

Energy conservation for weak solutions of incompressible fluid equations: The Hölder case and connections with Onsager's conjecture

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Abstract

In this paper we give elementary proofs of energy conservation for weak solutions to the Euler and Navier-Stokes equations in the class of Hölder continuous functions, relaxing some of the assumptions on the time variable (both integrability and regularity at initial time) and presenting them in a unified way.

Then, in the final section we prove (for the Navier-Stokes equations) a result of energy conservation in presence of a solid boundary and with Dirichlet boundary conditions. This result seems the first one –in the viscous case– with Hölder type hypotheses, but without additional assumptions on the pressure.

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1. Introduction

In this paper we start proving (and we slightly improve) in an elementary way –that is without resorting to Paley-Littlewood decomposition or other tools from harmonic analysis– results about energy conservation for weak solutions to incompressible fluids. We consider weak solutions to the Euler equations

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$$\begin{aligned}
\partial_t v^E + (v^E \cdot \nabla) v^E + \nabla p^E &= 0 & (t, x) \in (0, T) \times \mathbb{T}^3, \\
\operatorname{div} v^E &= 0 & (t, x) \in (0, T) \times \mathbb{T}^3, \\
v^E(0, x) &= v_0^E(x) & x \in \mathbb{T}^3,
\end{aligned} \tag{1}$$

or Leray-Hopf weak solutions of the Navier-Stokes equations (NSE in the sequel)

$$\begin{aligned}
\partial_t v + (v \cdot \nabla) v - \nu \Delta v + \nabla p &= 0 & (t, x) \in (0, T) \times \mathbb{T}^3, \\
\operatorname{div} v &= 0 & (t, x) \in (0, T) \times \mathbb{T}^3, \\
v(0, x) &= v_0(x) & x \in \mathbb{T}^3,
\end{aligned} \tag{2}$$

for fixed $\nu > 0$. We begin from the space-periodic case, under the additional assumption of having a $C^{0,\alpha}(\mathbb{T}^3)$ velocity, as suggested by Onsager conjecture [29]. In the sequel we will assume $f = 0$ for simplicity, but results can be easily adapted to consider also non-zero external forces.

The Onsager conjecture (only recently solved) suggested the value $\alpha = 1/3$, especially for the case of the Euler equations (1), but the conjecture was mainly considering only the Hölder regularity with respect the *space variables*. Here, we consider a combination of space-time conditions, proving families of criteria depending on the Hölder exponent.

The first rigorous results about Onsager conjecture are probably those of Eyink [21,22] and Constantin, E, and Titi [15] in Fourier setting and in Besov spaces (slightly larger than Hölder ones), respectively. A well-known result is that if v^E is a weak solution to (1) is such that

$$v^E \in L^3(0, T; B_3^{\alpha,\infty}(\mathbb{T}^3)) \cap C(0, T; L^2(\mathbb{T}^3)) \quad \text{with } \alpha > \frac{1}{3},$$

then $\|v^E(t)\| = \|v_0^E\|$, for all $t \in [0, T]$. As explained in [15] “...This is basically the content of Onsager’s conjecture, except Onsager stated his conjecture in Hölder spaces rather than Besov spaces. Obviously the above theorem implies similar results in Hölder spaces...”

Remark 1.1. In this paper we want to consider the Hölder case, to keep the results the most elementary as possible, to be understood also by an audience familiar with classical spaces of mathematical analysis. This will also create links with the most recent developments of the theory, especially after the work of De Lellis and Székelyhidi [16], which originated an intense activity to prove the Onsager conjecture, mainly the non-conservation below the critical space $C^{1/3}$, with endpoint reached in Isett [25] and Buckmaster *et al.* [10]. We wish also to mention the recent result of Cheskidov and Luo [14], again in the setting of Hölder continuous solutions, in connection with non-uniqueness, which treats scaling very similar to ones we obtain here. We avoid using Besov spaces (which can produce some technical improvements), both to keep the results elementary and to have a functional setting well-adapted also to the problem in a domain with boundary.

The first five sections contain both a summary of known results and the simplification/extension of several theorems. The most original result of the paper is presented in the last section, where we consider the Dirichlet problem for the NSE in the half-space. We prove energy conservation for a class of (partial) Hölder-type solutions, which is nevertheless larger than the scaling invariant class, which provides uniqueness of weak solution and regularity of Leray-Hopf solutions.

The results we prove here in the Hölder case follow easily from the method used in [15], and we are here trying to focus on the hypotheses in the time variable (instead of the spatial ones), showing how a slight more stringent assumption on the space variables (Hölder instead of Besov), produces some slight improvement in the time variable. We are not sure that all the results we prove in the first five sections could be considered as fully original (even if we could not find in the published literature), but nevertheless they are collected here in a unified way, providing also a summary of the state of the art.

The result in [15], but in the setting of Hölder functions has been recently extended also to the boundary value problems for the Euler equations in Bardos and Titi [2], proving energy conservation for weak solutions such that

$$v^E \in L^3(0, T; C^{0,\alpha}(\overline{\Omega})) \quad \text{with } \alpha > \frac{1}{3},$$

while technical improvements in the space variables can be found in Beirão da Veiga and Yang [6]. See also Duchon and Robert [19].

We observe that the situation for the NSE is a little different and in the final section we will also prove some results in the case of a domain with solid boundaries and Dirichlet boundary conditions for the NSE, for which the treatment of the pressure is problematic.

In the space periodic case recent results for the NSE are those by Cheskidov [12] in the setting of standard (fractional) Sobolev spaces of and Cheskidov Luo [13], see also Farwig and Taniuchi [23]. In the case of Leray-Hopf weak solutions to the NSE, the properties in the time variable are a little improved and also the range of exponents is wider, if compared with the Euler equations. The sharpest known result in the Besov setting seems to be the following one: Suppose that $1 \leq \beta < p \leq \infty$ are such that $\frac{2}{p} + \frac{1}{\beta} < 1$. If v is a weak solution such that

$$v \in L_w^\beta(0, T; B_{p,\infty}^{\frac{2}{\beta} + \frac{2}{p} - 1}(\mathbb{T}^3)), \quad (3)$$

then v satisfies the energy equality. This result contains as a particular case the classical ones by Prodi/Lions [30,26] and $v \in L_w^\beta(0, T; B_{3,\infty}^{1/3}(\mathbb{T}^3))$, for all $\beta > 3$ is the borderline case. Recent extensions (valid also for the Dirichlet problem) are those based on the condition on the gradient of v in [9] and in Beirão da Veiga and Yang [5]. The results therein can be “measured” in terms of Hölder spaces as follows

$$v \in L^{\frac{5}{3+2\alpha}}(0, T; C^{0,\alpha}(\overline{\Omega})), \quad (4)$$

which are the same as in (3), in terms of scaling. The results are in fact derived from $\nabla v \in L^{5q/(5q-6)}(0, T; L^q(\Omega))$, with $q > 3$, by Morrey inclusion, which “almost” implies the conservation of energy for $v \in L^{\frac{15}{11}}(0, T; C^{0,1/3}(\overline{\Omega}))$. Nevertheless, results obtained by embedding –even if valid also for the boundary value problem– are far away from being sharp. Hence, this is why we prove here results directly in the class of Hölder continuous functions.

A traditional way to take advantage from the additional Hölder continuity of the solution is to use properties of mollification operators and not Hilbert-space methods. For divergence-free functions, this is particularly delicate to be handled in the presence of a boundary.

Plan of the paper: In Section 4 we first prove some results for the Euler equations, improving the conditions on the time-variable and extending the range of allowed Hölder exponent.

The next result in Section 5 is about the viscous case, where we again give a new condition in the time variable, working in spaces with the same scaling as previous results but with different hypotheses at $t = 0$. Finally, in the last and more original Section 6 we treat the viscous problem supplemented with Dirichlet conditions in the half-space, proving energy conservation for velocities which are only partially Hölder continuous.

