

Endogenous preferences in a dynamic Cournot duopoly

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Abstract

This research combines two strands of economic literature in a dynamic setting: endogenous preferences (on the consumer side) and strategic competitive markets (on the production side). This is done by considering a model in which aggregate demand depends on past consumption (Benhabib and Day, 1981; Gaertner and Jungeilges, 1988), whereas firms naively react to the rival's past decisions in determining aggregate supply (Puu, 1991). The interaction between the two sides of the market generates nonlinear and chaotic dynamics that could not be observed by considering each side separately. From an analytical point of view, the article shows that the stationary-state equilibrium can be destabilised either by flip bifurcations or by Neimark-Sacker bifurcations. Interestingly, the destabilisation of the stationary-state equilibrium can also occur when firms are homogeneous.

Keywords Endogenous preferences; Nonlinear duopoly; Bifurcations

JEL Classification C62; D43; L13

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1 Introduction

The economic equilibrium analysis is a cornerstone of market interaction where consumers and firms can match their conflicting interests (the maximum price consumers are willing to pay versus the minimum price firms are willing to collect), showing, in turn, the existence of a vector of prices that guarantees market-clearing (i.e., prices always adjust up or down to ensure supply and demand equate) in each market and a Pareto efficient result for the society if markets are competitive (Arrow and Debreu, 1954). This phenomenon was generally studied by considering price taking¹, rational and optimising agents with *exogenous* preferences.

In a different route, the pioneering work of Benhabib and Day (1981) contributed to the development of a growing literature assuming that rational, price-taking and optimising consumers – given stationary prices and wages – can lead to erratic behaviour² when preferences are *endogenous*. Endogenous preferences refer to the case in which the consumer’s utility function depends on the consumer’s past consumption experience (Benhabib and Day, 1981) and/or the consumption experiences of a reference group (Leibenstein, 1950; Gaertner and Jungeilges, 1988; Matsuyama, 1992) or, alternatively, there exists habit formation (e.g., de la Croix, 1996; Alonso-Carrera et al., 2004). There are several articles focusing on the demand side of the market, assuming in some way that consumers’ preferences are not fixed. These works have different purposes. First, Naimzada and Tramontana (2010) considered a model where consumers are not optimising agents that aim at choosing, through an adaptive mechanism, the consumption bundle period by period. Though in their model preferences are de facto exogenous, the adaptive behavioural mechanism (gradient-like decisional process) allows the consumer to let his choices converge over time to the rational path due to an endogenous reactivity parameter, helping the agent to learn from his past consumption experience by adjusting the consumption choices. This adjustment is based on the rule that the more the actual level of consumption is far from the estimated optimum, the more reactive it becomes. In the same vein as the previous work, the contributions of Naimzada and Tramontana (2008, 2012) introduced an evolutionary mechanism comparing two trajectories: the gradient-like dynamic versus non-optimising agents aimed at evolutionarily choosing between several heuristics. Unlike the papers discussed so far, Gaertner and Jungeilges (1988) modified the pioneering work of Benhabib and Day (1981) by considering that consumers’ choices may be affected by the consumption behaviour of some social groups (neighbours). More recently, Naimzada and Tramontana (2009) studied a dynamic model with endogenous preferences assuming that consumption in the current period stimulates the consumption activity in the future and used a gradient-like evolution of the main variables of the problem.³ An-

¹An individual (or a firm) with no market power takes the price as given (price taker). This happens when the number of agents in the market is very large and the size of each of them (in terms of individual consumption or production compared to the corresponding aggregate values) is very small (e.g., perfect competition). At the opposite, an individual (or a firm) with market power chooses the (price maker) for the good to be consumed or produced. This happens when the number of agents in the market is very small and the size of each of them (in terms of individual consumption or production compared to the corresponding aggregate values) is very large (e.g., monopsony/monopoly or oligopoly).

²Following Benhabib and Day (1981, p. 459), erratic behaviour is used to identify "choice sequences that do not converge to a long-run stationary value or to any periodic pattern".

³See also Jungeilges et al. (2021) for a recent extension of the model of consumer behaviour with endogenous preferences in a stochastic framework and a review of the related literature.

other branch of this literature concentrates on the emergence of possible complex social behaviours in a consumption model that incorporates snob and bandwagon preferences that are endogenously determined (Di Giovinazzo and Naimzada, 2015; Cavalli et al., 2016) or the formation of fashion cycles by introducing heterogeneity of consumers characterised by different structures of endogenous preferences (Caravaggio and Sodini, 2020). As it was already pinpointed, all these works adopt a partial equilibrium analysis with fixed prices. Finally, there are contributions considering the problem of endogenous preferences in pure exchange economy where prices adjust to achieve the equilibrium (Naimzada and Pireddu, 2018a, 2018b).

On the production side, there are some relevant works that have pioneered a growing literature that assumed a dynamic setting where firms behave strategically in imperfectly competitive markets (Puu, 1991; Bischi et al., 1998). In the former paper, the author assumed that each producer has perfect knowledge about the market demand but behaves naively in a dynamic setting. Differently, the authors of the latter contribution considered the case of producers with an imperfect knowledge about the market demand that, in turn, adjust adaptively through a gradient-like mechanism in a dynamic setting (see also Puu, 1995). From these two works, an intense research agenda has developed on the dynamic interaction between producers in oligopolistic markets. In this vast literature (see Askar, 2022, for a recent and detailed review), we mention here the works of Tramontana (2010) in which the author introduces heterogeneity in firms' decision-making processes (bounded rational and naïve), Bischi et al. (2015) in which the authors consider an endogenous mechanism to select some heuristics, and Tramontana (2021) that includes some considerations about the Prospect Theory in the formulation of firms' decision-making heuristics. The aim of this work is to combine the literature on endogenous preferences (on the consumption side) with the literature on nonlinear oligopoly (on the production side) considering that the market demand is not invariant but rather it adjusts according to an adaptive mechanism and each producer has perfect knowledge about it but behaves naively in a dynamic setting. Specifically, we aim at understanding how the weight of past decisions in consumer choices and the cost structure of firms can affect the dynamics of the economic system. Results show that the economic equilibrium interaction between the demand side and the supply side is responsible for the emergence of several nonlinear dynamic outcomes that cannot be instead observed by considering each side separately. The stationary-state equilibrium of the discrete-time dynamic system can be destabilised either by a flip bifurcation or by a Neimark-Sacker bifurcation. Interestingly, the destabilisation of the equilibrium can also occur when the two firms are homogeneous on the cost side. In this case, the stationary-state becomes unstable when the ratio of marginal and average costs to the reactivity of preferences to past consumption is relatively low. This holds when a firm does not have to increase the total cost too much by augmenting production and, conversely, when the weight of past consumption experience is relevant in the decisions that consumers should make in the current period. Indeed, the presence of heterogeneities in production costs can dramatically change the dynamic results of the model giving rise to (local) bifurcations that cannot be observed under the assumption of homogeneous costs.

