

Non-isogenous abelian varieties sharing the same division fields

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Abstract

Two abelian varieties A_1, A_2 over a number field K are *strongly iso-Kummerian* if the torsion fields $K(A_1[d])$ and $K(A_2[d])$ coincide for all $d \geq 1$. For all $g \geq 4$ we construct geometrically simple, strongly iso-Kummerian g -dimensional abelian varieties over number fields that are not geometrically isogenous. We also discuss related examples and put significant constraints on any further iso-Kummerian pair.

1 Introduction

The present paper is motivated by the following question of Kowalski (see [11, Problem 1.2 and Remark 3.5]). Let A_1, A_2 be two abelian varieties over a number field K and suppose that

$$K(A_1[d]) = K(A_2[d]) \text{ holds for all } d \geq 1. \quad (1)$$

What can be said about A_1, A_2 ? In particular, is it true that (1) implies that A_1, A_2 have the same (geometric) isogeny factors?

A very similar question, formulated in terms of Mumford-Tate groups, was also asked by D. Zywinia during his talk at the conference ‘Abelian Varieties & Galois Actions’, held at the University of Poznań in 2017. The special case when A_1, A_2 are elliptic curves had also been considered before (see for example the *strong isogeny theorem for elliptic curves* in [9]).

Inspired by the terminology in [11], we say that two abelian varieties A_1, A_2 defined over a common number field K are *strongly iso-Kummerian* if condition (1) is satisfied. Our purpose is to provide examples of strongly iso-Kummerian abelian varieties A_1, A_2 that share no common isogeny factors (even over $\bar{K}$), and to show that in many cases such examples are explained by the presence of a CM field in the endomorphism algebra. For the first of these objectives, we show how a construction based on the theory of complex multiplication yields examples of strongly iso-Kummerian abelian varieties of almost all positive dimensions for which (1) is not explained by isogenies:

Theorem 1.1 (Section 5.1). *Let $g \geq 4$ be an integer. There exist infinitely many (geometrically non-isomorphic) pairs of strongly iso-Kummerian abelian varieties $A_1/K, A_2/K$, where K is a number field, with A_1, A_2 geometrically simple and not isogenous over $\bar{K}$, and such that $\dim(A_1) = \dim(A_2) = g$.*

The condition $g \geq 4$ is sharp: in fact, the situation in dimension $g \leq 3$ is significantly different, since all strongly iso-Kummerian pairs with A_1, A_2 of dimension at most 3 are explained by isogenies, independently of whether they admit complex multiplication or not. More precisely, we show the following:

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Theorem 1.2 (Corollary 5.4). *Let K be a number field and $A_1, A_2/K$ be strongly iso-Kummerian abelian varieties with $\dim(A_1) \leq 3, \dim(A_2) \leq 3$. Then A_1 and A_2 have the same isogeny factors (possibly with different multiplicities) over $\overline{K}$.*

We also analyse in detail a non-simple example, originally considered by Shioda [33] in a slightly different context, which gives two fairly concrete strongly iso-Kummerian abelian varieties over $\mathbb{Q}$ with different isogeny factors over $\overline{\mathbb{Q}}$ (notice that in this example $\dim A_1 = 4, \dim A_2 = 3$):

Theorem 1.3 (Theorem 5.5). *Let $J/\mathbb{Q}$ be the Jacobian of the smooth projective curve C with hyperelliptic equation $y^2 = x^9 - 1$. Then J is isogenous over $\mathbb{Q}$ to $X \times E$, where X is an absolutely simple abelian threefold with CM by $\mathbb{Q}(\zeta_9)$ and E is the CM elliptic curve $y^2 = x^3 - 1$. The abelian varieties $X \times E$ and X are strongly iso-Kummerian over $\mathbb{Q}$.*

In Section 6 we also point out how the Kuga-Satake construction may be used to obtain ‘near miss’ examples, where the two sides of the desired equality (1) differ at most by quadratic extensions (see Corollary 6.4 for a precise statement).

The problem of characterising *all* strongly iso-Kummerian pairs is of course much subtler. One would wish to understand to which extent the above examples are representative, and to investigate what restrictions strongly iso-Kummerian (and non-isogenous) abelian varieties should satisfy. We make some partial progress in this direction by recasting condition (1) in the language of Galois representations. Notice that the Chinese remainder theorem implies that it suffices to check (1) when d is a prime power: this suggests to work with the ℓ -adic representations attached to A_1, A_2 for a fixed prime ℓ , and to rephrase the question in terms of ℓ -adic monodromy groups. We carry this out in Section 3. This reformulation makes it clear that – at least conditionally on the Mumford-Tate conjecture – the problem of classifying non-trivial strongly iso-Kummerian pairs is closely related to the question of describing all pairs of non-isogenous abelian varieties A_1, A_2 for which the Hodge group of the product $A_1 \times A_2$ is isomorphic to the Hodge group of each of its factors (see Lemma 3.2 and Proposition 3.4 for details). This is a very restrictive condition, as one generically expects the Hodge group of a product to be isomorphic to the product of the Hodge groups of the factors, unless the two factors are isogenous. In particular, in [16] the author has given sufficient criteria for the equality $\mathrm{Hg}(A_1 \times A_2) \cong \mathrm{Hg}(A_1) \times \mathrm{Hg}(A_2)$ to hold: the fact that these criteria have wide applicability explains to a certain extent the difficulty of finding non-isogenous abelian varieties that satisfy (1). Based on the results of [16], we extend work of Kowalski concerning iso-Kummerian elliptic curves [11, Theorem 3.4 (4), (5)] (see also Theorem A in [9] for a related result by different authors) to show that for a large class of abelian varieties all strongly iso-Kummerian pairs are explained by isogenies:

Theorem 1.4. *Let A_1, A_2 be strongly iso-Kummerian abelian varieties over a number field K . Write $(A_1)_{\overline{K}} \sim B_1^{e_1} \times \cdots \times B_h^{e_h}$ and $(A_2)_{\overline{K}} \sim C_1^{f_1} \times \cdots \times C_k^{f_k}$ for the $\overline{K}$ -isogeny decompositions of A_1, A_2 into simple factors. Suppose that one of the following holds:*

1. *each simple isogeny factor B_i, C_j has dimension 1 or 2;*
2. *each simple isogeny factor B_i, C_j is of type I, II or III in the sense of Albert and has odd relative dimension¹ (see [16, Definition 1.3]).*

Then $h = k$, and, up to renumbering, B_i is isogenous to C_i for each $i = 1, \dots, k$.

By definition, all (geometrically) simple abelian varieties with an action of an order in a CM field are of type IV in the classification of Albert. Thus, the combination of Theorems 1.1 and 1.4 suggests that most strongly iso-Kummerian pairs not explained by isogenies are indeed connected to the presence of a CM field inside the endomorphism algebra of A_1, A_2 . We also remark that the criteria of [16], especially Theorem 1.1 in op. cit., would give further extensions of Theorem 1.4, but these are too technical to state concisely.

¹the relative dimension of an abelian variety is an integer dividing its dimension, so this condition is in particular satisfied by all abelian varieties of odd dimension (provided they are of type I, II or III in the Albert classification).

The paper is organised as follows. In Section 2 we collect some necessary preliminaries on Hodge groups and Galois representations attached to abelian varieties, and introduce some notation relevant to the theory of complex multiplication. We also prove a result that will be used in Section 6 to construct pairs of abelian varieties over number fields with a prescribed Mumford-Tate group (Proposition 2.7). In Section 3 we describe constraints satisfied by all strongly iso-Kummerian pairs over number fields, and in particular we connect condition (1) with properties of the Hodge and Mumford-Tate groups of A_1, A_2 . This allows us to prove Theorem 1.4. In Section 4 we describe the Mumford-Tate group of a product of simple CM abelian varieties in terms of linear algebra. In Section 5.1 we prove Theorem 1.1, discuss CM abelian varieties of dimension at most 3, and give an isolated example of a strongly iso-Kummerian pair over $\mathbb{Q}$. Finally, in Section 6 we describe an interesting family of pairs of non-isogenous abelian varieties that have no nontrivial endomorphisms, but still come very close to satisfying (1).

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2 Preliminaries

2.1 The Hodge group

We briefly review the notion of Hodge group of a complex abelian variety X , referring the reader to [22] for more details. The $\mathbb{Q}$ -vector space $V(X) := H_1(X(\mathbb{C}), \mathbb{Q})$ is naturally endowed with a Hodge structure of type $(-1, 0) \oplus (0, -1)$, that is, a decomposition of $\mathbb{C}$ -vector spaces $V(X) \otimes \mathbb{C} \cong V(X)^{-1,0} \oplus V(X)^{0,-1}$ such that $\overline{V(X)^{-1,0}} = V(X)^{0,-1}$. Let $\mu_\infty : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathrm{GL}_{V(X)\mathbb{C}}$ be the unique cocharacter through which $z \in \mathbb{C}^* = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$ acts as multiplication by z on $V(X)^{-1,0}$ and trivially on $V(X)^{0,-1}$. The *Mumford-Tate group* $\mathrm{MT}(X)$ of X is the $\mathbb{Q}$ -Zariski closure of the image of μ_∞ , that is to say, the smallest $\mathbb{Q}$ -algebraic subgroup $\mathrm{MT}(X)$ of $\mathrm{GL}_{V(X)}$ such that μ_∞ factors through $\mathrm{MT}(X)_\mathbb{C}$. The *Hodge group* $\mathrm{Hg}(X)$ of X is the identity component of $\mathrm{MT}(X) \cap \mathrm{SL}_{V(X)}$. The group $\mathrm{MT}(X)$ contains the torus of homotheties in $\mathrm{GL}_{V(X)}$, which implies both the equality $\dim \mathrm{MT}(X) = \dim \mathrm{Hg}(X) + 1$ and the fact that $\mathrm{Hg}(X), \mathrm{MT}(X)$ determine each other. The group $\mathrm{Hg}(X)$ is connected and reductive, and the $\mathrm{Hg}(X)$ -equivariant endomorphisms of the rational vector space $V(X)$ are canonically isomorphic to $\mathrm{End}^0(X) := \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Finally, the next proposition describes the behaviour of $\mathrm{Hg}(X)$ with respect to products:

Proposition 2.1. *1. Let X_1, X_2 be complex abelian varieties. The group $\mathrm{Hg}(X_1 \times X_2)$ is contained in $\mathrm{Hg}(X_1) \times \mathrm{Hg}(X_2)$ and projects surjectively onto each factor.*

2. Let $X_1, \dots, X_k$ be simple complex abelian varieties that are pairwise non-isogenous, and let $n_1, \dots, n_k$ be positive integers. The groups $\mathrm{Hg}(X_1^{n_1} \times \dots \times X_k^{n_k})$ and $\mathrm{Hg}(X_1 \times \dots \times X_k)$ are isomorphic.

2.2 Galois representations

Let A be an abelian variety defined over a number field K . For every prime ℓ and every $n \geq 1$ there is a natural continuous action of $\mathrm{Gal}(\overline{K}/K)$ on $A[\ell^n]$, giving rise to a representation

$$\rho_{\ell^n} : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{Aut}(A[\ell^n]);$$

the extension $K(A[\ell^n])/K$ is Galois, and its Galois group can be identified with the image G_{ℓ^n} of ρ_{ℓ^n} . Taking the inverse limit of this system of representations gives rise to the (continuous) ℓ -adic representation on the Tate module $T_\ell A$,

$$\rho_{\ell^\infty} : \mathrm{Gal}(\overline{K}/K) \rightarrow \mathrm{Aut}(T_\ell A).$$

Setting $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell$ we obtain in particular a representation $\rho_{\ell^\infty} : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}(V_\ell(A)) \cong \text{GL}_{2 \dim A}(\mathbb{Q}_\ell)$. We will call the Zariski closure of the image G_{ℓ^∞} of ρ_{ℓ^∞} the *algebraic monodromy group at ℓ* , and we will denote it by $G_\ell(A)$. As in the Hodge-theoretical situation, it is known that $G_\ell(A)$ contains the homotheties [3]. In analogy to the complex Hodge group we then define $H_\ell(A)$ as the identity component of $G_\ell(A) \cap \text{SL}_{V_\ell(A)}$.

Suppose now that we have fixed an embedding $K \hookrightarrow \mathbb{C}$, so that we can consider A as a complex abelian variety and speak of its Hodge group. The Mumford-Tate conjecture predicts that (for any choice of the embedding $K \hookrightarrow \mathbb{C}$) the group $H_\ell(A)$ should be obtained from $\text{Hg}(A)$ by base-change from $\mathbb{Q}$ to $\mathbb{Q}_\ell$ (equivalently, that $\text{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_\ell \cong G_\ell(A)^0$). Notice that this statement makes sense since by the comparison isomorphism of étale cohomology we can identify $V_\ell(A)$ with $H_1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The groups $\text{Hg}(A)$ and $H_\ell(A)$ are known to share many properties. We will only use some basic ones: the group $H_\ell(A)$ is unchanged by finite extensions of the base field K , and if A, B are K -abelian varieties that are $\overline{K}$ -isogenous we have $H_\ell(A) \cong H_\ell(B)$. The groups H_ℓ also satisfy an exact analogue of Proposition 2.1.

