} \sum_{N\geq 0} \frac{\lambda^N}{N!} \mu_N.$$

Equivalently, μ can be realised by considering a Poisson point process on $\mathbb{T}^2 \times \mathbb{R}$ with intensity measure $\lambda dx \otimes d\nu$, the samples of which are vectors $(x_1, \xi_1, \dots x_N, \xi_N)$, and setting μ to be the image law under the map $\gamma = \sum_{i=1}^N \xi_i \delta_{x_i}$. We refer to [6] for a complete discussion of Poisson processes and the configuration space.

By Theorem 2.1, for μ-almost every $\gamma \in \sum_{i=1}^N \xi_i \delta_{x_i} \Gamma$ the point vortices flow with initial positions x_i and intensities ξ_i is globally well-defined. Moreover, the flow defines a group of invertible measurable maps $\mathcal{T}_t : \Gamma \to \Gamma$, the cursive to distinguish it from the flow T_t on $L^2(\mathbb{T}^{2\times N})$ in section 2. The map $\mathcal{T}_t$ preserves μ since it leaves each Γ_N invariant, and for fixed N the point vortices evolution does not change intensities and preserves the product measure on the torus.

The main contribution of [5] is an explicit expression of the generator of the Koopman group U_t on $L^2(\Gamma, \mu)$ associated to $\mathcal{T}_t$, on the set of cylinder functions of Fourier modes. In order to comment the problem of essential self-adjointness in this setting we now repeat their result: we do so perhaps in a more concise way, by means of an important symmetrisation first introduced in the works of Delort and Schochet [11, 33] to give meaning to weak solutions of Euler equations in low regularity regimes.

Indeed, the weak formulation for Euler equations in vorticity form Equation 1.2 against a smooth test $\phi \in C^\infty(\mathbb{T}^2)$ is given by

$$\begin{aligned} \langle \phi, \omega_t \rangle - \langle \phi, \omega_0 \rangle &= \int_0^t \int_{\mathbb{T}^{2\times 2}} K(x-y) \omega_s(y) \omega_s(x) \nabla \phi(x) dx dy ds \\ &= \int_0^t \langle (K * \omega_s) \omega_s, \nabla \phi \rangle \, ds, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of $L^2(\mathbb{T}^2)$. For smooth solutions of Euler equations, one can symmetrise the variables x, y —keeping in mind that K is skew-symmetric— to obtain the expression

$$\begin{aligned} \langle\phi,\omega_t\rangle - \langle\phi,\omega_0\rangle &= \int_0^t \int_{\mathbb{T}^{2\times 2}} H_\phi(x,y)\omega_s(x)\omega_s(y)dxdyds \\ &= \int_0^t \langle H_\phi, \omega_s\otimes\omega_s\rangle\, ds, \end{aligned} \tag{3.5}$$

$$H_\phi(x,y) = \frac{1}{2}(\nabla\phi(x) - \nabla\phi(y))\cdot K(x-y), \quad x,y\in\mathbb{T}^2, \tag{3.6}$$

where $H_\phi(x,y)$ is a symmetric function with zero average in both variables and smooth outside the diagonal set $\triangle^2$, where it has a jump discontinuity. Because of this, by interpreting brackets $\langle\cdot,\cdot\rangle$ as suitable duality couplings, one can give meaning to Euler equations when vorticity ω has low space regularity.

One such example is the point vortices system: as detailed in [34], the empirical measure $\omega = \sum_{i=1}^N \xi_i\delta_{x_i}$ with x_i evolving as in (1.1) satisfies (3.5) if we assume that $H_\phi(x,x) = 0$, thas is if we neglect self-interactions of vortices. More precisely, brackets $\langle H_\phi,\cdot\rangle$ are to be interpreted as duality couplings between continuous functions and measures on $\mathbb{T}^{2\times 2}\setminus\triangle^2$. We will discuss this further below, here we also recall that (3.5) was also used as an alternative to the Fourier approach of [7] to give meaning to white noise and Poissonian solutions in [14, 15, 16, 20].