The rest of the article proceeds as follows. In Section 2, we present the modelling framework by characterising the demand side (including endogenous preferences) and the production side (duopoly). In Section 3, we study the dynamics of the model and present the main analytical results by also

showing the relevant simulations. In Section 4, we conclude the article.

2 The model

Let us assume the existence of an economy populated by firms and consumers. There exists a two-sector economy with a competitive sector producing product y and a duopolistic sector in which firm i , with $i = \{1, 2\}$, produces an homogeneous good x^i .

Consumers. There exists a continuum of identical consumers with preferences described by a Cobb-Douglas utility function $U(x, y) = x^a y^{1-a}$, where $a \in (0, 1)$ is a utility weight (see Benhabib and Day, 1981) or, alternatively, the elasticity of the utility U with respect to x . The representative consumer maximises U subject to the budget constraint $px + y = R$, where p represents the price of (i.e., the marginal willingness to pay of the representative consumer for) product x , $R = 1$ is the consumer's exogenous (normalised) nominal income and y is a numeraire good whose price is normalised to 1. By standard utility maximisation we get the solution of the representative consumer's problem, which is given by

$$x^* = \frac{a}{p}, \quad (1)$$

and

$$y^* = 1 - a. \quad (2)$$

By assuming that the consumer solves the optimisation problem at every time $t \in \mathbb{N}$, we now follow several recent works in the literature about endogenous preferences (e.g., Di Giovinazzo and Naimzada, 2015; Cavalli et al., 2016; Caravaggio and Sodini, 2020) and consider that the weight a is not a constant but rather $a(t)$ depends on the consumer's past decisions on x (endogenous preferences). This implies that at a generic time t

$$a(t) = e^{-\alpha x_{t-1}}, \quad (3)$$

where $\alpha > 0$ measures the reactivity of preferences to past consumption contributing to reduce the quality of the experience of consuming x period by period. Though the related literature has further explored the opposite hypothesis by stressing the fact that consuming in the current period stimulates the consumption activity in the future (e.g., Benhabib and Day, 1981; Naimzada and Tramontana, 2009; Caravaggio and Sodini, 2020), in this work we consider the phenomenon that the pleasure of consuming a good in the current period is negatively correlated with the experience of consuming it in the past.⁴

By using the expression in (3) and substituting a in Eq. (1) we obtain the direct and indirect market demand of x_t (D), respectively, that is

$$x_t^D = \frac{e^{-\alpha x_{t-1}}}{p_t}, \quad (4)$$

and

$$p_t = \frac{e^{-\alpha x_{t-1}}}{x_t^D}. \quad (5)$$

⁴Think about the consumption of meat and fish in nutrition. If a consumer eats meat today, he will want to eat fish tomorrow as continuing eating meat may start reduce the pleasure of consumption due to a saturating effect.

Firms (duopoly). On the production side, there exists an oligopolistic sector producing the homogeneous good x . This industry consists of two firms that compete à la Cournot serving the market with a total supply (S) $x_t = x_t^S = x_{1,t} + x_{2,t}$. The production function of each duopolistic firm has constant marginal returns to labour, that is $x_{i,t} = L_{i,t}$, where $L_{i,t}$ represents the labour force employed at time t (Correa-López and Naylor, 2004). Then, the total cost function of firm i is linear and it is given by $C_i(x_i) = c_i L_i = c_i x_i$, so that the (average and marginal) cost of producing an additional unit of output is $c_i > 0$ for firm i . The profit function of firm i at time t can easily be expressed as $\Pi_{i,t} = (p_t - c_i)x_{i,t}$. Then, the maximisation of expected profits of firm 1 reads as follows:

$$\max_{x_{1,t}} \left(\frac{e^{-\alpha(x_{1,t-1} + x_{2,t-1})}}{x_{1,t} + x_{2,t}^e} - c_1 \right) x_{1,t}, \quad (6)$$

where $x_{2,t}^e$ represents the expectation of firm 1 on the amount of output produced by firm 2 at time t . As each firm reacts naively, the condition $x_{2,t}^e = x_{2,t-1}$ holds. This means that firm 1 expects that firm 2 will produce at time t the same amount produced at time $t - 1$. The same problem applies to firm 2. Consequently, from the first order conditions we get the reaction curves of firm 1 and firm 2, that is:

$$x_{1,t} = \sqrt{\frac{x_{2,t-1}}{c_1} e^{-\alpha(x_{1,t-1} + x_{2,t-1})}} - x_{2,t-1}, \quad (7)$$

and

$$x_{2,t} = \sqrt{\frac{x_{1,t-1}}{c_2} e^{-\alpha(x_{1,t-1} + x_{2,t-1})}} - x_{1,t-1}. \quad (8)$$

By comparing the reaction functions of the original model of Puu (1991) with the expressions in (7) and (8) one can note the presence of the extra term $e^{-\alpha(x_{1,t-1} + x_{2,t-1})}$, which will play a crucial role in determining the dynamic properties of the map that will be shown later. This extra term makes the map no longer separable in its second iterate, as instead was the case in Puu (1991). From (7) and (8) the Puu's map is replicated by setting $\alpha = 0$.

3 Dynamics

To analyse the dynamic system in a neat analytical form, it is convenient to introduce the linear change of variables $q_{i,t} := \alpha x_{i,t}$ and also consider the new parameters $k_i := \frac{c_i}{\alpha}$ ($i = \{1, 2\}$). Therefore, from (7) and (8), and knowing that $x_{i,t} := \frac{q_{i,t}}{\alpha}$ the dynamics become:

$$T : \begin{cases} q'_1 = \sqrt{\frac{q_2}{k_1} e^{-(q_1 + q_2)}} - q_2 \\ q'_2 = \sqrt{\frac{q_1}{k_2} e^{-(q_1 + q_2)}} - q_1 \end{cases}, \quad (9)$$

where q'_i is the unit-time advancement of variable q_i . The following proposition shows that map (9) admits a unique interior fixed point.

Proposition 1 *Map (9) admits a unique interior fixed point $E = (q_1^*, q_2^*)$, where*

$$q_1^* = \frac{k_2}{k_1 + k_2} W_0 \left(\frac{1}{k_1 + k_2} \right) \text{ and } q_2^* = \frac{k_1}{k_1 + k_2} W_0 \left(\frac{1}{k_1 + k_2} \right), \quad (10)$$

and W_0 is the principal branch of the Lambert function.