2.3 CM fields and CM types

A number field E is said to be a *CM field* if it is a totally imaginary quadratic extension of a totally real number field E_0 . Denote by $\Sigma_E := \text{Hom}_{\text{fields}}(E, \mathbb{C})$ the set of field embeddings of E into $\mathbb{C}$. We briefly recall the notions of CM type, of reflex type, and of reflex norm; we refer the reader to [27, §3] and [17] for further details. Let $L/\mathbb{Q}$ be the Galois closure of $E/\mathbb{Q}$, let G be the Galois group of $L/\mathbb{Q}$, and let $H < G$ be the subgroup corresponding to E . We think of G as acting on L on the right, and identify Σ_E with $H \backslash G$. Any complex conjugation determines the same automorphism of L , hence we have a well-defined (central) involution ρ of G that sends E to itself. For $g \in G$ we denote by $\bar{g} \in G$ the composition $g \circ \rho$, and similarly for $\varphi \in \Sigma_E$ we write $\bar{\varphi}$ for the composition $\varphi \circ \rho$. A *CM type* of E is a subset Φ of $\Sigma_E = H \backslash G$ such that $\Phi \cap \bar{\Phi} = \emptyset$ and $\Phi \cup \bar{\Phi} = \Sigma_E$, where $\bar{\Phi} = \{\bar{\phi} : \phi \in \Phi\}$. In other words, a CM type is the choice of a distinguished element in each pair of complex conjugate embeddings of E into $\mathbb{C}$. Given a CM type $\Phi \subset H \backslash G$ we let $\tilde{\Phi} = \{\varphi \in G : H\varphi \in \Phi\}$. The CM type Φ is called *simple* or *primitive* if $H = \{g \in G : g\tilde{\Phi} = \tilde{\Phi}\}$. The reflex group of Φ is $H' = \{g \in G : \tilde{g}\tilde{\Phi} = \tilde{\Phi}\}$ and the reflex field E^* is the fixed field of H' . The reflex CM type Φ^* of the reflex field E^* is obtained as follows: letting $\tilde{\Phi}^{-1} = \{g^{-1} : g \in \tilde{\Phi}\}$, the CM type Φ^* is the image of $\tilde{\Phi}^{-1}$ in $H' \backslash G = \Sigma_{E^*}$. The *reflex norm* corresponding to Φ is the map

$$\begin{aligned} N_{\Phi^*} : (E^*)^\times &\rightarrow E^\times \\ x &\mapsto \prod_{\phi \in \Phi^*} x^\phi. \end{aligned}$$

We will also say that two CM types Φ_1, Φ_2 for a CM field E are *essentially equal* if there exists an automorphism σ of E such that Φ_2 coincides with $\Phi_1 \circ \sigma := \{\phi \circ \sigma : \phi \in \Phi_1\}$. Otherwise we say that Φ_1, Φ_2 are *essentially different*.

Let A/K be a g -dimensional absolutely simple abelian variety, admitting complex multiplication over $\overline{K}$ by the CM field E , of degree $2g$ over $\mathbb{Q}$ (that is, we have $\text{End}_{\overline{K}}^0(A) \cong E$). We will simply say that A is a CM abelian variety, or more precisely that it has CM by E . The tangent space at the identity of $A_{\overline{K}}$ is both a $\overline{K}$ -module and an E -module, and the two actions are compatible: it follows that this tangent space is a $(E \otimes \overline{K})$ -bimodule, so it decomposes as $T_{\text{id}} A_{\overline{K}} \cong \prod_{\varphi \in \Phi} \overline{K}_\varphi$, where $\overline{K}_\varphi$ is a 1-dimensional $\overline{K}$ -vector space on which E acts through the embedding $\varphi : E \hookrightarrow \overline{K}$. The set Φ of embeddings that appear in this decomposition can be shown to be a simple CM type for E , and we say that A admits complex multiplication by the CM type (E, Φ) .

Remark 2.2. Notice that (with our conventions) the CM type of a CM abelian variety is only well-defined up to the choice of the isomorphism $E \cong \text{End}_{\overline{K}}^0(A)$; equivalently, it is only well-defined up to automorphisms of E . Two geometrically simple abelian varieties A_1, A_2 with CM by the same field E are geometrically isogenous if and only if their respective CM types are essentially equal, see [20, Corollary 3.13].

The following result is well-known, see for example [32, Theorem 3 in Section II.6.2, Proposition 26 in Section II.8.2, and Proposition 26 in Section III.12.4] or [13, Chapter I, §4 and Chapter V, Proposition 1.1]:

Theorem 2.3. *Let E be a CM field and let Φ be a simple CM type of E . There exist a number field K and a geometrically simple CM abelian variety A/K such that $\text{End}_{\overline{K}}^0(A) \cong E$ and the CM type of A is essentially equal to Φ .*

2.3.1 The reflex norm as a map on tori

For any number field E let $T_E := \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ be the corresponding algebraic torus over $\mathbb{Q}$. We have in particular $T_E(\mathbb{Q}) = E^\times$. The torus T_E can equivalently be described as the unique $\mathbb{Q}$ -torus with character group $\mathbb{Z}[\Sigma_E]$, where Σ_E is the set of embeddings of E in $\overline{\mathbb{Q}}$, endowed with its natural Galois action. In the context of CM types, one can easily check that the reflex norm $(E^*)^\times = T_{E^*}(\mathbb{Q}) \rightarrow T_E(\mathbb{Q}) = E^\times$ is induced by a map of algebraic tori $N_{\Phi^*} : T_{E^*} \rightarrow T_E$. Denoting by $\hat{T}$ the character group of the torus T , the map on character groups corresponding to N_{Φ^*} is

$$\begin{aligned} \hat{T}_E &\rightarrow \hat{T}_{E^*} \\ [\varphi] &\mapsto \sum_{\psi \in \Phi^*} [\psi\varphi], \end{aligned} \tag{2}$$

where $[\varphi]$ denotes the character of T_E corresponding to the embedding $\varphi : E \rightarrow \overline{\mathbb{Q}}$. It can be checked that this map is Galois-equivariant and therefore the reflex norm (seen as a map of algebraic tori) is defined over $\mathbb{Q}$.

2.4 Galois representations in the CM case

Let A be a CM abelian variety defined over a number field K . We briefly review the well-known description of the Galois representations attached to A in the language of class field theory. Recall that the idèles I_K of a number field K are the restricted product $\prod'_{v \in \Omega_K} K_v^\times$, where Ω_K is the set of all (finite and infinite) places of K and the restricted product is taken with respect to the subgroups U_v of local units (so that $U_v = K_v^\times$ if v is infinite and $U_v = \mathcal{O}_{K_v}^\times$ if v is finite). For every number field F we denote by F_ℓ the algebra $F \otimes \mathbb{Q}_\ell$, and for an idèle $a \in I_K$ we write a_ℓ for the component of a in $K_\ell \cong \prod_{\text{char}(v)=\ell} K_v^\times$. If L/F is an extension of number fields, for each

prime number ℓ the norm map $N_{L/F} : L^\times \rightarrow F^\times$ induces local norms $N_{L/F} : L_\ell^\times \rightarrow F_\ell^\times$. The following result, which combines [32, Proposition 30 in Section II.8.5] with [31, Theorems 6, 10 and 11, Corollary 2 to Theorem 5, and Remark on page 502], gives a description of the Galois representations in the CM case. Note that the results quoted are actually more general, but for simplicity we only state the version we will need.

Theorem 2.4. *Let E be a CM field and let A be an absolutely simple CM abelian variety over a number field K with $\text{End}_{\overline{K}}(A) \cong \text{End}_K(A) \cong \mathcal{O}_E$. Let Φ be the CM type of A . The reflex field E^* of (E, Φ) is contained in K , and the following hold for each prime ℓ :*

- $T_\ell(A)$ is a free $(\mathcal{O}_E \otimes \mathbb{Z}_\ell)$ -module of rank 1;
- the ℓ -adic Galois representation ρ_{ℓ^∞} attached to A lands in $\text{Aut}_{\mathcal{O}_E \otimes \mathbb{Z}_\ell} T_\ell(A) \cong (\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times$;
- the representation $\rho_{\ell^\infty} : \text{Gal}(\overline{K}/K) \rightarrow (\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times$ factors through $\text{Gal}(\overline{K}/K)^{\text{ab}}$, so it induces a map from I_K to $(\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times$ (which we still denote by ρ_{ℓ^∞}). There is a unique continuous homomorphism $\varepsilon : I_K \rightarrow E^\times$ such that

$$\rho_{\ell^\infty}(a) = \varepsilon(a) N_{\Phi^*}(N_{K/E^*}(a_\ell))^{-1} \quad \forall a \in I_K.$$

- let v be a finite place of K . The restriction of ε to U_v has image contained in $\mu(E)$, the group of roots of unity in E , and is trivial if and only if v is a place of good reduction of A .

Combined with the Néron-Ogg-Shafarevich criterion, this also has the following well-known consequence:

Corollary 2.5. *In the setting of the previous theorem, there exists a finite extension K' of K such that $A_{K'}$ has good reduction at all finite places of K' .*

Finally we make the following observation, that is used implicitly both in [17] and in [2]:

Remark 2.6. With notation as in Theorem 2.4, let v be an infinite place of K . The restriction of the homomorphism ε to $K_v^\times$ (considered as the subgroup of idèles (a_w) with $a_w = 1$ for $w \neq v$) is trivial. Indeed, since K contains the CM field E^* , all its infinite places are complex, so that $K_v^\times \cong \mathbb{C}^\times$ is a divisible group. The image $\varepsilon(K_v^\times)$ is then a divisible subgroup of $E^\times$, where E is a number field, and is therefore trivial. The same result can also be established more directly by recalling that the connected component of $K^\times \cdot (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \subseteq I_K$ lies in the kernel of the global Artin reciprocity map (see [1, Chapter 9]).

2.5 Descending from $\mathbb{C}$ to number fields

In Section 6 we will need to construct abelian varieties defined over a number field with given ℓ -adic monodromy groups; more precisely, we will need to control their ℓ -adic monodromy in terms of corresponding Mumford-Tate groups. This motivates our interest in the following result:

Proposition 2.7. *Let A, B be complex abelian varieties. There exist abelian varieties A', B' , defined over a common number field K , such that the Mumford-Tate conjecture holds for $A', B', A' \times B'$, and such that A' (resp. B' , resp. $A' \times B'$) has the same dimension, Hodge group, and endomorphism algebra as A (resp. B , resp. $A \times B$).*

Proposition 2.7 is a minor variation of the following result, due to Serre and Noot:

Theorem 2.8 (Serre, Noot [25]). *Let $\text{MT}(X)$ be the Mumford-Tate group of an Abelian variety X over $\mathbb{C}$. Then there exists an Abelian variety Y , defined over a number field K , such that $\dim(Y) = \dim(X)$ and $\text{MT}(Y) \cong \text{MT}(X)$. Furthermore, if ℓ is a fixed rational prime, then we may choose Y/K so that the image of $\text{Gal}(\bar{K}/K)$ in $\text{Aut}(T_\ell Y)$ is open and Zariski dense in $\text{MT}(Y) \times_{\mathbb{Q}} \mathbb{Q}_\ell$; in particular, $G_\ell(Y) = \text{MT}(Y) \times_{\mathbb{Q}} \mathbb{Q}_\ell$ and the Mumford-Tate conjecture holds for Y .*

We shall also need the following fact, whose proof is immediate:

Lemma 2.9. *If the Mumford-Tate conjecture holds for a product $A \times B$, then it holds for both A and B .*

Proof of Proposition 2.7. By applying Theorem 2.8 to $X = A \times B$ and to any prime ℓ we get an abelian variety Y defined over a number field K , satisfying the Mumford-Tate conjecture, and such that $\text{MT}(Y) \cong \text{MT}(X)$ (hence $\text{Hg}(Y) \cong \text{Hg}(X)$). From the proof of [25, Theorem 1.7] we see that $V_Y := H_1(Y(\mathbb{C}), \mathbb{Q})$ and $V_X := H_1(X(\mathbb{C}), \mathbb{Q})$ are isomorphic as representations of $\text{MT}(Y) \cong \text{MT}(X)$. In particular, $\text{End}_{\mathbb{Q}}^0(Y) \cong V_Y^{\text{Hg}(Y)} \cong V_X^{\text{Hg}(X)} \cong \text{End}_{\mathbb{Q}}^0(X)$. Let $\pi_A, \pi_B \in \text{End}_{\mathbb{Q}}^0(X)$ be the projectors on the factors A, B . Under the above isomorphism, these correspond to nontrivial orthogonal idempotents $\pi_{A'}, \pi_{B'} \in \text{End}_{\mathbb{Q}}^0(Y)$, which shows that Y is isogenous to a product $A' \times B'$. Moreover, the Hodge groups of A' and B' can be described as $\pi_{A'}(\text{Hg}(Y)) \cong \pi_A(\text{Hg}(X)) = \text{Hg}(A)$, $\pi_{B'}(\text{Hg}(Y)) \cong \pi_B(\text{Hg}(X)) = \text{Hg}(B)$. Since the defining representation of the Hodge group determines the dimension and endomorphism algebra of an abelian variety, we obtain that A and A' (resp. B and B') have the same dimension and isomorphic endomorphism algebras. Finally, the previous lemma shows that the Mumford-Tate conjecture holds for both A' and B' (since they are factors of Y and Mumford-Tate holds for Y). $\square$

3 Constraints satisfied by strongly iso-Kummerian pairs

In this section we find necessary conditions that every pair of strongly iso-Kummerian abelian varieties over a number field must satisfy. This doesn't quite lead to a complete classification, but shows that non-trivial examples must be quite special. In particular, the second part of Theorem 1.4, that we will prove below, shows that no non-trivial examples may be found among abelian varieties all of whose simple factors have odd dimension, unless some of these factors admit an action by an order in a CM field (i.e., are of type IV in the Albert classification). Given the examples we construct in the next section, this result is thus in a sense the best possible.