2. Notation and preliminaries

In the sequel we will use classical Lebesgue $(L^p(\mathbb{T}^3), \|\cdot\|_p)$ and Sobolev $(W^{1,p}(\mathbb{T}^3), \|\cdot\|_{1,p})$ spaces; for simplicity we denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the L^2 scalar product and norm (while the other norms are explicitly indicated). Moreover, we will use the Banach space of (uniformly) Hölder continuous functions $C^\alpha(\mathbb{T}^3) = C^{0,\alpha}(\mathbb{T}^3)$, for $0 < \alpha \leq 1$, with the norm

$$\|u\|_{C^\alpha} = \max_{x \in \mathbb{T}^3} |u(x)| + [u]_\alpha,$$

where

$$[u]_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

We will focus on space-time properties of functions and what will be relevant in the sequel is the “homogeneous behavior”, that is the one of the Hölder semi-norm $[\cdot]_\alpha$, and we denote by $\dot{C}^\alpha$ the space of measurable functions such that this quantity is bounded.¹

We say that

$$u \in L^\beta(0, T; \dot{C}^\alpha(\mathbb{T}^3)),$$

if there exists $f_\alpha : [0, T] \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} a) \quad & |u(t, x) - u(t, y)| \leq f_\alpha(t) |x - y|^\alpha \quad \forall x, y \in \mathbb{T}^3, \text{ for a.e. } t \in [0, T], \\ b) \quad & \int_0^T f_\alpha^\beta(t) dt < \infty, \end{aligned}$$

and note that $f_\alpha(t) = [u(t)]_\alpha$ for almost all $t \in [0, T]$. This space will be endowed with the following semi-norm

$$\|u\|_{L^\beta(0, T; \dot{C}^\alpha(\mathbb{T}^3))} := \left[\int_0^T f_\alpha^\beta(t) dt \right]^{1/\beta}.$$

We define properly the notions of weak solution which we will use in the sequel. We focus now on the space periodic setting and we will explain needed changes in the last section. We denote

¹ Note that for bounded domains the uniform Hölder property implies also boundedness.

by H and V the closure of smooth, periodic, divergence-free, and with zero mean-value vector fields in $L^2(\mathbb{T}^3)$ and $W^{1,2}(\mathbb{T}^3)$, respectively. As space for test functions we use

$$\mathcal{D}_T := \left\{ \varphi \in C_0^\infty([0, T] \times \mathbb{T}^3) : \operatorname{div} \varphi = 0 \right\}.$$

We define the notion of weak solution in the inviscid case.

Definition 2.1 (*Weak solutions for the Euler equations*). Let be given $v_0 \in H$. A measurable function $v^E : (0, T) \times \mathbb{T}^3 \rightarrow \mathbb{R}^3$ is called a weak solution to the Euler equations (2) if the following hold true:

$$v^E \in L^\infty(0, T; H),$$

and if v^E solves the equations in the weak sense:

$$\int_0^\infty \left[(v^E, \partial_t \varphi) + ((v^E \otimes v^E), \nabla \varphi) \right] dt = -(v_0, \varphi(0)), \quad (5)$$

for all $\varphi \in \mathcal{D}_T$.

We also recall the definition of weak solution for the viscous problem.

Definition 2.2. (Space-periodic Leray-Hopf weak solution) Let be given $v_0 \in H$. A measurable function $v : (0, T) \times \Omega \rightarrow \mathbb{T}^3$ is called a Leray-Hopf weak solution to the space-periodic NSE (2) if $v \in L^\infty(0, T; H) \cap L^2(0, T; V)$; and the following hold true:
The function v solves the equations in the weak sense:

$$\int_0^\infty \left[(v, \partial_t \varphi) - v(\nabla v, \nabla \varphi) - ((v \cdot \nabla) v, \varphi) \right] dt = -(v_0, \varphi(0)), \quad (6)$$

for all $\varphi \in \mathcal{D}_T$;

It holds the (global) energy inequality

$$\frac{1}{2} \|v(t)\|_2^2 + \nu \int_0^t \|\nabla v(\tau)\|_2^2 d\tau \leq \frac{1}{2} \|v_0\|_2^2 \quad \forall t \in [0, T]; \quad (7)$$

The initial datum is strongly attained

$$\lim_{t \rightarrow 0^+} \|v(t) - v_0\| = 0. \quad (8)$$

It is well-known that for all $v_0 \in H$ there exists at least a Leray-Hopf weak solution in any time interval $(0, T)$.

The energy inequality (7) can be rewritten as an equality

$$\frac{1}{2}\|v(t)\|^2 + \nu \int_0^t \int_{\mathbb{T}^3} \epsilon[v(\tau, x)] \, dx d\tau = \frac{1}{2}\|v_0\|^2 \quad \forall t \in [0, T], \quad (9)$$

where the total energy dissipation rate is given by

$$\epsilon[v] := \nu |\nabla v|^2 + D(v),$$

with $D(v)$ a non-negative distribution (Radon measure); if $D(v) = 0$, then energy dissipation arises entirely from viscosity.

Remark 2.1. Other equivalent weak formulations can be used (especially for the NSE, see Temam [34, Ch. III, §3]) employing the time-derivative as an object at least in $L^1(0, T; V^*)$. This requires to find $v \in L^\infty(0, T; H) \cap L^2(0, T; V)$ with $\partial_t v \in L^1(0, T; V^*)$ —which in fact it will turn out to be in $L^{4/3}(0, T; V^*)$ —such that the energy inequality is satisfied, $v(0) = v_0$, and

$$\int_0^T \left[\langle \partial_t v, \varphi \rangle + \nu (\nabla v, \nabla \varphi) + ((v \cdot \nabla) v, \varphi) \right] dt = 0,$$

for all divergence-free $\varphi \in C_0^\infty([0, T] \times \mathbb{T}^3)$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and its dual V^* . The latter formulation will be particularly useful when using as test function a regularized velocity.

As for what concerns the Euler equations, we will essentially apply the same classical strategy as in [15, 19, 21]), based on mollifications to justify the calculations. Note that, *very roughly speaking* the introduction of the Hölder space $C^{1/3}$ (or Besov generalizations) is based on the idea of “equally distributing” in the integral

$$\int_0^T \int_{\mathbb{T}^3} (v \cdot \nabla) v \cdot v \, dx dt,$$

a single space derivative, into $1/3$ -spatial derivative on all terms. Despite the approach being the same, results can be greatly improved for the NSE, since one can take advantage of the already known L^2 -integrability of the gradient and a different combination of estimates (which are ν -dependent) could be used.

3. Mollification and Hölder spaces

To fully use the features of the Hölder continuity extra-assumption on the solution, the approximation should be done by a mollification argument. To this end we fix a symmetric $\rho \in C_0^\infty(\mathbb{R}^3)$ such that

$$\rho \geq 0, \quad \text{supp } \rho \subset B(0, 1) \subset \mathbb{R}^3, \quad \int_{\mathbb{R}^3} \rho(x) \, dx = 1,$$

and we define, for $\varepsilon \in (0, 1]$, the family of Friedrichs mollifiers $\rho_\varepsilon(x) := \varepsilon^{-3} \rho(\varepsilon^{-1}x)$. Then, for any function $f \in L^1_{loc}(\mathbb{R}^3)$ we set, by using the usual convolution (and not the circular convolution made by periodic kernels),

$$f_\varepsilon(x) := \int_{\mathbb{R}^3} \rho_\varepsilon(x - y) f(y) \, dy = \int_{\mathbb{R}^3} \rho_\varepsilon(y) f(x - y) \, dy.$$

It turns out that the last integral is evaluated only for $\{y : |y| < \varepsilon\} \subset]-\pi, \pi[^3$, for small $\varepsilon > 0$. So, if needed, we can restate the definition as follows

$$f_\varepsilon(x) := \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x - y) \, dy.$$

If $f \in L^1(\mathbb{T}^3)$, then $f \in L^1_{loc}(\mathbb{R}^3)$, and it turns out that f_ε is 2π -periodic along the x_j -direction, for $j = 1, 2, 3$ since

$$f_\varepsilon(x + 2\pi e_j) = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x + 2\pi e_j - y) \, dy = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x - y) \, dy,$$

by the periodicity of f , where e_j , $j = 1, 2, 3$, is the unit vector in the x_j direction, and it turns out that $f_\varepsilon \in C^\infty(\mathbb{T}^3)$.

We report now the basic calculus estimates which we will use in the sequel.

Lemma 3.1. *Let ρ as above and let $u \in \dot{C}^\alpha(\mathbb{T}^3) \cap L^1_{loc}(\mathbb{T}^3)$, then it follows*

$$\max_{x \in \mathbb{T}^3} |u(x + y) - u(x)| \leq [u]_\alpha |y|^\alpha, \quad (10)$$

$$\max_{x \in \mathbb{T}^3} |u(x) - u_\varepsilon(x)| \leq [u]_\alpha \varepsilon^\alpha, \quad (11)$$

$$\max_{x \in \mathbb{T}^3} |\nabla u_\varepsilon(x)| \leq C [u]_\alpha \varepsilon^{\alpha-1}, \quad (12)$$

where $C := \int_{\mathbb{R}^3} |\nabla \rho(x)| \, dx$.

Proof. The first one is just the statement of Hölder continuity. Concerning the second one we can write that

$$|u(x) - u_\varepsilon(x)| = \left| u(x) - \int_{B(0, \varepsilon)} \rho_\varepsilon(y) u(x - y) \, dy \right|$$

$$\begin{aligned}
&= \left| \int_{B(0,\varepsilon)} \rho_\varepsilon(y)(u(x) - u(x-y)) \, dy \right| \\
&\leq [u]_\alpha \int_{B(0,\varepsilon)} \rho_\varepsilon(y) |y|^\alpha \, dy \\
&\leq C \varepsilon^\alpha \int_{B(0,\varepsilon)} \rho_\varepsilon(y) \, dy,
\end{aligned}$$

hence the thesis, where we used that $\rho \geq 0$.