Proof. From the equations in (9), the fixed points of the map are solutions of the system:

$$q_1 + q_2 = \sqrt{\frac{q_2}{k_1} e^{-(q_1+q_2)}}, \quad (11)$$

$$q_1 + q_2 = \sqrt{\frac{q_1}{k_2} e^{-(q_1+q_2)}}. \quad (12)$$

By considering the ratio between the left-hand and right-hand sides of (11) and (12), the following relationship holds at an interior fixed point:

$$q_2 k_2 = k_1 q_1. \quad (13)$$

Therefore, we have the fixed points as the solutions of the following expression:

$$q_i \left(\frac{k_1 + k_2}{k_j} \right) e^{q_i \left(\frac{k_1 + k_2}{k_j} \right)} = \frac{1}{(k_1 + k_2)}, \quad (14)$$

where $i = 1, 2$, $j = 1, 2$ and $j \neq i$. Existence and uniqueness follow from the behaviour of the left-hand side of (14) for $q_i \geq 0$. The expressions in (10) are obtained by using the principal branch of the Lambert function W_0 (see, e.g., Rocha and Taha, 2020), which is depicted in Figure 1 for clarity purposes. We recall that given $f(x) = xe^x$, the main branch of the Lambert function is given by the unique real solution with respect to x of the equation $xe^x = y$ with $y > -1/e$. Loosely speaking, this means that the Lambert function is the inverse of xe^x . As we are only interested in studying the solutions $xe^x = y$ with $y > 0$, W_0 is positive. ■

We note that there is also the origin $O = (0, 0)$ as a fixed point of map T , which is however locally unstable (as can be ascertained by looking at Figure 3, Panel A). No other fixed points do exist.

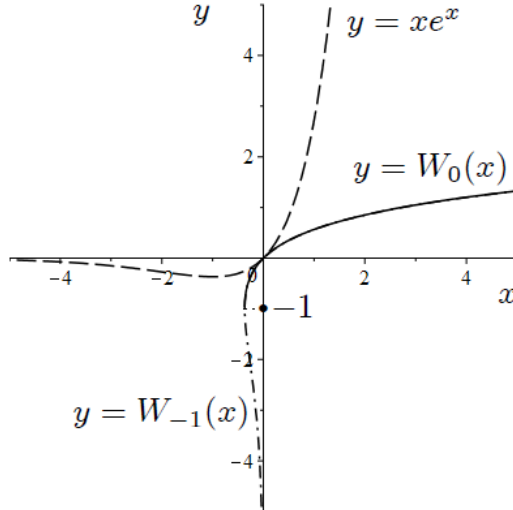


Figure 1. The function $y = xe^x$ and the two real branches of the Lambert W function, $W_0(x)$ (the black solid one) and $W_{-1}(x)$ (the black dash-dot one, which is not involved in our analysis).

To classify the stability properties of the fixed point we first consider the Jacobian matrix of the map in (9) evaluated at E . Then, we have

$$J(q_1^*, q_2^*) = \begin{pmatrix} -\frac{1}{2}W_0\left(\frac{1}{k_1+k_2}\right) & \frac{k_2 - \left[W_0\left(\frac{1}{k_1+k_2}\right) + 1\right]k_1}{2k_1} \\ \frac{k_1 - \left[W_0\left(\frac{1}{k_1+k_2}\right) + 1\right]k_2}{2k_1} & -\frac{1}{2}W_0\left(\frac{1}{k_1+k_2}\right) \end{pmatrix}. \quad (15)$$

Now, in order to analyse the stability of the fixed point E , from (15) one can get the Jury's conditions (Elaydi, 2007):

$$1 - \mathbf{Tr}(J(q_1^*, q_2^*)) + \mathbf{Det}(J(q_1^*, q_2^*)) = \frac{(k_1 + k_2)^2 \left[W_0\left(\frac{1}{k_1+k_2}\right) + 1\right]}{4k_1k_2} > 0, \quad (16)$$

$$1 + \mathbf{Tr}(J(q_1^*, q_2^*)) + \mathbf{Det}(J(q_1^*, q_2^*)) = \frac{(k_1^2 - 6k_1k_2 + k_2^2)W_0\left(\frac{1}{k_1+k_2}\right) + (k_1 + k_2)^2}{4k_1k_2} > 0, \quad (17)$$

$$1 - \mathbf{Det}(J(q_1^*, q_2^*)) = \frac{-(k_1 - k_2)^2W_0\left(\frac{1}{k_1+k_2}\right) - (k_1^2 - 6k_1k_2 + k_2^2)}{4k_1k_2} > 0. \quad (18)$$

The first condition is trivially fulfilled for any couple (k_1, k_2) . By studying the numerator of the second expression $1 + \mathbf{Tr}(J(q_1^*, q_2^*)) + \mathbf{Det}(J(q_1^*, q_2^*))$ we can see that it defines an oval curve, which is symmetrical with respect to the bisector of the first orthant, starting and turning back asymptotically to the origin of the axes. The other vertex of the region has instead coordinates $k_1 = k_2 = 1/(2e)$. The third expression $1 - \mathbf{Det}(J(q_1^*, q_2^*)) = 0$ defines a symmetric curve with respect to the bisector of the first orthant consisting of two branches. Unlike the previous one, this curve is not limited and considering the behaviour of $(k_1 - k_2)^2W_0\left(\frac{1}{k_1+k_2}\right)$ we deduce that the two branches are asymptotic to the straight lines $k_2 = (3 \pm 2\sqrt{2})k_1$. The stability region bounded by this curve is illustrated in Figure 2, where Panel B represents an enlargement view of Panel A close to the origin. The figure shows the existence of two intersections of the blue line and the dashed black line for low values of k_1 and k_2 (about the points $(0.017, 0.05)$ and $(0.05, 0.017)$). These intersections are symmetrical with respect to the bisector of the first and third quadrants.

