

Short-time existence for the network flow

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Abstract

This paper contains a new proof of the short-time existence for the flow by curvature of a network of curves in the plane. Appearing initially in metallurgy and as a model for the evolution of grain boundaries, this flow was later treated by Brakke [4] using varifold methods. There is good reason to treat this problem by a direct PDE approach, but doing so requires one to deal with the singular nature of the PDE at the vertices of the network. This was handled in cases of increasing generality by Bronsard-Reitich [5], Mantegazza-Novaga-Tortorelli [18] and eventually, in the most general case of irregular networks by Ilmanen-Neves-Schulze [11]. Although the present paper proves a result similar to the one in [11], the method here provides substantially more detailed information about how an irregular network ‘resolves’ into a regular one. Either approach relies on the existence of self-similar expanding solutions found in [20]. As a precursor to the main theorem, we also prove an unexpected regularity result for the mixed Cauchy-Dirichlet boundary problem for the linear heat equation on a manifold with boundary.

1 Introduction

This paper contains a new proof of short-time existence for the curve-shortening flow for networks of curves in the plane, or more generally in any Riemannian surface. This problem has been treated before, and the history is recalled below, but our approach draws out some important features and precise information not accessible by previous methods. The inspiration for the constructions here draws on the methods of geometric microlocal analysis, and in particular their application to the study of geometric flows on spaces with conic singularities, cf. [19]. Beyond the geometric naturality of this proof, it should be well adapted for proving analogous results in higher dimensions, e.g. for the flow by mean curvature of ‘foams’ of surfaces in $\mathbb{R}^3$.

The *Mean Curvature Flow* (MCF) is surely one of the best studied geometric evolution problems; it requires a submanifold to evolve with normal velocity equal to its mean curvature. It is the gradient flow of the length/area/volume functional, and hence, the length of a curve evolving by this flow decreases in time; this explains the name curve-shortening flow. The normal velocity at any (p, t) is the curvature of the curve at that point of space and time:

$$v^\perp = \vec{\kappa}.$$

The complete analysis of the curve-shortening flow for embedded closed curves, due to Gage-Hamilton [8] and Grayson [10], was the one of earliest successes in the theory of geometric flows. However, some years before, motivated in part by the desire to understand dynamics of grain boundaries, Brakke [4] defined a weak form of this curvature flow in the class of varifolds, which

thus provides one avenue for studying the evolution of certain singular geometric objects. This weak formulation yields global existence of solutions, but on the other hand, does not allow one to control the geometry of these evolving objects. This has motivated much further study. We mention the work of Kim and Tonegawa [12], who showed that one can obtain a nontrivial mean curvature flow even when the initial datum is singular, cf. also the recent monograph [24]. However, it is worth emphasizing that questions concerning uniqueness or non-uniqueness of solutions are mostly inaccessible in this measure theoretic context.

It is of interest to recast Brakke's results if one is interested in objects which are nearly smooth; in one dimension, this leads to the study of networks of curves. By definition, a network of curves Γ in some domain $\Omega \subset \mathbb{R}^2$ is a collection of smooth embedded curves $\{\gamma^{(j)}(x)\}$ with the following properties. First, the interiors of these curves are disjoint. Each end of any $\gamma^{(j)}$ is either an 'external' or 'internal' vertex. The external vertices are disjoint and lie on $\partial\Omega$ and are regarded as termini of the network. We impose some boundary conditions at these, e.g. they might be fixed or evolve remaining normal to the boundary, etc. At each interior vertex, however, three or more of the $\gamma^{(j)}$ meet non-tangentially. We always assume that there is at least one interior vertex. For simplicity we always assume that $\Omega \subset \mathbb{R}^2$ and Dirichlet (fixed) boundary conditions are imposed at the internal vertices. However, everything we say here can be generalized in a straightforward way if Ω is either a closed Riemannian surface, or a surface with boundary, or if other boundary conditions are imposed. In addition, we may also consider the situation where Γ is a finite but noncompact network in $\mathbb{R}^2$, where the 'external' arcs are asymptotic to rays at infinity (see § 2.1 below for a bit more on this).

As indicated above, we wish to study this problem in a strong sense, using the methods of PDE rather than geometric measure theory. Thus with respect to an arbitrary parametrization of each $\gamma^{(j)}$, consider the equations

$$\langle \partial_t \gamma^{(j)}, \nu^{(j)} \rangle = \kappa^{(j)} ;$$

here $\nu^{(j)}$ is the unit normal vector to $\gamma^{(j)}$ and the curvature vector is $\vec{\kappa}^{(j)} = \kappa^{(j)} \nu^{(j)}$.