The third estimate follows by observing that

$$\begin{aligned}
\frac{\partial u_\varepsilon(x)}{\partial x_i} &= \frac{\partial}{\partial x_i} \frac{1}{\varepsilon^3} \int_{\mathbb{R}^3} \rho\left(\frac{x-y}{\varepsilon}\right) u(y) \, dy = \int_{\mathbb{R}^3} \frac{1}{\varepsilon^4} \frac{\partial \rho}{\partial x_i} \Big|_{\frac{x-y}{\varepsilon}} u(y) \, dy \\
&= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} g_\varepsilon^i(x-y) u(y) \, dy,
\end{aligned}$$

where $g_\varepsilon^i(x) := \frac{1}{\varepsilon^3} \frac{\partial \rho}{\partial x_i}\left(\frac{x}{\varepsilon}\right)$. Note that $\int_{\mathbb{R}^3} g_\varepsilon^i(x) \, dx = 0$, hence we can write

$$\begin{aligned}
\frac{\partial u_\varepsilon(x)}{\partial x_i} &= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} g_\varepsilon^i(x-y) u(y) \, dy - \frac{u(x)}{\varepsilon} \int_{\mathbb{R}^3} g_\varepsilon^i(x-y) \, dy, \\
&= \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \left(g_\varepsilon^i(x-y) u(y) - g_\varepsilon^i(x-y) u(x) \right) \, dy,
\end{aligned}$$

and then

$$\begin{aligned}
\left| \frac{\partial}{\partial x_i} u_\varepsilon(x) \right| &= \frac{1}{\varepsilon} \int_{|x-y|<\varepsilon} |g_\varepsilon^i(x-y)| |u(y) - u(x)| \, dy, \\
&\leq \frac{[u]_\alpha}{\varepsilon} \int_{|x-y|<\varepsilon} |g_\varepsilon^i(x-y)| |y-x|^\alpha \, dy \\
&\leq \frac{[u]_\alpha \varepsilon^\alpha}{\varepsilon} \int_{|x-y|<\varepsilon} |g_\varepsilon^i(x-y)| \, dy,
\end{aligned}$$

hence the thesis. $\square$

3.1. A technical extension concerning the modulus of continuity

We observe, that some improvements in the results about energy conservation can be obtained considering the following spaces $C_\omega^\alpha(\mathbb{T}^3) \subset C^\alpha(\mathbb{T}^3)$ defined through the norm

$$\|u\|_{C_\omega^\alpha} = \max_{x \in \mathbb{T}^3} |u(x)| + [u]_{\omega, \alpha},$$

where

$$[u]_{\omega, \alpha} := \sup_{x \neq y} \frac{|u(x) - u(y)|}{\omega(|x - y|)|x - y|^\alpha},$$

with $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a non-decreasing function such that $\lim_{s \rightarrow 0^+} \omega(s) = 0$. These spaces have been already considered by Duchon and Robert [19] and, more recently in [6] for $\alpha = 1/3$. See also the Besov counterpart in [11]. With the same approach used in the previous lemma one can prove also the following extension of the properties of convolutions.

Lemma 3.2. *Let ρ as above and let $u \in \dot{C}_\omega^\alpha(\mathbb{T}^3) \cap L_{loc}^1(\mathbb{T}^3)$, then it follows*

$$\max_{x \in \mathbb{T}^3} |u(x + y) - u(x)| \leq [u]_{\omega, \alpha} \omega(|y|) |y|^\alpha, \quad (13)$$

$$\max_{x \in \mathbb{T}^3} |u(x) - u_\varepsilon(x)| \leq [u]_{\omega, \alpha} \omega(\varepsilon) \varepsilon^\alpha, \quad (14)$$

$$\max_{x \in \mathbb{T}^3} |\nabla u_\varepsilon(x)| \leq C [u]_{\omega, \alpha} \omega(\varepsilon) \varepsilon^{\alpha-1}. \quad (15)$$

4. Energy conservation for the Euler equations

The results in this section concern energy conservation for weak solutions to the Euler equations: Theorem 4.1 gives a small improvement in the time integrability, and extends the results to the full range of $\alpha \in]1/3, 1[$, see [2, 15]. Note that the result about non-conservation of energy below the exponent $1/3$ in Isett [25] are proved with L^∞ -bounds in time. Even if the original conjecture detected the value $\alpha = 1/3$ as a borderline case, we identify $L^{1/\alpha}(0, T; \dot{C}^\alpha)$ as a critical space (for all $\alpha > 1/3$).

On the other hand, the second result, namely Theorem 4.2, was already known for $\alpha = 1/3$ and here we prove just an extension to the full range of exponents, see [6].

Theorem 4.1. *Let v^E be a weak solution to the Euler equation (1) such that*

$$v^E \in L^{\frac{1}{\alpha} + \delta}(0, T; \dot{C}^\alpha(\mathbb{T}^3)) \quad \text{with } \alpha \in \left] \frac{1}{3}, 1[\right], \quad (16)$$

and with $\delta > 0$. Then, the weak solutions v^E conserves the energy.

Proof. We define $v_\varepsilon^E = \rho_\varepsilon * v^E$ and consider the following equality obtained by testing with $\rho_\varepsilon * (\rho_\varepsilon * v^E) = \rho_\varepsilon * v_\varepsilon^E$ (which is justified for instance in [2])

$$\frac{1}{2} \|v_\varepsilon^E(t)\|^2 - \frac{1}{2} \|v_\varepsilon^E(0)\|^2 = \int_0^t \int_{\mathbb{T}^3} (v^E \otimes v^E)_\varepsilon : \nabla v_\varepsilon^E \, dx d\tau. \quad (17)$$

The key observation to handle the convective term from the right-hand side is the following (Constantin-E-Titi) commutator-type identity, valid for any $u \in L^2(\mathbb{T}^3)$

$$(u \otimes u)_\varepsilon = u_\varepsilon \otimes u_\varepsilon + r_\varepsilon(u, u) - (u - u_\varepsilon) \otimes (u - u_\varepsilon), \quad (18)$$

with

$$r_\varepsilon(u, u) := \int_{\mathbb{T}^3} \rho_\varepsilon(y) (\delta_y u(x) \otimes \delta_y u(x)) \, dy,$$

where (using a now common notation borrowed from [15]) we set

$$\delta_y u(x) := u(x - y) - u(x).$$

We apply the decomposition (18) to a mollified weak solution v_ε^E . By integration by parts, it follows that

$$\int_{\mathbb{T}^3} (v_\varepsilon^E \otimes v_\varepsilon^E) : \nabla v_\varepsilon^E \, dx = 0,$$

so we are reduced to study only the contribution of the remaining two terms in (18), which are coming from the decomposition of the right-hand side of (17).

We start from the last one and we split the integrand as follows

$$|v^E - v_\varepsilon^E|^2 |\nabla v_\varepsilon^E| = |v^E - v_\varepsilon^E|^\eta |v^E - v_\varepsilon^E|^{2-\eta} |\nabla v_\varepsilon^E| \quad \text{for some } 0 \leq \eta \leq 2.$$

Hence, by using (16), we get

$$\int_{\mathbb{T}^3} |v^E - v_\varepsilon^E|^2 |\nabla v_\varepsilon^E| \, dx \leq f_\alpha^{1+\eta}(t) \varepsilon^{\alpha\eta+\alpha-1} \int_{\mathbb{T}^3} |v^E - v_\varepsilon^E|^{2-\eta} \, dx,$$

and, by Hölder inequality (since the measure of the domain is finite), we get $\int_{\mathbb{T}^3} |g|^{2-2\eta} \, dx \leq C \|g\|_2^{2-2\eta}$. Using the fact that $\|v_\varepsilon^E(t)\| \leq \|v^E(t)\| \leq C$, by the properties of convolution and the Definition 2.1 of weak solution, one obtains

$$\int_0^T \int_{\mathbb{T}^3} |v^E - v_\varepsilon^E|^2 |\nabla v_\varepsilon^E| \, dx \, dt \leq C \varepsilon^{\alpha\eta+\alpha-1} \int_0^T f_\alpha^{1+\eta}(t) \, dt.$$

If we want that this term vanishes as $\varepsilon \rightarrow 0$ we have to satisfy the following two facts:

i) to fix $\eta \in [0, 2]$ such that $\alpha\eta + \alpha - 1 > 0$. The constraints to be satisfied are

$$\frac{1-\alpha}{\alpha} < \eta \leq 2,$$

and such a choice of η is possible only if $\alpha > 1/3$;

ii) the integral from the right-hand side should be finite, which means $f_\alpha \in L^{1+\eta}(0, T)$ and consequently the inequality $1 + \eta \leq 1/\alpha + \delta$ should be verified.

Collecting the two estimates we find that we have to choose η such that $1/\alpha - 1 < \eta \leq 1/\alpha - 1 + \delta$.