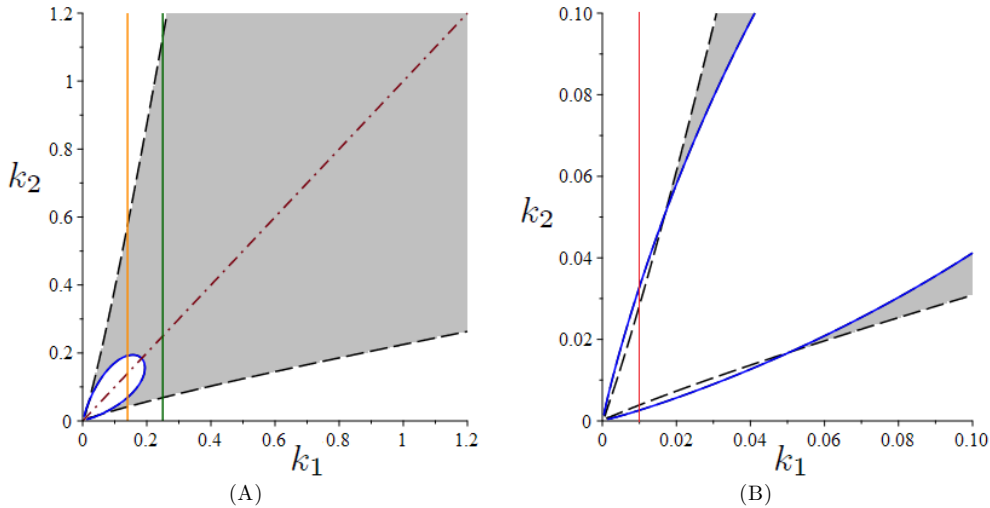


Figure 2. Panel A. The stability region (grey) with respect to k_1 and k_2 . The blue solid line represents the flip bifurcation curve. The black dotted line is the Neimark-Sacker bifurcation curve(s). The red dash-dotted line shows the stability properties when firms are homogeneous. The orange solid line shows the existence of two flip and two Neimark-Sacker bifurcations for increasing values of k_2 when k_1 is sufficiently small. The green solid line shows the existence of only two Neimark-Sacker bifurcations for increasing values of k_2 when k_1 is sufficiently high. Panel B. An enlargement view of Panel A close to the origin.

Given the definition of k_i at the beginning of Section 3, the heterogeneity (resp. homogeneity) in the production costs, i.e. $c_i \neq c_j$ (resp. $c_i = c_j$), is equivalent to assume that parameters k_i and k_j are different (resp. equal), i.e., $k_i \neq k_j$ (resp. $k_i = k_j$). Specifically, when firms are homogeneous on the costs side, i.e., $k_1 = k_2 = k$, then map T is symmetric, so that the map does not change if q_1 and q_2 are swapped up. In other terms, we have that $T \circ S = S \circ T$, where $S : (q_1, q_2) \rightarrow (q_2, q_1)$. Under the assumption of the symmetry of the map, $\Delta = \{(q_1, q_2) : q_1 = q_2 = q\}$ is an invariant manifold, that is $T(\Delta) \subseteq \Delta$. This implies that by starting with the same initial condition $q_1(0) = q_2(0) = q(0)$, the dynamics lie on Δ for every t . In this case, the behaviour of the dynamic system is described by the restriction of map T on Δ , and the synchronised trajectories (i.e., $q_1 = q_2$, for every t) are governed by $T_\Delta : \Delta \rightarrow \Delta$, where

$$T_\Delta : q' = f(q) := \sqrt{\frac{q}{k}e^{-2q}} - q. \quad (19)$$

The following result holds.

Proposition 2 *Map T_Δ has a unique fixed point $q^* = \frac{1}{2} \cdot W_0(\frac{1}{2k})$, which is stable for $k > \frac{1}{2e}$ and unstable for $k < \frac{1}{2e}$. The fixed point undergoes a flip bifurcation for $k = \frac{1}{2e}$.*

Map T_Δ is uni-modal whose maximum point is given by the unique solution, q_{\max} , of the equation $2e^q\sqrt{qk} + 2q - 1 = 0$. In addition, $q' = 0$ for $q = 0$ and $q = \hat{q} := \frac{1}{2} \cdot W_0(\frac{2}{k})$. Then, the dynamics generated by an initial condition $q_0 \in (0, \hat{q})$ remain positive for any t if $f(q_{\max}) \leq \hat{q}$ (Figure 3, Panel A). Indeed, both terms of the previous inequality depend on k . In any case, there exists a threshold k^* such that the previous inequality is (resp. is not) fulfilled for $k \geq k^*$ (resp. $k < k^*$).

The dynamics of map T_Δ are summarised by the bifurcation diagram in Figure 3, Panel B. Going back to the original parameters, this result implies that if the strength of the memory of past consumption α is sufficiently large, then – unlike Puu (1991) – the dynamics turn out to be destabilised also when firms are symmetric. Now, it would be interesting to pinpoint – following the original contribution of Benhabib and Day (1981) – that assuming the expression in (4) holds regarding the consumer side, but not considering the price formation process ($p_t = p$, for any t), the dynamics of x would be defined by the monotonic decreasing map:

$$x_{t+1} = \frac{e^{-\alpha x_t}}{p}, \quad (20)$$

from which only dynamics converging to the fixed point or to a two-period cycle emerge. This implies that the nonlinear dynamics emerging from the model are actually generated by the interaction between the demand side and the supply side of the market.

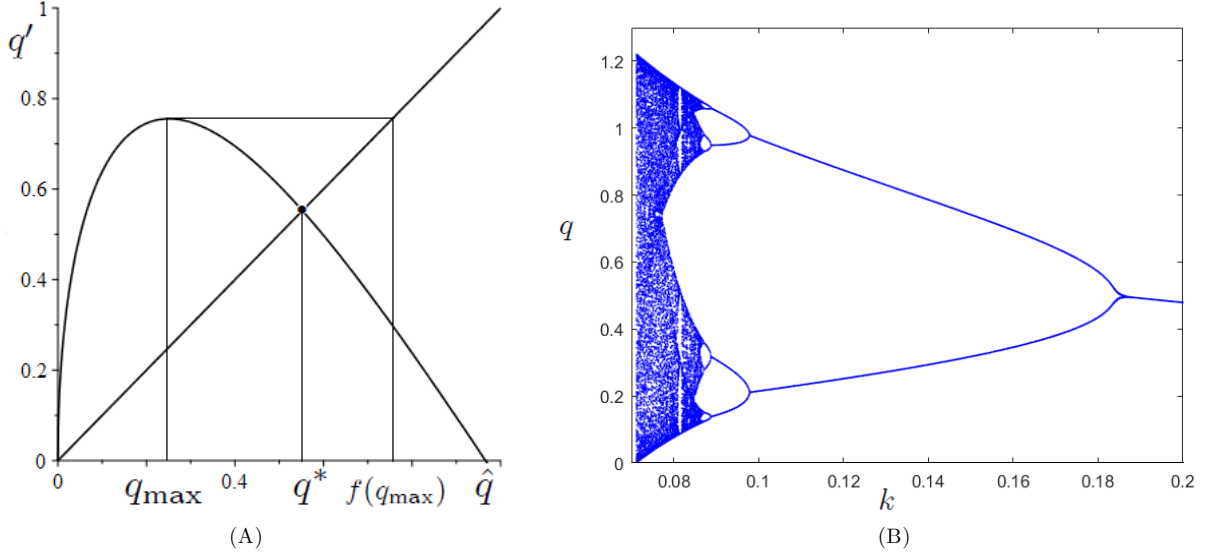


Figure 3. Panel A. Map restricted on the diagonal T_Δ (the 45 degrees line). Panel B. Bifurcation diagram for k ($0.07 \leq k \leq 0.2$).