We define *local observables* on Γ as the family $\mathcal{F}$ of functions of the form

$$F(\gamma) = f(\langle\phi_1,\gamma\rangle, \dots \langle\phi_n,\gamma\rangle), \tag{3.7}$$

where $f\in C_c^\infty(\mathbb{C}^n,\mathbb{C})$ and $\phi_1,\dots\phi_n\in C^\infty(\mathbb{T}^2,\mathbb{C})$, the brackets $\langle\cdot,\cdot\rangle$ denoting coupling of continuous functions and measures. In [5] the functions ϕ_k were chosen in the Fourier orthonormal basis, but this would not change anything in our discussion.

Proposition 3.1. *Let $\mathcal{U}_t$ be the Koopman group on $L^2(\mu)$ associated with $\mathcal{T}_t$, and $\mathcal{A}$ be its generator. For any $F\in\mathcal{F}$ of the form (3.7), the following expression defines an observable in $L^2(\mu)$,*

$$\mathcal{L}F(\gamma) = -\mathrm{i}\sum_{k=1}^n \partial_k f(\langle\phi_1,\gamma\rangle, \dots \langle\phi_n,\gamma\rangle)\langle H_{\phi_k}, \gamma\otimes\gamma\rangle. \tag{3.8}$$

The operator $(\mathcal{L},\mathcal{F})$ is symmetric, $\mathcal{F}\subseteq D(\mathcal{A})$, and $\mathcal{A}|_{\mathcal{F}} = \mathcal{L}$. Moreover, $(\mathcal{L},\mathcal{F})$ is Markov unique, that is $\mathcal{A}$ is the unique self-adjoint extension generating a strongly continuous, positivity and unit preserving group of unitaries, which is $\mathcal{U} = e^{\mathrm{i}t\mathcal{A}}$.

Proof. To show that $\mathcal{L}F\in L^2(\mu)$, since $\partial_k f$ is uniformly bounded, we just need to compute for $\phi\in C^\infty(\mathbb{T}^2,\mathbb{C})$,

$$\begin{aligned} \int \langle H_\phi,\gamma\otimes\gamma\rangle^2 d\mu_N(\gamma) &= \int_{\mathbb{T}^{2\times N}}\int_{\mathbb{R}^N}\left(\sum_{i\neq j}\xi_i\xi_j H_\phi(x_i,x_j)\right)^2 dx^N d\nu^N(\underline{\xi}) \\ &= \int_{\mathbb{T}^{2\times N}}\int_{\mathbb{R}^N}\sum_{i\neq j,\ell\neq k}\xi_i\xi_j\xi_\ell\xi_k H_\phi(x_i,x_j)H_\phi(x_\ell,x_k)dx^N d\nu^N(\underline{\xi}) \\ &= 2\sum_{i\neq j}\int_{\mathbb{R}^2}\xi_i^2\xi_j^2 d\nu(\xi_i)d\nu(\xi_j)\int_{\mathbb{T}^{2\times 2}} H_\phi(x,y)^2 dxdy \leq C_{\phi,\nu}N^2, \end{aligned}$$

where we made essential use of the fact that H_ϕ is zero-averaged in both variables, so the only non vanishing terms in the double sum are the ones with $i=\ell, j=k$ (or vice-versa). We also recall that the ξ_i's are independent with finite second moments. We mention that the latter was a crucial computation in the works [14, 20, 21]. From here,

$$\int \langle H_\phi, \gamma \otimes \gamma \rangle^2 \, d\mu(\gamma) \leq e^{-\lambda} \sum_{N \geq 0} \frac{\lambda^N}{N!} C_{\phi,\nu} N^2 < \infty,$$

from which we easily conclude $\mathcal{L}F \in L^2(\mu)$.