The other term arising from the commutator is estimated as follows, by fixing η in the same way as above:

$$\begin{aligned}
 |r_\varepsilon(v^E, v^E)| &\leq \int_{\mathbb{R}^3} \rho_\varepsilon(y) |v^E(x-y) - v^E(x)|^2 dy \\
 &\leq \int_{B(0, \varepsilon)} \rho_\varepsilon(y) |v^E(x-y) - v^E(x)|^\eta |v^E(x-y) - v^E(x)|^{2-\eta} dy \\
 &\leq f_\alpha^\eta(t) \int_{B(0, \varepsilon)} \rho_\varepsilon(y) |y|^{\alpha\eta} |v^E(x-y) - v^E(x)|^{2-\eta} dy, \\
 &\leq C(\eta) f_\alpha(t)^\eta \varepsilon^{\alpha\eta} \int_{\mathbb{T}^3} \rho_\varepsilon(y) (|v^E(x-y)|^{2-\eta} + |v^E(x)|^{2-\eta}) dy.
 \end{aligned}$$

Next,

$$\int_{\mathbb{T}^3} \int_{\mathbb{T}^3} \rho_\varepsilon(y) dy |v_\varepsilon^E(x)|^{2-\eta} dx = \int_{\mathbb{T}^3} |v_\varepsilon^E(x)|^{2-\eta} dx \leq C(\eta),$$

again by using the Hölder inequality and the fact that $\rho \geq 0$ is a mollifier. By the properties of the convolution in L^p we have also

$$\left\| \int_{\mathbb{T}^3} \rho_\varepsilon(y) |v^E(x-y)|^{2-\eta} dy \right\|_{L^1(\mathbb{T}^3)} \leq \|\rho_\varepsilon\|_1 \| |v^E|^{2-\eta} \|_1 \leq C(\eta),$$

where we used Hölder inequality and the uniform bound in L^2 . Consequently

$$\begin{aligned}
 \left| \int_{\mathbb{T}^3} r_\varepsilon(v^E, v^E) : \nabla v_\varepsilon^E dx \right| &\leq \int_{\mathbb{T}^3} |r_\varepsilon(v, v)| |\nabla v_\varepsilon^E| dx \\
 &\leq C f_\alpha(t)^{1+\eta} \varepsilon^{\alpha\eta+\alpha-1},
 \end{aligned}$$

showing that also this term can be treated as the previous one. Hence, we obtain that

$$\left| \int_0^t \int_{\mathbb{T}^3} (v^E \otimes v^E)_\varepsilon : \nabla v_\varepsilon^E dx d\tau \right| \leq C \varepsilon^\gamma \int_0^T f_\alpha^{1/\alpha+\delta}(t) dt \rightarrow 0,$$

under the hypothesis (16), being the integral finite and $\gamma = \alpha\eta + \alpha - 1 > 0$. $\square$

Remark 4.1. The above result means that, apart some “ δ -technical conditions”, the energy conservation holds for weak solutions $v^E \in L^{1/\alpha}(0, T; \dot{C}^\alpha)$, for $\alpha > 1/3$. Observe also that if $\alpha > 1/3$, then $1/\alpha < 3$, and beside having a full-range of scaled space/time results, we have a technical-improvement in the time variable with respect to the previous results: we can have $v^E \in L^q(0, T; \dot{C}^\alpha(\mathbb{T}^3))$ for some $q < 3$, even for $\alpha > 1/3$ arbitrarily close to $1/3$.

By using the slightly smaller space $C_\omega^\alpha(\mathbb{T}^3)$ we can get a little gain in the time variable. The space $C_\omega^\alpha(\mathbb{T}^3)$ can be considered as a counterpart of the space $B_{p,c(\mathbb{N})}^\alpha$, introduced in [11], and nevertheless in the same spirit of the results in [19], which were limited to the case $\alpha = 1/3$. The result which holds is the following

Theorem 4.2. *Let v^E be a weak solution to the Euler equation (1) such that*

$$v^E \in L^{1/\alpha}(0, T; \dot{C}_\omega^\alpha(\mathbb{T}^3)) \quad \text{with } \alpha \in \left[\frac{1}{3}, 1\right]. \quad (19)$$

Then v^E conserves the energy.

Proof. The proof is exactly the same as before, the only difference is the following estimation (and an analogous one for the other term in the commutator)

$$\int_0^T \int_{\mathbb{T}^3} |v^E - v_\varepsilon^E|^2 |\nabla v_\varepsilon^E| \, dx \, dt \leq \omega(\varepsilon)^{1+\eta} \varepsilon^{\alpha\eta+\alpha-1} \int_0^T f_\alpha^{\eta+1}(t) \, dt,$$

which allows us to fix $\eta = \frac{1-\alpha}{\alpha} \leq 2$, a choice which is possible now also in the case $\alpha = 1/3$. Due to the presence of the term $\omega(\varepsilon)$, the space-time integral vanishes, as $\varepsilon \rightarrow 0$, if $f_\alpha \in L^{1/\alpha}(0, T)$. $\square$

Remark 4.2. Results in this section can be extended to the boundary value problem for the Euler equations, with the assumption that $v^E \cdot n = 0$ at the boundary $\partial\Omega$, with n the unit outward vector. This is possible but with a certain amount of technicalities, by following word-by-word the approach in [2]. Mainly, the localization does not preserve the divergence-free condition and consequently an estimate on the pressure should be obtained by solving in the Hölder spaces the (formal) steady Poisson-Neumann problem

$$\begin{aligned} -\Delta p^E &= \operatorname{div} \operatorname{div}(v^E \otimes v^E) && \text{in } \Omega, \\ -n \cdot \nabla p^E &= (v^E \otimes v^E) : \nabla n && \text{on } \partial\Omega, \end{aligned}$$

where the time $t \in [0, T]$ is just a parameter. See also the recent work by Bardos, Boutros, and Titi [3] for a precise interpretation of this problem.

5. Energy conservation for the Navier-Stokes equations

In this section we focus on the viscous case and recall that the classical results on scale invariant spaces by Ladyženskaya-Prodi-Serrin imply regularity within the class $v \in L^r(0, T; L^s)$,

with $\frac{2}{r} + \frac{3}{s} = 1$, $s > 3$. Even the limiting case $s = 3$ is a regularity class, but in our setting it is relevant the other limiting case

$$v \in L^2(0, T; L^\infty(\Omega)).$$

In the light of this observation, full classical regularity (and consequently also energy conservation) holds true in the class of weak solutions such that $v \in L^2(0, T; C^{0,\alpha}(\overline{\Omega}))$, for all $\alpha \geq 0$. In the bounded domain this implies that conditions involving an exponent for the time-integrability strictly smaller than two will make the results proved here non-trivial (The role of this limiting space is also stressed in [14, Thm. 1.6] in connection with non-uniqueness).

The main result we prove (improving the known ones in terms of properties in the time variable) is the following.

Theorem 5.1. *Let v be a Leray-Hopf weak solution in $(0, T) \times \mathbb{T}^3$ such that for all $\lambda \in]0, T[$*

$$v \in L^{\frac{2}{1+\alpha}}(\lambda, T; \dot{C}^\alpha(\mathbb{T}^3)) \quad \text{with } \alpha \in]0, 1[. \quad (20)$$

Then, v satisfies the energy equality in $[0, T]$.

The condition (20) means that there exists $f : [0, T] \rightarrow \mathbb{R}^+$ such that $|v(t, x) - v(t, y)| \leq f_\alpha(t)|x - y|^\alpha$ for almost all $t \in]0, T]$ and $f_\alpha \in L^{\frac{2}{1+\alpha}}(\lambda, T)$ for all $\lambda \in]0, T[$, but it is not excluded that

$$\int_0^T f_\alpha^{\frac{2}{1+\alpha}}(t) dt = +\infty,$$

which can be restated as $f_\alpha \in L_{loc}^{\frac{2}{1+\alpha}}([0, T])$.

Remark 5.1. Since for all $0 < \alpha < 1$ we have $\frac{2}{1+\alpha} < \frac{1}{\alpha}$, the conditions proved in Theorem 5.1 are less restrictive than those proved for the Euler equations in the previous section. This is due to the fact we have at disposal some additional regularity, being v a Leray-Hopf weak solution. Recent results considering the vanishing viscosity limit in connection with energy conservation are also reported in [8].

Note also that the conditions in Theorem 5.1 have the same scaling of the recent results from [13] (recalled in (3)) in the case $p = \infty$. Nevertheless, in this case the two results are not completely comparable, since the space $L_w^\beta(0, T)$ is larger than $L^\beta(0, T)$. On the other hand, our results allow for less regularity near $t = 0$, using the properties of Leray-Hopf solutions, as exploited also in Maremonti [27].

