# Quenched linear response for smooth expanding on average cocycles

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January 22, 2025

**Abstract**

We establish an abstract quenched linear response result for random dynamical systems, which we then apply to the case of smooth expanding on average cocycles on the unit circle. In sharp contrast to the existing results in the literature, we deal with the class of random dynamics that does not necessarily exhibit uniform decay of correlations. Our techniques rely on the infinite-dimensional ergodic theory and in particular, on the study of the top Oseledets space of a parametrized transfer operator cocycle. Finally, we exhibit a surprising phenomenon: a random system and a smooth observable for which quenched linear response holds, but annealed response fails.

## 1 Introduction

### 1.1 Linear response for deterministic and random dynamical systems

Let $M$ be a (compact) Riemannian manifold and consider a family $(T_{\omega,\varepsilon})$ of sufficiently smooth maps acting on $M$ and indexed by $\omega \in \Omega$ and $\varepsilon \in I$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $0 \in I \subset \mathbb{R}$ is an interval. One can view $T_{\omega,\varepsilon}$ as a 'small' perturbations of $T_{\omega,0}$. Endowing the probability space $\Omega$ with an invertible map $\sigma \colon \Omega \to \Omega$ that is measure-preserving and ergodic, we may form the *random products* over $\sigma$, defined by

$$T^n_{\omega,\varepsilon} := T_{\sigma^{n-1}\omega,\varepsilon} \circ \ldots \circ T_{\sigma\omega,\varepsilon} \circ T_{\omega,\varepsilon}.$$

Let us assume that for each $\varepsilon \in I$, the cocycle $(T_{\omega,\varepsilon})_{\omega\in\Omega}$ admits a unique physical equivariant measure, that is, a measurable family of probability measures $(h_{\omega,\varepsilon})_{\omega\in\Omega}$, such that

$$T^*_{\omega,\varepsilon} h_{\omega,\varepsilon} = h_{\sigma\omega,\varepsilon} \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega,$$

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where $T^*_{\omega,\varepsilon} h_{\omega,\varepsilon}$ denotes the push-forward of $h_{\omega,\varepsilon}$ with respect to $T_{\omega,\varepsilon}$, and such that the ergodic basin of $h_{\omega,\varepsilon}$ has a positive Riemannian volume. Then, it is natural to ask the following questions: is the map $\varepsilon \mapsto h_{\omega,\varepsilon}$ differentiable at $\varepsilon = 0$ (in a suitable sense)? If so, can one give an explicit formula for its derivative? These questions are known as the *linear response* problem.

We emphasize that in the context of *deterministic* dynamical systems (which corresponds, in our setting, to the case when $\Omega$ is a singleton), the linear response problem has been thoroughly studied: it was discussed for smooth expanding systems (either on the unit circle or in higher dimensions) [7, 8, 36], piecewise expanding maps of the interval [6, 10], unimodal maps [11], intermittent maps [3, 12, 33], hyperbolic diffeomorphisms and flows [15, 16, 29, 34], as well as for large classes of partially hyperbolic systems [20]. We refer to [7] for a detailed survey of the linear response theory for deterministic dynamical systems.

The setting of random dynamical systems can be divided in two different subcases, the annealed case and the quenched case. The annealed case may be studied by methods very similar to the deterministic case, via weak spectral perturbation for the associated family of transfer operators, or quantitative stability statements for fixed points of Markov operators, and often enjoy a convenient 'regularization property': we refer to [4, 26, 27, 29, 31] for details.

Quenched statistical stability (i.e. continuity, in a suitable sense, of the map $\varepsilon \mapsto h_{\omega,\varepsilon}$ in $\varepsilon = 0$) has been studied for some time: see, e.g., [13] for random subshift of finite type, [9] for smooth expanding maps, [18, 23] for Anosov systems. Closer to the focus of the present paper, quenched statistical stability for an expanding on average cocycles of piecewise expanding systems, exhibiting non-uniform decay of correlations (as considered by Buzzi [17]) was established in [25].

On the other hand, quenched linear response has begun to receive adequate attention only very recently. More precisely, the quenched linear response for (smooth) random dynamical systems was discussed in [35] for expanding dynamics, in [22] for hyperbolic dynamics, and finally in [19] for some classes of partially hyperbolic dynamics.

A common feature of all those results is that they are restricted to the case when the 'unperturbed' cocycle $(T_{\omega,0})_{\omega\in\Omega}$ exhibits uniform (with respect to $\omega$) decay of correlations (see [35, Remark 4.20], [22, eq. (25)], [19, Definition 3.4] and [19, eq. (QR0)]). In addition, various other assumptions, such as those on appropriate Lasota-Yorke inequalities are of 'uniform'-type, i.e. the associated constants are not allowed to depend on the random parameter $\omega$.

However, there are many interesting classes of random dynamical systems which, in general, do not exhibit uniform decay of correlations. Those include smooth or piecewise smooth random expanding on average maps [17, 32], as well as random distance expanding maps [30]. For some recent results dealing with limit laws for such systems, we refer to [23, 24, 30] and references therein.

### 1.2 Contributions of the present paper

The main objective of the present paper is to establish a linear response result (see Theorem 20) for the so-called parameterized smooth expanding on average cocycles on the unit circle (see Definition 14). For this purpose, we formulate abstract statistical stability and linear response results for random dynamical systems (see Theorem 8 and 11), and (see Section 4) verify all of their assumptions in the case of parameterized smooth expanding on average cocycles. In sharp contrast with the previously discussed results in [19, 23, 35], our approach *does not require stochastic uniformity* (i.e. uniformity w.r.t $\omega$) and deal with systems exhibiting nonuniform decay of correlations. We also note that our Theorem 8 allows to recover the statistical stability results of [25], for a class of piecewise monotone, expanding on average random systems introduced by Buzzi [17], which is, to the best of our knowledge, the only result dealing with quenched stability in a stochastically non-uniform setting.

Our methods rely on the infinite-dimensional version of the multiplicative ergodic theorem (MET) established in [28], coupled with some perturbative estimates on transfer operators generalizing those of [26, 27] and close to those in [22]. In particular, our approach requires a detailed analysis of the so-called top Oseledets space of a parameterized transfer operator cocycle (see for example Corollary 36). We stress that unlike [19, 22], we do not use the so-called Mather operator, since in our setting it is unclear on which space would this operator act on (due to nonuniform behavior).

The reader will notice that our linear response results (Theorems 11 and 20) require a suitable discretization of the parameter $\varepsilon$. The explanation behind this is the following: for each $\varepsilon \in I$, the set of random parameters $\omega$ for which certain estimates (as well as the existence) on the equivariant measure $h_{\omega,\varepsilon}$ hold is some set $\Omega_\varepsilon$ with full probability. Hence, in order to study the behavior of $h_{\omega,\varepsilon}$ when $\varepsilon \to 0$, we would need to deal with the set $\bigcap_{\varepsilon \in I} \Omega_\varepsilon$, which could fail to even be measurable. Our statements of Theorems 11 and 20 are tailored to overcome this subtle issue. We refer to [22, p.3] for additional discussion.

Finally, in an appendix, we present an example of a surprising phenomenon: a random system and an observable such that our Theorem 20 applies, so that quenched response holds; but the annealed formula does not converge, implying that annealed linear response cannot hold. We hope that this example convince the reader of the richness of the class of examples considered in the present paper.

### 1.3 Organization of the paper

In Section 2, we recall some basic material from the infinite-dimensional ergodic theory, which will be used in the paper. In Section 3, we formulate our abstract version of the linear response result for random dynamical systems (see Theorem 11). Finally, in Section 4, we introduce and study parameterized smooth expanding on average cocycles on the unit circle. We establish several auxiliary results whose aim is to verify that all assumptions of Theorem 11 are satisfied in this setting. We conclude by establishing an explicit linear response result for parameterized smooth expanding on average cocycles (see Theorem 20), as a corollary of Theorem 11.
In the Appendix, we present an example of a random system and a smooth observable for which quenched response holds, but annealed response does not.

## 2 Preliminaries from multiplicative ergodic theory

The purpose of this section is to recall basic notions of the multiplicative ergodic theory that will be used throughout this paper. Our presentation follows closely [23, Section 2.1].

We begin by recalling the notion of a linear cocycle.

**Definition 1.** *A tuple* $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ *is said to be a* linear cocycle *or simply a* cocycle *if the following conditions hold:*

- $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* $\sigma \colon \Omega \to \Omega$ *is an invertible and ergodic* $\mathbb{P}$*-preserving transformation;*
- $(\mathcal{B}, \|\cdot\|)$ *is a Banach space and* $\mathcal{L} \colon \Omega \to L(\mathcal{B})$ *is a family of bounded linear operators.*

*We say that* $\mathcal{L}$ *is the* generator *of* $\mathcal{R}$*.*

We will often identify a cocycle $\mathcal{R}$ with its generator $\mathcal{L}$. Moreover, we will write $\mathcal{L}_\omega$ instead of $\mathcal{L}(\omega)$.
Recall that a cocycle $\mathcal{R}$ is said to be *strongly measurable* if $\Omega$ is a Borel subset of a separable, complete metric space, $\sigma$ is a homeomorphism and $\omega \mapsto \mathcal{L}_\omega h$ is measurable for each $h \in \mathcal{B}$.
Set, for $\omega \in \Omega$ and $n \in \mathbb{N}$:

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

If $\mathcal{L}$ is strongly measurable and such that

$$\int_\Omega \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P}(\omega) < +\infty, \tag{1}$$

where $\|\mathcal{L}_\omega\|$ denotes the operator norm of $\mathcal{L}_\omega$, it follows from Kingman's subadditive ergodic theorem that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the following limits exist:

$$\Lambda(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n\|$$

and

$$\kappa(\mathcal{R}) := \lim_{n\to\infty} \frac{1}{n} \log ic(\mathcal{L}_\omega^n),$$

where

$$\text{ic}(A) := \inf \Big\{ r > 0 : \; A(B_\mathcal{B}) \text{ admits a finite covering by balls of radius } r \Big\},$$

$B_\mathcal{B}$ is the unit ball of $\mathcal{B}$, and

$$-\infty \le \kappa(\mathcal{R}) \le \Lambda(\mathcal{R}) < +\infty.$$

We recall that $\Lambda(\mathcal{R})$ and $\kappa(\mathcal{R})$ are called the *top Lyapunov exponent* and *index of the compactness* of $\mathcal{R}$ respectively.

**Definition 2.** *Let* $\mathcal{R}$ *be a strongly measurable cocycle such that* (1) *holds. We say that* $\mathcal{R}$ *is* quasi-compact *if* $\kappa(\mathcal{R}) < \Lambda(\mathcal{R})$*.*

The following result gives sufficient conditions under which a cocycle is quasi-compact (see [21, Lemma 2.1]).

**Lemma 3.** *Let* $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ *be a strongly measurable cocycle such that* (1) *holds. Furthermore, let* $(\mathcal{B}', |\cdot|)$ *be a Banach space such that* $\mathcal{B} \subset \mathcal{B}'$ *and that the inclusion* $(\mathcal{B}, \|\cdot\|) \hookrightarrow (\mathcal{B}', |\cdot|)$ *is compact. Finally, assume the following:*

- $\mathcal{L}_\omega$ *can be extended continuously to* $(\mathcal{B}', |\cdot|)$ *for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*;*

- *there are measurable functions* $\alpha_\omega, \beta_\omega, \gamma_\omega : \Omega \to \mathbb{R}$ *such that for* $f \in \mathcal{B}$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$,
$$\|\mathcal{L}_\omega f\| \le \alpha_\omega \|f\| + \beta_\omega |f|$$
*and*
$$\|\mathcal{L}_\omega\| \le \gamma_\omega;$$
- *we have that*
$$\int_\Omega \log \alpha_\omega \, d\mathbb{P}(\omega) < \Lambda(\mathcal{R}) \text{ and } \int_\Omega \log \gamma_\omega \, d\mathbb{P}(\omega) < \infty.$$

*Then,*
$$\kappa(\mathcal{R}) \le \int_\Omega \log \alpha_\omega \, d\mathbb{P}(\omega).$$
*In particular,* $\mathcal{R}$ *is quasi-compact.*

We are now in a position to recall the version of the multiplicative ergodic theorem (MET) established in [28].

**Theorem 4.** *Let* $\mathcal{R} = (\Omega, \mathcal{F}, \mathbb{P}, \sigma, \mathcal{B}, \mathcal{L})$ *be a quasi-compact strongly measurable cocyle such that* $\mathcal{B}$ *is separable. Then, there exists* $1 \le l \le \infty$ *and a sequence of exceptional Lyapunov exponents*
$$\Lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{R}) \quad \text{(if } 1 \le l < \infty\text{)}$$
*or*
$$\Lambda(\mathcal{R}) = \lambda_1 > \lambda_2 > \ldots \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \kappa(\mathcal{R}) \quad \text{(if } l = \infty\text{)}.$$
*Furthermore, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *there exists a unique splitting (called the Oseledets splitting) of* $\mathcal{B}$ *into closed subspaces*
$$\mathcal{B} = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega), \tag{2}$$
*depending measurably on* $\omega$ *and such that:*

1. *For each* $1 \le j \le l$, $Y_j(\omega)$ *is finite-dimensional (i.e.* $m_j := \dim Y_j(\omega) < \infty$*),* $Y_j$ *is equivariant i.e.* $\mathcal{L}_\omega Y_j(\omega) = Y_j(\sigma\omega)$ *and for every* $y \in Y_j(\omega) \setminus \{0\}$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n y\| = \lambda_j.$$
2. $V$ *is weakly equivariant i.e.* $\mathcal{L}_\omega V(\omega) \subseteq V(\sigma\omega)$ *and for every* $v \in V(\omega)$,
$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n v\| \le \kappa(\mathcal{R}).$$

## 3 An abstract quenched linear response result for random dynamics

The purpose of this section is to establish an abstract linear response result for random dynamics that in principle could be applied in a variety of situations. However, our result is tailored to be applicable in the case of smooth expanding on average cocycles, which will be discussed in Section 4.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\sigma\colon \Omega \to \Omega$ be an arbitrary invertible $\mathbb{P}$-preserving transformation. Moreover, let us assume that $\sigma$ is ergodic. We recall that a random variable $K\colon \Omega \to (0, \infty)$ is said to be *tempered* if

$$\lim_{n\to\pm\infty} \frac{1}{n} \log K(\sigma^n \omega) = 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

By $\mathcal{K}$ we denote the set of all tempered random variables $K\colon \Omega \to (0, \infty)$.