It is useful to take the following point of view: if (1) holds, then in particular we have

$$K((A_1 \times A_2)[\ell^n]) = K(A_1[\ell^n]) = K(A_2[\ell^n])$$

for all positive integers n . We now show that this implies $\dim G_\ell(A_1 \times A_2) = \dim G_\ell(A_1) = \dim G_\ell(A_2)$. This is based on the following lemma:

Lemma 3.1. *Let A be an abelian variety defined over a number field K and let ℓ be a prime number. Let d be the dimension of $G_\ell(A)$. There exist positive constants c_1, c_2 , depending on A, ℓ and K , such that for all $n \geq 1$*

$$c_1 \ell^{nd} \leq [K(A[\ell^n]) : K] \leq c_2 \ell^{nd}.$$

Proof. Recall that G_{ℓ^n} denotes the image of the modulo- ℓ^n representation attached to A/K . By [19, Lemma 12], there exists n_0 such that for $n \geq n_0$ we have $\#G_{\ell^{n+1}} = \ell^d \#G_{\ell^n}$ for all $n \geq n_0$. Thus for sufficiently large n we have $\#G_{\ell^n} = \#G_{\ell^{n_0}} \cdot \ell^{(n-n_0)d}$. Since $[K(A[\ell^n]) : K] = \#G_{\ell^n}$, this concludes the proof. $\square$

Set $A = A_1 \times A_2$. As remarked above, the equality $K(A_1[\ell^n]) = K(A_2[\ell^n])$ implies $K(A_1[\ell^n]) = K(A_2[\ell^n]) = K(A[\ell^n])$. By the previous lemma, the degrees over K of these fields grow (up to a bounded factor) as $\ell^{n \dim G_\ell(A_1)}$, $\ell^{n \dim G_\ell(A_2)}$ and $\ell^{n \dim G_\ell(A)}$ respectively. As $n \rightarrow \infty$, this implies $\dim G_\ell(A) = \dim G_\ell(A_1) = \dim G_\ell(A_2)$ as claimed; the analogous equality also holds with G_ℓ replaced by H_ℓ . Since we also have canonical surjective maps $H_\ell(A) \rightarrow H_\ell(A_i)$ for $i = 1, 2$, the dimension condition implies that both maps $H_\ell(A) \rightarrow H_\ell(A_i)$ have finite kernel: this is then a *necessary* condition for (1) to hold. This discussion implies the following lemma (notice that if Mumford-Tate holds for A_1 and A_2 , then it also holds for $A_1 \times A_2$ [5]):

Lemma 3.2. *Let A_1, A_2 be two strongly iso-Kummerian abelian varieties over a number field K . Suppose that the Mumford-Tate conjecture holds for A_1 and A_2 . Then $\dim \text{Hg}(A_1 \times A_2) = \dim \text{Hg}(A_1) = \dim \text{Hg}(A_2)$, and the canonical projections $\text{Hg}(A_1 \times A_2) \rightarrow \text{Hg}(A_i)$, $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ are isogenies for $i = 1, 2$.*

We can now prove Theorem 1.4.

Proof of Theorem 1.4. By symmetry, it suffices to show that the simple isogeny factor B_i of $(A_1)_{\overline{K}}$ is also an isogeny factor of $(A_2)_{\overline{K}}$. Suppose by contradiction that this is not the case. Notice that over $\overline{K}$ there is a surjective homomorphism of abelian varieties $A_1 \times A_2 \rightarrow B_i \times A_2$, which gives $\dim H_\ell(A_1 \times A_2) \geq \dim H_\ell(B_i \times A_2)$. Under the assumptions of the theorem, [16, Corollaries 4.4 and 4.5] give

$$\dim H_\ell(A_1 \times A_2) \geq \dim H_\ell(B_i \times A_2) = \dim H_\ell(B_i) + \dim H_\ell(A_2) > \dim H_\ell(A_2),$$

contradicting the necessary condition we found above. Notice that we have used that the group $H_\ell(B_i)$ is positive-dimensional for any abelian variety B_i over any number field. This follows for example from the fact that if $H_\ell(B_i)$ is trivial, then Tate's conjecture on endomorphisms (proved by Faltings) implies

$$\text{End}_{\overline{K}}(B_i) \otimes \mathbb{Q}_\ell \cong \text{End}_{H_\ell(B_i)}(T_\ell B_i \otimes \mathbb{Q}_\ell) \cong \text{Mat}_{2 \dim(B_i) \times 2 \dim(B_i)}(\mathbb{Q}_\ell),$$

which is impossible for dimension reasons (here we use that K is of characteristic 0). $\square$

Remark 3.3. Let $A_1/K, A_2/K$ be abelian varieties satisfying the hypotheses of Theorem 1.4. The theorem guarantees that A_1 and A_2 have the same simple isogeny factors over $\overline{K}$, or equivalently, over a suitable finite extension of K . Generalising [11, Example 8.5], we now show that – in general – A_1 and A_2 need not have the same simple isogeny factors over K , so that replacing K with a finite extension is necessary for the conclusion to hold. Indeed, consider two abelian varieties A, B over $\mathbb{Q}$ and a non-trivial quadratic character χ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Denote by A^χ (respectively B^χ) the quadratic twist of A (resp. B) with character χ . Finally set $A_1 = A \times A^\chi \times B$ and $A_2 = A \times B^\chi \times B$. For generic choices of A and B , there are no non-trivial isogenies between A, B, A^χ, B^χ defined over $\mathbb{Q}$. The modulo- ℓ^n Galois representation attached to A_1 sends $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to $(\rho_{A, \ell^n}(\sigma), \chi(\sigma)\rho_{A, \ell^n}(\sigma), \rho_{B, \ell^n}(\sigma))$, while for A_2 we have $\rho_{A_2, \ell^n}(\sigma) = (\rho_{A, \ell^n}(\sigma), \chi(\sigma)\rho_{B, \ell^n}(\sigma), \rho_{B, \ell^n}(\sigma))$. It is easy to check that σ belongs to the kernel of ρ_{A_1, ℓ^n} if and only if $\rho_{A, \ell^n}(\sigma) = \text{Id}$, $\rho_{B, \ell^n}(\sigma) = \text{Id}$, and $\chi(\sigma) = +1$, and that the same conditions are also equivalent to σ being in the kernel of ρ_{A_2, ℓ^n} . Since the torsion fields $\mathbb{Q}(A_1[\ell^n]), \mathbb{Q}(A_2[\ell^n])$ correspond by Galois theory to the kernels of $\rho_{A_1, \ell^n}, \rho_{A_2, \ell^n}$ respectively, this shows that $\mathbb{Q}(A_1[\ell^n]) = \mathbb{Q}(A_2[\ell^n])$ holds for all primes ℓ and all positive integers n . In particular, A_1, A_2 are strongly iso-Kummerian without having the same $\mathbb{Q}$ -isogeny factors.

We have shown above that strongly iso-Kummerian abelian varieties have isogenous Mumford-Tate groups. As a partial converse, we have the following proposition:

Proposition 3.4. *Let $A_1/K_1, A_2/K_2$ be abelian varieties defined over number fields K_1, K_2 . Suppose that the canonical projections $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ are isomorphisms, for $i = 1, 2$. There exists a number field L , containing both K_1 and K_2 , such that the base-changes of A_1, A_2 to L satisfy*

$$L(A_1[\ell^\infty]) = L(A_2[\ell^\infty])$$

for all prime numbers ℓ .

Proof. Let K be any number field containing both K_1 and K_2 . For any abelian variety A over a number field K , by [7, I, Proposition 6.2] we know that the connected component of the Zariski closure of G_{A, ℓ^∞} is contained in $\text{MT}(A)_{\mathbb{Q}_\ell}$ for all ℓ . By [29, p. 17], or equivalently [14, Proposition 6.14], we also know that there is a finite extension of K for which the Zariski closure of G_{A, ℓ^∞} is connected for all ℓ . Letting L be such an extension for the abelian variety $A_1 \times A_2/K$, we obtain that for all primes ℓ the ℓ -adic Galois representations attached to $A_1 \times A_2, A_1$, and A_2 over L land in the $\mathbb{Q}_\ell$ -points of the respective Mumford-Tate groups.

Let $A := A_1 \times A_2$ and denote by $\rho_{A_1, \ell^\infty}, \rho_{A_2, \ell^\infty}$ and ρ_{A, ℓ^∞} the corresponding Galois representations. For $i = 1, 2$ we have a commutative diagram

$$\begin{array}{ccccc} \text{Gal}(\overline{L}/L) & \xrightarrow{\rho_{A, \ell^\infty}} & G_{A, \ell^\infty} & \hookrightarrow & \text{MT}(A)(\mathbb{Q}_\ell) \\ & \searrow \rho_{A_i, \ell^\infty} & \downarrow & & \downarrow \\ & & G_{A_i, \ell^\infty} & \hookrightarrow & \text{MT}(A_i)(\mathbb{Q}_\ell). \end{array}$$

By assumption, the vertical map $\text{MT}(A)(\mathbb{Q}_\ell) \rightarrow \text{MT}(A_i)(\mathbb{Q}_\ell)$ is an isomorphism. By an easy diagram chasing, this implies that $\pi_i : G_{A, \ell^\infty} \rightarrow G_{A_i, \ell^\infty}$ is injective (hence, again, an isomorphism). It follows that ρ_{A_i, ℓ^∞} and ρ_{A, ℓ^∞} have the same kernel for $i = 1, 2$, hence that $L(A_1[\ell^\infty]) = L(A[\ell^\infty]) = L(A_2[\ell^\infty])$ as claimed. $\square$

4 Mumford-Tate groups of products of CM abelian varieties

Let A_1, A_2 be geometrically simple CM abelian varieties (defined over a number field, or over $\mathbb{C}$), having complex multiplication by two CM fields E_1 and E_2 , with simple CM types Φ_1, Φ_2 respectively. Write $g_i = \dim A_i$. We now explain how to describe the Mumford-Tate group of $A_1 \times A_2$ in terms of integral linear algebra, and in particular how to check whether one of the

canonical projections $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ is an isogeny or an isomorphism (which, in the light of Lemma 3.2 and Proposition 3.4, is closely related to the problem of deciding whether A_1, A_2 are strongly iso-Kummerian over some number field). It is straightforward to extend all we are going to describe to the product of an arbitrary number of CM abelian varieties, but the notation quickly becomes cumbersome, so for simplicity of exposition we limit ourselves to the case of two factors. For a related but distinct approach see also Section 6 of [8], which however only applies to certain hyperelliptic Jacobians.

Let L_1, L_2 be the Galois closures of $E_1/\mathbb{Q}, E_2/\mathbb{Q}$ respectively, and for $i = 1, 2$ let H_i be the subgroup of $G_i = \text{Gal}(L_i/\mathbb{Q})$ given by $\text{Gal}(L_i/E_i)$. As explained in Section 2.3, the CM types Φ_1, Φ_2 can be considered as subsets of $H_i \backslash G_i$, and we obtain reflex subgroups $H'_i < G_i$, reflex fields $E_i^* \subseteq L_i$, and reflex CM types Φ_1^*, Φ_2^* . Fix a finite Galois extension L of $\mathbb{Q}$ containing both L_1 and L_2 (hence, a fortiori, both E_1^* and E_2^*) and denote by $N_i : L \rightarrow E_i^*$ the norm function. We also denote by the same letter the induced map $T_L \rightarrow T_{E_i^*}$ between the corresponding algebraic tori. Finally, let $\phi_i = N_{\Phi_i^*} : T_{E_i^*} \rightarrow T_{E_i}$ be the reflex norm corresponding to Φ_i^* . The following is a consequence of the theory of complex multiplication, and essentially a reformulation of part of Theorem 2.4:

Lemma 4.1. *The image of the map $T_L \xrightarrow{(N_1, N_2)} T_{E_1^*} \times T_{E_2^*} \xrightarrow{(\phi_1, \phi_2)} T_{E_1} \times T_{E_2}$ is $\text{MT}(A_1 \times A_2)$.*

Proof. Fix a common (number) field of definition K for A_1 and A_2 . Enlarging K if necessary, and replacing each A_i by an isogenous variety (which does not alter their Mumford-Tate groups), we may assume that K contains L , that $\text{End}_K(A_i) = \mathcal{O}_{E_i}$ for $i = 1, 2$, and that A_i has good reduction at all finite places of K . Let M be the finite extension of K corresponding to the open subgroup $K^\times \cdot \left(\prod_{v \text{ finite place of } K} U_v \times \prod_{v \text{ infinite}} K_v^\times \right)$ of I_K . Finally fix a prime ℓ for which T_M, T_L, T_{E_i} and $T_{E_i^*}$ all have good reduction (any prime unramified in M will work). For such a prime it makes sense to take the $\mathbb{Z}_\ell$ -points of the tori T_M, T_L, T_{E_i} and $T_{E_i^*}$, and $T_F(\mathbb{Z}_\ell) = (\mathcal{O}_F \otimes \mathbb{Z}_\ell)^\times = \prod_{v \text{ place of } F} U_v$ for $F = M, L, E_i, E_i^*$.