Consider first the situation where the initial configuration Γ_0 is *regular*, i.e., each interior vertex is trivalent, with the three curves meeting at each such vertex in equal angles of $2\pi/3$ (this is called the *Herring condition*). Regular vertices are one feature of stable critical points of the length functional in the class of networks. If the curves meeting at an interior vertex are labelled $\gamma^{(i_\ell)}$, $\ell = 1, 2, 3$, then for every $t > 0$,

$$\begin{aligned} \gamma^{(i_1)}(t, 0) &= \gamma^{(i_2)}(t, 0) = \gamma^{(i_3)}(t, 0), \quad \text{and} \\ \tau^{(i_1)}(t, 0) + \tau^{(i_2)}(t, 0) + \tau^{(i_3)}(t, 0) &= 0, \end{aligned}$$

where $\tau^{(j)}$ is the unit tangent vector to $\gamma^{(j)}(t, 0)$.

Short-time existence of this flow for regular networks was obtained in the papers of Kinderlehrer and Liu [14], Bronsard and Reitich [5], and Mantegazza, Novaga and Tortorelli [18]. In this last paper, as well as in the more recent paper by Magni, Mantegazza and Novaga [16], certain long-time existence results are also proved. We refer also to the survey by Mantegazza et al. [17].

Our main interest here is the situation where Γ_0 is not regular. In other words, we are interested in the short-time existence of this flow when the initial network has interior vertices which are either trivalent but for which the Herring conditions are not satisfied, or else k -valent for some $k > 3$.

There are several motivations for studying this problem. The most basic one is the inherent interest in enlarging the class of admissible initial data. However, another is that the class of

regular networks is not preserved by the flow. An initial regular network can evolve into one which at some later time is not regular. For example, two triple junctions might coalesce, or an enclosed region bounded by a loop of curves in the network might collapse to a point. The best scenario is when the curvature of the constituent curves remains bounded in this singularity formation (as happens when two triple points collide). At the very least, we would like to know whether this limiting network can evolve past this singular time. The short time existence result for irregular networks ‘restarts’ the evolution.

Short time existence for irregular networks was established in the paper of Ilmanen, Neves and Schulze [11]. They prove that if $\Gamma_0 = \{\gamma_0^{(j)}\}$ is an irregular initial network, then there exists a time $T > 0$ and a solution of the flow $\Gamma(t) = \{\gamma_t^{(j)}\}_{t \in [0, T]}$ such that $\Gamma(t)$ converges to Γ_0 in the sense of varifolds. (As we explain later, $\Gamma(t)$ may have a larger number of arcs than Γ_0 itself.) Although this convergence is already slightly stronger than requiring that $\mathcal{H}^1 \llcorner \Gamma_t \rightarrow \mathcal{H}^1 \llcorner \Gamma_0$ as $t \rightarrow 0$ in the sense of measures, it is desirable to have an even more detailed understanding of the geometry of this ‘singularity resolution’.

The key point in all of this has relatively little to do with the regularity of the initial curves, but is more closely related to their geometry. We must, in particular, show precisely how a single multi-point gives birth to a cluster of triple junctions. This already appears in [11], but our approach provides a more detailed understanding. Our main result is the following

Theorem 1.1. *Let Γ_0 be an initial network where all component arcs are of class $\mathcal{C}^2$ and at least one interior vertex is irregular (even in the mildest sense that the curves which meet at the vertex do not satisfy the matching conditions at that point needed to obtain maximal regularity). Then there exists a time $T > 0$ and an evolving family of regular networks $\Gamma(t)$, $0 < t < T$, with the property that $\Gamma(t) \rightarrow \Gamma_0$ in a strong sense to be described below.*

The set of these possible flowouts is classified by the collection of all (appropriate) expanding soliton solutions of the flow at each interior vertex.

The method developed here was initially inspired by the paper of the second author, Rubinstein and Sesum [19], which studies the Ricci flow on surfaces with isolated conical singularities. The main results there are a set of sharp estimates for linear heat equations on spaces with conic singularities, from which the short-time existence of the flow is readily proved. (That paper also considers a number of long-time and limiting results.) At first we had planned to set things up to be able to invoke those estimates, but in fact we ultimately decided that it is more transparent to approach the present problem by an explicit construction of a very accurate approximate solution to the problem, followed by some more standard-looking analytic estimates which are used to solve away the (rapidly decaying) error term.