Remark 5. It is straightforward to verify that for $K_1, K_2 \in \mathcal{K}$, we have that $K_1 + K_2$, $K_1 \cdot K_2$ and $\max\{K_1, K_2\}$ also belong to $\mathcal{K}$. Moreover, as a simple consequence of Birkhoff's ergodic theorem, we conclude that each random variable $K\colon \Omega \to (0, \infty)$ such that $\log K \in L^1(\Omega, \mathbb{P})$, belongs to $\mathcal{K}$. We will use these properties often throughout this paper, and mostly without explicitly referring to this remark.

In addition, we will often use the following well-known result (see [1, Proposition 4.3.3]).

**Proposition 6.** *Let $K \in \mathcal{K}$. Then, for each $a > 0$ there exists $K_a \in \mathcal{K}$ such that*

$$K(\omega) \le K_a(\omega) \quad \text{and} \quad K_a(\sigma^n \omega) \le K_a(\omega) e^{a|n|}, \tag{3}$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$.*

Remark 7. For the convenience of the reader, we recall that $K_a$ is given by

$$K_a(\omega) = \sup_{n\in\mathbb{Z}} (K(\sigma^n\omega) e^{-a|n|}), \quad \omega \in \Omega.$$

### 3.1 Quenched statistical stability

Let $\mathcal{B}_w = (\mathcal{B}_w, \|\cdot\|_w)$ and $\mathcal{B}_s = (\mathcal{B}_s, \|\cdot\|_s)$ be two Banach spaces such that $\mathcal{B}_s$ is embedded in $\mathcal{B}_w$ and that $\|\cdot\|_w \le \|\cdot\|_s$ on $\mathcal{B}_s$.

In addition, let $I \subset (-1, 1)$ be an arbitrary interval such that $0 \in I$ and assume that for $\omega \in \Omega$ and $\varepsilon \in I$, $\mathcal{L}_{\omega,\varepsilon}$ is a bounded operator on both spaces $\mathcal{B}_w$ and $\mathcal{B}_s$. We will denote $\mathcal{L}_{\omega,0}$ simply by $\mathcal{L}_\omega$. Finally, we assume that $\psi \in \mathcal{B}_s'$ is a nonzero functional, that admits a bounded extension to $\mathcal{B}_w$ (that we still denote by $\psi$) such that

$$\mathcal{L}_{\omega,\varepsilon}^* \psi = \psi \quad \text{for } \omega \in \Omega \text{ and } \varepsilon \in I. \tag{4}$$

For $\omega \in \Omega$, $\varepsilon \in I$ and $n \in \mathbb{N}$, set

$$\mathcal{L}_{\omega,\varepsilon}^n := \mathcal{L}_{\sigma^{n-1}\omega,\varepsilon} \circ \ldots \mathcal{L}_{\sigma\omega,\varepsilon} \circ \mathcal{L}_{\omega,\varepsilon}. \tag{5}$$

The following is our quenched statistical stability result.

**Theorem 8.** *Let* $(\varepsilon_k)_{k\in\mathbb{N}}$ *be a sequence in* $I\setminus\{0\}$ *such that* $\lim_{k\to\infty}\varepsilon_k = 0$ *and write* $\mathcal{L}_{\omega,k}$ *instead of* $\mathcal{L}_{\omega,\varepsilon_k}$*. We assume that there exists a* $\sigma$*-invariant set of full measure* $\Omega'\subset\Omega$ *and* $C\in\mathcal{K}$ *such that:*

- *for* $\omega\in\Omega'$ *and* $k\in\mathbb{N}_0$*, there exists* $h_{\omega,k}\in\mathcal{B}_s$ *such that* $\psi(h_{\omega,k})=1$*,*

$$\mathcal{L}_{\omega,k}h_{\omega,k}=h_{\sigma\omega,k}, \tag{6}$$

  *and*

$$\|h_{\omega,k}\|_s\le C(\omega); \tag{7}$$

- *for* $\omega\in\Omega'$*,* $h\in\mathcal{B}_s$ *and* $n\in\mathbb{N}$*,*

$$\|\mathcal{L}^n_{\sigma^{-n}\omega}h\|_w\le C(\omega)\|h\|_w; \tag{8}$$

- *there exists* $\lambda>0$ *such that*

$$\|\mathcal{L}^n_{\omega}h\|_w\le C(\omega)e^{-\lambda n}\|h\|_w, \tag{9}$$

  *for* $\omega\in\Omega'$*,* $n\in\mathbb{N}$ *and* $h\in V_s$*, where*

$$V_s:=\{h\in\mathcal{B}_s:\psi(h)=0\};$$

- *for* $\omega\in\Omega'$*,* $k\in\mathbb{N}$ *and* $h\in\mathcal{B}_s$*,*

$$\|(\mathcal{L}_{\omega,k}-\mathcal{L}_\omega)h\|_w\le C(\omega)|\varepsilon_k|\|h\|_s. \tag{10}$$

*Then, there exist* $\tilde{C}\in\mathcal{K}$ *and* $r>0$ *such that*

$$\|h_{\omega,k}-h_\omega\|_w\le\tilde{C}(\omega)|\varepsilon_k|^r, \tag{11}$$

*for* $\mathbb{P}$*-a.e.* $\omega\in\Omega$ *and* $k\in\mathbb{N}$*, where* $h_\omega:=h_{\omega,0}$*. In particular,*

$$\lim_{k\to\infty}h_{\omega,k}=h_\omega \text{ in } \mathcal{B}_w, \text{ for } \mathbb{P}\text{-a.e. } \omega\in\Omega.$$

Remark 9. Our Theorem 8 can be applied to expanding on average cocycles of piecewise monotone maps as considered by [17] and thus recover the results of [25].

*Proof.* Take an arbitrary $a$ such that $0<a<\min\{\frac{1}{2},\frac{\lambda}{2}\}$, and let $C_a$ be the tempered random variable given by Proposition 6 (corresponding to $C$). For an arbitrary $N\in\mathbb{N}$, by using the property (6), we have that

$$\begin{aligned}\|h_{\omega,k}-h_\omega\|_w&=\|\mathcal{L}^N_{\sigma^{-N}\omega,k}h_{\sigma^{-N}\omega,k}-\mathcal{L}^N_{\sigma^{-N}\omega}h_{\sigma^{-N}\omega}\|_w\\&\le\|(\mathcal{L}^N_{\sigma^{-N}\omega,k}-\mathcal{L}^N_{\sigma^{-N}\omega})h_{\sigma^{-N}\omega,k}\|_w+\|\mathcal{L}^N_{\sigma^{-N}\omega}(h_{\sigma^{-N}\omega,k}-h_{\sigma^{-N}\omega})\|_w,\end{aligned} \tag{12}$$

for $\omega\in\Omega'$ and $k\in\mathbb{N}$. Since $\psi(h_{\sigma^{-N}\omega,k})=\psi(h_{\sigma^{-N}\omega})=1$, we have that $h_{\sigma^{-N}\omega,k}-h_{\sigma^{-N}\omega}\in V_s$, and thus it follows from (7) and (9) that

$$\|\mathcal{L}^N_{\sigma^{-N}\omega}(h_{\sigma^{-N}\omega,k}-h_{\sigma^{-N}\omega})\|_w\le 2C(\sigma^{-N}\omega)^2e^{-\lambda N}, \tag{13}$$

for $\omega\in\Omega'$ and $k\in\mathbb{N}$. Moreover, since

$$\mathcal{L}^N_{\sigma^{-N}\omega,k}-\mathcal{L}^N_{\sigma^{-N}\omega}=\sum_{j=1}^N\mathcal{L}^{N-j}_{\sigma^{-(N-j)}\omega}(\mathcal{L}_{\sigma^{-N+j-1}\omega,k}-\mathcal{L}_{\sigma^{-N+j-1}\omega})\mathcal{L}^{j-1}_{\sigma^{-N}\omega,k},$$

by (6), (7), (8) and (10) we have that

$$\begin{aligned}
&\|(\mathcal{L}^N_{\sigma^{-N}\omega,k}-\mathcal{L}^N_{\sigma^{-N}\omega})h_{\sigma^{-N}\omega,k}\|_w \\
&\le \sum_{j=1}^N \|\mathcal{L}^{N-j}_{\sigma^{-(N-j)}\omega}\|_{B_w\to B_w}\|(\mathcal{L}_{\sigma^{-N+j-1}\omega,k}-\mathcal{L}_{\sigma^{-N+j-1}\omega})h_{\sigma^{-N+j-1}\omega,k}\|_w \\
&\le C(\omega)|\varepsilon_k|\ \sum_{j=1}^N C(\sigma^{-N+j-1}\omega)^2 \\
&\le C(\omega)|\varepsilon_k|N \max_{0\le j\le N-1} C(\sigma^{-N+j}\omega)^2.
\end{aligned}$$

Hence, by using (3) (for $C$ and $C_a$ instead of $K$ and $K_a$ respectively), we conclude that

$$\|(\mathcal{L}^N_{\sigma^{-N}\omega,k}-\mathcal{L}^N_{\sigma^{-N}\omega})h_{\sigma^{-N}\omega,k}\|_w \le C(\omega)C_a(\omega)^2|\varepsilon_k|Ne^{2aN}, \tag{14}$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$ and $k\in\mathbb{N}$. It follows from (3), (12), (13) and (14) that

$$\|h_{\omega,k}-h_\omega\|_w \le 2C_a(\omega)^2e^{-(\lambda-2a)N}+C(\omega)C_a(\omega)^2|\varepsilon_k|Ne^{2aN},$$

for $\mathbb{P}$-a.e. $\omega\in\Omega$ and $k\in\mathbb{N}$. Choosing $N=\lfloor|\log|\varepsilon_k||\rfloor$, yields the desired result. □

Remark 10. We stress that the requirement that conditions (7), (8), (9) and (10) hold with the same $C\in\mathcal{K}$ does not impose any restriction. Indeed, if conditions (7)-(10) are fulfilled with $C_i\in\mathcal{K}$, for $i\in\{1,2,3,4\}$ respectively, then we can take $C=\max\{C_1,C_2,C_3,C_4\}$ (see Remark 5).

### 3.2 Quenched linear response

In order to formulate our abstract quenched linear response result for random dynamics, besides requiring the existence of spaces $\mathcal{B}_w$ and $\mathcal{B}_s$ as in Subsection 3.1, we also require the existence of a third space $\mathcal{B}_{ss}=(\mathcal{B}_{ss},\|\cdot\|_{ss})$ that can be embedded in $\mathcal{B}_s$ and such that $\|\cdot\|_s\le\|\cdot\|_{ss}$ on $\mathcal{B}_{ss}$.

As in Subsection 3.1, we assume that $\psi$ is a nonzero functional on $\mathcal{B}_s$, and we shall also assume that it admits a bounded extension to $\mathcal{B}_w$. We still denote its restriction (resp. extension) to $\mathcal{B}_{ss}$ (resp. $\mathcal{B}_w$) by $\psi$. We denote $V_{ss}:=\{h\in\mathcal{B}_{ss}:\psi(h)=0\}$ and let $V_s$ and $V_w$ denote images of $V_{ss}$ via the natural injections $B_{ss}\hookrightarrow B_s$ and $B_{ss}\hookrightarrow B_w$ respectively.

Furthermore, we let $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega,\varepsilon\in I}$ be a family such that each $\mathcal{L}_{\omega,\varepsilon}$ is a bounded operator on each of those three spaces satisfying (4), where $I\subset(-1,1)$ is an interval that contains 0. We continue to denote $\mathcal{L}_{\omega,0}$ by $\mathcal{L}_\omega$. Moreover, we suppose that $\omega\mapsto\mathcal{L}_\omega$ is strongly measurable on $\mathcal{B}_w$, i.e. for each $h\in\mathcal{B}_w$ the map $\omega\mapsto\mathcal{L}_\omega h$ is measurable.

**Theorem 11.** *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be a sequence in $I\setminus\{0\}$ such that $\lim_{k\to\infty}\varepsilon_k=0$ and write $\mathcal{L}_{\omega,k}$ instead of $\mathcal{L}_{\omega,\varepsilon_k}$. We assume that there exists a $\sigma$-invariant set of full measure $\Omega'\subset\Omega$ and $C\in\mathcal{K}$ such that:*

- *for $\omega\in\Omega'$ and $k\in\mathbb{N}_0$, there exists $h_{\omega,k}\in\mathcal{B}_{ss}$ such that $\mathcal{L}_{\omega,k}h_{\omega,k}=h_{\sigma\omega,k}$, $\psi(h_{\omega,k})=1$ and*

$$\|h_{\omega,k}\|_{ss}\le C(\omega). \tag{15}$$

*Moreover, suppose that $\omega\mapsto h_{\omega,0}$ is measurable;*

- *for $\omega \in \Omega'$, $h \in \mathcal{B}_i$, $i \in \{w, s\}$ and $n \in \mathbb{N}$,*
$$\|\mathcal{L}^n_{\sigma^{-n}\omega} h\|_i \le C(\omega)\|h\|_i; \tag{16}$$
- *there exists $\lambda > 0$ such that for any $\omega \in \Omega'$, $n \in \mathbb{N}$ and $i \in \{w, s\}$,*
$$\|\mathcal{L}^n_\omega h\|_i \le C(\omega)e^{-\lambda n}\|h\|_i \quad \text{for } n \in \mathbb{N} \text{ and } h \in V_i; \tag{17}$$
- *for $\omega \in \Omega'$, $h \in \mathcal{B}_s$ and $k \in \mathbb{N}$,*
$$\|(\mathcal{L}_{\omega,k} - \mathcal{L}_\omega)h\|_w \le C(\omega)|\varepsilon_k| \cdot \|h\|_s. \tag{18}$$
*Moreover,*
$$\|(\mathcal{L}_{\omega,k} - \mathcal{L}_\omega)h\|_s \le C(\omega)|\varepsilon_k| \cdot \|h\|_{ss}, \tag{19}$$
*for $\omega \in \Omega'$, $h \in \mathcal{B}_{ss}$ and $k \in \mathbb{N}$;*
- *for $\omega \in \Omega'$, there exists a bounded operator $\hat{\mathcal{L}}_\omega : B_{ss} \to V_w$ such that*
$$\left\| \frac{1}{\varepsilon_k}(\mathcal{L}_{\omega,k} - \mathcal{L}_\omega)h - \hat{\mathcal{L}}_\omega h \right\|_w \le C(\omega)|\varepsilon_k|\|h\|_{ss}, \tag{20}$$
*for $\omega \in \Omega'$, $h \in \mathcal{B}_{ss}$ and $k \in \mathbb{N}$. Moreover, assume that for each $k \in \mathbb{N}$ and $h \in \mathcal{B}_{ss}$, $\omega \mapsto \hat{\mathcal{L}}_\omega h$ is measurable.*

*Then, there exist a measurable $\omega \in \Omega \mapsto \hat{h}_\omega \in V_w$, $K \in \mathcal{K}$ and $r > 0$ such that*
$$\left\| \frac{1}{\varepsilon_k}(h_{\omega,k} - h_\omega) - \hat{h}_\omega \right\|_w \le K(\omega)|\varepsilon_k|^r, \tag{21}$$
*for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $k \in \mathbb{N}$, where $h_\omega := h_{\omega,0}$. In addition,*
$$\hat{h}_\omega := \sum_{n=0}^{\infty} \mathcal{L}^n_{\sigma^{-n}\omega} \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega} h_{\sigma^{-(n+1)}\omega}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{22}$$

Remark 12. Theorem 11 is a direct generalization to the quenched case of the 'quantitative stability for fixed points of Markov operators' result in [27]: when $\Omega = \{\omega_0\}$ is reduced to a singleton, we recover [27, Theorem 1] (see also [26, Theorem 3]).