We now use Theorem 2.4 to describe the ℓ -adic Galois representations attached to A_1, A_2 over M . Notice that the characters ε_i are trivial: they are trivial on U_v for all finite v since A_i has good reduction at v , and at all the infinite places by Remark 2.6. Thus these representations are obtained as the maps on $\mathbb{Z}_\ell$ -points induced by the morphisms of algebraic tori

$$T_M \xrightarrow{N_{M/L}} T_L \xrightarrow{N_i} T_{E_i^*} \xrightarrow{\phi_i} T_{E_i}.$$

The simultaneous Galois action on $A_1 \times A_2$ is then the map on $\mathbb{Z}_\ell$ -points induced by $T_M \xrightarrow{N_{M/L}} T_L \rightarrow T_{E_1^*} \times T_{E_2^*} \rightarrow T_{E_1} \times T_{E_2}$. Given that the Mumford-Tate conjecture holds for $A_1 \times A_2$ by [26], the Mumford-Tate group of $A_1 \times A_2$ is the smallest subtorus T of $T_{E_1} \times T_{E_2}$ with the property that (for all sufficiently large number fields M and all primes ℓ) the image of the ℓ -adic Galois representation attached to $A_1 \times A_2/M$ factors via $T(\mathbb{Q}_\ell)$. Since $T_M \xrightarrow{N_{M/L}} T_L$ is surjective, it is clear from the above description that the minimal such torus is precisely the image of the map $T_L \rightarrow T_{E_1^*} \times T_{E_2^*} \rightarrow T_{E_1} \times T_{E_2}$, as claimed. $\square$

We consider the commutative diagram (well-defined by the previous lemma)

$$\begin{array}{ccccc} T_L & \xrightarrow{(\phi_1 N_1, \phi_2 N_2)} & \text{MT}(A_1 \times A_2) & \xrightarrow{\pi_i} & \text{MT}(A_i) \\ \downarrow (N_1, N_2) & & \downarrow & & \downarrow \\ T_{E_1^*} \times T_{E_2^*} & \xrightarrow{(\phi_1, \phi_2)} & T_{E_1} \times T_{E_2} & \xrightarrow{p_i} & T_{E_i} \end{array}$$

and its dual

$$\begin{array}{ccccc}
\widehat{T}_L & \xleftarrow{N_1^* \phi_1^* + N_2^* \phi_2^*} & \widehat{\text{MT}}(A_1 \times A_2) & \xleftarrow{\pi_i^*} & \widehat{\text{MT}}(A_i) \\
\uparrow N_1^* + N_2^* & & \uparrow & & \uparrow \\
\widehat{T}_{E_1}^* \times \widehat{T}_{E_2}^* & \xleftarrow{(\phi_1^*, \phi_2^*)} & \widehat{T}_{E_1} \times \widehat{T}_{E_2} & \xleftarrow{p_i^*} & \widehat{T}_{E_i}
\end{array}$$

where π_i, p_i are the canonical projections on the i -th factor, and we denote by f^* the map induced on characters by a morphism f of algebraic tori.

The condition that $\pi_i : \text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ be an isomorphism (respectively, an isogeny) translates into π_i^* being an isomorphism (respectively, having finite kernel and cokernel), and since $\pi_i : \text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ is certainly surjective (hence π_i^* is injective), it suffices to check that π_i^* is onto (resp. has finite cokernel). By diagram chasing, this is equivalent to the composition $\widehat{T}_{E_i} \rightarrow \widehat{T}_{E_1} \times \widehat{T}_{E_2} \rightarrow \widehat{\text{MT}}(A_1 \times A_2)$ being surjective (resp. having finite cokernel).

As $\widehat{\text{MT}}(A_1 \times A_2) \rightarrow \widehat{T}_L$ is injective, the kernel of the map $\widehat{T}_{E_1} \times \widehat{T}_{E_2} \rightarrow \widehat{\text{MT}}(A_1 \times A_2)$ coincides with the kernel of $\widehat{T}_{E_1} \times \widehat{T}_{E_2} \xrightarrow{N_1^* \phi_1^* + N_2^* \phi_2^*} \widehat{T}_L$. Thus we may identify $\widehat{\text{MT}}(A_1 \times A_2)$ with $\frac{\widehat{T}_{E_1} \times \widehat{T}_{E_2}}{\ker(N_1^* \phi_1^* + N_2^* \phi_2^*)}$, and under this identification π_i^* is surjective (resp. has finite cokernel) if and only if $p_i^* \left(\widehat{T}_{E_i} \right)$ – which is $\widehat{T}_{E_1} \times \{0\}$ or $\{0\} \times \widehat{T}_{E_2}$ according to whether $i = 1$ or 2 – surjects onto $\frac{\widehat{T}_{E_1} \times \widehat{T}_{E_2}}{\ker(N_1^* \phi_1^* + N_2^* \phi_2^*)}$ (resp. maps to it with finite cokernel). We have thus established:

Lemma 4.2. *The canonical projection π_1 , respectively π_2 , is an isomorphism if and only if*

$$\widehat{T}_{E_1} \times \{0\} + \ker(N_1^* \phi_1^* + N_2^* \phi_2^*) = \widehat{T}_{E_1} \times \widehat{T}_{E_2},$$

respectively $\{0\} \times \widehat{T}_{E_2} + \ker(N_1^ \phi_1^* + N_2^* \phi_2^*) = \widehat{T}_{E_1} \times \widehat{T}_{E_2}$. Similarly, the canonical projection π_1 , respectively π_2 , is an isogeny if and only if $\{0\} \times \widehat{T}_{E_2} + \ker(N_1^* \phi_1^* + N_2^* \phi_2^*)$, respectively $\widehat{T}_{E_1} \times \{0\} + \ker(N_1^* \phi_1^* + N_2^* \phi_2^*)$, has finite index in $\widehat{T}_{E_1} \times \widehat{T}_{E_2}$.*

We now rephrase this lemma in terms of matrices. Write G for the Galois group of L over $\mathbb{Q}$, and let $f_i : G \rightarrow G_i$ be the canonical projections. The character group $\widehat{T}_L$ is canonically isomorphic to $\mathbb{Z}[G] \cong \mathbb{Z}^{|G|}$, while $\widehat{T}_{E_i} \cong \mathbb{Z}[H_i \backslash G_i] \cong \mathbb{Z}^{2g_i}$ and $\widehat{T}_{E_i}^* \cong \mathbb{Z}[H_i' \backslash G_i] \cong \mathbb{Z}^{|H_i'| \backslash G|}$. In order to make Lemma 4.2 explicit, it now suffices to write down a matrix representing $N_1^* \phi_1^* + N_2^* \phi_2^*$ with respect to the bases of $\widehat{T}_{E_1} \times \widehat{T}_{E_2}$ and $\widehat{T}_L$ induced by the groups $H_i \backslash G_i$ and G . Clearly it suffices to do so for $N_1^* \phi_1^*$ and $N_2^* \phi_2^*$ separately, and this is achieved by noticing that ϕ_i^* is described by Equation (2), while

$$\begin{array}{ccc}
N_i^* : & \widehat{T}_{E_i}^* & \rightarrow \widehat{T}_L \\
& [H_i' g] & \mapsto \sum_{f_i(h) \in H_i'} [hg].
\end{array}$$

This determines the matrices M_1, M_2 representing $N_1^* \phi_1^*, N_2^* \phi_2^*$ with respect to the bases above, thus allowing Lemma 4.2 to be used algorithmically. In the important special case $E_1 = E_2 = E$ we may take $L = L_1 = L_2$, $G = G_1 = G_2$ and $H_1 = H_2 =: H$, and the composition $N_i^* \phi_i^*$ is then described by

$$\begin{array}{ccc}
N_i^* \phi_i^* : & \widehat{T}_E & \rightarrow \widehat{T}_L \\
& [Hg] & \mapsto \sum_{r \in \widehat{\Phi}_i^*} [rg].
\end{array}$$

The corresponding matrix M_i (with rows indexed by G and columns indexed by $H \backslash G$) has coefficient in position (g_1, Hg_2) given by $\begin{cases} 1, & \text{if } g_1 g_2^{-1} \in \widehat{\Phi}_i^* \\ 0, & \text{otherwise} \end{cases}$. The matrix M of $N_1^* \phi_1^* + N_2^* \phi_2^*$ has rows indexed by G and columns indexed by $H \backslash G \times \{1, 2\}$, and is obtained by concatenating the matrices of $N_1^* \phi_1^*, N_2^* \phi_2^*$ horizontally. Lemma 4.2 then takes the following completely explicit form:

Lemma 4.3. *Suppose A_1, A_2 are g -dimensional simple CM abelian varieties with complex multiplication by the same CM field E . The projection $\mathrm{MT}(A_1 \times A_2) \rightarrow \mathrm{MT}(A_1)$ is an isomorphism if and only if the $\mathbb{Z}$ -submodule of $\mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g} \cong \mathbb{Z}[H \setminus G] \oplus \mathbb{Z}[H \setminus G]$ spanned by $\ker M$ and by $\mathbb{Z}^{2g} \oplus \{0\}$ is all of $\mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g}$. Similarly, $\mathrm{MT}(A_1 \times A_2) \rightarrow \mathrm{MT}(A_1)$ is an isogeny if and only if $\ker M$ and $\mathbb{Z}^{2g} \oplus \{0\}$ span a finite-index submodule of $\mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g}$.*

As an application of this circle of ideas we show the following statement concerning CM abelian threefolds:

Proposition 4.4. *Let A_1, A_2 be simple CM abelian threefolds over $\mathbb{C}$ for which $\dim \mathrm{MT}(A_1 \times A_2) = \dim \mathrm{MT}(A_1) = \dim \mathrm{MT}(A_2)$. Then A_1 and A_2 are isogenous.*

Proof. The Mumford-Tate group of a simple abelian threefold with CM by the field E is equal to the $\mathbb{Q}$ -torus given as a functor by

$$U_E(R) = \{x \in T_E(R) : xx^* \in R^\times\}$$

for all $\mathbb{Q}$ -algebras R , see [23, §2.3]. Let L be the Galois closure of $E/\mathbb{Q}$, let $G = \mathrm{Gal}(L/\mathbb{Q})$ and write H for the subgroup of G corresponding to E . The group G has nontrivial centre of even order (complex conjugation is a central element of order 2) and is isomorphic to a subgroup of S_6 of order divisible by 6 (since it is the Galois group of the normal closure of the field E , of degree 6). Up to isomorphism, there are precisely 4 such groups: $\mathbb{Z}/6\mathbb{Z}, D_6, \mathbb{Z}/2\mathbb{Z} \times A_4$ and $\mathbb{Z}/2\mathbb{Z} \times S_4$. Notice that each of these groups has a single central involution ρ , so complex conjugation is uniquely identified by the group structure.

The rational character group $\hat{U}_E \otimes_{\mathbb{Z}} \mathbb{Q}$ can be identified with $\mathbb{Q}[H \setminus G]$, and in particular is equipped with the natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, which is nothing but the composition of the natural projection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow G$ with the permutation action of G on the coset space $H \setminus G$. Since L is the Galois closure of E , the core of H in G is trivial, hence the kernel of the permutation action is trivial. Translating this group-theoretic statement to Galois theory, we obtain that the kernel of the Galois action on $\hat{U}_E \otimes_{\mathbb{Z}} \mathbb{Q}$ defines the extension $L/\mathbb{Q}$, hence that the Mumford-Tate group determines L .

Let now E_1, E_2 be the fields of complex multiplication of A_1, A_2 respectively. The hypothesis implies that the projections $\mathrm{MT}(A_1 \times A_2) \rightarrow \mathrm{MT}(A_i)$ are isogenies, so the rational character groups

$$\widehat{\mathrm{MT}}(A_1) \otimes \mathbb{Q}, \quad \widehat{\mathrm{MT}}(A_2) \otimes \mathbb{Q}, \quad \widehat{\mathrm{MT}}(A_1 \times A_2) \otimes \mathbb{Q}$$

are all isomorphic (also as Galois modules). In particular, E_1 and E_2 have the same Galois closure $L/\mathbb{Q}$. The group $G = \mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to one of the finitely many possibilities listed above. For each such group, there are only finitely many possibilities for the subgroups H_1, H_2 (not containing ρ) corresponding to the sub-extensions $L/E_1, L/E_2$, and for each H_i there are only finitely many possible CM types for $H_i \setminus G$. We wrote a short script [18] that loops over all G in the above list, all subgroups H_1, H_2 not containing ρ and satisfying $[G : H_1] = [G : H_2] = 6$, and all pairs (Φ_1, Φ_2) of CM types of $H_1 \setminus G, H_2 \setminus G$. For each such possibility we have then determined the dimensions of $\mathrm{MT}(A_1), \mathrm{MT}(A_2)$ and $\mathrm{MT}(A_1 \times A_2)$ (this uses the identification of $\widehat{\mathrm{MT}}(A_1 \times A_2)$

with $\frac{\hat{T}_{E_1} \times \hat{T}_{E_2}}{\ker(N_1^* \phi_1^* + N_2^* \phi_2^*)}$ proved above). In all cases we have thus been able to check that the equality $\dim \mathrm{MT}(A_1 \times A_2) = \dim \mathrm{MT}(A_1) = \dim \mathrm{MT}(A_2)$ implies that H_1, H_2 are conjugated in G (hence that the fields E_1, E_2 are isomorphic), and that under the identification $E_1 \cong E_2$ induced by conjugation the CM types Φ_1, Φ_2 are essentially equal. We conclude as desired that A_1, A_2 are isogenous. $\square$

Remark 4.5. In the context of the previous proof, we know a priori that $\dim \mathrm{MT}(A_i) = 4$ by [27, Examples 3.7]. Our computations confirm this independently.