We are left to prove that $\mathcal{U}_t$ is differentiable on $\mathcal{F}$ and that its derivative at time $t = 0$ is given by $\mathcal{L}$. However, this is equivalent to show that $\omega_t = \mathcal{T}_t \gamma$ solves (3.5), for which we already referred to [34]. □

Local observables $\mathcal{F}$ are not invariant for $\mathcal{U}_t$: this is due to the nonlinearity of the dynamics, not to singularity of the interaction. Our techniques thus does not seem to be suited to this setting.

We conclude by mentioning an idea of [34], from which we quote: *"Considering point vortices to be solutions of the weak vorticity formulation allows us to extend their dynamics beyond collisions simply by merging vortices that collide into a single vortex whose strength is the algebraic sum of the colliding vortices. Clearly this defines a solution for times less than and for times greater than the collision time, and the resulting vorticity is continuous in time in the weak-* topology of measures, so that there is no contribution [...] from the "jump" at the collision time. Of course, this extended notion of point-vortex dynamics is horribly nonunique since the time-reversibility of the Euler equations implies that a single vortex can split equally well into several vortices at any time."* Non uniqueness for the weak formulation of Euler equation in the point vortices case might be a clue that $(\mathcal{L}, \mathcal{F})$ is not essentially self-adjoint or even $L^2(\mu)$ unique. However, producing counterexamples with collisions or splitting of vortices is a difficult problem: explicit examples of collisions rely on integrability properties of the Hamiltonian dynamics. Whether $(\mathcal{L}, \mathcal{F})$ is essentially self-adjoint thus remains an interesting open question.

References

[1] Michael Aizenman. A sufficient condition for the avoidance of sets by measure preserving flows in $\mathbf{R}^n$. *Duke Math. J.*, 45(4):809–813, 1978.

[2] S. Albeverio, V. Barbu, and B. Ferrario. Uniqueness of the generators of the 2D Euler and Navier-Stokes flows. *Stochastic Process. Appl.*, 118(11):2071–2084, 2008.

[3] S. Albeverio, V. Barbu, and B. Ferrario. Erratum to "Uniqueness of the generators of 2D Euler and Stokes flows" [Stochastic Process. Appl. 118 (11) (2008) 2071–2084] [mr2462289]. *Stochastic Process. Appl.*, 120(10):2102, 2010.

[4] S. Albeverio and B. Ferrario. Uniqueness results for the generators of the two-dimensional Euler and Navier-Stokes flows. The case of Gaussian invariant measures. *J. Funct. Anal.*, 193(1):77–93, 2002.

[5] S. Albeverio and B. Ferrario. 2D vortex motion of an incompressible ideal fluid: the Koopman-von Neumann approach. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(2):155–165, 2003.

[6] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.

[7] Sergio Albeverio and Ana Bela Cruzeiro. Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two-dimensional fluids. *Comm. Math. Phys.*, 129(3):431–444, 1990.

[8] S. Albeverio, M. Ribeiro de Faria, and R. Höegh-Krohn. Stationary measures for the periodic Euler flow in two dimensions. *J. Statist. Phys.*, 20(6):585–595, 1979.

[9] Hassan Aref. Point vortex dynamics: a classical mathematics playground. *J. Math. Phys.*, 48(6):065401, 23, 2007.

[10] Lawrence Conlon. *Differentiable manifolds*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, second edition, 2001.

[11] Jean-Marc Delort. Existence de nappes de tourbillon en dimension deux. *J. Amer. Math. Soc.*, 4(3):553–586, 1991.

[12] D. Dürr and M. Pulvirenti. On the vortex flow in bounded domains. *Comm. Math. Phys.*, 85(2):265–273, 1982.

[13] Andreas Eberle. *Uniqueness and non-uniqueness of semigroups generated by singular diffusion operators*, volume 1718 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

[14] Franco Flandoli. Weak vorticity formulation of 2D Euler equations with white noise initial condition. *Comm. Partial Differential Equations*, 43(7):1102–1149, 2018.