Remark 13. The main novelty of Theorem 11 when compared with the similar results in the literature (see for example [22, Theorem 12]) is that we allow for $C$ to depend on $\omega$. Our requirement that $\lambda$ appearing in (17) is independent on $\omega$ does not represent a serious restriction: indeed, in applications $\lambda$ can be expressed in terms of the second Lyapunov exponent associated with the cocycle $(\mathcal{L}_\omega)_{\omega\in\Omega}$ acting on $\mathcal{B}_s$ and $\mathcal{B}_w$, which due to our ergodicity assumption for the base space $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ is a deterministic quantity. Finally, we note that in our applications, the requirement that $C \in \mathcal{K}$ will again be fulfilled by applying known results from the multiplicative ergodic theory. We refer to Lemma 28 and Proposition 31 for details.

*Proof.* We first show that $\hat{h}_\omega$ given by (22) is well-defined for $\mathbb{P}$-a.e. $\omega \in \Omega$. Indeed, observe that (17), (18) and (20) imply that

$$
\begin{aligned}
&\left\| \sum_{n=0}^{\infty} \mathcal{L}^n_{\sigma^{-n}\omega} \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega} h_{\sigma^{-(n+1)}\omega} \right\|_w \\
&\le \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} \|\hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega} h_{\sigma^{-(n+1)}\omega}\|_w \\
&\le \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} \left\| \frac{1}{\varepsilon_1} (\mathcal{L}_{\sigma^{-(n+1)}\omega,1} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) h_{\sigma^{-(n+1)}\omega} \right\|_w \\
&\quad + \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} \left\| \left( \frac{1}{\varepsilon_1} (\mathcal{L}_{\sigma^{-(n+1)}\omega,1} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) - \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega} \right) h_{\sigma^{-(n+1)}\omega} \right\|_w \\
&\le \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} C(\sigma^{-(n+1)}\omega) \|h_{\sigma^{-(n+1)}\omega}\|_{ss} \\
&\quad + |\varepsilon_1| \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} C(\sigma^{-(n+1)}\omega) \|h_{\sigma^{-(n+1)}\omega}\|_{ss} \\
&\le (1+|\varepsilon_1|) \sum_{n=0}^{\infty} C(\sigma^{-n}\omega) e^{-\lambda n} C(\sigma^{-(n+1)}\omega)^2 \\
&\le (1+|\varepsilon_1|) e^2 C_{\lambda/4}(\omega)^3 \sum_{n=0}^{\infty} e^{-\frac{\lambda}{4} n} < +\infty,
\end{aligned}
$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, where $C_{\lambda/4}$ is given by Proposition 6. Hence, $\hat{h}_\omega$ is well-defined for $\mathbb{P}$-a.e. $\omega \in \Omega$. In addition, due to our measurability assumptions for maps $\omega \mapsto \mathcal{L}_\omega$, $\omega \mapsto \hat{\mathcal{L}}_\omega$ and $\omega \mapsto h_\omega$, we conclude that $\omega \mapsto \hat{h}_\omega$ is measurable.

In order to establish the statement of the theorem, we begin by observing that by introducing $\mathcal{L}_{\sigma^{-1}\omega} h_{\sigma^{-1}\omega,k}$, we have that

$$
\begin{aligned}
h_{\omega,k} - h_\omega &= \mathcal{L}_{\sigma^{-1}\omega,k} h_{\sigma^{-1}\omega,k} - \mathcal{L}_{\sigma^{-1}\omega} h_{\sigma^{-1}\omega} \\
&= \mathcal{L}_{\sigma^{-1}\omega} (h_{\sigma^{-1}\omega,k} - h_{\sigma^{-1}\omega}) + (\mathcal{L}_{\sigma^{-1}\omega,k} - \mathcal{L}_{\sigma^{-1}\omega}) h_{\sigma^{-1}\omega,k},
\end{aligned} \tag{23}
$$

for $\omega \in \Omega'$ and $k \in \mathbb{N}$. By iterating (23) and noting that $\mathcal{L}_{\sigma^{-1}\omega} \circ \mathcal{L}_{\sigma^{-2}\omega} \circ \ldots \circ \mathcal{L}_{\sigma^{-N}\omega} = \mathcal{L}^N_{\sigma^{-N}\omega}$, we have that

$$
h_{\omega,k} - h_\omega = \mathcal{L}^N_{\sigma^{-N}\omega} (h_{\sigma^{-N}\omega,k} - h_{\sigma^{-N}\omega}) + \sum_{n=0}^{N-1} \mathcal{L}^n_{\sigma^{-n}\omega} (\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) h_{\sigma^{-(n+1)}\omega,k}, \tag{24}
$$

for $\omega \in \Omega'$ and $k, N \in \mathbb{N}$. On the other hand, (15) and (17) imply that

$$
\|\mathcal{L}^N_{\sigma^{-N}\omega} (h_{\sigma^{-N}\omega,k} - h_{\sigma^{-N}\omega})\|_w \le 2C(\sigma^{-N}\omega)^2 e^{-\lambda N} \le 2C_{\lambda/4}(\omega)^2 e^{-\frac{\lambda}{2} N}, \tag{25}
$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $k, N \in \mathbb{N}$. Dividing (24) by $\varepsilon_k$, letting $N \to \infty$ and using (25), yields that

$$
\frac{1}{\varepsilon_k} (h_{\omega,k} - h_\omega) = \sum_{n=0}^{\infty} \mathcal{L}^n_{\sigma^{-n}\omega} \left( \frac{1}{\varepsilon_k} (\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) h_{\sigma^{-(n+1)}\omega,k} \right), \tag{26}
$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $k \in \mathbb{N}$. Thus, it follows from (22) and (26) that

$$\frac{1}{\varepsilon_k}(h_{\omega,k} - h_\omega) - \hat{h}_\omega = (I) + (II),$$

where

$$(I) := \sum_{n=0}^{\infty} \mathcal{L}^n_{\sigma^{-n}\omega}\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) - \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega}\right)h_{\sigma^{-(n+1)}\omega}$$

and

$$(II) := \sum_{n=0}^{\infty} \mathcal{L}^n_{\sigma^{-n}\omega}\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega})(h_{\sigma^{-(n+1)}\omega,k} - h_{\sigma^{-(n+1)}\omega})\right).$$

Now, applying Theorem 8 for the pair of spaces $(\mathcal{B}_{ss}, \mathcal{B}_s)$ and using (17) and (18), we observe that

$$\begin{aligned}
&\left\|\mathcal{L}^n_{\sigma^{-n}\omega}\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega})(h_{\sigma^{-(n+1)}\omega,k} - h_{\sigma^{-(n+1)}\omega})\right)\right\|_w \\
&\le C(\sigma^{-n}\omega)e^{-\lambda n}\left\|\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega})(h_{\sigma^{-(n+1)}\omega,k} - h_{\sigma^{-(n+1)}\omega})\right\|_w \\
&\le C(\sigma^{-n}\omega)e^{-\lambda n}C(\sigma^{-(n+1)}\omega)\|h_{\sigma^{-(n+1)}\omega,k} - h_{\sigma^{-(n+1)}\omega}\|_s \\
&\le C(\sigma^{-n}\omega)e^{-\lambda n}C(\sigma^{-(n+1)}\omega)\tilde{C}(\sigma^{-(n+1)}\omega)|\varepsilon_k|^r,
\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \in \mathbb{N}$ and $k \in \mathbb{N}$, where $\tilde{C}$ and $r > 0$ are given by Theorem 8. By applying Proposition 6, we find that

$$\begin{aligned}
&\left\|\mathcal{L}^n_{\sigma^{-n}\omega}\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega})(h_{\sigma^{-(n+1)}\omega,k} - h_{\sigma^{-(n+1)}\omega})\right)\right\|_w \\
&\le e^2 C_{\lambda/4}(\omega)^2 \tilde{C}_{\lambda/4}(\omega)e^{-\frac{\lambda}{4}n}|\varepsilon_k|^r,
\end{aligned}$$

and therefore we conclude that there exists $D_1 \in \mathcal{K}$ such that

$$\|(II)\|_w \le D_1(\omega)|\varepsilon_k|^r, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } k \in \mathbb{N}. \tag{27}$$

On the other hand, by (15), (17), (20) and Proposition 6 we have that

$$\begin{aligned}
&\left\|\mathcal{L}^n_{\sigma^{-n}\omega}\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) - \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega}\right)h_{\sigma^{-(n+1)}\omega}\right\|_w \\
&\le C(\sigma^{-n}\omega)e^{-\lambda n}\left\|\left(\frac{1}{\varepsilon_k}(\mathcal{L}_{\sigma^{-(n+1)}\omega,k} - \mathcal{L}_{\sigma^{-(n+1)}\omega}) - \hat{\mathcal{L}}_{\sigma^{-(n+1)}\omega}\right)h_{\sigma^{-(n+1)}\omega}\right\|_w \\
&\le |\varepsilon_k|C(\sigma^{-n}\omega)e^{-\lambda n}C(\sigma^{-(n+1)}\omega)\|h_{\sigma^{-(n+1)}\omega}\|_{ss} \\
&\le |\varepsilon_k|C(\sigma^{-n}\omega)e^{-\lambda n}C(\sigma^{-(n+1)}\omega)^2 \\
&\le e^2|\varepsilon_k|C_{\lambda/4}(\omega)^3 e^{-\frac{\lambda}{4}n},
\end{aligned}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n, k \in \mathbb{N}$. Thus, there exists $D_2 \in \mathcal{K}$ such that

$$\|(I)\|_w \le D_2(\omega)|\varepsilon_k|, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega \text{ and } k \in \mathbb{N}. \tag{28}$$

Finally, we observe that it follows from the estimates (27) and (28) that (21) holds with $K = D_1 + D_2 \in \mathcal{K}$. The proof of the theorem is completed. □

## 4 Parameterized smooth expanding on average cocycles

In this section, we introduce and study parameterized smooth expanding on average cocycles. More precisely, we establish several auxiliary results in which we essentially verify that assumptions of Theorem 11 are fulfilled in our setting.

Throughout this section, $\mathbb{S}^1$ will denote the unit circle endowed with the Lebesgue measure $m$.

We begin by introducing the notion of a parameterized smooth expanding on average cocycle.

**Definition 14.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space such that $\Omega$ is a Borel subset of a separable complete metric space. Furthermore, let $\sigma \colon \Omega \to \Omega$ be a homeomorphism that preserves $\mathbb{P}$ and that $\mathbb{P}$ is ergodic. Moreover, take an interval $I \subset (-1, 1)$ that contains 0 and let $r \in \mathbb{N}$, $r \geq 4$. We say that a measurable map $\mathbf{T} \colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ is a* parameterized smooth expanding on average cocycle *if the following holds:*

- *there exists a log-integrable random variable $K \colon \Omega \to (0, \infty)$ such that for $\varepsilon \in I$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,*
$$\|T_{\omega,\varepsilon}\|_{C^r} \leq K(\omega) \tag{29}$$
*and, for $i \in \{1, 2\}$,*
$$\|\partial^i_\varepsilon T_{\omega,\varepsilon}\|_{C^{r-i}} \leq K(\omega), \tag{30}$$
*where $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon, \cdot) \in C^r(\mathbb{S}^1, \mathbb{S}^1)$;*
- *for each $\varepsilon \in I$, $\omega \mapsto \lambda_{\omega,\varepsilon} := \min |T'_{\omega,\varepsilon}|$ and $\omega \mapsto \int_{\mathbb{S}^1} \frac{|T''_{\omega,\varepsilon}|}{(T'_{\omega,\varepsilon})^2}\, dm$ are measurable;*
- *there exists a random variable $\underline{\lambda} \colon \Omega \to (0, +\infty)$ such that for any $\varepsilon \in I$, $\lambda_{\omega,\varepsilon} \geq \underline{\lambda}(\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover,*
$$\int_\Omega \log \underline{\lambda}(\omega)\, d\mathbb{P}(\omega) > 0. \tag{31}$$

Remark 15. It follows from (31) that $\underline{\lambda} \in \mathcal{K}$ (see Remark 5).

Remark 16. The class of systems introduced in Definition 14 admits a natural higher dimensional counterpart: on the $d$-dimensional torus $\mathbb{T}^d$, endowed with its Haar-Lebesgue measure $m$, one may consider a measurable mapping $\mathbf{T} : \Omega \to C^r(I \times \mathbb{T}^d, \mathbb{T}^d)$ for some $r > 4$. Setting $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon, \cdot)$, we assume that:

- (29) and (30) hold.
- for each $\varepsilon \in I$, $\lambda_{\omega,\varepsilon} := \min_{x \in \mathbb{T}^d} \min_{\substack{v \in \mathbb{R}^d \\ \|v\|=1}} \|DT_{\omega,\varepsilon}(x) \cdot v\|$ is measurable.
  Furthermore, there exists a random variable $\underline{\lambda} : \Omega \to (0, +\infty)$ such that for any $\varepsilon \in I$, $\lambda_{\omega,\varepsilon} \geq \underline{\lambda}(\omega)$ for $\mathbb{P}$-a.e $\omega \in \Omega$. In addition, we suppose that (31) holds;
- setting $g_{\omega,\varepsilon} := \det(DT_{\omega,\varepsilon}(\cdot))$, we assume that $\omega \mapsto \int_{\mathbb{T}^d} \|Dg_{\omega,\varepsilon}(x)\| dm(x)$ is log-integrable (with respect to $\mathbb{P}$), for each $\varepsilon \in I$.