5 Examples of strongly iso-Kummerian pairs

5.1 CM abelian varieties

Fix a positive integer g not equal to 1, 2, 3, 4, 6. We describe the construction of infinitely many pairs of abelian varieties (A_1, A_2) with $\dim A_1 = \dim A_2 = g$ that satisfy (1), are absolutely simple, and geometrically non-isogenous. It is easy to see that there exist infinitely many non-isomorphic totally real number fields E_0 , Galois over $\mathbb{Q}$, with $\text{Gal}(E_0/\mathbb{Q}) \cong \mathbb{Z}/g\mathbb{Z}$ (simply consider appropriate real subfields of cyclotomic fields). We further fix a quadratic imaginary field F and let $E = E_0 F$ be a CM field. We will tacitly identify $\text{Hom}_{\text{fields}}(E, \mathbb{C})$ with $\text{Gal}(E/\mathbb{Q})$. Notice that this group can be canonically identified with the product $\text{Gal}(E_0/\mathbb{Q}) \times \text{Gal}(F/\mathbb{Q})$, and that the generator ι of the direct factor $\text{Gal}(F/\mathbb{Q})$ is complex conjugation on E . We also fix a generator ψ of the cyclic group $\text{Gal}(E/F) \cong \text{Gal}(E_0/\mathbb{Q})$ and write the $2g$ elements of $\text{Gal}(E/\mathbb{Q}) = \text{Hom}_{\text{fields}}(E, \mathbb{C})$ as $\sigma_0, \dots, \sigma_{g-1}, \overline{\sigma}_0, \dots, \overline{\sigma}_{g-1}$, where $\sigma_i = \psi^i$ and $\overline{\sigma}_i = \iota \circ \psi^i$. It will be useful to use the notation σ_i also for $i \geq g$, interpreting indices modulo g . Fix two integers r, h smaller than g and prime to g (to be chosen below; we will only consider $r = h$, but other choices would also work). We consider the two CM types Φ_1, Φ_2 defined by the conditions

$$\Phi_1^* = \{\overline{\sigma}_i : i = 0, \dots, r-1\} \cup \{\sigma_i : i = r, \dots, g-1\}$$

and

$$\Phi_2^* = \{\overline{\sigma}_{hi} : i = 0, \dots, r-1\} \cup \{\sigma_{hi} : i = r, \dots, g-1\}.$$

Notice that $\Phi \mapsto \Phi^*$ is an involution on the set of CM-types, so specifying Φ^* is equivalent to describing Φ itself. One can check easily that choosing $r = h$ to be the smallest prime that does not divide g yields CM types Φ_1, Φ_2 that are primitive and essentially different. Also notice that for these CM types the reflex field E^* coincides with E .

By Theorem 2.3 there exist abelian varieties A_1, A_2 , defined over some common number field K , such that $\text{End}_{\overline{K}}(A_i) \otimes \mathbb{Q} = E$ and the CM type of A_i is (essentially equal to) Φ_i . By enlarging K and replacing A_i with some isogenous abelian variety if necessary, we can also assume that $\text{End}_K(A_i) = \mathcal{O}_E$ (see [36, Proposition 2.5.4]) and that A_1, A_2 have good reduction at all places of K (Corollary 2.5). Since $\text{End}_{\overline{K}}(A_i) \otimes \mathbb{Q} \cong E$ is a field, both A_1 and A_2 are geometrically simple, and since their CM types are essentially different we have $\text{Hom}_{\overline{K}}(A_1, A_2) = (0)$ (Remark 2.2). We now prove that condition (1) holds for our abelian varieties A_1, A_2 over a finite extension K' of K . Notice that for each dimension g we have constructed infinitely many geometrically non-isomorphic abelian varieties: indeed, their geometric endomorphism algebras fall into infinitely many distinct isomorphism classes. Since as already pointed out it suffices to check (1) in case d is the power of a prime, the following result then proves Theorem 1.1 for $g \neq 4, 6$.

Theorem 5.1. *There is a finite extension K' of K with the following property: for every prime ℓ and every $n \in \mathbb{N} \cup \{\infty\}$ we have $K'(A_1[\ell^n]) = K'(A_2[\ell^n])$.*

Proof. Denote by ρ_1, ρ_2 the ℓ -adic Galois representations attached to A_1, A_2 . By Theorem 2.4 we may consider ρ_i as a map $I_K \rightarrow (\mathcal{O}_E \otimes \mathbb{Z}_\ell)^\times$, and we get corresponding homomorphisms $\varepsilon_1, \varepsilon_2 : I_K \rightarrow \mathcal{O}_E^\times$. The open subgroup $K^\times \cdot (\prod_{v \text{ finite}} U_v \times \prod_{v \text{ infinite}} K_v^\times)$ of I_K defines a finite extension K' of K . We restrict ρ_1, ρ_2 to $K^\times \cdot (\prod_{v \text{ finite}} U_v \times \prod_{v \text{ infinite}} K_v^\times)$ or, which is the same, to $\text{Gal}(\overline{K}/K')$. The representation ρ_i is trivial on $K^\times$ (because it factors via $\text{Gal}(\overline{K}/K)^{\text{ab}}$) and on $\prod_{v \text{ infinite}} K_v^\times$, because for an idèle $a = (a_v)$ that is nontrivial only along the infinite components we have both $a_\ell = 1$ and $\varepsilon(a) = 1$, see Theorem 2.4 and Remark 2.6. It follows that the representation $\rho_i|_{\text{Gal}(\overline{K}/K')}$ is completely determined by the restriction of ρ_i to the subgroup $H := \prod_{v \text{ finite}} U_v$.

Since the homomorphisms ε_i are trivial on U_v for each place v of good reduction, and since by construction A_i has good reduction at all finite places of K , by Theorem 2.4 we see that for $a = (a_v) \in H$ we have

$$\rho_1(a) = \prod_{\phi \in \Phi_1^*} \phi(N_{K/E}(a_\ell))^{-1} \quad \text{and} \quad \rho_2(a) = \prod_{\phi \in \Phi_2^*} \phi(N_{K/E}(a_\ell))^{-1}.$$

Let $b_\ell = N_{K/E}(a_\ell)$ and for each integer i let $\beta_i = \overline{\sigma_i(b_\ell)}/\sigma_i(b_\ell)$. We have $\psi \circ \sigma_i = \sigma_{i+1}$, and since ι commutes with ψ we also have $\psi \circ \overline{\sigma_i} = \psi \circ \iota \circ \psi^i = \iota \circ \psi^{i+1} = \overline{\sigma_{i+1}}$. In particular, $\psi(\beta_i) = \beta_{i+1}$. Then

$$\begin{aligned} \rho_2(a)^{-1} &= \prod_{\phi \in \Phi_2^*} \phi(b_\ell) = \prod_{i=0}^{r-1} \beta_{hi} \cdot \prod_{i=0}^{g-1} \sigma_i(b_\ell) \\ &= \prod_{i=0}^{r-1} \beta_{hi} \cdot \prod_{\sigma \in \text{Gal}(E/F)} \sigma(b_\ell) = \prod_{i=0}^{r-1} \beta_{ri} \cdot N_{E/F}(N_{K/E}(a_\ell)) \\ &= \prod_{i=0}^{r-1} \beta_{hi} \cdot N_{K/F}(a_\ell). \end{aligned}$$

We remark that for simplicity of notation we are now writing the action of $\text{Gal}(E/\mathbb{Q})$ on the left. This does not conflict with our previous conventions since $\text{Gal}(E/\mathbb{Q})$ is abelian.

Consider $\ker \rho_2|_H$: it consists of the idèles $a = (a_\ell) \in H$ such that $\prod_{i=0}^{r-1} \beta_{hi} = N_{K/F}(a_\ell)^{-1}$. However, since $N_{K/F}(a_\ell)^{-1} \in (F \otimes \mathbb{Q}_\ell)^\times$ is invariant under $\psi \in \text{Gal}(E/F)$, applying ψ^h to the previous equality we find

$$(a_\ell) \in \ker \rho_2|_H \iff \prod_{i=0}^{r-1} \beta_{hi} = N_{K/F}(a_\ell)^{-1} \iff \prod_{i=0}^{r-1} \beta_{hi} = \prod_{i=1}^r \beta_{hi} = N_{K/F}(a_\ell)^{-1}.$$

As β_i is invertible for all i (being the norm of an idèle), the last equivalent condition above implies $\beta_{hr} = \beta_0$. By repeatedly applying ψ^{hr} we then obtain $\beta_{hrk} = \beta_0$ for all integers k . As hr is assumed to be prime with g , this implies that all β_i are equal, and since $\beta_i = \psi^i(\beta_0)$ this condition is also equivalent to $\psi(\beta_0) = \beta_0$. We have thus proved that an idèle $(a_\ell) \in H$ is in $\ker \rho_2|_H$ if and only if the two conditions $\psi(\beta_0) = \beta_0$ and $\beta_0^r = N_{K/F}(a_\ell)^{-1}$ hold. Exactly the same argument applies to $\rho_1|_H$ (it is simply the case $h = 1$ of the above), which shows that $\rho_1|_H$ and $\rho_2|_H$ have the same kernel. We have then showed that the kernels of the ℓ -adic representations attached to A_1, A_2 over K' are the same, hence that $K'(A_1[\ell^\infty]) = K'(A_2[\ell^\infty])$. Finally, the same argument works verbatim for modulo- ℓ^n representations (simply replace equalities with congruences modulo ℓ^n), so the equality $K'(A_1[\ell^n]) = K'(A_2[\ell^n])$ holds for all primes ℓ and all positive integers n . $\square$

5.1.1 Small dimensions

We now analyse the remaining cases $g = 1, 2, 3, 4, 6$. Theorem 1.4 completely settles the case when A_1, A_2 have dimension at most 2 (independently of whether they have complex multiplication or not). Examples for $g = 4, 6$ (the cases missing to complete the proof of Theorem 1.1) can be obtained as follows. Let $E/\mathbb{Q}$ be a CM field having Galois group isomorphic to $\mathbb{Z}/2g\mathbb{Z}$ (with generator σ), and consider the CM types

$$\begin{aligned} \Phi_1 &= \{\sigma^2, \sigma^3, \sigma^4, \sigma^5\}, \quad \Phi_2 = \{\sigma^0, \sigma^2, \sigma^5, \sigma^7\} \quad \text{if } g = 4 \\ \Phi_1 &= \{\sigma^0, \sigma, \sigma^2, \sigma^9, \sigma^{10}, \sigma^{11}\}, \quad \Phi_2 = \{\sigma^0, \sigma^2, \sigma^5, \sigma^7, \sigma^9, \sigma^{10}\} \quad \text{if } g = 6. \end{aligned}$$

Let moreover A_1, A_2 be CM abelian varieties (defined over a suitable number field) with CM by E and by the CM types Φ_1, Φ_2 respectively.

Remark 5.2. The above CM types have been found by looping through all possible CM types on $\mathbb{Z}/2g\mathbb{Z}$ and using the algorithm of Lemma 4.3 to check when the projections $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ are isomorphisms, see [18].

We proceed as in the proof of Theorem 5.1 to define the field K' . One can then check that A_1, A_2 are strongly iso-Kummerian over K' . For example, for $g = 6$ (which is the slightly harder case), Theorem 2.4 shows that $(a_v) \in \prod_{v \text{ finite}} U_v$ is in the kernel of the modulo- ℓ^n representation ρ_i attached to A_i if and only if $e_i \equiv 1 \pmod{\ell^n}$, where

$$e_1 := b_\ell \sigma^{11}(b_\ell) \sigma^{10}(b_\ell) \sigma^3(b_\ell) \sigma^2(b_\ell) \sigma(b_\ell), \quad \text{resp. } e_2 := b_\ell \sigma^{10}(b_\ell) \sigma^7(b_\ell) \sigma^5(b_\ell) \sigma^3(b_\ell) \sigma^2(b_\ell)$$

and $b_\ell = N_{K/E}(a_\ell)^{-1}$. Straightforward linear algebra leads to the identities $e_1 = \frac{\sigma^8(e_2)\sigma^9(e_2)\sigma^{10}(e_2)}{\sigma^5(e_2)\sigma^6(e_2)}$ and $e_2 = \frac{\sigma^5(e_1)\sigma^7(e_1)\sigma^9(e_1)\sigma^{11}(e_1)}{\sigma^6(e_1)\sigma^8(e_1)\sigma^{10}(e_1)}$, which show that e_1 is congruent to 1 if and only if e_2 is. As in the proof of Theorem 5.1, this implies that A_1, A_2 are strongly iso-Kummerian over K' . Computations for $g = 4$ are analogous but simpler, and we omit them. Finally, since one can easily check that Φ_1, Φ_2 are simple and essentially different, this gives examples of 4- and 6-dimensional absolutely simple abelian varieties that are strongly iso-Kummerian but geometrically non-isogenous. Notice that in this way we may construct infinitely many geometrically distinct examples: indeed, one can find CM fields with cyclic Galois group of order $2g$ by considering the unique subfield of $\mathbb{Q}(\zeta_p)$ of degree $2g$ for the (infinitely many) primes p such that $p - 1 \equiv 2g \pmod{4g}$.

It remains to consider the case $g = 3$. Recall that an absolutely simple abelian threefold A is of type IV in Albert's classification if $\text{End}_{\overline{K}}(A) \otimes \mathbb{Q}$ is either a quadratic imaginary field or a sextic CM field (in the latter case A is a CM abelian variety).