To be more precise, suppose that p is an irregular vertex of our initial network Γ_0 which is k -valent for some $k > 3$ (the construction near vertices of valence 3 is slightly simpler). Assume for simplicity that Γ lies in $\mathbb{R}^2$ and p is placed at the origin. Enumerate the curves meeting at p by $\gamma^{(1)}, \dots, \gamma^{(k)}$, sequentially going counterclockwise. Denote by τ_j the unit tangent vector to γ_j at p , and let $\ell_j = \mathbb{R}^+ \tau_j$ be the ray in $\mathbb{R}^2$ in the direction τ_j . The incoming edges at p thus determine a fan of rays in $\mathbb{R}^2$ emanating from the origin. As proved in [20], there exists at least one expanding soliton solution for the curve-shortening flow which has this fan as its initial condition. In fact, [20] shows that there are typically many such solutions corresponding to a given fan when $k > 3$; for generic fans, and conjecturally for all fans, there are only finitely many such solitons. Choose one of these solutions, and denote it by S_p . Having made this choice for every irregular interior vertex, we prove that there exists a unique solution $\Gamma(t)$ of the network flow, defined on some interval $0 < t < T$, whose combinatorics are the same as if we were to replace a small ball around each irregular vertex p with the corresponding choice of soliton S_p .

This solution converges as $t \rightarrow 0$ to Γ_0 , but this convergence happens in a slightly odd fashion since $\Gamma(t)$ may have more curves than Γ_0 , so certain curves in $\Gamma(t)$ converge to curves in Γ_0 (as smoothly as the regularity of those curves in Γ_0 allow), while the remaining curves in $\Gamma(t)$ shrink in size and ‘disappear into’ these irregular vertices as $t \searrow 0$. There is a sense in which the entire convergence can be regarded as $\mathcal{C}^\infty$ if the initial network Γ_0 is smooth. At each irregular vertex p , the soliton S_p is the tangent flow at $(p, 0)$. This replacement of a neighborhood of an irregular vertex p with a soliton S_p appears also in [11], but their analysis only describes the convergence as $t \searrow 0$ in a much weaker sense.

The key to this construction (as well as the one in [11]) is the existence of expanding solitons. For trivalent vertices, their existence and uniqueness was proved by Schnürer and Schulze [21]. For more general vertices, their existence and classification was obtained by the second and fourth authors here [20]. One striking feature and outcome of this is that if any interior vertex has valence greater than 3, then the flow is ill-posed: there exists more than one solution emanating from the given initial configuration. The chosen soliton at each p emerges in a very natural way, as we shall explain. This illustrates yet again that nonuniqueness is an expected phenomenon in the realm of flows on spaces with singular initial data.

After describing the general features of the problem in §2, we introduce in §3 our key tool: the blowups of the domain and range spaces. These blowups (which are a central tool in geometric microlocal analysis) provide sufficient ‘room’ to explain the $t \searrow 0$ asymptotics. The main work is the construction of an approximate solution, i.e., a family of networks $\widehat{\Gamma}(t)$ which converges to Γ_0 and which satisfy the flow equations up to an error which vanishes to all orders at $t = 0$. This is rather delicate, and as a warm-up to it, we carry out this construction for a similar but quite classical problem in §4, namely the solution of the linear heat equation on an interval, where the prescribed initial and boundary conditions do not satisfy the compatibility conditions which would allow the solution to be smooth down to $t = 0$. The construction of the approximate solution for the nonlinear problem appears in §5. There is still a need to solve away the rapidly vanishing error term. This could be done using the heat space theory as in [19], but we have chosen to do this using a classical approach based on a priori estimates. Due to some unusual features of the equation, this takes a bit of work, and is done in §6. The short-time existence is then proved quickly in §7, and §8 contains a number of remarks concerning extensions of this theorem.

This plan for this paper, and the realization that a proof of this nature should be possible, was envisioned over a decade ago, motivated by the then-recent work [20] but also by conversations between the second author and Felix Schulze during a sequence of quarter-long visits by Schulze to Stanford. Those conversations were perhaps precursors to [11], but certainly also to the present paper. The details have been slow in coming, not for any particularly good reason, but in any event, gradually crystalized during a number of visits of varying lengths to Stanford by the first, third and fourth authors in the intervening years, and later, as the project came together, in other visits by the second author to Rome and Pisa.

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2 The evolution equation for networks

2.1 Networks

We now develop language to state our precise results.