The two limiting cases $\alpha = 0$ and $\alpha = 1$ in condition (20) correspond to

- a) $L^2(\lambda, T; L^\infty(\Omega)),$
- b) $L^1(\lambda, T; W^{1,\infty}(\Omega)),$

and if $\lambda = 0$ the case $a)$ is the case covered by the scaling invariant condition of Ladyžhenskaya-Prodi-Serrin [24], while $b)$ corresponds to the Beale-Kato-Majda criterion. In terms of scaling (using DeVore diagrams to “measure” the regularity of Sobolev and Hölder spaces) we can associate to the space $C^{0,\alpha}$ the Sobolev space $W^{1, \frac{3}{1-\alpha}} \subset C^{0,\alpha}(\overline{\Omega})$ by Morrey theorem. Results from Theorem 5.1 are comparable with $\nabla v \in L^{\frac{2}{1+\alpha}}(\lambda, T; L^{\frac{3}{1-\alpha}}(\Omega))$ which is, in terms of scaling, such that

$$\frac{2}{\frac{2}{1+\alpha}} + \frac{3}{\frac{3}{1-\alpha}} = 2,$$

hence with the same scaling-invariant class of regularity from [4, 7].

Moreover, note that the relation between Ladyžhenskaya-Prodi-Serrin results and non-uniqueness has been recently considered in [14] obtaining as by product non-uniqueness in the class $L^{3/2-\delta}(0, T; C^{1/3}(\mathbb{T}^3) \cap L^1(0, T; C^{1-\delta'}(\mathbb{T}^3)))$ for any small $\delta, \delta' > 0$. Note that this has exactly the same scaling as our results for $\alpha = 1/3$ and $\alpha = 1$, hence showing the critical role of the condition (at least for very-weak solutions of the NSE).

Proof of Theorem 5.1. First, we restrict to some interval $[s, t]$, with $0 < s < t \leq T$. We use as test function a double smoothed velocity $\rho_\varepsilon * v_\varepsilon := \rho_\varepsilon * (\rho_\varepsilon * v)$, obtaining the following equality

$$\frac{1}{2} \|v_\varepsilon(t)\|^2 - \frac{1}{2} \|v_\varepsilon(s)\|^2 + \nu \int_s^t \|\nabla v_\varepsilon(\tau)\|^2 d\tau = \int_s^t \int_{\mathbb{T}^3} (v \otimes v)_\varepsilon : \nabla v_\varepsilon dx d\tau. \quad (21)$$

To justify the calculations, observe that for weak solution it holds that $\partial_t v \in L^{4/3}(0, T; V')$, hence the coupling

$$\langle \partial_t v, \rho_\varepsilon * (\rho_\varepsilon * v) \rangle_{L^{4/3}(0, T; V'), L^4(0, T; V)},$$

is well defined, being the right-hand side at least in $L^4(s, T; V)$. By means of a further time-mollification one can immediately deduce that

$$\int_s^t \langle \partial_t v, \rho_\varepsilon * (\rho_\varepsilon * v) \rangle d\tau = \frac{1}{2} \|v_\varepsilon(t)\|^2 - \frac{1}{2} \|v_\varepsilon(s)\|^2.$$

Hence, see for instance Galdi [24],

$$\lim_{\varepsilon \rightarrow 0} \int_s^t \langle \partial_t v, \rho_\varepsilon * (\rho_\varepsilon * v) \rangle d\tau = \frac{1}{2} \|v(t)\|^2 - \frac{1}{2} \|v(s)\|^2.$$

Next, the same classical arguments show also that

$$\int_s^t \int_{\mathbb{T}^3} \nabla v_\varepsilon : \nabla v_\varepsilon dx d\tau \xrightarrow{\varepsilon \rightarrow 0} \int_s^t \int_{\mathbb{T}^3} |\nabla v|^2 dx d\tau.$$

Next, since by integration by parts

$$\int_{\mathbb{T}^3} (v_\varepsilon \otimes v_\varepsilon) : \nabla v_\varepsilon \, dx = 0,$$

we are reduced (as in the previous section) to study only the contribution of the remaining two terms from (18) in the decomposition of the right-hand side of (21). By using the properties (11)-(12) of the convolution we get

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (v - v_\varepsilon) \otimes (v - v_\varepsilon) : \nabla v_\varepsilon \, dx \right| &\leq \int_{\mathbb{T}^3} |v - v_\varepsilon|^2 |\nabla v_\varepsilon| \, dx \\ &\leq \int_{\mathbb{T}^3} |v - v_\varepsilon|^{1+\alpha} |v - v_\varepsilon|^{1-\alpha} |\nabla v_\varepsilon|^\alpha |\nabla v_\varepsilon|^{1-\alpha} \, dx \\ &\leq \int_{\mathbb{T}^3} |v - v_\varepsilon|^{1+\alpha} f_\alpha(t)^{1-\alpha} \varepsilon^{\alpha(1-\alpha)} f_\alpha(t)^\alpha \varepsilon^{\alpha(\alpha-1)} |\nabla v_\varepsilon|^{1-\alpha} \, dx \\ &\leq f_\alpha(t) \int_{\mathbb{T}^3} |v - v_\varepsilon|^{1+\alpha} |\nabla v_\varepsilon|^{1-\alpha} \, dx. \end{aligned}$$

By using the Hölder inequality with exponents $r = \frac{2}{1+\alpha}$ and $r' = \frac{2}{1-\alpha}$ we get

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (v - v_\varepsilon) \otimes (v - v_\varepsilon) : \nabla v_\varepsilon \, dx \right| &\leq f_\alpha(t) \|v - v_\varepsilon\|^{1+\alpha} \|\nabla v_\varepsilon\|^{1-\alpha} \\ &\leq (2\|v_0\|)^{1+\alpha} f_\alpha(t) \|\nabla v_\varepsilon\|^{1-\alpha}, \end{aligned}$$

where in the last step we used elementary properties of convolutions and the energy inequality to get $\|v_\varepsilon(t)\| \leq \|v(t)\| \leq \|v_0\|$.

This proves, by using again Hölder inequality, now with respect to the time variable, that

$$\begin{aligned} \left| \int_s^t \int_{\mathbb{T}^3} (v - v_\varepsilon) \otimes (v - v_\varepsilon) : \nabla v_\varepsilon \, dx \, d\tau \right| &\leq C \int_s^t f_\alpha(\tau) \|\nabla v_\varepsilon(\tau)\|_2^{1-\alpha} \, d\tau \\ &\leq C \|f_\alpha\|_{\frac{2}{2+\alpha}} \left(\int_s^t \|\nabla v_\varepsilon(\tau)\|_2^2 \, d\tau \right)^{1-\alpha} \\ &\leq C \|f_\alpha\|_{\frac{2}{2+\alpha}} \left(\int_s^t \|\nabla v(\tau)\|_2^2 \, d\tau \right)^{1-\alpha}, \end{aligned}$$

which is uniform in $\varepsilon > 0$, since $\nabla v \in L^2(0, T; L^2(\mathbb{T}^3))$. Concerning the *remainder* term $r_\varepsilon(v, v)$ we can use the same argument based on Young theorem on convolutions to prove that

$$\left| \int_s^t \int_{\mathbb{T}^3} r_\varepsilon(v, v) : \nabla v_\varepsilon \, dx d\tau \right| \leq C_1 \|f_\alpha\|_{\frac{2}{2+\alpha}} \left(\int_s^T \|\nabla v(\tau)\|_2^2 \, d\tau \right)^{1-\alpha}.$$

The above calculations prove that the family of functions $F_\varepsilon : (0, T] \rightarrow \mathbb{R}^+$, indexed by $\varepsilon > 0$, and defined by

$$F_\varepsilon(\tau) := \int_{\mathbb{T}^3} (v(\tau, x) \otimes v(\tau, x))_\varepsilon : \nabla v_\varepsilon(\tau, x) \, dx \quad \forall \tau \in]0, T],$$

is such that for any fixed $v > 0$ and for any $s \in (0, T)$

1. $|F_\varepsilon(\tau)|$ is uniformly bounded in (s, T) by the function $G(\tau) := C_2 f_\alpha(\tau) \|\nabla v(\tau)\|_2^{1-\alpha}$ which belongs to $L^1(s, T)$ by hypothesis;
2. Due to the a.e. in $(0, T)$ convergence of $v_\varepsilon(\tau)$ towards $v(\tau) \in V$ (valid being $v \in L^2(0, T; V)$) it follows that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(\tau) = \int_{\mathbb{T}^3} (v(\tau, x) \otimes v(\tau, x)) : \nabla v(\tau, x) \, dx = 0 \quad \text{for a.e. } \tau \in (0, T).$$

By using the Lebesgue dominated convergence theorem, the two properties imply that, along a sub-sequence,

$$\lim_{\varepsilon \rightarrow 0} \int_s^t F_\varepsilon(\tau) \, d\tau = \lim_{\varepsilon \rightarrow 0} \int_s^t \int_{\mathbb{T}^3} (v(\tau, x) \otimes v(\tau, x))_\varepsilon : \nabla v_\varepsilon(\tau, x) \, dx d\tau = 0,$$

for all $t \in]s, T]$, hence finally proving that

$$\frac{1}{2} \|v(t)\|^2 + v \int_s^t \|\nabla v(\tau)\|^2 \, d\tau = \frac{1}{2} \|v(s)\|^2 \quad \forall s, t \text{ s.t. } 0 < s < t \leq T.$$