[15] Franco Flandoli and Dejun Luo. ρ-white noise solution to 2D stochastic Euler equations. *Probab. Theory Related Fields*, 175(3-4):783–832, 2019.

[16] Franco Flandoli and Dejun Luo. Energy conditional measures and 2D turbulence. *J. Math. Phys.*, 61(1):013101, 22, 2020.

[17] Stefan Fleischer and Andreas Knauf. Improbability of Collisions in n-Body Systems. *Arch. Ration. Mech. Anal.*, 234(3):1007–1039, 2019.

[18] G. Gallavotti and M. Pulvirenti. Classical KMS condition and Tomita-Takesaki theory. *Comm. Math. Phys.*, 46(1):1–9, 1976.

[19] K. Goodrich, K. Gustafson, and B. Misra. On converse to Koopman's lemma. *Phys. A*, 102(2):379–388, 1980.

[20] Francesco Grotto. Stationary Solutions of Damped Stochastic 2-dimensional Euler's Equation. *arXiv e-prints*, page arXiv:1901.06744, Jan 2019.

[21] Francesco Grotto and Marco Romito. A Central Limit Theorem for Gibbsian Invariant Measures of 2D Euler Equation. *arXiv e-prints*, page arXiv:1904.01871, Apr 2019. To appear in Comm. Math. Phys.

[22] Massimiliano Gubinelli and Nicolas Perkowski. The infinitesimal generator of the stochastic Burgers equation. *arXiv e-prints*, page arXiv:1810.12014, Oct 2018.

[23] Rangachari Kidambi and Paul K. Newton. Motion of three point vortices on a sphere. *Phys. D*, 116(1-2):143–175, 1998.

[24] Alexander A. Kilin and Elizaveta M. Artemova. Integrability and chaos in vortex lattice dynamics. *Regul. Chaotic Dyn.*, 24(1):101–113, 2019.

[25] Mariusz Lemańczyk. Spectral theory of dynamical systems. In *Mathematics of complexity and dynamical systems. Vols. 1–3*, pages 1618–1638. Springer, New York, 2012.

[26] Pierre-Louis Lions. *On Euler equations and statistical physics*. Cattedra Galileiana. [Galileo Chair]. Scuola Normale Superiore, Classe di Scienze, Pisa, 1998.

[27] C. Marchioro, A. Pellegrinotti, and M. Pulvirenti. Selfadjointness of the Liouville operator for infinite classical systems. *Comm. Math. Phys.*, 58(2):113–129, 1978.

[28] Carlo Marchioro and Mario Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[29] Lorenzo M. Polvani and David G. Dritschel. Wave and vortex dynamics on the surface of a sphere. *J. Fluid Mech.*, 255:35–64, 1993.

[30] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[31] Donald G. Saari. Improbability of collisions in Newtonian gravitational systems. II. *Trans. Amer. Math. Soc.*, 181:351–368, 1973.

[32] Donald Gene Saari. Improbability of collisions in Newtonian gravitational systems. *Trans. Amer. Math. Soc.*, 162:267–271; erratum, ibid. 168 (1972), 521, 1971.

[33] Steven Schochet. The weak vorticity formulation of the 2-D Euler equations and concentration-cancellation. *Comm. Partial Differential Equations*, 20(5-6):1077–1104, 1995.

[34] Steven Schochet. The point-vortex method for periodic weak solutions of the 2-D Euler equations. *Comm. Pure Appl. Math.*, 49(9):911–965, 1996.

[35] Mark A. Stremler and Hassan Aref. Motion of three point vortices in a periodic parallelogram. *J. Fluid Mech.*, 392:101–128, 1999.

[36] A. F. M. ter Elst and M. Lemańczyk. On one-parameter Koopman groups. *Ergodic Theory Dynam. Systems*, 37(5):1635–1656, 2017.

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