Proposition 5.3. *Let K be a number field. If $A_1, A_2/K$ are absolutely simple, strongly iso-Kummerian abelian threefolds, then they are geometrically isogenous.*

Proof. Let $F_i := \text{End}_{\overline{K}}(A_i) \otimes \mathbb{Q}$. The proof is divided into several cases:

1. at least one of A_1, A_2 is not of type IV in Albert's classification.
2. both A_1, A_2 are of type IV in Albert's classification, and at least one of them has CM.
3. both A_1 and A_2 are of type IV, and neither has CM.

We recall that abelian threefolds satisfy the Mumford-Tate conjecture [28], [16, Theorem 6.1]. We will also need the following facts: the Mumford-Tate group of an abelian variety A is a torus if and only if A has CM [21, §5.3]; the Hodge group of a simple abelian threefold A has nontrivial centre if and only if A is of type IV in the sense of Albert [23, §2.3]; if $\text{End}_{\overline{K}}(A)$ is an order in a quadratic imaginary field F , then $\text{MT}(A)$ is the general unitary group of $V := H_1(A(\mathbb{C}), \mathbb{Q})$, considered as an F -vector space of dimension 3 endowed with a suitable F -Hermitian form ψ [23, §2.3]. Recall that, given a quadratic extension $F/\mathbb{Q}$, an F -Hermitian form on an F -vector space V is a map $\psi : V \times V \rightarrow F$ that is F -linear in the first argument and satisfies $\psi(v, w) = \sigma(\psi(w, v))$ for all $v, w \in V$, where σ is a generator of $\text{Gal}(F/\mathbb{Q})$. Denote by $V_{\mathbb{Q}}$ the $\mathbb{Q}$ -vector space underlying the F -vector space V . The *general unitary group* $\text{GU}_F(V, \psi)$ is the $\mathbb{Q}$ -algebraic subgroup of $\text{GL}_{V_{\mathbb{Q}}}$ consisting of those F -linear transformations f that, for some nonzero scalar λ , satisfy $\psi(fv, fw) = \lambda\psi(v, w)$ for all $v, w \in V$. Finally, given a reductive group G we denote by G^{ss} its derived (semisimple) subgroup.

1. If neither A_1 nor A_2 are of type IV, then (since both A_1 and A_2 have odd dimension, hence odd relative dimension) Theorem 1.4 shows that A_1, A_2 are geometrically isogenous. If only one of the two is of type IV, say A_1 , then $\text{Hg}(A_2)$ is semisimple, while the centre of $\text{Hg}(A_1)$ is nontrivial. It follows that $\dim \text{Hg}(A_1 \times A_2) = \dim \text{Hg}(A_1 \times A_2)^{\text{ss}} + \dim Z(\text{Hg}(A_1 \times A_2)) \geq \dim \text{Hg}(A_2)^{\text{ss}} + \dim Z(\text{Hg}(A_1)) \geq \dim \text{Hg}(A_2) + 1$, which contradicts Lemma 3.2.

2. By symmetry we may assume that A_1 has CM. If A_2 does not have CM, then its Mumford-Tate group is not a torus, so $\dim \text{Hg}(A_2)^{\text{ss}} > 0$. Recalling that $\text{Hg}(A_1)$ is a torus, so that $\dim \text{Hg}(A_1) = \dim Z(\text{Hg}(A_1))$, as in part (1) it follows that $\dim \text{Hg}(A_1 \times A_2) > \dim \text{Hg}(A_1)$, contradiction. Hence both A_1 and A_2 are CM, and the conclusion follows from Proposition 4.4.

3. Suppose now that A_1, A_2 are both of type IV, but neither has CM. Then as recalled above F_i is a quadratic imaginary field and we have $\text{MT}(A_i) = \text{GU}_{F_i}(V_i, \psi_i)$, where $V_i = H_1(A_i, \mathbb{Q})$ and ψ_i is a suitable F_i -Hermitian form. By Lemma 3.2, the projection $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ is an isogeny, so the central tori of $\text{MT}(A_1) \cong \text{GU}_{F_1}(V_1, \psi_1)$ and $\text{MT}(A_2) \cong \text{GU}_{F_2}(V_2, \psi_2)$ are isogenous. Since these tori are T_{F_1}, T_{F_2} respectively, one sees easily that $F_1 = F_2$. Let $\mathfrak{g}$ be the Lie algebra of $\text{MT}(A_1 \times A_2)$. Using again the fact that $\text{MT}(A_1 \times A_2) \rightarrow \text{MT}(A_i)$ is an isogeny, we have

$$\mathfrak{g}_{\mathbb{C}} \cong \text{Lie}(\text{MT}(A_i)) \otimes_{\mathbb{Q}} \mathbb{C} \cong \text{Lie}(\text{GU}_{F_i}(V_i, \psi_i)) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^2 \oplus \mathfrak{sl}_{3, \mathbb{C}},$$

where $\mathbb{C}^2$ can be identified with $F \otimes_{\mathbb{Q}} \mathbb{C}$ (here $F = F_1 = F_2$) via

$$\begin{aligned} F \otimes_{\mathbb{Q}} \mathbb{C} &\rightarrow \mathbb{C}^2 \\ f \otimes z &\mapsto (zf, z\bar{f}). \end{aligned}$$

We now describe the structure of the 6-dimensional cohomology groups $V_i \otimes_{\mathbb{Q}} \mathbb{C}$ as representations of $\mathfrak{g}_{\mathbb{C}} \cong \mathbb{C}^2 \oplus \mathfrak{sl}_{3,\mathbb{C}}$; for further details, see also [28, Proof of Theorem 3]. The $\mathbb{Q}$ -representation V_i of $\mathrm{MT}(A_i) \cong \mathrm{GU}_F(V_i, \psi_i)$ is the defining representation of this group. The centraliser of $\mathrm{GU}_F(V_i, \psi_i)$ in this representation is the centre of $\mathrm{GU}_F(V_i, \psi_i)$ itself, which is isomorphic to T_F (the action is such that $T_F(\mathbb{Q}) = F^\times$ acts on V_i via its F -vector space structure). Upon base-change to $\mathbb{C}$, the Lie algebra $\mathrm{Lie}(T_F) \otimes_{\mathbb{Q}} \mathbb{C} \cong F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^2$ acts on $V_i \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^6$ via the two characters χ_1, χ_2 given by the two natural projections $\mathbb{C}^2 \rightarrow \mathbb{C}$. The subspace $W_{i,j}$ of $V_i \otimes_{\mathbb{Q}} \mathbb{C}$ on which $\mathrm{Lie}(T_F) \otimes_{\mathbb{Q}} \mathbb{C}$ acts via χ_j is 3-dimensional, for $j = 1, 2$. Since $\mathrm{Lie}(T_F)$ is the centre of $\mathrm{Lie}(\mathrm{GU}_F(V_i, \psi_i))$, the subspace $W_{i,j}$ is stable under the action of $\mathrm{Lie}(\mathrm{GU}_F(V_i, \psi_i)) \otimes \mathbb{C}$, that is, it is a sub-representation. On each subspace $W_{i,j}$, the simple Lie algebra $\mathfrak{sl}_{3,\mathbb{C}}$ acts via its standard representation Std : this follows from the fact that the base-change to $\mathbb{C}$ of the defining representation of $\mathrm{Lie}(\mathrm{SU}_F(V_i, \psi_i)) = \mathrm{Lie}(\mathrm{GU}_F(V_i, \psi_i)^{\mathrm{ss}})$ on V_i is isomorphic to two copies of the standard representation of $\mathfrak{sl}_{3,\mathbb{C}}$. Summarising, under the action of $\mathfrak{g}_{\mathbb{C}} \cong \mathbb{C}^2 \oplus \mathfrak{sl}_{3,\mathbb{C}}$, the representation $V_i \otimes_{\mathbb{Q}} \mathbb{C} \cong W_{i,1} \oplus W_{i,2}$ is isomorphic to $\chi_1 \otimes \mathrm{Std} \oplus \chi_2 \otimes \mathrm{Std}$. In particular, we have $V_1 \otimes_{\mathbb{Q}} \mathbb{C} \cong V_2 \otimes_{\mathbb{Q}} \mathbb{C}$ as representations of $\mathfrak{g}_{\mathbb{C}}$. Using the fact that $\mathrm{MT}(A_1 \times A_2)$ is connected, hence that (for any representation) the invariants for $\mathrm{MT}(A_1 \times A_2)$ and for $\mathfrak{g}$ coincide, we obtain

$$\begin{aligned} \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathbb{C}}(A_1, A_2) \otimes \mathbb{Q} &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathrm{MT}(A_1 \times A_2)}(V_1, V_2) \\ &= \dim_{\mathbb{Q}} \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) \\ &= \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V_1 \otimes_{\mathbb{Q}} \mathbb{C}, V_2 \otimes_{\mathbb{Q}} \mathbb{C}) > 0, \end{aligned}$$

because the two $\mathfrak{g}_{\mathbb{C}}$ -representations $V_1 \otimes_{\mathbb{Q}} \mathbb{C}$ and $V_2 \otimes_{\mathbb{Q}} \mathbb{C}$ are isomorphic. Since A_1, A_2 are simple, any non-zero map between them is an isogeny, and we are done. $\square$

Corollary 5.4. *Let K be a number field and $A_1, A_2/K$ be strongly iso-Kummerian abelian varieties with $\dim(A_1) \leq 3, \dim(A_2) \leq 3$. Then A_1 and A_2 have the same isogeny factors (possibly with different multiplicities) over $\bar{K}$.*

Proof. If all the absolutely simple isogeny factors of A_1, A_2 have dimension at most 2 we are done by Theorem 1.4. Thus up to symmetry we may assume that A_1 is absolutely simple of dimension 3. It now suffices to show that A_2 is also absolutely simple of dimension 3, because the result will then follow from the previous proposition. By contradiction, let B be an absolutely simple factor of A_2 of dimension at most 2. By [23, §5.4] we have $\mathrm{Hg}(A_1 \times B) = \mathrm{Hg}(A_1) \times \mathrm{Hg}(B)$, unless B is a CM elliptic curve with CM by the quadratic imaginary field F and F embeds in $\mathrm{End}_{\bar{K}}^0(A_1)$. The case $\mathrm{Hg}(A_1 \times B) = \mathrm{Hg}(A_1) \times \mathrm{Hg}(B)$ is impossible by Lemma 3.2. In the remaining case, every geometrically simple isogeny factor of A_2 is an elliptic curve with CM by the same quadratic imaginary field, and therefore A_2 is geometrically isogenous to E^3 for a suitable CM elliptic curve E . In this case, $\dim \mathrm{Hg}(A_2) = \dim \mathrm{Hg}(E^3) = 1$, while $\dim \mathrm{Hg}(A_1) > 1$ by [23], again contradicting Lemma 3.2 and finishing the proof. $\square$

5.1.2 An example of Shioda

We conclude our discussion of strongly iso-Kummerian CM abelian varieties by constructing an explicit strongly iso-Kummerian pair defined over $\mathbb{Q}$. This example was first noticed by Shioda [33] in a slightly different context, and was also considered by Zywina [37, §1.7].

Theorem 5.5. *Let $J/\mathbb{Q}$ be the Jacobian of the smooth projective curve C with hyperelliptic equation $y^2 = x^9 - 1$. Then J is isogenous over $\mathbb{Q}$ to $X \times E$, where X is an absolutely simple abelian threefold with CM by $\mathbb{Q}(\zeta_9)$ and E is the CM elliptic curve $y^2 = x^3 - 1$. The abelian varieties $X \times E$ and X are strongly iso-Kummerian over $\mathbb{Q}$.*

The proof of Theorem 5.5 will occupy the rest of this section. We start by describing the abelian variety X . The map $\pi : C \rightarrow E$ given by $(x, y) \mapsto (x^3, y)$ shows that E appears as an isogeny factor of J . We let $X \subseteq J$ be the identity component of the kernel of $\pi_* : J \rightarrow E$: it is a 3-dimensional abelian subvariety of J , absolutely simple by results of Shioda [33, Example 6.1]. We may identify the complex uniformisation of J with $H^0(C_{\mathbb{C}}, \Omega_C^1) = \mathbb{C}\langle \frac{dx}{2y}, x\frac{dx}{2y}, x^2\frac{dx}{2y}, x^3\frac{dx}{2y} \rangle$. Pulling back regular differentials shows that π^*E corresponds to the vector subspace $\mathbb{C}\langle \frac{x^2 dx}{2y} \rangle$ of $H^0(C_{\mathbb{C}}, \Omega_C^1)$, so J/E (which is isogenous to X) has natural analytic uniformisation given by the quotient

$$\mathbb{C}\langle \frac{dx}{2y}, x\frac{dx}{2y}, x^2\frac{dx}{2y}, x^3\frac{dx}{2y} \rangle / \mathbb{C}\langle \frac{x^2 dx}{2y} \rangle = \mathbb{C}\langle \frac{dx}{2y}, x\frac{dx}{2y}, x^3\frac{dx}{2y} \rangle.$$

The automorphism $\alpha : (x, y) \mapsto (\zeta_9 x, y)$ induces an action of $F := \mathbb{Q}(\zeta_9)$ on this vector space. Since $\alpha^* x^i \frac{dx}{y} = \zeta_9^{1+i} \frac{dx}{y}$, the characters appearing in this representation are $\zeta_9 \mapsto \zeta_9^i$ for $i = 1, 2, 4$. This identifies the CM type $\Phi = \{\sigma_1, \sigma_2, \sigma_4\}$ of $\text{Gal}(\mathbb{Q}(\zeta_9)/\mathbb{Q}) = \{\sigma_i : \zeta_9 \mapsto \zeta_9^i \mid (i, 9) = 1\}$, which is the CM type of both X and J/E . The reflex field is F itself, and the dual CM type is $\Phi^* = \{\sigma_1, \sigma_5, \sigma_7\}$.