Next, we take a sequence $\{s_m\}$ of strictly positive times such that $s_m \rightarrow 0$. By the definition of weak solution it holds that: i)

$$\lim_{m \rightarrow +\infty} \|v(s_m)\| = \|v_0\|,$$

being the initial datum strongly attained; ii)

$$\lim_{m \rightarrow +\infty} \int_{s_m}^t \|\nabla v(\tau)\|^2 \, d\tau = \int_0^t \|\nabla v(\tau)\|^2 \, d\tau,$$

by the absolute continuity of the time-integral of $\|\nabla v(t)\|^2$. Hence, passing to the limit as $m \rightarrow +\infty$ in the energy equality over $[s_m, t]$ is justified and we get that

$$\frac{1}{2}\|v(t)\|^2 + \nu \int_0^t \|\nabla v(\tau)\|^2 d\tau = \frac{1}{2}\|v_0\|^2 \quad \forall t \in [0, T]. \quad \square$$

6. Energy conservation in presence of boundaries: the half-space case

In this section we prove a result concerning the energy conservation in Hölder spaces, in the case of a domain with boundary, and with homogeneous Dirichlet conditions. The presence of a solid boundary makes the problem more complex: obtaining smooth and with compact support approximations is generally subject to a localization process, which is not preserving the divergence-free constraint and requires the derivation of appropriate estimates on the pressure. Correction of the divergence is a non-local process, which seems at present not easily obtainable preserving all the pointwise estimates, as needed to take benefit from the extra Hölder-continuity assumptions on the solution.

For the (time evolution) NSE the estimates on the pressure are much more technical to be proved than for the Euler equations: the estimates for the problem in domain Ω with boundary $\partial\Omega$ are generally obtainable through the solution of a linear evolution Stokes problem

$$\begin{aligned} \partial_t v - \nu \Delta v + \nabla p &= -(v \cdot \nabla) v & (t, x) \in (0, T) \times \Omega, \\ \operatorname{div} v &= 0 & (t, x) \in (0, T) \times \Omega, \\ v &= 0 & (t, x) \in (0, T) \times \partial\Omega, \\ v(0, x) &= v_0(x) & x \in \Omega. \end{aligned}$$

This requires the construction of a strong solution (v, p) , with right-hand side $-(v \cdot \nabla) v$ which has to belong at least to some Lebesgue space (and a negative Sobolev is not enough); the pressure cannot be simply constructed time-by-time solving a family of Poisson problem, (see [32, p. 247]), as for the Euler equations, cf. [2, Prop. 1]. In addition, another technical obstruction is that if $v \in L^\beta(0, T; C^\alpha(\Omega))$ (with $\beta < 2$ not to fall into classical regularity classes) then the right-hand side is not time-integrable, since at most $\nabla v \in L^2(0, T; L^2(\Omega))$. This excludes the possibility of using fully and directly the additional Hölder regularity of the solution.

Other possibilities of constructing directly divergence-free approximations (as for instance in Masuda [28, Appendix] or by transversal expansion of Lipschitz domains as in Ern and Guermond [20]) seem at the moment not to produce conclusive results. If we call R_ε a smoothing operator preserving zero boundary data and divergence, a critical technical point would be that of using the vanishing of $\int (v_\varepsilon \otimes v_\varepsilon) : \nabla v_\varepsilon dx = 0$, where $u_\varepsilon = R_\varepsilon u$. Recall that the latter integral is obtained, after using the commutation identity on the term

$$\int (u \otimes u) R_\varepsilon R_\varepsilon u dx = \int R_\varepsilon^* (u \otimes u) R_\varepsilon u dx,$$

requiring $R_\varepsilon^* = R_\varepsilon$, which is not valid for the above operators. Note that this property is valid for the localize (by means of bump function χ) and mollify operator, which results in testing by $\chi \rho_\varepsilon * (\rho_\varepsilon * (\chi u))$, as in [2], but which does not preserve divergence-free constraint.

Up to our best knowledge, the results of energy conservation in presence of boundaries (in the Hölder/Besov case for the NSE) are always subject to extra assumptions on the pressure. In this respect (with additional condition on p) we wish to mention the results of Drivas and Nguyen [18] where, to study weak limits, conditions uniform in $v \in]0, 1]$ are identified. See also Bardos, Titi, and Wiedemann [1], with additional conditions on the pressure, and the results in Drivas and Nguyen [17] with an extra L^3 -equi-continuity near boundary.

In this paper we prove a new result in the half-space, by using a suitable decomposition of variables, into horizontal and vertical ones. We consider the NSE in the half-space $\mathbb{R}_+^3 := \{x \in \mathbb{R}^3 : x_3 > 0\}$ with vanishing Dirichlet condition $v = 0$ on $\{x \in \mathbb{R}^3 : x_3 = 0\}$. In the sequel denote by H and V the closure of smooth, periodic, divergence-free and with zero mean-value vector fields in $L^2(\mathbb{R}_+^3)$ and $W_0^{1,2}(\mathbb{R}_+^3)$. The notion of Leray-Hopf weak solution remains the same as in Definition 2.2, just changing the function spaces and taking test functions with compact support in $[0, T) \times \mathbb{R}_+^3$.

The particular geometry of the domain will make the problem tractable, by means of a natural splitting of variables and unknowns. To this end, any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ will be considered as $x = (x_h, x_3)$ and the same decomposition will be applied also to vector valued functions $u = (u^h, u^3)$. We will also denote by $\nabla_h u := (\partial_{x_1} u, \partial_{x_2} u)$ and $\operatorname{div}_h u^h := \partial_{x_1} u^1 + \partial_{x_2} u^2$ the gradient and the divergence in the horizontal variables, respectively.

Remark 6.1. Different from what done for the Euler equations by Robinson, Rodrigo, and Skipper [31], for the NSE the reflection principle does not apply in the Dirichlet case. On the other hand it is well-known that for the problem with *Navier boundary conditions in the half-space* the reflection principle can be used, giving a direct way to adapt Theorem 5.1 to the half-space.

We introduce the following class of functions:

Definition 6.1. We say that $u \in \dot{C}_{\omega, hor}(\mathbb{R}_+^3)$ if for almost all $z_3 > 0$

$$\sup_{x_h \neq y_h} |u(x_h, z_3) - u(y_h, z_3)| \leq \omega(|x_h - y_h|),$$

with $\omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a non-decreasing function such that $\lim_{s \rightarrow 0^+} \omega(s) = 0$. This means that, for almost all $z_3 > 0$, the function u is uniformly continuous with modulus of continuity $\omega(\cdot)$, with respect to the horizontal variables x_h .

The main result of this section is the following one:

Theorem 6.1. *Let v be a Leray-Hopf weak solution of the NSE in the half space, such that*

$$v \in L^2(0, T; \dot{C}_{\omega, hor}(\mathbb{R}_+^3)). \quad (22)$$

Then, v conserves the energy.

Remark 6.2. The result can be interpreted as (especially near to a flat boundary) energy is conserved if some regularity of the translations parallel to the boundary is assumed. No conditions on the pressure or on vertical increments are required.

The same result applies also to a channel flow, that is if the domain is $\mathbb{R}^2 \times (0, 1)$, or a periodic strip $\mathbb{T}^2 \times (0, 1)$, as studied in [31] with a reflection principle. Note that the regularity assumed in the time variable is more restrictive than in the previous section.

The proof of the main result of this section is based on the use of a partial mollification. We fix a symmetric $\tilde{\rho} \in C_0^\infty(\mathbb{R}^2)$ such that

$$\tilde{\rho} \geq 0, \quad \text{supp } \tilde{\rho} \subset B(0, 1) \subset \mathbb{R}^2, \quad \int_{\mathbb{R}^2} \tilde{\rho}(x_h) \, dx_h = 1,$$

and we define, for $\varepsilon \in (0, 1]$, $\tilde{\rho}_\varepsilon(x_h) := \varepsilon^{-2} \tilde{\rho}(\varepsilon^{-1} x_h)$.

Definition 6.2 (*Horizontal mollification*). We set, for $u \in L^p(K \times \mathbb{R}_+)$, for all $K \subset \subset \mathbb{R}^2$

$$\tilde{u}_\varepsilon(x_h, x_3) := (\tilde{\rho}_\varepsilon *_{\text{h}} u)(x_h, x_3) = \int_{\mathbb{R}^2} \rho_\varepsilon(y_h) u(x_h - y_h, x_3) \, dy_h,$$

where $*_{\text{h}}$ is a mollification only with respect to the horizontal variables, obtained by the convolution with a 2d kernel.