Definition 2.1. *A planar network Γ is a finite collection of embedded regular curves $\{\gamma^{(j)}\}_{j=1}^N$, each of class $\mathcal{C}^2$, which meet in groups of at least 3 at each interior vertex and which have some fixed ends (or else Γ is noncompact and each end is a curve asymptotic to a half-line at infinity).*

We also make the convention that we consider only networks where the curves meet nontangentially at each interior vertex.

Since the focus in this paper is the behavior near interior vertices, we assume that Γ is compact and lying in a smoothly bounded domain $\Omega \subset \mathbb{R}^2$. Were we to be studying long-term behavior of the flow, it would be highly advantageous to assume that Ω is convex, but this is irrelevant for our purposes. The methods here apply equally well to noncompact networks, with noncompact edges asymptotic to straight lines, or also to networks lying in a Riemannian surface (Ω, g) , with or without boundary.

A single curve (closed or not) is a particularly trivial network without vertices. Networks with only two curves meeting in a vertex have been studied from the point of view of curves with corners or cusps, cf. [1, 2, 3, 7, 8, 10].

Definition 2.2. *A network Γ is said to be regular if the only vertices are triple junctions where the unit tangent vectors of the curves meeting there form angles of $2\pi/3$ (this is the Herring condition), and in addition the compatibility conditions on higher derivatives are satisfied at each vertex, cf. the beginning of §3. (If the network is only $\mathcal{C}^2$, these compatibility conditions state only that the curvatures of the three curves are equal.) Otherwise the network is called irregular.*

The regular networks which are also trees have some particularly nice properties, as we shall mention later.

2.2 Evolution equation

Given a network composed of m curves we define its *length* L as the sum of the length of each curve. The curve-shortening flow is formally the L^2 -gradient flow of the length. We aim to define the motion using classical PDE theory, so it is helpful to parametrize each curve of the network rather than to look at the network as a set. To avoid confusion we denote by $\Gamma = \{\gamma^{(j)}\}_{j=1}^m$ a *parametrized* network if each $\gamma^{(j)}$ is a $\mathcal{C}^2$ map from $[0, 1]$ to Ω with derivative everywhere nonvanishing, and by $\mathcal{N}$ the corresponding set: $\mathcal{N} = \bigcup_{j=1}^m \gamma^{(j)}([0, 1])$. The notion of regular network is geometric: if a parametrized network Γ is regular, then any smooth reparametrization of Γ is still a regular network.

Consider first a family of smooth parametrized curves $\gamma(t, x)$ where x is an arbitrary strictly monotone smooth parametrization (i.e., $\partial_x \gamma \neq 0$). As already noted, the basic curve-shortening flow equation is

$$\langle \partial_t \gamma, \nu \rangle = \kappa, \quad \gamma(0, \cdot) = \gamma_0(\cdot), \quad (2.1)$$

where ν is the unit normal to γ at any (t, x) , or equivalently, denoting the unit tangent vector to γ by τ ,

$$\partial_t \gamma = \kappa \nu + \zeta \tau, \quad (2.2)$$

for some function ζ . Here κ is the oriented curvature and equations (2.1) and (2.2) can also be written in terms of the curvature vector as $(\partial_t \gamma)^\perp = \vec{\kappa}$ where $(\partial_t \gamma)^\perp$ denotes the normal component of the velocity.

We next generalize the definition of curve shortening flow to a family of time-dependent networks $\Gamma(t) = \{\gamma^{(j)}(t, x)\}$. We ask the network to be regular for $t > 0$, at least for some short time interval: if $\gamma^{(i_\ell)}(0, \mathbf{x})$, $\ell = 1, 2, 3$, intersect at an interior vertex, then for every $t > 0$

$$\begin{aligned}\gamma^{(i_1)}(t, 0) &= \gamma^{(i_2)}(t, 0) = \gamma^{(i_3)}(t, 0), \\ \tau^{(i_1)}(t, 0) + \tau^{(i_2)}(t, 0) + \tau^{(i_3)}(t, 0) &= 0,\end{aligned}\tag{2.3}$$

where $\tau^{(j)}$ is the unit tangent vector to $\gamma^{(j)}$. These matching boundary conditions (2.3) are called the Herring conditions.