It is clear that we have an action of $\mathbb{Z}[\zeta_9] = \mathcal{O}_F$ on J . We denote by $[\alpha] \in \text{End}_F(J)$ the endomorphism corresponding to $\alpha \in \mathcal{O}_F$. For every $\alpha \in \mathcal{O}_F$ we have $[\alpha](X) = X$ (indeed X is the unique abelian subvariety of J of dimension 3), so $[\alpha]$ induces an endomorphism of X , and it follows easily that $\text{End}_F(X) = \mathcal{O}_F$. Moreover, as $\pi_* \pi^* : E \rightarrow E$ is multiplication by $\deg \pi = 3$, we obtain that the canonical isogeny $X \times E \rightarrow J$ given by addition has degree a power of 3. In particular, for every positive integer d prime to 3 we have $J[d] \cong X[d] \oplus E[d]$.

We now prove that $A := X \times E$ and X are strongly iso-Kummerian over F . Once this is done, it will be easy to deduce that the same statement also holds over $\mathbb{Q}$. We certainly have $F(X[d]) \subseteq F(A[d])$, so we only need to show the other inclusion. Since $F(A[d]) = F(X[d], E[d])$, it suffices to show that $F(E[d]) \subseteq F(X[d])$. By Galois theory, this amounts to proving that every $\sigma \in \text{Gal}(\overline{F}/F)$ that fixes $X[d]$ also fixes $E[d]$. By the Chinese Remainder Theorem, it suffices to restrict to the case of d being a power ℓ^n of a prime ℓ . Proposition 2.3 in [24] shows that the open subgroup $F^\times \cdot (\prod_{v \text{ finite}} U_v \times \prod_{v \text{ infinite}} F_v^\times)$ of I_F corresponds to the abelianised absolute Galois group of the Hilbert class field of F . Since F has class number 1, we find that this subgroup maps surjectively onto $\text{Gal}(\overline{F}/F)^{\text{ab}}$. As in the proof of Theorem 5.1, the ℓ -adic Galois representations attached to X and E over F are then completely determined by their restrictions to $\prod_{v \text{ finite}} U_v$. More precisely, these representations are described by Theorem 2.4: given a finite idèle $a = (a_v) \in \prod_v U_v$ we have

$$\rho_{E, \ell^\infty}(a) = \varepsilon_E(a) N_{F/\mathbb{Q}(\zeta_3)}(a_\ell)^{-1}, \quad \rho_{X, \ell^\infty}(a) = \varepsilon_X(a) N_{\Phi^*}(a_\ell)^{-1},$$

where we denote by N_{Φ^*} both the reflex norm $F^\times \rightarrow F^\times$ attached to the CM type of X described above and its extension to $(F \otimes \mathbb{Q}_\ell)^\times \rightarrow (F \otimes \mathbb{Q}_\ell)^\times$. Here ε_E and ε_X are characters of $\prod_v U_v$ with values in the roots of unity in $\mathbb{Q}(\zeta_3)$ and F , respectively. Since it is easily checked that C has good reduction away from 2 and 3, it follows from Theorem 2.4 that ε_X and ε_E factor via the product $\prod_{w|6} U_w$ of local units U_w at the places w of $\mathcal{O}_F$ above 2 and 3. We now prove that $\varepsilon_E(a) = \varepsilon_X(a)^3$, or equivalently $\varepsilon_E(a_w) = \varepsilon_X(a_w)^3$ for every place w of bad reduction.

In order to do this, recall from [31, p. 508] that the restriction of ε_X to U_w coincides with the homomorphism φ_w described in the proof of [31, Theorem 6 (b)]. This homomorphism is obtained as follows: let L/F be a finite extension over which X acquires good reduction at the places above w . The homomorphism φ_w is then given by the action of inertia on the special fibre of the Néron model of the abelian variety X/L . Observe that, since all the automorphisms of J send X to itself, one can equivalently consider the action of inertia on the Néron model of J instead. We now simplify this description by using the fact that J is a Jacobian.

An element σ in the inertia at w acts on the Néron model $\mathcal{J}_L$ of J/L via the finite group $\text{Aut}(\mathcal{J}_L) = \text{Aut}(J_F) = \mu(F)$. Since this group coincides with the automorphism group of the curve C , we may equivalently describe the action of σ on $\tilde{C}$, and obtain $\varphi_{J, w}(\sigma)$ as follows. We take as L a finite extension of F over which C admits a semistable model $\tilde{C}$ (in particular, J/L is semistable and hence has good reduction everywhere), and fix an isomorphism $\psi : C \rightarrow \tilde{C}$ defined

over L . Then $\psi^{-1} \circ \sigma \psi$ is an automorphism of C , which is precisely $\varphi_{J,w}(\sigma)$ as it can be seen by embedding C into its Jacobian. Since J has good reduction over L , so do its subvariety X and its quotient E . Thus we may use the same extension L in order to compute the automorphisms of X and E induced by σ ; these will be our $\varepsilon_X(\sigma)$ and $\varepsilon_E(\sigma)$. To describe them, we first notice that we have already proved that the restriction map $\mathcal{O}_F = \text{End}(J) \rightarrow \text{End}(X) = \mathcal{O}_F$ is the identity. Thus $\varepsilon_X(\sigma) = \varphi_{J,w}(\sigma)$. On the other hand, since the map $\pi : C \rightarrow E$ is $(x, y) \mapsto (x^3, y)$, the map $\mu(F) = \text{Aut}_F(J) \rightarrow \text{Aut}_F(E) = \mu(\mathbb{Q}(\zeta_3))$ is easily seen to be $\zeta \mapsto \zeta^3$. Thus $\varepsilon_E(\sigma) = \varphi_{J,w}(\sigma)^3$, and we conclude, as desired, that $\varepsilon_E(a_w) = \varepsilon_X(a_w)^3$ for all $a_w \in U_w$.

We can finally prove the inclusion $F(E[\ell^n]) \subseteq F(X[\ell^n])$. Indeed, let $a \in \prod_{v \text{ finite}} U_v$ be an element whose image in $\text{Gal}(\overline{F}/F)^{\text{ab}}$ fixes $X[\ell^n]$. By what we already proved, this amounts to the condition

$$\varepsilon_X(a_3)\sigma_1(a_\ell)^{-1}\sigma_5(a_\ell)^{-1}\sigma_7(a_\ell)^{-1} \equiv 1 \pmod{\ell^n}. \quad (3)$$

Applying σ_5 to both sides of this congruence we find $\varepsilon_X(a_3)^5\sigma_5(a_\ell)^{-1}\sigma_7(a_\ell)^{-1}\sigma_{-1}(a_\ell)^{-1} \equiv 1 \pmod{\ell^n}$, and comparing with Equation (3) we obtain $\varepsilon_X(a_3)^5\sigma_{-1}(a_\ell)^{-1} \equiv \varepsilon_X(a_3)\sigma_1(a_\ell)^{-1} \pmod{\ell^n}$. Applying σ_5 again we get $\varepsilon_X(a_3)^{20}\sigma_4(a_\ell)^{-1} \equiv \sigma_5(a_\ell)^{-1} \pmod{\ell^n}$, and replacing this congruence in Equation (3) we finally obtain

$$\varepsilon_X(a_3)^{21}\sigma_1(a_\ell)^{-1}\sigma_4(a_\ell)^{-1}\sigma_7(a_\ell)^{-1} \equiv 1 \pmod{\ell^n}.$$

As $\varepsilon_X(a_3)$ is an 18-th root of unity we have $\varepsilon_X(a_3)^{21} = \varepsilon_X(a_3)^3 = \varepsilon_E(a_3)$, and therefore

$$1 \equiv \varepsilon_X(a_3)^{21}\sigma_1(a_\ell)^{-1}\sigma_4(a_\ell)^{-1}\sigma_7(a_\ell)^{-1} \equiv \varepsilon_E(a_3)N_{F/\mathbb{Q}(\zeta_3)}(a_\ell)^{-1} = \rho_{E,\ell^n}(a) \pmod{\ell^n},$$

hence a also acts trivially on $E[\ell^n]$. By Galois correspondence, this shows that $F(E[\ell^n]) \subseteq F(X[\ell^n])$ for all ℓ and n , and therefore that A and X are strongly iso-Kummerian over F .

It remains to show that the same holds over $\mathbb{Q}$. As above, it suffices to prove that $\mathbb{Q}(E[\ell^n]) \subseteq \mathbb{Q}(X[\ell^n])$. By a theorem of Silverberg [34], for $\ell^n \geq 3$ the ℓ^n -torsion field $\mathbb{Q}(X[\ell^n])$ contains the field of definition of the endomorphisms of X , which is F (notice that by Theorem 2.4 every field of definition of the endomorphisms contains the reflex field of the CM type of X). Hence we obtain $\mathbb{Q}(X[\ell^n]) = F(X[\ell^n]) \supseteq F(E[\ell^n]) \supseteq \mathbb{Q}(E[\ell^n])$ as desired. Finally we consider the case $\ell^n = 2$. Since the isogeny $E \times X \sim J$ has odd degree we have $J[2] \cong E[2] \oplus X[2]$. By well-known properties of hyperelliptic Jacobians, the 2-torsion $J[2]$ is the $\mathbb{F}_2$ -vector space spanned by the divisor classes $e_i = [(\zeta_3^i, 0) - (\infty)]$ for $i = 0, \dots, 8$, subject to the only relation $\sum_{i=0}^8 e_i = 0$. Similarly, $E[2]$ is spanned by $f_i = [(\zeta_3^i, 0) - (\infty)]$ for $i = 0, 1, 2$, with the only relation $\sum_{i=0}^2 f_i = 0$. The pullback $\pi^* : E \rightarrow J$ sends f_i to $e_i + e_{i+3} + e_{i+6}$. Thus $X[2] = J[2]/E[2]$ is free with basis the classes of $e_0, \dots, e_5$, and from this one checks that $\mathbb{Q}(E[2])$ is $\mathbb{Q}(\zeta_3)$, while $\mathbb{Q}(X[2])$ is F . Thus also for $\ell^n = 2$ we have $\mathbb{Q}(E[\ell^n]) \subseteq \mathbb{Q}(X[\ell^n])$, which concludes the proof of Theorem 5.5.

Remark 5.6. The projection $\text{MT}(X \times E) \rightarrow \text{MT}(X)$ is an isomorphism. This fact can be deduced from the above proof, but we will show it using the algorithm suggested by Lemmas 4.2 and 4.3. In the notation of those lemmas, we can take $E_1 = F, E_2 = \mathbb{Q}(\zeta_3)$ and $L = F$. The character group $\widehat{T}_F$ has natural basis $\{\sigma_i : (i, 9) = 1\}$, while $\widehat{T}_{E_2}$ has basis $\{\sigma, \bar{\sigma}\}$ given by the two complex conjugate embeddings of $\mathbb{Q}(\zeta_3)$ into $\mathbb{C}$. The morphism N_1^* induced on character groups by the norm N_1 is the identity, while N_2^* sends σ to $\sigma_1 + \sigma_4 + \sigma_7$ and $\bar{\sigma}$ to $\sigma_2 + \sigma_5 + \sigma_8$. Concerning the reflex norms, we have that ϕ_2^* is the identity, while $\phi_1^* : \widehat{T}_L = \mathbb{Z}[\sigma_i] \rightarrow \widehat{T}_L = \mathbb{Z}[\sigma_i]$ is multiplication by $\sigma_1 + \sigma_5 + \sigma_7$. Thus $N_1^*\phi_1^* + N_2^*\phi_2^*$ is represented by the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Here the first six columns are indexed by the σ_i with $(i, 9) = 1$, and the last two by $\sigma, \bar{\sigma}$. The rows are indexed by the σ_i with $(i, 9) = 1$. The column vectors $(0, 0, 1, 0, 1, -1, -1, 0)^T$ and

$(0, 0, 0, 1, -1, 1, 0, -1)^T$ are in the (right) kernel of this matrix, and together with $\widehat{T}_F \times \{0\}$ they clearly generate $\widehat{T}_F \times \widehat{T}_{\mathbb{Q}(\zeta_3)}$. By Lemma 4.2, this proves the claim.

6 Abelian varieties with $\text{End}(A_{\overline{K}}) = \mathbb{Z}$

We now consider the other end of the spectrum with respect to the CM case, namely, abelian varieties with trivial geometric endomorphism ring. We will show that there exist abelian varieties A, B with $\text{End}_{\overline{K}}(A) = \text{End}_{\overline{K}}(B) = \mathbb{Z}$ and without common isogeny factors for which the canonical projections $\text{Hg}(A \times B) \rightarrow \text{Hg}(A), \text{Hg}(A \times B) \rightarrow \text{Hg}(B)$ both have finite kernel. We are unfortunately unable to construct examples for which the kernels in question are actually trivial (see also Remark 6.6). Our main tool is the Kuga-Satake construction, introduced in [12], and for which we refer the reader to [35]. See also [22] for the notion of Mumford-Tate group in the more general context of Hodge structures of arbitrary weight.

Let n be a fixed positive integer and V be a $\mathbb{Q}$ -vector space of dimension $n+2$. Fix a quadratic form Q on V of signature $(2-, n+)$. By [35, Lemma 4.7] we can equip V with a Hodge structure of weight 2 in such a way that $\dim V^{2,0} = 1$ and $\text{MT}(V) = \text{GO}(Q)$, the group of orthogonal similitudes of Q . Let $h : \mathbb{S} \rightarrow \text{GL}_{V_{\mathbb{R}}}$ be the morphism defining any such Hodge structure on V , where $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{R})$ is the Deligne torus.