As it was pinpointed by Bischi et al. (1998), if there exists an invariant subset of the phase plane it is useful to wonder whether the dynamics of the system can be efficiently summarised by studying the dynamics on this subset when there exists an attractor for the dynamics restricted on such a subset. For doing so, we can study the transverse attractiveness of attractors located on Δ . Therefore, we consider the following Jacobian matrix associated with map (9)

$$J(q, q) = \begin{pmatrix} l(q) & m(q) \\ m(q) & l(q) \end{pmatrix} \quad (21)$$

where $l(q) = -\frac{\sqrt{q}e^{-q}}{2\sqrt{k}}$ and $m(q) = -\frac{(2\sqrt{k}qe^q + q - 1)e^{-q}}{2\sqrt{qk}}$. The related eigenvalues are the following:

$$\lambda_{||} = (l(q) + m(q)) = \frac{(\frac{1}{2} - q)e^{-q}}{2\sqrt{qk}} - 1, \quad (22)$$

with eigenvector $(1, 1)$ and

$$\lambda_{\perp} = l(q) - m(q) = 1 - \frac{e^{-q}}{2\sqrt{qk}} < 1, \quad (23)$$

with eigenvector $(1, -1)$. The eigenvalue $\lambda_{||}$ is associated to the eigenvector parallel to the invariant manifold Δ and coincides with the multiplier of the restriction of the map on Δ . The eigenvector associated with the eigenvalue λ_{\perp} is always orthogonal to Δ regardless of q . Now, from Eqs. (22) and (23), it is possible to classify the stability of cycles that occur on Δ . Indeed, for a z -cycle $\{(q_1, q_1), (q_2, q_2), \dots, (q_z, q_z)\}$ of map (9), corresponding to cycle $\{q_1, q_2, \dots, q_z\}$ of map (19), the two multipliers are:

$$\lambda_{||}^{(z)} = \sum_{i=1}^z (l(q_i) + m(q_i)) = \sum_{i=1}^z \left(\frac{(\frac{1}{2} - q_i)e^{-q_i}}{2\sqrt{q_i k}} - 1 \right), \quad (24)$$

related to the eigenvector $(1, 1)$, and

$$\lambda_{\perp}^{(z)} = \sum_{i=1}^z (l(q_i) - m(q_i)) = \sum_{i=1}^z \left(1 - \frac{e^{-q_i}}{2\sqrt{q_i k}} \right), \quad (25)$$

related to the eigenvector $(1, -1)$.

Focusing on transverse stability we have that the task is trivial when a cycle of finite period exists on the diagonal Δ , as only the eigenvalue $\lambda_{\perp}^{(z)}$ needs to be evaluated. The problem becomes more interesting when the dynamics restricted to the invariant sub-manifold show the emergence of a chaotic set A_s especially when this set is attracting for the restricted map. To study this issue, the main tool is the transverse Lyapunov exponent, which characterises the “average” local behaviour of the trajectories in a neighbourhood of the invariant set $A_s \subset \Delta$. We now recall its definition (Maistrenko et al., 1998):

Definition 3 Let $\{q_i = f^i(q_0), i \geq 0\}$ be a trajectory of Eq. (19) embedded in $A_s \subset \Delta$. Then,

$$\Lambda_{\perp} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln |\lambda_{\perp}(q_i)| = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \ln \left| 1 - \frac{e^{-q_i}}{2\sqrt{q_i k}} \right|, \quad (26)$$

is called a transverse Lyapunov exponent.

If x_0 belongs to a z -cycle, then $\Lambda_{\perp} = \ln |\lambda_{\perp}^{(z)}|$ and if $\Lambda_{\perp} < 0$, then the z -cycle is transversely stable. Otherwise, if A_s is a chaotic set and $\Lambda_{\perp} < 0$ for each trajectory starting inside A_s , then A_s is asymptotically stable. In addition, the following classical definition of attractiveness can be introduced (Bischi et al., 1998; 2016):

Definition 4 A_s is an asymptotically stable attractor (or topological attractor) if it is Lyapunov stable, i.e. for every neighbourhood I of A_s , a neighbourhood V of A_s exists such that $T^n(V) \subset I$ for every $n \geq 0$ and the basin of attraction $B(A_s)$ contains a neighbourhood of A_s .

According to Definition 4, if A_s is a topological attractor, then there exists a neighbourhood (with positive Lebesgue measure) Q surrounding A_s , which is included in the basin of attraction of A_s , such that $\lim_{n \rightarrow +\infty} T^n(x) = A_s$, for any $x \in Q$. We recall that the basin of attraction of A_s is defined as $B(A_s) = \bigcup_{n \geq 0} T^{-n}(Q)$. This implies that this basin is characterised by a positive Lebesgue measure.

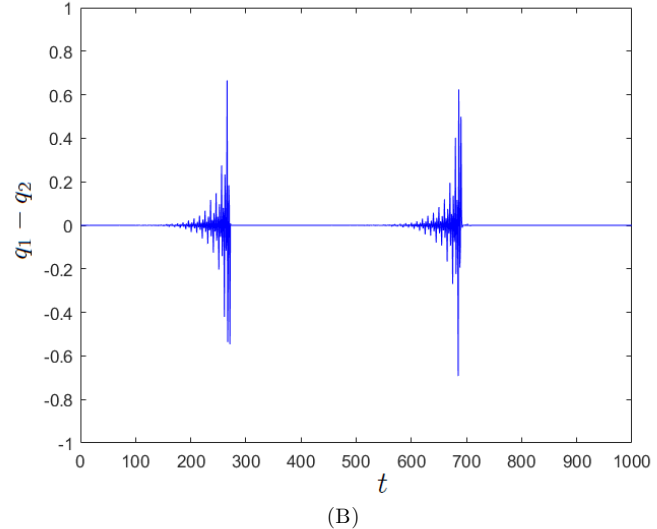
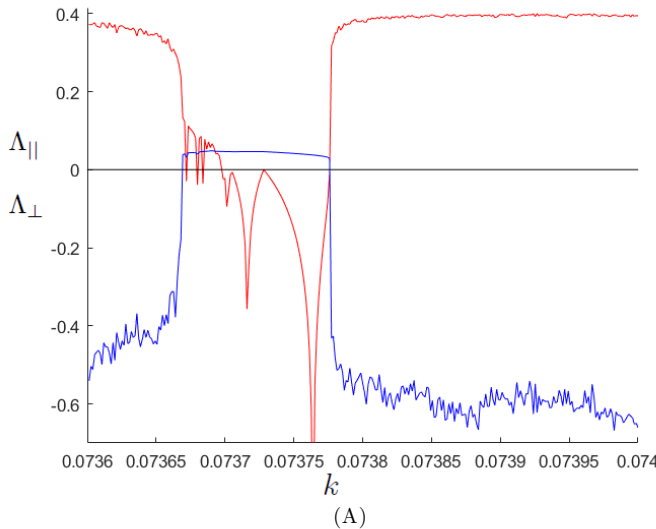
It is important to note that it is possible to have a set of initial conditions with positive Lebesgue measure converging to the attractor with no neighbourhood of A_s in the basin of attraction A_s . Now, we can give a weaker definition of attractor (Milnor, 1985):

Definition 5 A closed invariant set A_s is said to be a weak attractor in Milnor sense (or simply Milnor attractor) if its basin of attraction $B(A_s)$ has positive Lebesgue measure.