All the results of the present section apply also in this setting, modulo some obvious changes.

Example 17. Let $(\Omega, \mathcal{F}, \mathbb{P})$, $\sigma\colon \Omega \to \Omega$ and $r \in \mathbb{N}$ be as in Definition 14. Choose a log-integrable random variable $\beta\colon \Omega \to \mathbb{N}$ such that $\mathbb{P}(\{\beta > 1\}) > 0$. Hence,

$$\int_\Omega \log \beta(\omega)\, d\mathbb{P}(\omega) > 0.$$

For $\omega \in \Omega$, we define $T_\omega\colon \mathbb{S}^1 \to \mathbb{S}^1$ by

$$T_\omega(x) = \beta(\omega)x \,(mod\ 1), \quad x \in \mathbb{S}^1.$$

Furthermore, let $D\colon \Omega \to C^r(\mathbb{S}^1, \mathbb{S}^1)$ be a measurable map such that $\omega \mapsto \|D_\omega\|_{C^r}$ is log-integrable and $D'_\omega(x) \geq 0$ for $x \in \mathbb{S}^1$ and $\omega \in \Omega$. We define $\mathbf{T}\colon \Omega \to C^r((-1,1) \times \mathbb{S}^1, \mathbb{S}^1)$ by

$$\mathbf{T}(\omega)(\varepsilon, \cdot) = (\mathrm{Id} + \varepsilon^2 D_\omega) \circ T_\omega.$$

It is straightforward to verify that $\mathbf{T}$ is a parameterized smooth expanding on average cocycle. In particular, using the same notation as in Definition 14, we have $\underline{\lambda} = \beta$.

Example 18. Consider, for $m \in \mathbb{N}$ and $r > 4$, a family $\{T_0, \dots, T_m\}$ of local $C^r$ diffeomorphisms of $\mathbb{S}^1$, and a family $\{d_0, \dots, d_m\}$ of $C^r$ mappings $d_i : \mathbb{S}^1 \to \mathbb{R}$. Setting $\lambda_i := \min_{x \in \mathbb{S}^1} |T'_i(x)| > 0$, we assume that $\sum_{i=0}^{m-1} \log \lambda_i > 0$.
Consider now the full-shift on the alphabet $\{0, \dots, m\}$, i.e. set $\Omega := \{0, \dots, m\}^{\mathbb{Z}}$, and for $\omega = (\omega_n)_{n \in \mathbb{Z}}$, $(\sigma\omega)_n = \omega_{n+1}$, endowed with a Markov measure $\mathbb{P}$ associated to a probability vector $(p_0, \dots, p_m)$. Let $\varepsilon_0 > 0$ be small enough, and set $I = (-\varepsilon_0, \varepsilon_0)$. We then define a measurable mapping $\mathbf{T} : \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ by

$$\mathbf{T}(\omega)(\varepsilon, \cdot) := T_i + \varepsilon d_i \text{ if } \omega_0 = i.$$

This data defines a cocycle over $\sigma$, which is simply the i.i.d composition of maps $T_{\omega,\varepsilon} = \mathbf{T}(\omega)(\varepsilon, \cdot)$. It is easy to see that it satisfies the requirements of Definition 14: in particular, (29), (30) are satisfied with constant $K$, and the expansion on average condition is satisfied with $\underline{\lambda}(\omega) := \lambda_i - \varepsilon_0 \min |d'_i|$ if $\omega_0 = i$; (31) holds up to shrinking $\varepsilon_0$.

The following auxiliary result shows that parameterized smooth expanding on average cocycles exhibit the so-called random covering condition (of uniform type with respect to the parameter $\varepsilon$). For the proof, we refer to [24, Example 6].

**Proposition 19.** *Let $\mathbf{T}\colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ be a parameterized smooth expanding on average cocycle. Then, for any interval $J \subset \mathbb{S}^1$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $n_c = n_c(\omega, J) \in \mathbb{N}$ such that*

$$T^n_{\omega,\varepsilon}(J) = \mathbb{S}^1, \quad \text{for } n \geq n_c \text{ and } \varepsilon \in I.$$

The following is the main result of this section: a linear response result for parameterized smooth expanding on average cocycles.

**Theorem 20.** *Let $\mathbf{T}\colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ be an arbitrary parameterized smooth expanding on average cocycle. For $\varepsilon \in I$, let $(h_{\omega,\varepsilon})_{\omega \in \Omega}$ be the family given by Lemma 27. Then, there exists a measurable family $(\hat{h}_\omega)_{\omega \in \Omega} \subset W^{1,1}$, $r > 0$ and $K \in \mathcal{K}$ such that*

$$\left\| \frac{1}{\varepsilon_k}(h_{\omega,k} - h_\omega) - \hat{h}_\omega \right\|_{W^{1,1}} \leq K(\omega)|\varepsilon_k|^r,$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $k \in \mathbb{N}$, where $h_{\omega,k} := h_{\omega,\varepsilon_k}$, $h_\omega := h_{\omega,0}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ is a sequence in $I \setminus \{0\}$ such that $\varepsilon_k \to 0$. Moreover, $\hat{h}_\omega$ is given by (22), where $\mathcal{L}_\omega$ is the transfer operator associated to $\mathbf{T}(\omega)(0,\cdot)$ and $\hat{\mathcal{L}}_\omega$ is given by (59).*

The proof of Theorem 20 is given in §.4.5.

### 4.1 Lasota-Yorke inequalities

By $W^{\ell,1} := W^{\ell,1}(\mathbb{S}^1)$ we will denote the Sobolev space consisting of all $f \in L^1(\mathbb{S}^1)$ with the property that its derivatives $f^{(j)}$ exist in the weak sense for $j \le \ell$ and $\|f^{(j)}\|_{L^1} < +\infty$. Then, $W^{\ell,1}(\mathbb{S}^1)$ is a Banach space with respect to the norm

$$\|f\|_{W^{\ell,1}} := \sum_{0\le j\le \ell} \|f^{(j)}\|_{L^1}.$$

We start by recalling the following well-known lemma (see for example [18, Lemma 5.9]).

**Lemma 21.** *Let $T \in C^r(\mathbb{S}^1, \mathbb{S}^1)$ and let $\ell \in \mathbb{N}$ be such that $\ell + 1 \le r$. Then, for any $f \in W^{\ell,1}$ we have that*

$$\mathcal{L}_T(f)^{(\ell)} = \mathcal{L}_T\left(\frac{1}{(T')^{2\ell}}\sum_{j=0}^{\ell} G_{\ell,j}(T', \dots, T^{(\ell+1)})f^{(j)}\right),$$

*where $G_{\ell,j}(\cdot, \dots, \cdot) : \mathbb{R}^{\ell+1} \to \mathbb{R}$ are polynomials, $\mathcal{L}_T$ denotes the transfer operator associated to $T$ and $\mathcal{L}_T(f)^{(\ell)}$ is the $\ell$-derivative of $\mathcal{L}_T(f)$.*

Remark 22. We note that polynomials $G_{\ell,j}$ can be constructed explicitly via simple recursive relations. Indeed, setting $G_{0,0} := 1$, we have that

$$\begin{aligned}
G_{\ell+1,0} &= (1-2\ell)x_2 G_{\ell,0} + x_1 G'_{\ell,0} \\
G_{\ell+1,k} &= (1-2\ell)x_2 G_{\ell,k} + x_1(G'_{\ell,k} + G_{\ell,k-1}), \quad \text{for } k \in \{1, \dots, \ell\} \\
G_{\ell+1,\ell+1} &= x_1^{\ell+1}.
\end{aligned}$$

Here, we define $P'$ of an arbitrary polynomial $P$ by setting $(1)' = 0$, $(x_j)' = x_{j+1}$ and by requiring that $(P+Q)' = P' + Q'$ and $(PQ)' = P'Q + PQ'$ for arbitrary polynomials $P$ and $Q$. For example, $(-x_2^2 + x_1x_3)' = -2x_2x_3 + x_1x_4 + x_2x_3 = -x_2x_3 + x_1x_4$.

It is easy to verify that

$$(G_{\ell,j}(T', \dots, T^{(\ell+1)}))' = G'_{\ell,j}(T', \dots, T^{(\ell+2)}), \tag{32}$$

where $(G_{\ell,j}(T', \dots, T^{(\ell+1)}))'$ denotes the usual derivative of the map $G_{\ell,j}(T', \dots, T^{(\ell+1)})$.

**Lemma 23.** *Let $\mathbf{T}\colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ be a parameterized smooth expanding on average cocycle. For any $\ell \in \mathbb{N}$ such that $\ell+1 \le r$, there exist log-integrable random variables $B_\ell, C_\ell\colon \Omega \to (0,\infty)$ such that, for any $\varepsilon \in I$, $\mathbb{P}$-a.e $\omega \in \Omega$[1] and $f \in W^{\ell,1}$,*

$$\|\mathcal{L}_{\omega,\varepsilon} f\|_{W^{\ell,1}} \le C_\ell(\omega)\|f\|_{W^{\ell,1}} \tag{33}$$

[1]Here and throughout the paper this means that for each $\varepsilon \in I$, there exists a full measure set $\Omega_\varepsilon \subset \Omega$

*and*

$$\|\mathcal{L}_{\omega,\varepsilon} f\|_{W^{\ell,1}} \leq \frac{1}{\underline{\lambda}(\omega)^{\ell}} \|f\|_{W^{\ell,1}} + B_\ell(\omega)\|f\|_{W^{\ell-1,1}}, \tag{34}$$

*where $\mathcal{L}_{\omega,\varepsilon}$ denotes the transfer operator of $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon,\cdot)$, $\underline{\lambda}(\omega)$ is as in Definition 14 and $\|\cdot\|_{W^{0,1}} := \|\cdot\|_{L^1}$.*

*Proof.* Starting from Lemma 21, applying (29) and using the same notation as in Definition 14, we have that

$$\begin{aligned}\int_{\mathbb{S}^1} \left|(\mathcal{L}_{\omega,\varepsilon} f)^{(\ell)}\right| dm &\leq \int_{\mathbb{S}^1} \left| \frac{1}{(T'_{\omega,\varepsilon})^{2\ell}} \sum_{j=0}^{\ell} G_{\ell,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell+1)}_{\omega,\varepsilon}) f^{(j)} \right| dm \\ &\leq \lambda_{\omega,\varepsilon}^{-2\ell} \max_{0\leq j\leq \ell} |G_{\ell,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell+1)}_{\omega,\varepsilon})|_\infty \sum_{j=0}^{\ell} \int_{\mathbb{S}^1} |f^{(j)}|\, dm \\ &\leq \underline{\lambda}(\omega)^{-2\ell} \max_{0\leq j\leq \ell} \tilde{G}_{\ell,j}(K(\omega), \dots, K(\omega)) \cdot \|f\|_{W^{\ell,1}},\end{aligned}$$

for any $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{\ell,1}$, for some polynomials $\tilde{G}_{\ell,j}$ satisfying $\tilde{G}_{\ell,j}(x_1, \dots, x_{\ell+1}) \geq 0$ for $x_i \geq 0$, $1 \leq i \leq \ell+1$. We conclude that (33) holds with

$$C_\ell(\omega) := \sum_{i=0}^{\ell} \underline{\lambda}(\omega)^{-2i} \max_{0\leq j\leq i} \tilde{G}_{i,j}(K(\omega), \dots, K(\omega)).$$

Remarking that $\omega \mapsto \max_{0\leq j\leq i} \tilde{G}_{i,j}(K(\omega), \dots, K(\omega))$ is log-integrable and recalling that the same holds for $\underline{\lambda}$ , we have that $C_\ell$ is log-integrable.

Furthermore, we have (recall that $G_{\ell,\ell} = x_1^\ell$) that

$$\begin{aligned}&\int_{\mathbb{S}^1} \left|(\mathcal{L}_{\omega,\varepsilon} f)^{(\ell)}\right| dm \leq \int_{\mathbb{S}^1} \left| \frac{1}{(T'_{\omega,\varepsilon})^{2\ell}} \sum_{j=0}^{\ell} G_{\ell,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell+1)}_{\omega,\varepsilon}) f^{(j)} \right| dm \\ &\leq \frac{1}{\lambda_{\omega,\varepsilon}^{\ell}} \int_{\mathbb{S}^1} |f^{(\ell)}|\, dm + \sum_{j=0}^{\ell-1} \int_{\mathbb{S}^1} \left| \frac{1}{(T'_{\omega,\varepsilon})^{2\ell}} G_{\ell,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell+1)}_{\omega,\varepsilon}) f^{(j)} \right| dm \\ &\leq \underline{\lambda}(\omega)^{-\ell} \int_{\mathbb{S}^1} |f^{(\ell)}|\, dm + \underline{\lambda}(\omega)^{-2\ell} \max_{0\leq j\leq \ell-1} \tilde{G}_{\ell,j}(K(\omega), \dots, K(\omega)) \sum_{j=0}^{\ell-1} \int_{\mathbb{S}^1} |f^{(j)}|\, dm,\end{aligned}$$

for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{\ell,1}$, which implies that (34) holds with

$$B_\ell(\omega) := \underline{\lambda}(\omega)^{-2\ell} \max_{0\leq j\leq \ell-1} \tilde{G}_{\ell,j}(K(\omega), \dots, K(\omega)) + C_{\ell-1}(\omega).$$

The proof of the lemma is completed. □

### 4.2 Lyapunov exponents and Oseledets splittings

Throughout this subsection, we fix an arbitrary parameterized smooth expanding on average cocycle $\mathbf{T}\colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$. We continue to denote by $\mathcal{L}_{\omega,\varepsilon}$ the transfer operator associated to $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon,\cdot)$.