Lemma 6.1. *We have the following properties: Let $\tilde{u}_\varepsilon$ be the horizontal mollification of $u \in W_0^{1,2}(\mathbb{R}_+^3) \cap \dot{C}_{\omega, \text{hor}}(\mathbb{R}_+^3)$, then it holds*

$$\tilde{u}_\varepsilon(x_h, 0) = 0, \tag{23}$$

$$\text{if } \operatorname{div} u = 0, \text{ then } \operatorname{div} \tilde{u}_\varepsilon = 0, \tag{24}$$

$$\sup_{x \in \mathbb{R}_+^3} |u(x) - \tilde{u}_\varepsilon(x)| \leq \omega(\varepsilon). \tag{25}$$

Proof. The first statement comes from $u(x_h, 0)$, which implies directly $\tilde{\rho} *_{\text{h}} u(x_h, 0) = 0$, being the integral evaluated in a set where the function u vanishes. The divergence of the mollified function can be evaluated explicitly, to prove that is preserved by the horizontal mollification.

The third property follows by observing that

$$\begin{aligned} |u(x) - \tilde{u}_\varepsilon(x)| &= \left| u(x_h, x_3) - \int_{\mathbb{R}^2} \tilde{\rho}_\varepsilon(y_h) u(x_h - y_h, x_3) \, dy_h \right| \\ &= \left| \int_{B(0, \varepsilon)} \tilde{\rho}_\varepsilon(y_h) (u(x_h, x_3) - u(x_h - y_h, x_3)) \, dy_h \right| \\ &\leq \int_{B(0, \varepsilon)} \tilde{\rho}_\varepsilon(y_h) \omega(|y_h|) \, dy_h \end{aligned}$$

$$\leq \omega(\varepsilon) \int_{\mathbb{R}^2} \tilde{\rho}_\varepsilon(y_h) \, dy_h,$$

hence the thesis, where we used that $\tilde{\rho} \geq 0$ and the total mass equals one. $\square$

The main fact is that if v is a Leray-Hopf weak solution, then $\tilde{v}_\varepsilon$ is still vanishing at the boundary, divergence-free, and it is smooth “enough” with respect to the space variables x_h .

Remark 6.3. The special geometry of the domain allows for a neat and elementary treatment of the problem. Nevertheless the main difficulty in treating more general smooth and bounded domains would be that of having a smoothing operator which is: commuting with the standard differential operators as the divergence, self-adjoint, and with strong convergence properties in the entire domain.

Also in [20] it is noted how using a partition of unity subordinated to a covering of the boundary is (seemingly) incompatible with the above constraints. Nevertheless, if the boundary of the domain is not flat but it can be described by means of a single Lipschitz function $x_3 = H(x_h)$ (such that after rotation $\nabla_h H(0, 0) = 0$ and Ω is defined by $\{x : x_3 > H(x_h)\}$) one can probably adapt the same process by flattening the domain with the transformation $x^* = T(x)$ such that $(x_h)^* = x_h$, $(x_3)^* = x_3 - H(x_h)$. Next, one can transform vectors with the covariant (Piola type) map which preserves divergence

$$(u^h)^* = u^h \quad (u^3)^* = u^3 - u^h \cdot \nabla_h H,$$

where $u^*(x^*) = u(T^{-1}x^*)$ and the transformed function is defined in the upper half-space. On the new velocity u^* one can then apply the mollification only in the new horizontal variables x_h^* and then transform back to Ω . Technical details, to verify all the steps involve presumably lengthy calculations, which are outside the scope of the paper. Nevertheless, this extension will not improve significantly the result, since we think (for the reasons already explained concerning localization) that the case of a general smooth and bounded domain cannot be treated in this way and, if possible, different tools should be applied.

Proof of Theorem 6.1. To justify the use of $\tilde{v}_{\varepsilon, \varepsilon} := \tilde{\rho}_\varepsilon *_{x_h} (\tilde{\rho}_\varepsilon *_{x_h} v)$ as a test function, a further smoothing in the time-variable will be needed. As in the previous section we fix $0 < s < t \leq T$, to work (for the moment) in the fixed closed interval $[s, t] \subset]0, T]$. Next, we define $u_\kappa : [s, t] \times \mathbb{R}_+^3 \rightarrow \mathbb{R}^3$ by

$$u_\kappa(\sigma, x) := \int_s^t J_\kappa(\sigma - \tau) u(\tau, x) \, d\tau \quad \text{with } 0 < \kappa < t - s,$$

which is obtained through another mollification with respect to the time-variable (this time by means of the scaled non-negative, even with respect to $t = s$, smooth and compactly supported function J_κ , such that $\int_{s-1}^{s+1} J(\tau) \, d\tau = 1$, e.g. one can use $J(\tau) = \rho(\tau - s)$.)

Then, we will use as test function in the weak formulation of the NSE the function

$$\tilde{v}_{\varepsilon, \varepsilon, \kappa}(\sigma, x) := \int_s^t J_\kappa(\sigma - \tau) (\tilde{\rho}_\varepsilon *_{\tilde{h}} (\tilde{\rho}_\varepsilon *_{\tilde{h}} v))(\tau, x) d\tau.$$

Observe that, the use as test function is justified, because it is zero at the boundary, divergence-free and, from

$$\tilde{v}_{\varepsilon, \varepsilon, \kappa} \in W^{1, \infty}(s, t; V') \cap L^\infty(s, t; V),$$

the following estimates

$$\left| \int_s^t \langle v, \partial_t \tilde{v}_{\varepsilon, \varepsilon, \kappa} \rangle d\tau \right| \leq \|\partial_t v\|_{L^{4/3}(V')} \|\tilde{v}_{\varepsilon, \varepsilon, \kappa}\|_{L^4(V)},$$

$$\left| \int_s^t \int_{\mathbb{R}_+^3} (v \cdot \nabla) \tilde{v}_{\varepsilon, \varepsilon, \kappa} \cdot v dx d\tau \right| \leq C \|v\|_{L^{8/3}(L^4)} \|\nabla \tilde{v}_{\varepsilon, \varepsilon, \kappa}\|_{L^\infty(L^2)},$$

hold true. This shows that the left-hand sides are properly defined. Moreover, by using the regularity of solutions to the linear evolution Stokes problem, we can infer (as in Sohr and von Wahl [33]) that the pressure can be selected such that $\nabla p \in L^{5/4}((s_n, t) \times \mathbb{R}_+^3)$ for a sequence of times $s_n > 0$ such that $s_n \rightarrow 0$. These are chosen, for instance, such that $v(s_n) \in H_0^1(\mathbb{R}_+^3)$. This shows also that

$$\int_s^t \int_{\mathbb{R}_+^3} \nabla p \cdot \tilde{v}_{\varepsilon, \varepsilon, \kappa} dx d\tau = 0,$$

since

$$\tilde{v}_{\varepsilon, \varepsilon, \kappa} \in L^\infty(s, t; L^2(\mathbb{R}_+^3)) \cap L^\infty(s, t; \dot{W}_0^{1,2}(\mathbb{R}_+^3)) \subset L^\infty(s, t; L^2(\mathbb{R}_+^3) \cap L^6(\mathbb{R}_+^3)).$$

This finally proves that, by using the so-called Hopf-lemma (cf., e.g., [24]) that

$$\begin{aligned} & \int_s^t \langle \tilde{v}_\varepsilon, \partial_t \tilde{v}_{\varepsilon, \kappa} \rangle d\tau - \nu \int_s^t \int_{\mathbb{R}_+^3} \nabla \tilde{v}_\varepsilon : \nabla \tilde{v}_{\varepsilon, \kappa} dx d\tau \\ & + \int_s^t \int_{\mathbb{R}_+^3} \widetilde{(v \otimes v)}_\varepsilon : \nabla \tilde{v}_{\varepsilon, \kappa} dx d\tau = \int_{\mathbb{R}_+^3} \tilde{v}_\varepsilon(t) \cdot \tilde{v}_{\varepsilon, \kappa}(t) dx - \int_{\mathbb{R}_+^3} \tilde{v}_\varepsilon(s) \cdot \tilde{v}_{\varepsilon, \kappa}(s) dx. \end{aligned}$$

Note that

$$\int_s^t \langle \tilde{v}, \partial_t \tilde{v}_{\varepsilon, \varepsilon, \kappa} \rangle d\tau = \int_{\mathbb{R}_+^3} \int_s^t \int_s^t J'_\kappa(\sigma - \tau) \tilde{v}_\varepsilon(\sigma) \tilde{v}_\varepsilon(\tau) d\sigma d\tau dx = 0,$$

being the kernel $J'(\cdot)$ odd with respect to $t = s$. By a standard limiting procedure $\kappa \rightarrow 0$, by using the weak L^2 -continuity of Leray-Hopf weak solutions, and with the fact that $\int_s^{s+\kappa} J_\kappa(\tau) d\tau = \frac{1}{2}$, one obtains that

$$\int_{\mathbb{R}_+^3} \tilde{v}_\varepsilon(t) \cdot \tilde{v}_{\varepsilon, \kappa}(t) dx - \int_{\mathbb{R}_+^3} \tilde{v}_\varepsilon(s) \cdot \tilde{v}_{\varepsilon, \kappa}(s) dx \xrightarrow{\kappa \rightarrow 0} \frac{1}{2} \|\tilde{v}_\varepsilon(t)\|^2 - \frac{1}{2} \|\tilde{v}_\varepsilon(s)\|^2,$$

and also trivially $\int_s^t \int_{\mathbb{R}_+^3} \nabla \tilde{v}_\varepsilon : \nabla \tilde{v}_{\varepsilon, \kappa} dx d\tau \xrightarrow{\kappa \rightarrow 0} \int_s^t \|\nabla \tilde{v}_{\varepsilon, \kappa}(t)\|^2 d\tau$, since we know that $\tilde{v}_{\varepsilon, \kappa} \rightarrow \tilde{v}_\varepsilon$ at least in $L^2((s, t) \times \mathbb{R}_+^3)$. To handle the nonlinear term observe that for any Banach space X , if $u \in L^q(s, t; X)$, then $u_\kappa \rightarrow u$ strongly in $L^q(s, t; X)$, hence if the limiting object is finite, then it would follow

$$\int_s^t \int_{\mathbb{R}_+^3} (\widetilde{v \otimes v})_\varepsilon : \nabla \tilde{v}_{\varepsilon, \kappa} dx d\tau \xrightarrow{\kappa \rightarrow 0} \int_s^t \int_{\mathbb{R}_+^3} (\widetilde{v \otimes v})_\varepsilon : \nabla \tilde{v}_\varepsilon dx d\tau. \quad (26)$$