Given Definitions 4 and 5, a topological attractor is also a Milnor attractor, but the opposite is not generally true. Roughly speaking, a Milnor attractor may be unstable (in Lyapunov sense) while attracting a set of points with positive Lebesgue measure.

Proving that an attractor is asymptotically stable implies evaluating the Lyapunov exponent over an infinity of trajectories. It is therefore easier to show whether or not a chaotic set along the diagonal is an attractor in the Lyapunov sense, because it is sufficient to find a trajectory (for instance, a cycle) with respect to which the condition $\Lambda_{\perp} < 0$ is violated.

Figure 4, Panel A shows the Lyapunov exponents for a generic trajectory attracted by the attractor on the diagonal when k varies. The red line shows the nature of the attractor embedded in the diagonal (Λ_{\parallel}) with respect to map T restricted on the diagonal when k varies; the blue line instead describes the transverse stability of the invariant set located on the diagonal. Specifically, for $k \in (0.07367, 0.07378)$ there are several subintervals where a chaotic but not topological attractor⁵ on the diagonal ($\Lambda_{\parallel} > 0$ and $\Lambda_{\perp} > 0$) exists for the dynamics defined by map T . Figures 4B and 4C depict the consequences of this phenomenon. In particular, Figure 4, Panel B identifies the mismatch between q_1 and q_2 against time when the production costs are homogeneous, but the initial conditions are slightly different. The figure proves that the convergence towards the attractor embedded in the diagonal can require a very long transient because of the existence of time windows in which the dynamics are decoupled. The variables q_1 and q_2 stay permanent close to the diagonal only after about 700 iterates. Instead, Figure 4, Panel C, considers the case of slightly heterogeneous production costs. The structure of the attractor (depicted in blue) appears dramatically different compared to the case of homogeneous costs (depicted in red), in which the attractor is composed of five pieces embedded in the diagonal ($k = 0.073671$).



⁵The same phenomenon is not observed for values of k outside the range considered so far.

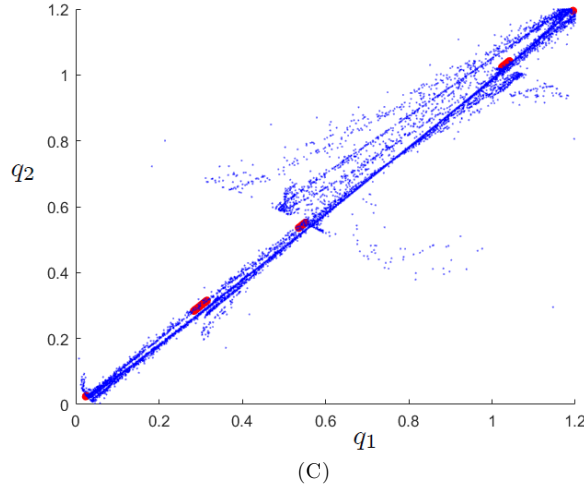


Figure 4. Panel A. Lyapunov exponent. Initial conditions: $q_1 = q_2 = q = 0.541$. Panel B. Time series of q_1 and q_2 describing the on-off intermittency phenomenon (for $k = 0.073671$): convergence towards the chaotic (Milnor) attractor of the system embedded in the diagonal occurs only after almost 700 iterates. Initial conditions: $q_1 = 0.541$ and $q_2 = 0.54101$. Panel C. Change of the shape of the attractor when a mismatch between k_1 and k_2 is introduced ($k_1 = 0.073671 - 0.001$ and $k_2 = 0.073671 + 0.001$). Initial conditions: $q_1 = q_2 = 0.541$.

The phenomena illustrated so far refer either to homogeneous firms (producing with the same technology and sustaining, in turn, the same marginal and average costs) or firms displaying small heterogeneities on the costs side. Differently, we will now look at the phenomena emerging by increasing the degree of heterogeneity on the costs side, that is by varying one of the two c 's and leaving the other one unchanged. Given the definition of k_i at the beginning of this section, this implies changing k_i while leaving k_j and the reactivity of preferences to past consumption (α) unaltered.

From a graphical point of view, we analyse the phenomena occurring when, letting k_1 unchanged, k_2 varies on the orange and green lines in Figure 2, Panel A. Moving along the green line from $k_2 = 0$ it is clear that the steady-state equilibrium is unstable for very low values of k_2 . Then, it becomes stable (for intermediate values of k_2) turning then back to be unstable for larger values of k_2 . Both switches are generated by a Neimark-Sacker bifurcation (the intersections of the green line with the dashed black line). This phenomenon resembles the results of the original contribution by Puu (1991). However, more interesting dynamic phenomena can be observed by varying k_2 along the orange line in Figure 2, Panel A. Starting again from $k_2 = 0$ and increasing the value of the parameter the initially unstable stationary state becomes stable through a Neimark-Sacker bifurcation. However, by increasing further the value of k_2 , the orange line intersects the blue ovoid region two times. At the first intersection, the flip bifurcation causes a destabilisation of the stationary state. At the second intersection, a new flip bifurcation makes the fixed point back to be stable. Increasing further k_2 , the fixed point undergoes a second Neimark-Sacker bifurcation (when the orange line intersects the steepest black dotted line) that makes the fixed point definitely unstable.

Figure 5, divided in two Panels for scaling purposes, shows the evolution of the dynamics of the system (for $k_1 = 0.14$) starting from an initial condition close enough to the fixed point. The figure identifies the existence of four different bifurcations of the fixed points as k_2 changes. No other

bifurcation arises for the dynamic system. Specifically, the initial part of the bifurcation diagram in Panel A shows the existence of an attractive closed invariant curve that is shrinking until it disappears by letting the stationary state become stable. This equilibrium undergoes a flip bifurcation through which it loses stability in favour to an attractive two-period cycle that persists until the second flip bifurcation ($k_2 \cong 0.17$). Finally, the stationary state is again destabilised through a Neimark-Sacker bifurcation causing the emergence of a new attractive cycle (these last two phenomena are depicted in Panel B). We recall that when a closed invariant curve is born, the dynamics on it can be quasi-periodic or periodic depending on the rotation number. We note that very small or very large values of k_2 let the closed invariant curve disappear.