**Proposition 24.** *For each $\varepsilon \in I$, the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ is strongly measurable and quasi-compact on $W^{\ell,1}$ for $1 \leq \ell \leq r-1$.*

*Proof.* Take an arbitrary $1 \leq \ell \leq r-1$. The strong measurability of $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$, for $\varepsilon \in I$ follows from [14, Prop 4.11] (by arguing as in [18, Proof of Proposition 5.2]) and Definition 14.

Observe that (33) implies that the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ satisfies integrability assumptions (1). The quasi-compactness now follows directly from (31), (34) and Lemma 3. □

REMARK 25. Observe that Proposition 24 enables us to apply Theorem 4 for the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ on $W^{\ell,1}$, for every $1 \leq \ell \leq r-1$ and $\varepsilon \in I$. Since $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ is a cocycle of transfer operators, we have that its largest Lyapunov exponent is zero on each $W^{\ell,1}$.

REMARK 26. Setting $\mathrm{Var}(f) = \|f'\|_{L^1}$, we note that the $BV$ space introduced in [23, p.10] coincides with $W^{1,1}$. Thus, it follows easily from Definition 14 and Proposition 19 that for each $\varepsilon \in I$, the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ on $W^{1,1}$ satisfies all requirements of [23, Definition 13] except $\mathbb{P}$-continuity (see [23, Definition 5]). Nevertheless, all the results from [23, Section 2] are applicable to the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ on $W^{1,1}$, for each $\varepsilon \in I$. Indeed, the $\mathbb{P}$-continuity assumption imposed in [23] ensured that in the context of [23, Definition 13], the version of the multiplicative ergodic theorem (dealing with cocycles acting on non-separable Banach spaces) as stated in [23, Theorem 9] can be applied. In the present context, one can easily check that the arguments in [23, Section 2] can be repeated verbatim by simply applying Theorem 4 instead of [23, Theorem 9] when needed.

Taking into account Remark 26, the following result is a direct consequence of [23, Theorem 20] and [23, Proposition 24].

**Lemma 27.** *For $\varepsilon \in I$, there exists a unique measurable family $(h_{\omega,\varepsilon})_{\omega\in\Omega} \subset W^{1,1}$ such that $h_{\omega,\varepsilon} \geq 0$, $\int_{\mathbb{S}^1} h_{\omega,\varepsilon}\, dm = 1$ and*

$$\mathcal{L}_{\omega,\varepsilon} h_{\omega,\varepsilon} = h_{\sigma\omega,\varepsilon}, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

*Moreover, there is $\rho(\varepsilon) \in (0,1)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n_{\omega,\varepsilon} f\|_{W^{1,1}} \leq \log(\rho(\varepsilon)),$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{1,1}$, $\int_{\mathbb{S}^1} f\, dm = 0$.*

Our next goal is to extend the second part of Lemma 27 to any $W^{\ell,1}$, $1 \leq \ell \leq r-1$.

**Lemma 28.** *For $\varepsilon \in I$ and $1 \leq \ell \leq r-1$, we have that*

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n_{\omega,\varepsilon} f\|_{W^{\ell,1}} \leq \log(\rho(\varepsilon)), \tag{35}$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{\ell,1}$, $\int_{\mathbb{S}^1} f\, dm = 0$, where $\rho(\varepsilon)$ is given by Lemma 27.*

*Proof.* Let us fix an arbitrary $\varepsilon \in I$. We note that the existence of the limit in (35) follows from Theorem 4. Moreover, without any loss of generality, we may assume that

$$-\log \rho(\varepsilon) - \int_\Omega \log \underline{\lambda}(\omega)\, d\mathbb{P}(\omega) < 0. \tag{36}$$

We proceed by induction on $\ell$. For $\ell = 1$, the desired conclusion follows from Lemma 27. Assume now that the conclusion holds for some $2 \le \ell \le r-2$ and take $f \in W^{\ell+1,1}$, $\int_{\mathbb{S}^1} f\, dm = 0$. By our induction hypothesis,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n_{\omega,\varepsilon} f\|_{W^{\ell,1}} \le \log(\rho(\varepsilon)), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{37}$$

We consider the cocycle $(\bar{\mathcal{L}}_{\omega,\varepsilon})_{\omega\in\Omega}$ over $\sigma$, where $\bar{\mathcal{L}}_{\omega,\varepsilon} = \frac{1}{\rho(\varepsilon)}\mathcal{L}_{\omega,\varepsilon}$. Observe that (37) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\bar{\mathcal{L}}^n_{\omega,\varepsilon} f\|_{W^{\ell,1}} \le 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{38}$$

Moreover, (33) and (34) give that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $h \in W^{\ell+1,1}$,

$$\|\bar{\mathcal{L}}_{\omega,\varepsilon} h\|_{W^{\ell+1,1}} \le \frac{1}{\rho(\varepsilon)} C_{\ell+1}(\omega) \|f\|_{W^{\ell+1,1}} \tag{39}$$

and

$$\|\bar{\mathcal{L}}_{\omega,\varepsilon} h\|_{W^{\ell+1,1}} \le \frac{1}{\rho(\varepsilon)\underline{\lambda}(\omega)^{\ell+1}} \|h\|_{W^{\ell+1,1}} + \frac{1}{\rho(\varepsilon)} B_{\ell+1}(\omega)\|h\|_{W^{\ell,1}}. \tag{40}$$

Furthermore, (36) implies that

$$\int_\Omega \log\left(\frac{1}{\rho(\varepsilon)\underline{\lambda}(\omega)^{\ell+1}}\right) d\mathbb{P}(\omega) < 0. \tag{41}$$

Since the inclusion $W^{\ell+1,1} \hookrightarrow W^{\ell,1}$ is compact, it follows from (38), (39), (40), (41) and [28, Lemma C.5] that

$$\lim_{n\to\infty} \frac{1}{n} \log \|\bar{\mathcal{L}}^n_{\omega,\varepsilon} f\|_{W^{\ell+1,1}} \le 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hence,

$$\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}^n_{\omega,\varepsilon} f\|_{W^{\ell+1,1}} \le \log \rho(\varepsilon), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The proof of the proposition is completed. □

Let $Y_1^{\ell,\varepsilon}(\omega) \subset W^{\ell,1}$, $\omega \in \Omega$ denote the Oseledets subspace corresponding to the largest Lyapunov exponent (which is zero) of the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ on $W^{\ell,1}$.

**Proposition 29.** *We have that* $\dim Y_1^{\ell,\varepsilon}(\omega) = 1$ *and* $Y_1^{\ell,\varepsilon}(\omega)$ *is spanned by* $h_{\omega,\varepsilon}$*, for* $\varepsilon \in I$*,* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and* $1 \le \ell \le r-1$*.*

*Proof.* Take an arbitrary $\varepsilon \in I$. For $\ell = 1$, the desired conclusions follow directly from Lemma 27. Assume now that $\ell > 1$. By arguing as in the proof of [21, Lemma 2.10], we find that there exists $h^\ell_{\omega,\varepsilon} \in Y_1^{\ell,\varepsilon}(\omega) \subset W^{\ell,1}$ satisfying $\int_{\mathbb{S}^1} h^\ell_{\omega,\varepsilon}\, dm = 1$, $h^\ell_{\omega,\varepsilon} \ge 0$ and such that $\mathcal{L}_{\omega,\varepsilon} h^\ell_{\omega,\varepsilon} = h^\ell_{\sigma\omega,\varepsilon}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Moreover, $\omega \mapsto h^\ell_{\omega,\varepsilon}$ is measurable. Since $W^{\ell,1} \subset W^{1,1}$, it follows from the uniqueness in Lemma 27 that $h^\ell_{\omega,\varepsilon} = h_{\omega,\varepsilon}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Finally, the fact that $h_{\omega,\varepsilon}$ spans $Y_1^{\ell,\varepsilon}$ follows from Lemma 28. □

Remark 30. From the previous observations, we find that for each $\varepsilon \in I$ and $1 \le \ell \le r-1$, the Oseledets splitting of the cocycle $(\mathcal{L}_{\omega,\varepsilon})_{\omega\in\Omega}$ on $W^{\ell,1}$ is given by

$$W^{\ell,1} = \operatorname{span}\{h_{\omega,\varepsilon}\} \oplus W_0^{\ell,1},$$

for $\mathbb{P}$-a.e $\omega \in \Omega$, where

$$W_0^{\ell,1} := \left\{ h \in W^{\ell,1} : \int_{\mathbb{S}^1} h\, dm = 0 \right\}.$$

By taking into account Lemma 28 and applying [2, Proposition 3.2], we obtain the following result.

**Proposition 31.** *For $\varepsilon \in I$, there exist $\lambda(\varepsilon) > 0$ and $K^\varepsilon \in \mathcal{K}$ such that*

$$\|\mathcal{L}^n_{\omega,\varepsilon} f\|_{W^{\ell,1}} \le K^\varepsilon(\omega) e^{-\lambda(\varepsilon)n} \|f\|_{W^{\ell,1}},$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$, $f \in W_0^{\ell,1}$, $n \ge 0$ and $1 \le \ell \le r-1$.*

**Proposition 32.** *For each $1 \le \ell \le r-1$, there exists $D_\ell \in \mathcal{K}$ such that*

$$\|\mathcal{L}^n_{\sigma^{-n}\omega} f\|_{W^{\ell,1}} \le D_\ell(\omega) \|f\|_{W^{\ell,1}}, \tag{42}$$

*for $\mathbb{P}$-a.e. $\omega \in \Omega$, $n \in \mathbb{N}$ and $f \in W^{\ell,1}$.*

*Proof.* Writing $K$ instead of $K^0$ and $\lambda$ instead of $\lambda(0)$, it follows from Propositions 6 and 31 that

$$\begin{aligned}
\|\mathcal{L}^n_{\sigma^{-n}\omega} f\|_{W^{\ell,1}} &= \left\| \mathcal{L}^n_{\sigma^{-n}\omega}\left( f - \left(\int_{\mathbb{S}^1} f\, dm\right) h_{\sigma^{-n}\omega,0} + \left(\int_{\mathbb{S}^1} f\, dm\right) h_{\sigma^{-n}\omega,0}\right)\right\|_{W^{\ell,1}} \\
&\le K(\sigma^{-n}\omega) e^{-\lambda n} \left\| f - \left(\int_{\mathbb{S}^1} f\, dm\right) h_{\sigma^{-n}\omega,0}\right\|_{W^{\ell,1}} + \|f\|_{W^{\ell,1}} \cdot \|h_{\omega,0}\|_{W^{\ell,1}} \\
&\le K_{\lambda/2}(\omega) e^{-\frac{\lambda}{2}n} (1 + \|h_{\sigma^{-n}\omega,0}\|_{W^{\ell,1}}) \|f\|_{W^{\ell,1}} + \|f\|_{W^{\ell,1}} \cdot \|h_{\omega,0}\|_{W^{\ell,1}}.
\end{aligned}$$

Since $\omega \mapsto \|h_{\omega,0}\|_{W^{\ell,1}}$ is tempered, by Proposition 6 there exists $Q \in \mathcal{K}$ such that

$$\|h_{\omega,0}\|_{W^{\ell,1}} \le Q(\omega), \quad Q(\sigma^n\omega) \le e^{\frac{\lambda}{2}|n|} Q(\omega),$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$. We conclude that (42) holds with

$$D_\ell(\omega) = K_{\lambda/2}(\omega) + K_{\lambda/2}(\omega) Q(\omega) + Q(\omega).$$

□

**Lemma 33.** *For each $1 \le \ell \le r-1$, there exists $P_\ell \colon \Omega \to (0,\infty)$ measurable such that*

$$\|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} 1\|_{W^{\ell,1}} \le P_\ell(\omega), \tag{43}$$

*for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$.*

*Proof.* Fix an arbitrary $1 \le \ell \le r-1$. It follows from Lemma 23 and the proof of [28, Lemma C.5] that there exist $\alpha \colon \Omega \to (0,\infty)$ measurable and $D > 0$[2] such that $\int_\Omega \log \alpha \, d\mathbb{P} < 0$ and

$$\|\mathcal{L}_{\omega,\varepsilon} h\|_{W^{\ell,1}} \le \alpha(\omega) \|h\|_{W^{\ell,1}} + D\|h\|_{W^{\ell-1,1}}, \tag{44}$$

for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $h \in W^{\ell,1}$. By iterating (44), we have that

$$\|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon} 1\|_{W^{\ell,1}} \le D\left(1 + \sum_{j=1}^n \alpha^{(j)}(\sigma^{-j}\omega)\right), \tag{45}$$

[2]We drop the $\ell$ dependency in both $\alpha, D$ to alleviate the notation.

for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$, where

$$\alpha^{(j)}(\sigma^{-j}\omega) := \alpha(\sigma^{-1}\omega)\cdots\alpha(\sigma^{-j}\omega).$$

Note that

$$\lim_{j\to\infty}\frac{1}{j}\log\alpha^{(j)}(\sigma^{-j}\omega) = \lim_{j\to\infty}\frac{1}{j}\sum_{t=1}^{j}\log\alpha(\sigma^{-t}\omega) = \int_\Omega \log\alpha\, d\mathbb{P} < 0, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

Hence, there exists $\delta > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\alpha^{(j)}(\sigma^{-j}\omega) \le e^{-\delta j}$ for $j \ge j(\omega)$. Set

$$\bar{C}(\omega) = \sup_j(\alpha^{(j)}(\sigma^{-j}\omega)e^{\delta j}), \quad \omega \in \Omega.$$

Now it follows readily from (45) that (43) holds with

$$P_\ell(\omega) := D\bigg(1 + \bar{C}(\omega)\frac{e^{-\delta}}{1-e^{-\delta}}\bigg), \quad \omega \in \Omega.$$

□

**Lemma 34.** *For each $1 \le \ell \le r-1$, we have that*

$$\|h_{\omega,\varepsilon}\|_{W^{\ell,1}} \le P_\ell(\omega),$$

*for $\varepsilon \in I$ and $\mathbb{P}$-a.e. $\omega \in \Omega$, where $P_\ell$ is given by Lemma 33.*

*Proof.* Take an arbitrary $\varepsilon \in I$. We claim that

$$h_{\omega,\varepsilon} = \lim_{n\to\infty}\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon}1, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{46}$$

Once we establish (46), the conclusion of the lemma follows directly from Lemma 33. Writing $K$ and $\lambda$ instead of $K^\varepsilon$ and $\lambda(\varepsilon)$ respectively, it follows from Proposition 31 that

$$\begin{aligned}\|h_{\omega,\varepsilon} - \mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon}1\|_{W^{\ell,1}} &= \|\mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon}(h_{\sigma^{-n}\omega,\varepsilon} - 1)\|_{W^{\ell,1}} \\ &\le K(\sigma^{-n}\omega)e^{-\lambda n}\|h_{\sigma^{-n}\omega,\varepsilon} - 1\|_{W^{\ell,1}} \\ &\le K_{\lambda/2}(\omega)e^{-\frac{\lambda}{2}n}\|h_{\sigma^{-n}\omega,\varepsilon} - 1\|_{W^{\ell,1}},\end{aligned} \tag{47}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$.