To show the validity of the limit (26) we write as usual

$$(\widetilde{u \otimes u})_\varepsilon = \tilde{u}_\varepsilon \otimes \tilde{u}_\varepsilon + \tilde{r}_\varepsilon(u, u) - (u - \tilde{u}_\varepsilon) \otimes (u - \tilde{u}_\varepsilon), \quad (27)$$

with

$$\tilde{r}_\varepsilon(u, u) := \int_{\mathbb{T}^3} \tilde{\rho}_\varepsilon(y) (\tilde{\delta}_y u(x) \otimes \tilde{\delta}_y u(x)) dy,$$

where we set

$$\tilde{\delta}_y u(x) := u(x - y_h) - u(x).$$

We first handle the resulting integral $\int_{\mathbb{T}^3} (\tilde{v}_\varepsilon \otimes \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon dx$ which we will prove to be finite by the following observations: by properties of smoothing we have that $\nabla_h \tilde{v}_\varepsilon = (\partial_{x_1} \tilde{v}_\varepsilon, \partial_{x_2} \tilde{v}_\varepsilon) \in L^2(0, T; H^1(\mathbb{R}_+^3))$. The same holds also for any “horizontal” derivative $D_h^\alpha \tilde{v} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \tilde{v}$, with $|\alpha| = \alpha_1 + \alpha_2 = k$, acting only on the horizontal variables and of arbitrary order $k \geq 1$.

This implies in particular that

$$\partial_{x_3} \tilde{v}_\varepsilon^3 = -\operatorname{div}_h \tilde{v}_\varepsilon^h \in L^2(0, T; H^1(\mathbb{R}_+^3)),$$

hence $\tilde{v}_\varepsilon^3$ is such that $(\nabla_h \tilde{v}_\varepsilon^3, \partial_{x_3} \tilde{v}_\varepsilon^3) = \nabla \tilde{v}_\varepsilon^3 \in L^2(0, T; H^1(\mathbb{R}_+^3))$. In particular, this implies that $\tilde{v}_\varepsilon^3 \in L^2(0, T; H^2(\mathbb{R}_+^3))$. By standard properties of convolution we also have that

$$\|\nabla_h \tilde{v}_\varepsilon\|_{H^1} \leq \frac{C}{\varepsilon} \|v\|_{H^1} \quad \text{and} \quad \|\tilde{v}_\varepsilon^3\|_{H^2} \leq \frac{C}{\varepsilon} \|v\|_{H^1}.$$

Next, observe that by using the decomposition

$$(\tilde{v}_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon \cdot \tilde{v}_\varepsilon = (\tilde{v}_\varepsilon^h \cdot \nabla_h) \tilde{v}_\varepsilon \cdot \tilde{v}_\varepsilon + \tilde{v}_\varepsilon^3 \partial_3 \tilde{v}_\varepsilon \cdot \tilde{v}_\varepsilon,$$

then we can estimate the following integral as

$$\begin{aligned} \left| \int_s^t \int_{\mathbb{R}_+^3} (\tilde{v}_\varepsilon \cdot \nabla) \tilde{v}_\varepsilon \cdot \tilde{v}_\varepsilon \, dx d\tau \right| &= \left| \int_s^t \int_{\mathbb{R}_+^3} (\tilde{v}_\varepsilon \otimes \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon \, dx d\tau \right| \\ &\leq \int_s^T \|\tilde{v}_\varepsilon^h\|_2 \|\nabla_h \tilde{v}_\varepsilon\|_6 \|\tilde{v}_\varepsilon\|_3 + \|\tilde{v}_\varepsilon^3\|_\infty \|\partial_3 \tilde{v}_\varepsilon\|_2 \|\tilde{v}_\varepsilon\|_2 \, d\tau \\ &\leq C \int_s^T \|\tilde{v}_\varepsilon\|_2 \|\nabla_h \tilde{v}_\varepsilon\|_{H^1} \|\nabla \tilde{v}_\varepsilon\|_2 + \|\tilde{v}_\varepsilon^3\|_{H^2} \|\nabla \tilde{v}_\varepsilon\|_2 \|\tilde{v}_\varepsilon\|_2 \, d\tau, \\ &\leq C \varepsilon^{-1} \|v\|_{L^\infty(H) \cap L^2(V)}^3. \end{aligned}$$

This shows that, for any fixed $\varepsilon > 0$,

$$\int_s^t \int_{\mathbb{T}^3} (\tilde{v}_\varepsilon \otimes \tilde{v}_\varepsilon) : \nabla \tilde{v}_{\varepsilon, \kappa} \, dx d\tau \xrightarrow{\kappa \rightarrow 0} \int_s^t \int_{\mathbb{T}^3} (\tilde{v}_\varepsilon \otimes \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon \, dx d\tau,$$

and, by integration by parts (which is legitimate since the integral is well defined), it follows that

$$\int_s^t \int_{\mathbb{T}^3} (\tilde{v}_\varepsilon \otimes \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon \, dx d\tau = 0.$$

Next, by using the same argument as before, we estimate the first term in the decomposition as follows

$$\begin{aligned} \left| \int_{\mathbb{R}_+^3} (v - \tilde{v}_\varepsilon) \otimes (v - \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon \, dx \right| &\leq \int_{\mathbb{R}^3} |v - \tilde{v}_\varepsilon| |v - \tilde{v}_\varepsilon| |\nabla \tilde{v}_\varepsilon| \, dx \\ &\leq f_\omega(t) \omega(\varepsilon) \|v - \tilde{v}_\varepsilon\|_2 \|\nabla \tilde{v}_\varepsilon\|_2 \\ &\leq 2 f_\omega(t) \omega(\varepsilon) \|v\|_2 \|\nabla v\|_2. \end{aligned}$$

This finally shows that, being the integral finite, then the limit process in $\kappa \rightarrow 0$ is valid also in the other term deriving from the commutator decomposition; moreover

$$\begin{aligned}
& \left| \int_s^t \int_{\mathbb{R}_+^3} (v - \tilde{v}_\varepsilon) \otimes (v - \tilde{v}_\varepsilon) : \nabla \tilde{v}_\varepsilon \, dx d\tau \right| \\
& \leq \omega(\varepsilon) \|v_0\|^2 \left[\int_s^T f_\omega^2(t) \, dt \right]^{1/2} \left[\int_s^T \|\nabla v(t)\|^2 \, dt \right]^{1/2} \\
& \leq C\omega(\varepsilon).
\end{aligned}$$

The term arising from $\tilde{r}_\varepsilon(u, u)$ is treated in the same way, hence showing that, for each fixed $\varepsilon > 0$, the following equality holds true

$$\frac{1}{2} \|\tilde{v}_\varepsilon(t)\|^2 + \nu \int_s^t \|\nabla \tilde{v}_\varepsilon(\tau)\|^2 d\tau - \int_s^t \int_{\mathbb{R}_+^3} \widetilde{(v \otimes v)_\varepsilon} : \nabla \tilde{v}_\varepsilon \, dx d\tau = \frac{1}{2} \|\tilde{v}_\varepsilon(s)\|^2.$$

Moreover, the bounds proved show also that in the limit $\varepsilon \rightarrow 0$

$$\left| \int_s^t \int_{\mathbb{R}_+^3} \widetilde{(v \otimes v)_\varepsilon} : \nabla \tilde{v}_\varepsilon \, dx d\tau \right| \leq C\omega(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

due to the hypothesis (22). This proves that

$$\frac{1}{2} \|v(t)\|^2 + \nu \int_s^t \|\nabla v(\tau)\|^2 d\tau = \frac{1}{2} \|v(s)\|^2, \quad \text{for all } 0 < s < t \leq T.$$

The arbitrariness of s, t and the strong limit (8) is used to end the proof. $\square$

Data availability

No data was used for the research described in the article.

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