Figure 6 depicts several dynamic scenarios emerging when k_2 varies considering different values of k_1 (Panel A: $k_1 = 0.14$; Panel B: $k_1 = 0.12$; Panel C: $k_1 = 0.11$). The figures show that the dynamics of the model can converge towards a cycle of given order or there can be chaotic dynamics.

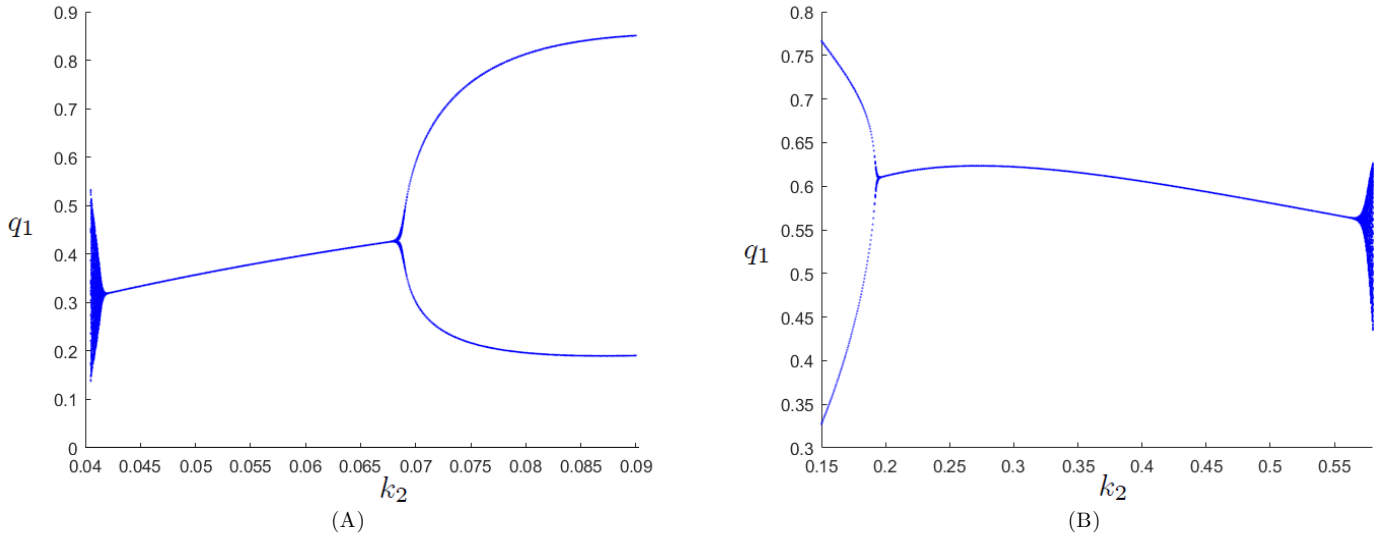


Figure 5. Bifurcation diagram for k_2 ($k_1 = 0.14$). Initial conditions: $q_1 = 0.2$ and $q_2 = 1.05$. Panel A and Panel B respectively show the flip and Neimark-Sacker bifurcations for low and high values of k_2 .

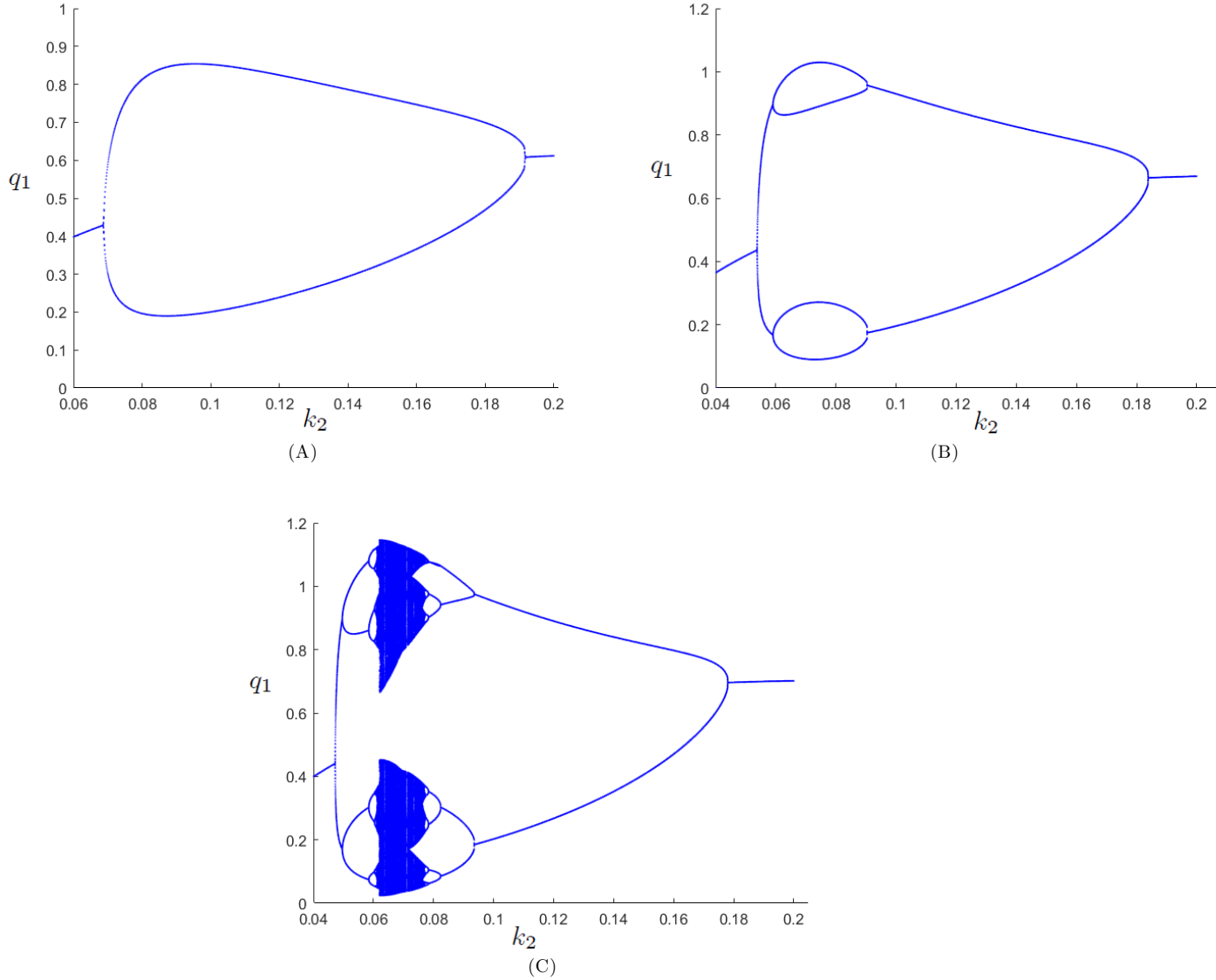


Figure 6. Bifurcation diagram for k_2 . Initial conditions: $q_1 = 0.6$ and $q_2 = 0.3$. Panel A. $k_1 = 0.14$. Panel B. $k_1 = 0.12$. Panel C. $k_1 = 0.11$.