On the other hand, since $\omega \mapsto \|h_{\omega,\varepsilon}\|_{W^{\ell,1}}$ is tempered, there exists $C \in \mathcal{K}$ (depending on $\varepsilon$) such that

$$\|h_{\omega,\varepsilon} - 1\|_{W^{\ell,1}} \le C(\omega) \quad \text{and} \quad C(\sigma^n\omega) \le C(\omega)e^{\frac{\lambda}{4}|n|}, \tag{48}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$. By (47) and (48), we have that

$$\|h_{\omega,\varepsilon} - \mathcal{L}^n_{\sigma^{-n}\omega,\varepsilon}1\|_{W^{\ell,1}} \le C(\omega)K_{\lambda/2}(\omega)e^{-\frac{\lambda}{4}n}, \tag{49}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{N}$. Letting $n \to \infty$ in (49) yields (46). The proof of the lemma is completed. □

**Proposition 35.** *Take an arbitrary $1 \leq \ell \leq r-1$. Then, for each $s > 0$ there exists a measurable $C\colon \Omega \to (0, \infty)$ such that*

$$\|h_{\sigma^n\omega,\varepsilon}\|_{W^{\ell,1}} \leq C(\omega)e^{s|n|}, \tag{50}$$

*for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $n \in \mathbb{Z}$.*

*Proof.* We follow closely the proof of [17, Lemma 4.2]. Since $C_\ell$ in (33) is log-integrable, we have that $\sigma_0 := \int_\Omega \log C_\ell(\omega)\, d\mathbb{P}(\omega)$ is finite. Without any loss of generality, we may assume that $\frac{s}{2(\sigma_0+1)} < 1$. By applying Birkhoff's ergodic theorem, we have that there exists $\Omega_2 \subset \Omega$, $\mathbb{P}(\Omega_2) > 1 - \frac{s}{4(\sigma_0+1)}$ such that for $\omega \in \Omega_2$, sufficiently large $n$ and each choice of the sign $\pm$ we have that

$$\sum_{0 \leq j < n} \log C_\ell(\sigma^{\pm j}\omega) \leq (\sigma_0 + 1)n.$$

Let $P_\ell$ be given by Lemma 33. Note that there exists $A > 1$ such that $\mathbb{P}(\{\omega \in \Omega : P_\ell(\omega) > A\}) < \frac{s}{4(\sigma_0+1)}$. By applying Birkhoff's ergodic theorem once again, we have that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and each choice of $\pm$,

$$\lim_{k\to\pm\infty} \frac{1}{|k|} \#\{0 \leq j < |k| : P_\ell(\omega) > A \text{ or } \sigma^{\pm j}\omega \notin \Omega_2\} < \frac{s}{2(\sigma_0+1)} < 1.$$

By using (33), Lemma 34 and arguing exactly as in the proof of [17, Lemma 4.2], we find that there exists $N_0\colon \Omega \to \mathbb{N}$ such that for each $\varepsilon \in I$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\|h_{\sigma^n\omega,\varepsilon}\|_{W^{\ell,1}} \leq Ae^{s|n|} \quad \text{for } n \in \mathbb{Z},\ |n| \geq N_0(\omega).$$

By writing $h_{\sigma^n\omega,\varepsilon} = \mathcal{L}^{n+N_0(\omega)}_{\sigma^{-N_0(\omega)}\omega} h_{\sigma^{-N_0(\omega)}\omega,\varepsilon}$ for $|n| < N_0(\omega)$ and invoking (33) again, we conclude that (50) holds with

$$C(\omega) = Ae^{2sN_0(\omega)} \prod_{i=-N_0(\omega)}^{N_0(\omega)-2} \max\{C_\ell(\sigma^i\omega), 1\}.$$

The proof of the proposition is completed. □

As a direct consequence of the previous result, we obtain the following:

**Corollary 36.** *Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be an arbitrary sequence in $I$. Then, for each $1 \leq \ell \leq r-1$, $\omega \mapsto \sup_k \|h_{\omega,\varepsilon_k}\|_{W^{\ell,1}}$ is a tempered random variable.*

### 4.3 Estimates in the triple norm

Let $\mathbf{T}\colon \Omega \to C^r(I \times \mathbb{S}^1, \mathbb{S}^1)$ be an arbitrary parameterized smooth expanding on average cocycle. We continue to use the same notation as in the previous subsection. In addition, we will write $T_\omega$ and $\mathcal{L}_\omega$ instead of $T_{\omega,0}$ and $\mathcal{L}_{\omega,0}$ respectively.

**Lemma 37.** *There exists $Q_1 \in \mathcal{K}$ such that*

$$\|(\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega)f\|_{L^1} \leq Q_1(\omega)d_{C^1}(T_{\omega,\varepsilon}, T_\omega)\|f\|_{W^{1,1}},$$

*for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{1,1}$.*

*Proof.* For every $f \in W^{1,1}$, $\varphi \in C^1(\mathbb{S}^1)$, $\varepsilon \in I$ and $\omega \in \Omega$, we have that

$$\int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} f - \mathcal{L}_\omega f)\varphi \, dm = \int_{\mathbb{S}^1} f(\varphi \circ T_{\omega,\varepsilon} - \varphi \circ T_\omega)\, dm.$$

Set $\Phi_{\omega,\varepsilon}(x) := \frac{1}{T_\omega'(x)} \int_{T_\omega x}^{T_{\omega,\varepsilon} x} \varphi(z)\, dm(z)$. Then,

$$\Phi_{\omega,\varepsilon}'(x) = -\frac{1}{T_\omega'(x)} T_\omega''(x)\Phi_{\omega,\varepsilon}(x) + \frac{T_{\omega,\varepsilon}'(x)}{T_\omega'(x)}\varphi(T_{\omega,\varepsilon}x) - \varphi(T_\omega x).$$

Hence substituting for $\varphi(T_\omega x)$ and integrating by parts,

$$\begin{aligned}\int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} f - \mathcal{L}_\omega f)\varphi \, dm &= \int_{\mathbb{S}^1} f\Phi_{\omega,\varepsilon}' \, dm + \int_{\mathbb{S}^1} f\left[\frac{T_\omega''}{T_\omega'}\Phi_{\omega,\varepsilon} + \left(1 - \frac{T_{\omega,\varepsilon}'}{T_\omega'}\right)\varphi \circ T_{\omega,\varepsilon}\right] dm \\ &= -\int_{\mathbb{S}^1} f'\Phi_{\omega,\varepsilon}\, dm + \int_{\mathbb{S}^1} f\left[\frac{T_\omega''}{T_\omega'}\Phi_{\omega,\varepsilon} + \left(1 - \frac{T_{\omega,\varepsilon}'}{T_\omega'}\right)\varphi \circ T_{\omega,\varepsilon}\right] dm.\end{aligned}$$

Observe that

$$|\Phi_{\omega,\varepsilon}|_\infty \le \underline{\lambda}(\omega)^{-1}|\varphi|_\infty d_{C^1}(T_{\omega,\varepsilon}, T_\omega) \quad \text{and} \quad \left|1 - \frac{T_{\omega,\varepsilon}'}{T_\omega'}\right|_\infty \le \underline{\lambda}(\omega)^{-1} d_{C^1}(T_{\omega,\varepsilon}, T_\omega).$$

Moreover, (29) implies that

$$\left|\frac{T_\omega''}{T_\omega'}\Phi_{\omega,\varepsilon}\right|_\infty \le \underline{\lambda}(\omega)^{-1}K(\omega)|\Phi_{\omega,\varepsilon}|_\infty \le \underline{\lambda}(\omega)^{-2}K(\omega) d_{C^1}(T_{\omega,\varepsilon}, T_\omega)|\varphi|_\infty.$$

Hence,

$$\begin{aligned}\left|\int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} f - \mathcal{L}_\omega f)\varphi \, dm\right| &\le \underline{\lambda}(\omega)^{-1}\|f'\|_{L^1}|\varphi|_\infty d_{C^1}(T_{\omega,\varepsilon}, T_\omega) \\ &\quad + \underline{\lambda}(\omega)^{-2}K(\omega)\|f\|_{L^1} d_{C^1}(T_{\omega,\varepsilon}, T_\omega)|\varphi|_\infty \\ &\quad + \|f\|_{L^1}|\varphi|_\infty \underline{\lambda}(\omega)^{-1} d_{C^1}(T_{\omega,\varepsilon}, T_\omega),\end{aligned}$$

which implies that the desired conclusion holds with

$$Q_1(\omega) = \underline{\lambda}(\omega)^{-1} + \underline{\lambda}(\omega)^{-2}K(\omega),$$

which due to log-integrability of $\underline{\lambda}$ and $K$ belongs to $\mathcal{K}$. □

The proof of the following lemma is inspired by the proof of [18, Proposition 5.11].

**Lemma 38.** *For each $\ell \in \mathbb{N}$ such that $2 \le \ell$ and $\ell + 1 \le r$, there exists $Q_\ell \in \mathcal{K}$ such that*

$$\|(\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega)f\|_{W^{\ell-1,1}} \le Q_\ell(\omega)\|f\|_{W^{\ell,1}} d_{C^\ell}(T_{\omega,\varepsilon}, T_\omega),$$

*for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{\ell,1}$.*

*Proof.* Throughout the proof, $C$ will denote a generic element of $\mathcal{K}$ (depending on $\ell$ but not on $\varepsilon$) that can change values from one occurrence to the next.

By Lemma 21, we have that for $g \in L^\infty(\mathbb{S}^1)$,

$$
\begin{aligned}
&\int_{\mathbb{S}^1} ((\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega) f)^{(\ell-1)} g \, dm \\
&= \int_{\mathbb{S}^1} \mathcal{L}_{\omega,\varepsilon} \left( \sum_{j=0}^{\ell-1} (T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon}) f^{(j)} \right) g \, dm \\
&\quad - \int_{\mathbb{S}^1} \mathcal{L}_\omega \left( \sum_{j=0}^{\ell-1} (T'_\omega)^{-2(\ell-1)} G_{\ell-1,j}(T'_\omega, \dots, T^{(\ell)}_\omega) f^{(j)} \right) g \, dm \\
&= \int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega) \left( \sum_{j=0}^{\ell-1} (T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon}) f^{(j)} \right) g \, dm \\
&\quad + \int_{\mathbb{S}^1} \sum_{j=0}^{\ell-1} \left( \frac{G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon})}{(T'_{\omega,\varepsilon})^{2(\ell-1)}} - \frac{G_{\ell-1,j}(T'_\omega, \dots, T^{(\ell)}_\omega)}{(T'_\omega)^{2(\ell-1)}} \right) f^{(j)} g \circ T_\omega \, dm \\
&=: (I) + (II).
\end{aligned}
$$

By Lemma 37, we have (using that $\|\psi h\|_{W^{1,1}} \le \|\psi\|_{C^1} \cdot \|h\|_{W^{1,1}}$ for $\psi \in C^1(\mathbb{S}^1)$ and $h \in W^{1,1}$) that

$$
\begin{aligned}
&\left| \int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega) \left( \sum_{j=0}^{\ell-1} (T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon}) f^{(j)} \right) g \, dm \right| \\
&\le C(\omega) d_{C^1}(T_{\omega,\varepsilon}, T_\omega) |g|_\infty \left\| \sum_{j=0}^{\ell-1} (T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon}) f^{(j)} \right\|_{W^{1,1}} \\
&\le C(\omega) d_{C^1}(T_{\omega,\varepsilon}, T_\omega) |g|_\infty \|f\|_{W^{\ell,1}} \sum_{j=0}^{\ell-1} \|(T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon})\|_{C^1} \\
&\le C(\omega) d_{C^1}(T_{\omega,\varepsilon}, T_\omega) |g|_\infty \|f\|_{W^{\ell,1}} \sum_{j=0}^{\ell-1} \|(T'_{\omega,\varepsilon})^{-2(\ell-1)}\|_{C^1} \cdot \|G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon})\|_{C^1}.
\end{aligned} \tag{51}
$$

Moreover, (29) implies that

$$
\|(T'_{\omega,\varepsilon})^{-2(\ell-1)}\|_{C^1} \le \underline{\lambda}(\omega)^{-2(\ell-1)} + 2(l-1)\underline{\lambda}(\omega)^{-2\ell+1} K(\omega).
$$

By taking into account (32), we conclude that

$$
\begin{aligned}
&\left| \int_{\mathbb{S}^1} (\mathcal{L}_{\omega,\varepsilon} - \mathcal{L}_\omega) \left( \sum_{j=0}^{\ell-1} (T'_{\omega,\varepsilon})^{-2(\ell-1)} G_{\ell-1,j}(T'_{\omega,\varepsilon}, \dots, T^{(\ell)}_{\omega,\varepsilon}) f^{(j)} \right) g \, dm \right| \\
&\le C(\omega) d_{C^1}(T_{\omega,\varepsilon}, T_\omega) |g|_\infty \|f\|_{W^{\ell,1}},
\end{aligned} \tag{52}
$$

for $\varepsilon \in I$, $\mathbb{P}$-a.e. $\omega \in \Omega$ and $f \in W^{\ell,1}$.