We now define the system of PDEs driving the motion by curvature of networks; it remains to discuss the freedom in the choice of the tangential component ζ in (2.2). At first sight, the choice of ζ in (2.2) does not seem important since it only corresponds to a reparametrization. However, observe that we cannot take $\zeta^{(j)} \equiv 0$. Indeed, following [18] (as well as [5], if the solution is sufficiently smooth in space and time, then the values of $\zeta^{(j)}(t, 0)$ are determined entirely by the values of $\kappa^{(j)}(t, 0)$ and vice versa. Different choices of $\zeta^{(j)}$ taking these required values at the vertices then do correspond to reparametrizations. However, suppose $\zeta^{(j)} \equiv 0$, and let $P(t)$ denote the location of any interior vertex. Then for each j , $P'(t) = \partial_t \gamma^{(j)}(t, 0) = \kappa^{(j)}(t, 0) \nu^{(j)}(t, 0)$; since the $\nu^{(j)}$ are independent, we deduce that $P'(t) \equiv 0$, i.e., the vertex remains fixed. However, imposing (2.3) and also requiring this vertex to be constant leads to an overdetermined system of PDEs. It is also the case that, due to the ‘gauge invariance’, i.e., the possibility of reparametrization, (2.2) with $\zeta \equiv 0$ is not strictly parabolic, see [8].

In any case, there are various natural choices for ζ , but the one we use here is $\zeta = |\partial_x \gamma|^{-3} \langle \partial_x^2 \gamma, \partial_x \gamma \rangle$ so that the flow equation becomes

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2}.$$

Definition 2.3 (Parametrized network flow). *Let $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^n$ be a (possibly irregular) initial network consisting of n regular curves $\gamma^{(j)} \in \mathcal{C}^2([0, 1], \overline{\Omega})$. A one-parameter family $\Gamma(t) = \{\gamma^{(j)}(t, \cdot)\}_{j=1}^m$, $0 < t < T$ and $m \geq n$, of regular networks is a solution to the network flow with initial condition Γ_0 and with Dirichlet boundary conditions at $\partial\Omega$ if the following is true: for each $t \in (0, T)$, $\gamma^{(j)}(t, \cdot)$ is a smoothly parametrized curve in Ω with $\partial_x \gamma^{(j)} \neq 0$ for all $(t, x) \neq (0, 0)$; each network $\Gamma(t)$ is regular; each $\gamma^{(j)}$ evolves according to the equation*

$$\partial_t \gamma^{(j)}(t, x) = \frac{\partial_x^2 \gamma^{(j)}(t, x)}{|\partial_x \gamma^{(j)}|^2};\tag{2.4}$$

at each triple junction where three curves meet, the matching conditions (2.3) are satisfied, while all exterior vertices, where any one of the $\gamma^{(j)}(t, 1) \in \partial\Omega$, remain fixed as t varies; finally, we require that $\Gamma(t)$ converges in Hausdorff distance as a family of sets to Γ_0 as $t \searrow 0$.

At this stage the convergence of $\Gamma(t)$ to Γ_0 as $t \searrow 0$ can be taken as locally uniform convergence of sets, but we shall give a more precise description of this convergence later.

Remark 2.4. *As checked in §6, the Herring boundary conditions satisfy the Lopatinski-Shapiro conditions for the linearization of the network flow. This, of course, was known much earlier, cf. [5], but here is quite simple. This implies that each $\gamma^{(j)}$ is $\mathcal{C}^\infty$ for $x \in [0, 1]$ and $t > 0$.*

2.3 Irregular initial data

Now let us turn to the main question in this paper. Suppose that Γ_0 is an irregular network. It has a collection of exterior vertices $q_1, \dots, q_r$ on $\partial\Omega$ as well as some number of interior vertices $p_1, \dots, p_s$. Associated to each p_i is a collection of indices $I_i \subset \{1, \dots, n\}$ such that each of the curves $\gamma_0^{(j)}$, $j \in I_i$, terminate at p_i at either $x = 0$ or $x = 1$. We set $k_i = |I_i|$; this is the valence of that vertex. Similarly, each end of any $\gamma_0^{(j)}$ is either an interior or exterior vertex. We assume for uniformity of notation that if an exterior vertex q_i is one end of the curve $\gamma_0^{(j)}$, then $\gamma_0^{(j)}(1) = q_i$.