So far the article concentrated on the study of the dynamics developing in the positive orthant without involving points that are permanently or temporarily on the axes (i.e., dynamic trajectories in which one of the two firms exits the market). In the pure Cournot model in which consumers' preferences are exogenous and do not depend on past consumption experiences (i.e., there is no memory in the decisions taken by consumers), the problem, which can be tackled from a mathematical point of view, has some difficulties about the related economic intuition: with an isoelastic demand curve the firm's revenues are independent of output and, when only one firm remains in the market it prefers to do not produce.

To illustrate the possibility of dynamics escaping the positive orthant in our model, let us consider the phenomena emerging when we move along the red line (Figure 2, Panel B). In this case, the stationary state is always unstable and from numerical simulations we obtain that no dynamics persist

in the inner part of the first orthant. This implies, due to the definition of the dynamic system, that the dynamics cease to exist (root of a negative number) within the real numbers. Of course, negative values of quantities do not make sense from an economic point of view either. Starting from map (9) and considering the non-negativity constraints on quantities, one would get the following dynamic system:

$$\tilde{T} : \begin{cases} q'_1 = \max \left\{ 0, \sqrt{\frac{q_2}{k_1} e^{-(q_1+q_2)}} - q_2 \right\} \\ q'_2 = \max \left\{ 0, \sqrt{\frac{q_1}{k_2} e^{-(q_1+q_2)}} - q_1 \right\} \end{cases}, \quad (27)$$

Figure 7, Panel A shows the fate of the dynamics of system (27) starting from different initial conditions. The light grey region represents the area of the initial conditions converging to the locally asymptotically stable stationary state (the black point in the figure). Initial conditions starting in the white area refer to trajectories that jump in $(0,0)$ in one iterate. Differently, initial conditions that belong to the dark grey region generate trajectories that involve points lying on one of the axes at each iteration converging towards a two-period (saddle) cycle (Tramontana et al., 2010), whose coordinates can be obtained by considering the second iterate of map \tilde{T} .

The existence of trajectories converging to $(0,0)$ causes problem of well definiteness of the economic model. This is because when the production chosen by both firms is null, the price is not well defined (due to the isoelastic demand function). Similar problems can arise also when the initial conditions are close to the stationary state in the case it is unstable (as it happens for all parameter values lying on the red line in Figure 2, Panel B). Figure 7, Panel B shows (light grey) the set of parameter values (k_1, k_2) corresponding to which there are internal dynamics. Differently, the white region of the figure shows combinations of the parameters such that the set of initial conditions that generate positive trajectories for any t has zero Lebesgue measure. For example, by setting $k_1 = 0.01$ and letting k_2 increase (see the red line plotted in Figure 2B) then almost all trajectories definitely lie on the boundary of the first orthant.

Details and possible alternative ways to overcome the null production problem by one of the two firms can be found in Tramontana et al. (2010), who consider a lower bound on production and present a global analysis of the resulting map. Other studies that have tackled out this issue are the works of Cánovas et al. (2008) that consider the dynamic Cournot model with a linear market demand, and Gori et al. (2017) that consider a different decision-making mechanism.

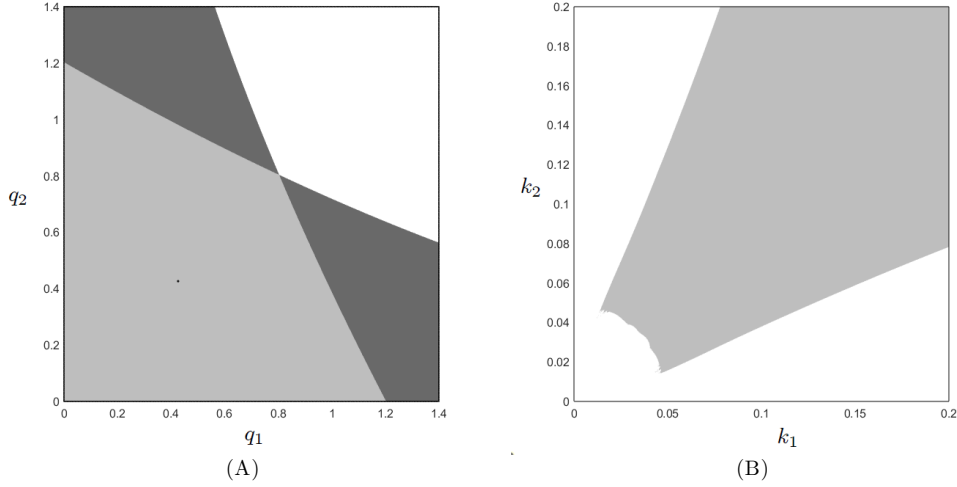


Figure 7. Panel A (plotted for $k_1 = k_2 = 0.25$). The light grey region defines the set of initial conditions that generate positive trajectories for any t . The dark grey region defines the set of initial conditions that generate trajectories for which, alternatively, one coordinate is zero and the other one is positive from the first iterate onwards. The white region defines the set of initial conditions that jump at $(0, 0)$ in one iterate. Panel B. The grey (resp. white) region represents the parameter space (k_1, k_2) such that an initial condition close to the stationary state generates a positive trajectory for any t (resp. a trajectory that develops on one of the axes from a given t onwards).

4 Conclusions

This article combined built on a dynamic model in which consumers have (endogenous) preferences depending on their own past consumption bundles, following the idea of the pioneering work of Benhabib and Day (1981), whereas quantity-setting firms produce in strategic competitive markets (duopoly). Firms have perfect knowledge about the market demand but behave naively in a dynamic setting à la Puu (1991). Unlike Benhabib and Day (1981), prices are determined endogenously. The model is able to generate dynamic phenomena that cannot be observed by considering the demand side and the supply side of the market separately by also considering the parameter values adopted in the pioneering contribution of Puu (1991).

From a mathematical point of view, the stationary state of the model can be either stable or unstable and the switches between these two stability properties can occur – unlike the seminal Puu (1991) – either by flip bifurcation or by Neimark-Sacker bifurcation.

Possible future research may explore the assumption that past consumption decisions positively affect consumers’ preferences in the current period, as in Caravaggio and Sodini (2020) or consider other market structures.

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Conflict of Interest The authors declare that they have no conflict of interest.

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