Similarly,

$$
\begin{aligned}
&\Big|\int_{\mathbb{S}^1}\sum_{j=0}^{\ell-1}\left(\frac{G_{\ell-1,j}(T'_{\omega,\varepsilon},\dots,T^{(\ell)}_{\omega,\varepsilon})}{(T'_{\omega,\varepsilon})^{2(\ell-1)}}-\frac{G_{\ell-1,j}(T'_{\omega},\dots,T^{(\ell)}_{\omega})}{(T'_{\omega})^{2(\ell-1)}}\right)f^{(j)}g\circ T_\omega\,dm\Big| \\
&\le |g|_\infty\sum_{j=0}^{\ell-1}\left|\frac{G_{\ell-1,j}(T'_{\omega,\varepsilon},\dots,T^{(\ell)}_{\omega,\varepsilon})}{(T'_{\omega,\varepsilon})^{2(\ell-1)}}-\frac{G_{\ell-1,j}(T'_{\omega},\dots,T^{(\ell)}_{\omega})}{(T'_{\omega})^{2(\ell-1)}}\right|_\infty\cdot\|f^{(j)}\|_{L^1} \\
&\le |g|_\infty\underline{\lambda}(\omega)^{-4(l-1)}\|f\|_{W^{\ell,1}}\sum_{j=0}^{\ell-1}P_{\omega,\varepsilon,j,\ell} \\
&\le C(\omega)|g|_\infty\|f\|_{W^{\ell,1}},
\end{aligned}
\tag{53}
$$

for $\varepsilon\in I$, $\mathbb{P}$-a.e. $\omega\in\Omega$ and $f\in W^{\ell,1}$, where

$$
P_{\omega,\varepsilon,j,\ell}=\left|G_{\ell-1,j}(T'_{\omega,\varepsilon},\dots,T^{(\ell)}_{\omega,\varepsilon})(T'_\omega)^{2(\ell-1)}-G_{\ell-1,j}(T'_{\omega},\dots,T^{(\ell)}_{\omega})(T'_{\omega,\varepsilon})^{2(\ell-1)}\right|_\infty.
$$

The conclusion of the lemma follows readily from (51), (52) and (53). □

The following result is a simple consequence of (30), Lemmas 37 and 38 and the mean-value theorem.

**Corollary 39.** *For each $\ell\in\mathbb{N}$, $\ell+1\le r$ there exists $\tilde{Q}_l\in\mathcal{K}$ such that*

$$
\|(\mathcal{L}_{\omega,\varepsilon}-\mathcal{L}_\omega)f\|_{W^{\ell-1,1}}\le|\varepsilon|\tilde{Q}_\ell(\omega)\|f\|_{W^{\ell,1}},
$$

*for $\varepsilon\in I$, $\mathbb{P}$-a.e. $\omega\in\Omega$ and $f\in W^{\ell,1}$, where $\|\cdot\|_{W^{0,1}}:=\|\cdot\|_{L^1}$.*

### 4.4 Derivative operator

We again take an arbitrary parameterized smooth expanding on average cocycle $\mathbf{T}\colon\Omega\to C^r(I\times\mathbb{S}^1,\mathbb{S}^1)$. We start this section by recalling some elementary facts. For $f\in C^\ell(\mathbb{S}^1,\mathbb{R})$, we set $M_f(\phi):=f\phi$. Then, $M_f$ is a bounded operator on $W^{\ell,1}(\mathbb{S}^1,\mathbb{R})$. More precisely, there exists a constant $C$ (depending only on $\ell$) such that

$$
\|M_f(\phi)\|_{W^{\ell,1}}=\|f\phi\|_{W^{\ell,1}}\le C\|f\|_{C^\ell}\cdot\|\phi\|_{W^{\ell,1}},\quad\text{for }\phi\in W^{\ell,1}. \tag{54}
$$

For $\phi\in C^r(\mathbb{S}^1,\mathbb{R})$, set

$$
g_{\omega,\varepsilon}:=\frac{1}{|T'_{\omega,\varepsilon}|}\in C^{r-1}(\mathbb{S}^1,\mathbb{R}) \tag{55}
$$

$$
V_{\omega,\varepsilon}(\phi):=-\frac{\phi'}{T'_{\omega,\varepsilon}}\cdot\partial_\varepsilon T_{\omega,\varepsilon}\in C^{r-1}(\mathbb{S}^1,\mathbb{R}). \tag{56}
$$

Remark that $\phi\mapsto V_{\omega,\varepsilon}(\phi)$ defines a bounded operator from $W^{\ell,1}$ to $W^{\ell-1,1}$, for any $\ell\le r$.
We also define

$$
J_{\omega,\varepsilon}:=\frac{\partial_\varepsilon g_{\omega,\varepsilon}+V_{\omega,\varepsilon}(g_{\omega,\varepsilon})}{g_{\omega,\varepsilon}}\in C^{r-2}(\mathbb{S}^1,\mathbb{R}). \tag{57}
$$

A direct consequence of Definition 14, equations (29) and (30) is that there exists $\tilde{K}\in\mathcal{K}$ such that for $\varepsilon\in I$ and $\mathbb{P}$-a.e. $\omega\in\Omega$,

$$
\|g_{\omega,\varepsilon}\|_{C^{r-1}}\le\tilde{K}(\omega),\quad\|\partial_\varepsilon g_{\omega,\varepsilon}\|_{C^{r-2}}\le\tilde{K}(\omega)\quad\text{and}\quad\|J_{\omega,\varepsilon}\|_{C^{r-2}}\le\tilde{K}(\omega). \tag{58}
$$

We note that $\tilde{K}$ is a polynomial in the tempered random variable $K$ appearing in (29) and (30). As before, let $\mathcal{L}_{\omega,\varepsilon}$ be the transfer operator associated to $T_{\omega,\varepsilon} := \mathbf{T}(\omega)(\varepsilon,\cdot)$. By formal differentiation (see also [22, p.18]), we have that

$$\partial_\varepsilon[\mathcal{L}_{\omega,\varepsilon}\phi] = \mathcal{L}_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi + V_{\omega,\varepsilon}\phi)$$
$$\partial_\varepsilon^2[\mathcal{L}_{\omega,\varepsilon}\phi] = \mathcal{L}_{\omega,\varepsilon}\left(J_{\omega,\varepsilon}^2\phi + J_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + [\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi + \partial_\varepsilon[V_{\omega,\varepsilon}\phi]\right).$$

For later usage we will need the differentiation in absence of the perturbation. Namely, we define $\hat{\mathcal{L}}_\omega$ by

$$\hat{\mathcal{L}}_\omega\phi := \mathcal{L}_\omega(J_{\omega,0}\phi + V_{\omega,0}\phi). \tag{59}$$

By Lemma 23, we have, for $1 \le j \le r-3$,

$$\begin{aligned}
&\|\partial_\varepsilon^2[\mathcal{L}_{\omega,\varepsilon}\phi]\|_{W^{j,1}} \\
&= \|\mathcal{L}_{\omega,\varepsilon}\left(J_{\omega,\varepsilon}^2\phi + J_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + [\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi + \partial_\varepsilon[V_{\omega,\varepsilon}\phi]\right)\|_{W^{j,1}} \\
&\le C_j(\omega)\|J_{\omega,\varepsilon}^2\phi + J_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi) + V_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi) + [\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi + \partial_\varepsilon[V_{\omega,\varepsilon}\phi]\|_{W^{j,1}}.
\end{aligned}$$

We will now estimate the $W^{j,1}$ norm of each term appearing on the R.H.S above. By $C$ we will denote a generic constant (that doesn't depend on $\omega$, $\varepsilon$ or $\phi$). Firstly, we have (see (54) and (58))

$$\|J_{\omega,\varepsilon}^2\phi\|_{W^{j,1}} \le C\|J_{\omega,\varepsilon}\|_{C^j}\|J_{\omega,\varepsilon}\phi\|_{W^{j,1}} \le C\|J_{\omega,\varepsilon}\|_{C^j}^2\|\phi\|_{W^{j,1}} \le C\tilde{K}^2(\omega)\|\phi\|_{W^{j,1}}.$$

Secondly,

$$\begin{aligned}
\|J_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi)\|_{W^{j,1}} &\le C\|J_{\omega,\varepsilon}\|_{C^j}\cdot\|\phi' g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{W^{j,1}} \\
&\le C\|J_{\omega,\varepsilon}\|_{C^j}\cdot\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\cdot\|\phi\|_{W^{j+1,1}} \\
&\le C\tilde{K}^2(\omega)K(\omega)\|\phi\|_{W^{j+1,1}}.
\end{aligned}$$

Thirdly,

$$\begin{aligned}
\|V_{\omega,\varepsilon}(J_{\omega,\varepsilon}\phi)\|_{W^{j,1}} &\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\cdot\|(J_{\omega,\varepsilon}\phi)'\|_{W^{j,1}} \\
&\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\|J_{\omega,\varepsilon}\phi\|_{W^{j+1,1}} \\
&\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\|J_{\omega,\varepsilon}\|_{C^{j+1}}\|\phi\|_{W^{j+1,1}} \\
&\le C\tilde{K}^2(\omega)K(\omega)\|\phi\|_{W^{j+1,1}}.
\end{aligned}$$

Fourthly,

$$\begin{aligned}
\|V_{\omega,\varepsilon}(V_{\omega,\varepsilon}\phi)\|_{W^{j,1}} &\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\cdot\|(V_{\omega,\varepsilon}\phi)'\|_{W^{j,1}} \\
&\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\|V_{\omega,\varepsilon}\phi\|_{W^{j+1,1}} \\
&\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^{j+1}}\cdot\|\phi'\|_{W^{j+1,1}} \\
&\le C\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^j}\|g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}(\cdot)\|_{C^{j+1}}\cdot\|\phi\|_{W^{j+2,1}} \\
&\le C\tilde{K}^2(\omega)K^2(\omega)\|\phi\|_{W^{j+2,1}}.
\end{aligned}$$

Fifthly,

$$\|[\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi\|_{W^{j,1}} \le C\|\partial_\varepsilon J_{\omega,\varepsilon}\|_{C^j}\cdot\|\phi\|_{W^{j,1}}.$$

Noting that

$$\partial_\varepsilon J_{\omega,\varepsilon} = \partial_\varepsilon T'_{\omega,\varepsilon}\Big(\partial_\varepsilon g_{\omega,\varepsilon} - g'_{\omega,\varepsilon} g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}\Big) + T'_{\omega,\varepsilon}\Big(\partial^2_\varepsilon g_{\omega,\varepsilon} - \partial_\varepsilon g'_{\omega,\varepsilon} g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon} - g'_{\omega,\varepsilon}\partial_\varepsilon g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon} - g'_{\omega,\varepsilon} g_{\omega,\varepsilon}\partial^2_\varepsilon T_{\omega,\varepsilon}\Big),$$

we obtain that

$$\|[\partial_\varepsilon J_{\omega,\varepsilon}]\cdot\phi\|_{W^{j,1}} \le C\bar{K}(\omega)\|\phi\|_{W^{j,1}},$$

where $\bar{K}$ is a tempered random variable, which is a polynomial in the tempered random variable $K$ appearing in (29) and (30).
Finally,

$$\begin{aligned}\|\partial_\varepsilon[V_{\omega,\varepsilon}\phi]\|_{W^{j,1}} &= \|\phi'\partial_\varepsilon g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon} + \phi' g_{\omega,\varepsilon}\partial^2_\varepsilon T_{\omega,\varepsilon}\|_{W^{j,1}} \\ &\le C\left(\|\partial_\varepsilon g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}\|_{C^j} + \|g_{\omega,\varepsilon}\partial^2_\varepsilon T_{\omega,\varepsilon}\|_{C^j}\right)\|\phi'\|_{W^{j,1}} \\ &\le C\left(\|\partial_\varepsilon g_{\omega,\varepsilon}\partial_\varepsilon T_{\omega,\varepsilon}\|_{C^j} + \|g_{\omega,\varepsilon}\partial^2_\varepsilon T_{\omega,\varepsilon}\|_{C^j}\right)\|\phi\|_{W^{j+1,1}} \\ &\le 2C\tilde{K}(\omega)K(\omega)\|\phi\|_{W^{j+1,1}}.\end{aligned}$$

Putting the last six estimates together, we conclude that for each $1 \le j \le r-3$, there exists $\tilde{C}_j \in \mathcal{K}$ such that

$$\|\partial^2_\varepsilon[\mathcal{L}_{\omega,\varepsilon}\phi]\|_{W^{j,1}} \le \tilde{C}_j(\omega)\|\phi\|_{W^{j+2,1}}.$$

We can now proceed as in [22, p. 18]: by Taylor's formula of order two, we have that

$$\begin{aligned}\|\mathcal{L}_{\omega,\varepsilon}\phi - \mathcal{L}_\omega\phi - \varepsilon\hat{\mathcal{L}}_\omega\phi\|_{W^{j,1}} &= \left\|\int_0^\varepsilon\int_0^\eta \partial^2_\varepsilon[\mathcal{L}_{\omega,\varepsilon}\phi]|_{\varepsilon=\xi}\,d\xi\,d\eta\right\|_{W^{j,1}} \\ &\le \frac{\varepsilon^2}{2}\tilde{C}_j(\omega)\|\phi\|_{W^{j+2,1}}.\end{aligned}$$

Hence, for $1 \le j \le r-3$, $\varepsilon \in I\setminus\{0\}$, $\phi \in W^{j+2,1}$ and $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\left\|\frac{1}{\varepsilon}\Big(\mathcal{L}_{\omega,\varepsilon}\phi - \mathcal{L}_\omega\phi\Big) - \hat{\mathcal{L}}_\omega\phi\right\|_{W^{j,1}} \le \frac{|\varepsilon|}{2}\tilde{C}_j(\omega)\|\phi\|_{W^{j+2,1}}. \tag{60}$$

### 4.5 Proof of Theorem 20

In order to establish Theorem 20, we apply Theorem 11 for the choice of spaces

$$\mathcal{B}_w = W^{1,1}, \quad \mathcal{B}_s = W^{2,1}, \quad \mathcal{B}_{ss} = W^{3,1},$$

and with the functional $\psi$ given by $\psi(h) = \int_{\mathbb{S}^1} h\,dm$. It remains to observe the following:

- (15) follows from Corollary 36;
- (16) is established in Proposition 32;
- (17) is a consequence of Proposition 31 (applied for $\varepsilon = 0$);
- (18) and (19) are proved in Corollary 39;
- (20) is established in (60), where $\hat{\mathcal{L}}_\omega$ is given by (59).