Let $C(Q)$ be the Clifford algebra of the quadratic form Q and $C^+(Q)$ be its even part. The Kuga-Satake construction takes as input the data (V, h, Q) , and gives as output a *polarisable* Hodge structure on the rational vector space $C^+(Q)$ of weight 1 and type $(1, 0) \oplus (0, 1)$, which we will denote by $(C^+(Q), h_s)$. By passing to the dual, such a Hodge structure identifies a complex abelian variety A (up to isogeny): indeed, the category of complex abelian varieties (up to isogeny) and that of polarisable Hodge structures of type $(-1, 0) \oplus (0, -1)$ are equivalent (see [35, §8] for details).

Recall that V is canonically a vector subspace of $C(Q)$, and that with this identification the group $\text{CSpin}(Q)$ is defined as

$$\text{CSpin}(Q)(R) = \{g \in (C^+(Q) \otimes R)^* \mid g(V \otimes 1)g^{-1} \subseteq V \otimes 1\}$$

for every $\mathbb{Q}$ -algebra R . For the general properties of the Spin group, the reader can refer to [10, pp. 304-308] and [6]. By its very definition, $\text{CSpin}(Q)$ admits a representation

$$\begin{aligned} \rho : \text{CSpin}(Q) &\rightarrow \text{GL}_{C^+(Q)} \\ g &\mapsto (x \mapsto gx), \end{aligned} \tag{4}$$

and it can be checked that $\text{End}_{\text{CSpin}(Q)}(C^+(Q)) \cong C^+(Q)$, the isomorphism being induced by the right action of $C^+(Q)$ on itself. Moreover, one has $\text{MT}(C^+(Q), h_s) = \text{CSpin}(Q)$ as soon as $\text{MT}(V) = \text{GO}(Q)$ [35, Proposition 6.3], and clearly the dual Hodge structure also has Mumford-Tate group $\text{GO}(Q)$. We will need to know that there is a certain *spinor norm* $\text{CSpin}(Q) \rightarrow \mathbb{G}_{m, \mathbb{Q}}$, and that the Spin group $\text{Spin}(Q)$ can be described both as the derived subgroup of $\text{CSpin}(Q)$ and as the kernel of the spinor norm. It is also easy to see the $\text{CSpin}(Q)$ is the almost-direct product of $\text{Spin}(Q)$ and $\mathbb{G}_m$.

We now turn to the study of the abelian variety A obtained from the Kuga-Satake construction. Its dimension is

$$\dim A = \frac{1}{2} \dim_{\mathbb{Q}} C^+(Q) = \frac{1}{4} \dim_{\mathbb{Q}} C(Q) = 2^n.$$

The endomorphism algebra of A is canonically identified with

$$\text{End}_{\text{MT}(C^+(Q))}(C^+(Q)) \cong \text{End}_{\text{CSpin}(Q)}(C^+(Q)) \cong C^+(Q).$$

Choose $n = 4k + 2$ and $Q = x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2 - x_{n+2}^2$. The structure of $C^+(Q)$ is as follows:

Proposition 6.1. *Let $n = 4k + 2$ and $Q = x_1^2 + x_2^2 + \dots + x_n^2 - x_{n+1}^2 - x_{n+2}^2$, and write $\mathbb{H}$ for the rational quaternions $\mathbb{Q} \oplus \mathbb{Q}i \oplus \mathbb{Q}j \oplus \mathbb{Q}k$ with $i^2 = j^2 = k^2 = -1, ij = -ji = k$.*

- *If k is odd we have $C^+(Q) \cong \text{Mat}_{2^{2k}}(\mathbb{H}) \oplus \text{Mat}_{2^{2k}}(\mathbb{H})$;*
- *If k is even we have $C^+(Q) \cong \text{Mat}_{2^{2k+1}}(\mathbb{Q}) \oplus \text{Mat}_{2^{2k+1}}(\mathbb{Q})$.*

Proof. This is proved exactly as [35, Theorem 7.7], the only difference being that we keep track of the quaternion algebra denoted D in op. cit. $\square$

Combining the Kuga-Satake construction with the above proposition we obtain a family of examples of abelian varieties A, B for which the two projections $\text{Hg}(A \times B) \rightarrow \text{Hg}(A)$ and $\text{Hg}(A \times B) \rightarrow \text{Hg}(B)$ have finite kernel:

Theorem 6.2. *Let k be a positive integer. There exist simple, non-isogenous $\mathbb{C}$ -abelian varieties A_+, A_- such that*

- $\text{End}^0(A_+) \cong \text{End}^0(A_-) \cong \begin{cases} \mathbb{H}, & \text{if } k \text{ is odd;} \\ \mathbb{Q}, & \text{if } k \text{ is even} \end{cases}$
- $\dim(A_+) = \dim(A_-) = \begin{cases} 2^{2k+1}, & \text{if } k \text{ is odd;} \\ 2^{2k}, & \text{if } k \text{ is even} \end{cases}$
- $\text{Hg}(A_+ \times A_-)$ is absolutely simple. The projections $\text{Hg}(A_+ \times A_-) \rightarrow \text{Hg}(A_+)$ and $\text{Hg}(A_+ \times A_-) \rightarrow \text{Hg}(A_-)$ have finite kernel of order 2.

Proof. Let $n = 4k + 2$, choose V, Q, h as above and let A be the $\mathbb{C}$ -abelian variety up to isogeny attached to the triple (V, Q, h) by the Kuga-Satake construction (that is, the abelian variety corresponding to the dual Hodge structure of $(C^+(Q), h_s)$). Combining the results above we see that $\dim(A) = 2^n$ and

$$\text{End}^0(A) \cong C^+(Q) \cong \begin{cases} \text{Mat}_{2^{2k}}(\mathbb{H}) \oplus \text{Mat}_{2^{2k}}(\mathbb{H}), & \text{if } k \text{ is odd} \\ \text{Mat}_{2^{2k+1}}(\mathbb{Q}) \oplus \text{Mat}_{2^{2k+1}}(\mathbb{Q}), & \text{if } k \text{ is even} \end{cases}$$

For the sake of brevity let $m = 2k$ (resp. $2k + 1$) if k is odd (resp. even). From the description of its endomorphism algebra we obtain that A is isogenous to $(A_+ \times A_-)^{2^m}$ for simple, non-isogenous Abelian varieties A_+, A_- whose endomorphism algebra is $\mathbb{H}$ or $\mathbb{Q}$, depending on the parity of k . Moreover,

$$\dim(A_+) = \dim(A_-) = \frac{1}{2} \frac{\dim(A)}{2^m} = 2^{n-m-1} = \begin{cases} 2^{2k+1}, & \text{if } k \text{ is odd} \\ 2^{2k}, & \text{if } k \text{ is even} \end{cases},$$

because the decomposition $A_+ \times A_-$ comes from the analogous decomposition of the spin representation as a direct sum of the two non-isomorphic half spin representations [10, p. 305]. Note now that $\text{MT}(A)$ is isomorphic to $\text{CSpin}(Q)$, so $\text{Hg}(A) \cong \text{Spin}(Q)$ is absolutely simple (over the complex numbers, the group $\text{Spin}(n + 2)$ is simple as soon as $n + 2 > 4$). The kernels of the surjective projections

$$\text{Hg}(A_+ \times A_-) \rightarrow \text{Hg}(A_+), \quad \text{Hg}(A_+ \times A_-) \rightarrow \text{Hg}(A_-)$$

must therefore be finite as claimed. More precisely, from the discussion above we see that these kernels may be identified with the kernels of the half-spin representations of $\text{Spin}(Q)$, hence they have order 2 [4, III.6.1]. $\square$

The above complex construction leads to the following version over number fields:

Theorem 6.3. *Let h be a positive integer. There exist geometrically simple abelian varieties A_+, A_- , defined over a number field K , non-isogenous over $\bar{K}$, and such that*

- $\text{End}_{\bar{K}}(A_+) \cong \text{End}_{\bar{K}}(A_-) \cong \mathbb{Z}$;
- $\dim(A_+) = \dim(A_-) = 2^{4h}$;
- the Mumford-Tate conjecture holds for A_+, A_- and $A_+ \times A_-$;
- for every rational prime ℓ the group $H_\ell(A_+ \times A_-)$ is a double cover (via the canonical projections) of both $H_\ell(A_+)$ and $H_\ell(A_-)$.

Proof. Let B_+, B_- be the abelian varieties whose existence is guaranteed by Theorem 6.2 with $k = 2h$. By Proposition 2.7, there exist abelian varieties A_+, A_- , defined over a number field K , with the same Mumford-Tate groups as B_+, B_- and such that the Mumford-Tate conjecture holds for A_+, A_- and $A_+ \times A_-$. The morphism $H_\ell(A_+ \times A_-) \rightarrow H_\ell(A_+)$ may then be identified with the base-change to $\mathbb{Q}_\ell$ of the corresponding morphism $\text{Hg}(A_+ \times A_-) \rightarrow \text{Hg}(A_+)$, and similarly for A_- . As discussed in the proof of Theorem 6.2, this morphisms are precisely the half-spin representations of a certain Spin group and have kernel of order 2. $\square$

The abelian varieties of the previous theorem are almost-examples of the situation considered in condition (1). For simplicity we only study the case $d = \ell^\infty$, but the result could also be adapted to finite values of d .

Corollary 6.4. *Let A_+, A_- be as in the previous theorem and write A_0 for the abelian variety $A_+ \times A_-$. There exists a finite extension K' of K such that for every prime ℓ we have*

$$[K'(A_0[\ell^\infty]) : K'(A_+[\ell^\infty])] \leq 2, \quad [K'(A_0[\ell^\infty]) : K'(A_-[\ell^\infty])] \leq 2.$$

Proof. Up to replacing K with a finite extension K' , we may assume that for each prime ℓ the image of the ℓ -adic Galois representation attached to A_+/K' (respectively A_0/K') lands inside the $\mathbb{Q}_\ell$ -points of $\text{MT}(A_+)$ (respectively $\text{MT}(A_0)$), see [30, 133] or [15]. Let $K_0 := K'(A_0[\ell^\infty])$ and $K_+ := K'(A_+[\ell^\infty])$. Also let ρ_{0,ℓ^∞} and ρ_{+,ℓ^∞} be the ℓ -adic Galois representations attached to A_0/K' and A_+/K' respectively, with images G_{0,ℓ^∞} and G_{+,ℓ^∞} . The extension K_0/K_+ is Galois, with Galois group isomorphic to the kernel of the natural surjective map $G_{0,\ell^\infty} \rightarrow G_{+,\ell^\infty}$. By construction we have $G_{0,\ell^\infty} \subseteq G_\ell(A_0)(\mathbb{Q}_\ell)$ and $G_{+,\ell^\infty} \subseteq G_\ell(A_+)(\mathbb{Q}_\ell)$, and the projection map corresponds to the algebraic map $G_\ell(A_0) \rightarrow G_\ell(A_+)$. Since by Lemma 6.5 below the kernel of this map has order 2 (even over $\overline{\mathbb{Q}_\ell}$; note that we are working with the Spin group $\text{Spin}(n+2) = \text{Spin}(4h+4)$), the projection $G_{0,\ell^\infty} \rightarrow G_{+,\ell^\infty}$ has kernel of order at most 2, as claimed. The proof for A_- is identical. $\square$

Lemma 6.5. *Let F be an algebraically closed field of characteristic different from 2. Consider the F -algebraic groups $\text{Spin}(4m)$ and $\text{CSpin}(4m)$, defined from any non-degenerate quadratic form Q on F^{4m} . The half-spin representations $\rho_\pm$ of $\text{Spin}(4m)$ extend to representations of $\text{CSpin}(4m) = \mathbb{G}_m \cdot \text{Spin}(4m)$ by setting $\rho_\pm(\lambda g) = \lambda \rho_\pm(g)$. The kernels of these extended representations have order 2.*

Proof. The fact that the half-spin representations extend is clear from their construction (see Equation (4)). It is well-known that the half-spin representations of $\text{Spin}(4m)$ are orthogonal (see for example [16, Table 2.1]), so for $g \in \text{Spin}(4m)$ the matrix $\rho_\pm(g)$ is a multiple of the identity if and only if it is $\pm \text{Id}$ (the only orthogonal matrices that are also scalars are $\pm \text{Id}$). Hence if $\rho_\pm(\lambda g) = \lambda \rho_\pm(g)$ is trivial for some $\lambda g \in \text{CSpin}(4m)$ we must have $\lambda = \pm 1$, so λg is in $\text{Spin}(4m)$. It follows that the kernels of the extended representations coincide with the kernels of the original representations of $\text{Spin}(4m)$, which have order 2. $\square$

Remark 6.6. The example of Corollary 6.4 is motivated on the one hand by the condition appearing in Lemma 3.2, and on the other by an examination of Table 2.1 in [16]: the two half-spin representations of a Lie algebra of type D_l provide the simplest example for which the criterion

of [16, Theorem 1.1] fails. However, in order for $H_\ell(A_+ \times A_-) \rightarrow H_\ell(A_+)$ to be an isomorphism, one would need the half-spin representations to be faithful, which happens only if l is odd (the present l is connected to the parameter n in the construction above by $l = (n + 2)/2$). On the other hand, when l is odd [16, Table 2.1] shows that the half-spin representations are not self-dual, complicating the construction of corresponding polarised Hodge structures.

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