# A An example where quenched response holds and annealed response fails

Let us remind the reader that whenever quenched linear response holds (see e.g. Theorem 20), one has, for any smooth observable $\phi$, that

$$\int_{\mathbb{S}^1} \phi \hat{h}_\omega dm = \sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \phi \cdot \mathcal{L}^n_{\sigma^{-n}\omega} \hat{\mathcal{L}}_{\sigma^{-n-1}\omega} h_{\sigma^{-n-1}\omega} dm$$
$$= \sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \phi \circ T^n_{\sigma^{-n}\omega} \cdot \hat{\mathcal{L}}_{\sigma^{-n-1}\omega} h_{\sigma^{-n-1}\omega} dm.$$

In particular, the integral on the L.H.S is well-defined.
If one is interested in *annealed* linear response, the object of interest becomes the (double) integral $\int_\Omega \int_{\mathbb{S}^1} \phi \hat{h}_\omega dm \, d\mathbb{P}$: when it holds this integral is well-defined, and one has

$$\int_\Omega \int_{\mathbb{S}^1} \phi \hat{h}_\omega dm \, d\mathbb{P} = \sum_{n=0}^{\infty} \int_\Omega \int_{\mathbb{S}^1} \phi \circ T^n_\omega \cdot \hat{\mathcal{L}}_{\sigma^{-1}\omega} h_{\sigma^{-1}\omega} dm \, d\mathbb{P}.$$

Given that quenched results concerns a.e trajectory, and that annealed one are on average, in general one expects that a quenched result implies its annealed counterpart. This intuition is validated in the context of response theory by the existing results [23, 35, 19]. However, when there is no stochastic uniformity, this intuition can fail, as the next example shows:

Consider $(\tilde{\Omega}, \mathcal{B}, \mathbb{Q}, S)$ the full-shift over $\{1, 2, \dots\}$, with probability vector $(Z, Z/2^{2+\delta}, \dots, Z/n^{2+\delta}, \dots)$, for some $0 \leq \delta \leq 1$, $Z$ being the normalization constant.
Let $h : \tilde{\Omega} \to \mathbb{R}$ be the (positive) observable defined by $h(\omega) = \omega_0$ if $\omega := (\omega_n)_{n\in\mathbb{Z}} \in \tilde{\Omega}$. Note that

$$\int_{\tilde{\Omega}} h \, d\mathbb{Q} = \sum_{i\geq 1} i \cdot \frac{Z}{i^{2+\delta}} = Z \sum_{i=1}^{\infty} \frac{1}{i^{1+\delta}} < +\infty,$$

when $0 < \delta \leq 1$.
Define $(\Omega, \mathcal{F}, \mathbb{P}, \sigma)$ to be the suspension over $S$ with roof function $h$, i.e. $\Omega := \{(\omega, i) \in \tilde{\Omega} \times \mathbb{N}, \ 0 \leq i < h(\omega)\}$, $\sigma : \Omega \circlearrowleft$ is given by

$$\sigma(\omega, i) := \begin{cases} (\omega, i+1) & \text{if } i < h(\omega) - 1 \\ (S\omega, 0) & \text{if } i = h(\omega) - 1 \end{cases}$$

and $\mathbb{P}(A) := \left(\int_{\tilde{\Omega}} h \, d\mathbb{Q}\right)^{-1} \sum_{i\geq 0} \mathbb{Q}\left(A \cap (\tilde{\Omega} \times \{i\})\right)$.
We can now define our random system: take $T_0 : \mathbb{S}^1 \circlearrowleft$ to be the doubling map, i.e. $T_0(x) = 2x \pmod 1$, and let $T_1$ be the identity map on $\mathbb{S}^1$. We consider the random circle map $T_{(\omega,i)}$, $(\omega, i) \in \Omega$ defined by

$$T_{(\omega,i)} := \begin{cases} T_1 & \text{if } i < h(\omega) - 1 \\ T_0 & \text{if } i = h(\omega) - 1. \end{cases}$$

Clearly, $(T_{(\omega,i)})_{(\omega,i)\in\Omega}$ is an expanding on average cocycle, i.e.

$$\int_\Omega \log \lambda_{(\omega,i)} \, d\mathbb{P}((\omega, i)) > 0.$$

Indeed, observe that $(T_{(\omega,i)})_{(\omega,i)\in\Omega}$ is a particular case of cocycles $(T_\omega)_\omega$ introduced in Example 17. Moreover, $\mu_{(\omega,i)} = m$ for $(\omega, i) \in \Omega$, where $m$ denotes the Lebesgue measure on $\mathbb{S}^1$.

Let $n_c$ be the (random) covering time for the interval $[0, 1/2]$:

$$n_c(\omega, i) := \min\{k \in \mathbb{N} : T^k_{(\omega,i)}([0, 1/2]) = \mathbb{S}^1\}, \quad (\omega, i) \in \Omega.$$

Then, it is easy to see that $n_c(\omega, i) = h(\omega) - i$. Observe that $n_c$ is not integrable. Indeed, for each $N \in \mathbb{N}$ we have that

$$\begin{aligned}\mathbb{P}(n_c(\omega, i) = N) &= \left(\int_{\tilde{\Omega}} h \, d\mathbb{Q}\right)^{-1} \sum_{i\geq 0} \mathbb{Q}(h(\omega) - i = N) \\ &= \left(\int_{\tilde{\Omega}} h \, d\mathbb{Q}\right)^{-1} \sum_{i\geq N} \mathbb{Q}(\omega_0 = i) \\ &= \left(\int_{\tilde{\Omega}} h \, d\mathbb{Q}\right)^{-1} \sum_{i\geq N} \frac{Z}{i^{2+\delta}} \sim \frac{C}{N^{1+\delta}},\end{aligned}$$

for some constant $C > 0$, which easily implies that $n_c$ is not integrable.
We now introduce the observable: consider a $\psi \in C^\infty(\mathbb{S}^1)$, such that:

(i) supp $\psi \subset I$, where $I \subset \mathbb{S}^1$ is an interval such that $I \cap \frac{1}{2}I = \emptyset$ (in particular, we can take any small enough $I \not\ni 0$).

(ii) $\int_{\mathbb{S}^1} \psi \, dm = 0$ and $\int_{\mathbb{S}^1} \psi^2 \, dm = 1$.

It is easy to see that $\int_{\mathbb{S}^1} \psi \cdot \psi \circ T_0 \, dm = 0$. Therefore, we have

$$\int_{\mathbb{S}^1} \psi \cdot \psi \circ T^n_{(\omega,i)} \, dm = \begin{cases} 1 \text{ if } n < n_c(\omega, i) \\ 0 \text{ otherwise.} \end{cases}$$

In particular, it follows from the non-integrability of $n_c$ that

$$\int_\Omega \sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \psi \cdot \psi \circ T^n_{(\omega,i)} \, dm \, d\mathbb{P} = +\infty \tag{61}$$

Let us now introduce the perturbed cocycle $T_{(\omega,i),\varepsilon} := D_\varepsilon \circ T_{(\omega,i)}$, where $D_\varepsilon : \mathbb{S}^1 \circlearrowleft$ is given by

$$D_\varepsilon(x) := x - \varepsilon \int_0^x \psi \, dm \ (mod\ 1).$$

Denote by $S(x) := -\int_0^x \psi \, dm$. Since this perturbation is smooth and deterministic ($D_\varepsilon$ does not depend on $\omega$), we may apply Theorem 20, and obtain that for a.e. $(\omega, i) \in \Omega$, the integral $\int_{\mathbb{S}^1} \psi \hat{h}_{(\omega,i)} dm$ is well defined, and satisfies

$$\begin{aligned}\int_{\mathbb{S}^1} \psi \hat{h}_{(\omega,i)} dm &= \sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \psi \mathcal{L}^n_{\sigma^{-n}(\omega,i)} \hat{\mathcal{L}}_{\sigma^{-n-1}(\omega,i)} h_{\sigma^{-n-1}(\omega,i)} dm \\ &= -\sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \psi \circ T^n_{\sigma^{-n}(\omega,i)} S' dm \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{S}^1} \psi \circ T^n_{\sigma^{-n}(\omega,i)} \psi dm,\end{aligned}$$

where we used that $\hat{\mathcal{L}}_{(\omega,i)}f = -(\mathcal{L}_{(\omega,i)}(f) \cdot S)'$ (which can be proved by arguing as in [27, Sec 6.2]) and $h_{(\omega,i)} = \mathbb{1}$ for our particular example.

Hence, from (61)

$$\int_{\Omega} \int_{\mathbb{S}^1} \psi \hat{h}_{\omega,i} dm d\mathbb{P} = +\infty. \tag{62}$$

In particular, annealed response cannot hold.

## Acknowledgments

D.D. was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-prirod-18-9 and uniri-pr-prirod-19-16.

J.S. was supported by the European Research Council (E.R.C.) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 787304).

P.G was partially supported by the PRIN Grant 2017S35EHN "Regular and stochastic behaviour in dynamical systems" and by the INDAM - GNFM 2020 grant "Deterministic and stochastic dynamical systems for climate studies".

## References

[1] L. Arnold, *Random dynamical systems*, Springer Monogr. Math., Springer, Berlin, 1998.

[2] L. Backes and D. Dragičević, *Periodic approximation of exceptional Lyapunov exponents for semi-invertible operator cocycles*, Ann. Acad. Sci. Fenn. Math. **44** (2019), 183–209.

[3] W. Bahsoun and B. Saussol, *Linear response in the intermittent family: differentiation in a weighted $C^0$-norm*, Discrete Contin. Dyn. Syst. **36** (2016), 6657–6668.

[4] W. Bahsoun, M. Ruziboev and B. Saussol, *Linear response for random dynamical systems*, Adv. Math. **364** (2020), 107011, 44pp.

[5] V.Baladi, *Correlation spectrum of quenched and annealed equilibrium states for random expanding maps*, Comm. Math. Phys. **186** (1997), 671–700.

[6] V. Baladi, *On the susceptibility function of piecewise expanding interval maps*, Comm. Math. Phys. **275** (2007), 839–859.

[7] V. Baladi, *Linear response, or else*, ICM Seoul. In: Proceedings, Volume III, 525–545 (2014).

[8] V. Baladi, *Dynamical zeta functions and dynamical determinants for hyperbolic maps. A functional approach*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 68. Springer, Cham, 2018. xv+291.

[9] V. Baladi, A. Kondah and B.Schmitt *Random correlations for small perturbations of expanding maps*, Random and Computational Dynamics **4** (1996), 179–204.

[10] V. Baladi and D. Smania, *Linear response formula for piecewise expanding unimodal maps*, Nonlinearity **21**, 677–711 (2008) (Corrigendum: Nonlinearity **25**, 2203–2205 (2012)).

[11] V. Baladi and D. Smania, *Linear response for smooth deformations of generic nonuniformly hyperbolic unimodal maps*, Ann. Sci. Éc. Norm. Supér. **45** (2012), 861–926.

[12] V. Baladi and M. Todd, *Linear response for intermittent maps*, Comm. Math. Phys. **347** (2016), 857–874.

[13] T. Bogenschütz, *Stochastic stability of invariant subspaces*, Ergodic Theory Dynam. Systems **20** (2000), 663–680.

[14] T.Bomfim, A. Castro and P. Varandas, *Differentiability of thermodynamical quantities in non-uniformly expanding dynamics*, Adv. Math. **292** (2016), 478–528.

[15] O. Butterley and C. Liverani, *Smooth Anosov flows: correlation spectra and stability*, J. Mod. Dyn. **1** (2007), 301–322.

[16] O. Butterley and C. Liverani, *Robustly invariant sets in fiber contracting bundle flows*, J. Mod. Dyn. **7** (2013), 255–267.

[17] J. Buzzi, *Exponential Decay of Correlations for Random Lasota–Yorke Maps*, Comm. Math. Phys. **208** (1999), 25–54.

[18] H. Crimmins, *Stability of hyperbolic Oseledets splittings for quasi-compact operator cocycles*, Discrete Contin. Dyn. Syst. **42** (2022), 2795–2857.

[19] H. Crimmins and Y. Nakano, *A spectral approach to quenched linear and higher-order response for partially hyperbolic dynamics*, arXiv e-prints (2021), arXiv:2105.11188.

[20] D. Dolgopyat, *On differentiability of SRB states for partially hyperbolic systems*, Invent. Math. **155** (2004), 389–449.

[21] D. Dragičević, G. Froyland, C. Gonzàlez-Tokman and S. Vaienti, *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Comm. Math. Phys. **360** (2018), 1121–1187.

[22] D. Dragičević and J. Sedro, *Statistical stability and linear response for random hyperbolic dynamics*, Ergodic Theory and Dynamical Systems, 1–30. doi:10.1017/etds.2021.153

[23] D. Dragičević and J. Sedro, *Quenched limit theorems for expanding on average cocycles*, arXiv:2105.00548

[24] D. Dragičević, Y. Hafouta and J. Sedro, *A vector-valued almost sure invariance principle for random expanding on average cocycles*, arXiv:2108.08714

[25] G. Froyland, C. Gonzalez-Tokman and A. Quas, *Stability and approximation of random invariant densities for Lasota-Yorke map cocycles*, Nonlinearity **27** (2014), 647–660.

[26] S. Galatolo and P. Giulietti, *A linear response for dynamical systems with additive noise*, Nonlinearity **32** (2019), 2269–2301.

[27] S. Galatolo and J. Sedro, *Quadratic response of random and deterministic dynamical systems*, Chaos **30** (2020), 023113, 15 pp.

[28] C. Gonzàlez-Tokman and A. Quas, *A semi-invertible operator Oseledets theorem*, Ergodic Theory Dynam. Systems **34** (2014), 1230–1272.

[29] S. Gouëzel and C. Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory Dynam. Systems **26** (2006), 123–151.

[30] Y. Hafouta and Yu. Kifer, *Nonconventional limit theorems and random dynamics*, World Scientific, Singapore, 2018.

[31] M. Hairer and A. Majda, *A simple framework to justify linear response theory*, Nonlinearity **23** (2010), 909–922.

[32] Y. Kifer, *Thermodynamic formalism for random transformations revisited*, Stochastics and Dynamics, Vol. 8, No. 1 (2008) 77–102.

[33] A. Korepanov, *Linear response for intermittent maps with summable and nonsummable decay of correlations*, Nonlinearity **29** (2016), 1735–1754.

[34] D. Ruelle, *Differentiation of SRB states*, Comm. Math. Phys. **187** (1997), 227–241.

[35] J. Sedro and H.H Rugh, *Regularity of characteristic exponents and linear response for transfer operator cocycles*, Comm. Math. Phys. 383, 1243–1289 (2021).

[36] J. Sedro, *A regularity result for fixed points, with applications to linear response*, Nonlinearity **31** (2018), 1417–1440.