For each internal vertex p_i , there is a set of rays emanating from p_i determined by the tangents $\tau^{(j)} = \partial_x \gamma_0^{(j)}$ of incoming arcs. This forms a *fan* $\mathcal{F}_i$. In the next subsection we recall an old result by two of us [20] that shows that for each fan $\mathcal{F}_i$ with at least 3 rays there exists one (and, if $k_i > 3$, many) regular self-similar expanding solutions of the network flow which converge to $\mathcal{F}_i$ as $t \searrow 0$. Denote this set of expanding solitons by $\{\Gamma_j^{\text{ss},(i)}\}_{j \in J}$. There is a unique such soliton for each combinatorial type of network. A priori there might be infinitely many, so we do not specify the enumeration precisely. Choosing one soliton for each i we can define a regular network obtained by inserting a truncated version of $\Gamma_j^{\text{ss},(i)}$ at each p_i . This determines a particular combinatorial type of regular network, and our eventual theorem shows that there exists an actual solution of the network flow with this topology, at least for some short interval of time. For these fans, and also for the original network Γ_0 , the resolution appears instantaneously in the flow at time 0; a number of new internal arcs are generated at each internal vertex.

For the truncation to a ball of large radius of each $\Gamma_j^{\text{ss},(i)}$ there are k_i external vertices. An elementary combinatorial argument proves that if this soliton network is a tree, then there are $k_i - 2$ interior vertices and $2k_i - 3$ edges, of which $k_i - 3$ are interior edges. The exterior edges are those which limit to the rays of the fan $\mathcal{F}_i$. These enumerations show that, assuming Γ_0 and each of the solitons are trees, then

$$m = n + \sum_i (k_i - 3).$$

The defect $m - n$ determines the number of new edges which appear instantaneously in the flowout. In general, however, if some of these networks are not trees, then $m - n$ may be much larger.

In the following we denote by $\mathcal{V}_0$ the set of all vertices; this set is the union $\mathcal{V}_{0,\text{int}} \cup \mathcal{V}_{0,\text{ext}}$ of interior and exterior vertices. The set of vertices of a flowout is denoted by $\mathcal{V}$, and decomposes into $\mathcal{V}_{\text{int,or}} \cup \mathcal{V}_{\text{int,new}} \cup \mathcal{V}_{\text{ext}}$, where $\mathcal{V}_{\text{ext}} = \mathcal{V}_{0,\text{ext}}$ are the exterior vertices, $\mathcal{V}_{\text{int,or}}$ is the set of vertices already present at $t = 0$, and $\mathcal{V}_{\text{int,new}}$ is the set of new vertices that appear because of the self-similar resolution. Similarly, the set $\mathcal{E}_0$ of all edges of an initial network Γ_0 decomposes into $\mathcal{E}_{0,\text{int}} \cup \mathcal{E}_{0,\text{ext}}$ of interior and exterior edges; for the set $\mathcal{E}$ of edges of a flowout, the decomposition becomes $\mathcal{E}_{\text{ext}} \cup \mathcal{E}_{\text{int,or}} \cup \mathcal{E}_{\text{int,new}}$, where $\mathcal{E}_{\text{int,or}}$ consists of the original edges which exist both at $t = 0$ and $t > 0$, while $\mathcal{E}_{\text{int,new}}$ contains those new edges which arise as internal edges of the self-similar resolution of each internal vertex.

2.4 Expanding solitons

It is now standard in any geometric flow that there is an intimate relationship between singularity formation in the flow and the existence and nature of self-similar solutions to the flow. Self-similar collapsing solutions determine, or at least predict, how singularities form as the network degenerates, whereas self-similar expanding solutions indicate how singular configurations (i.e.,

irregular networks in our setting) should flow to smooth (regular) ones. Beyond this, of course, self-similar solutions also provide an interest class of examples of solutions.

A network $\Gamma = \{\gamma^{(j)}\}$ is an expanding self-similar solution to network flow if each evolving curve $\gamma^{(j)}$ has the form

$$\gamma^{(j)}(t, x) = \lambda(t)\eta^{(j)}(x/\lambda(t)); \quad (2.5)$$

the expansion factor $\lambda(t)$ is determined below.

As in the last subsection, let $\mathcal{F} = \{\ell^{(1)}, \dots, \ell^{(k)}\}$ be a fan of rays emanating from the origin in $\mathbb{R}^2$. Inspired by [21], the paper [20] establishes the existence of at least one expanding self-similar solution of the network flow which limits to $\mathcal{F}$ as $t \searrow 0$. When $k = 3$, this solution is unique, but for $k > 3$ there are many (in fact, the number of solutions grows very rapidly as k increases).

First, a short calculation obtained by inserting (2.5) into (2.1) and fixing the normalization $\lambda(1/2) = 1$ leads to the *expander equation*

$$\kappa - \langle \eta, \nu \rangle = 0, \quad \lambda(t) = \sqrt{2t}. \quad (2.6)$$

This equation is equivalent to the geodesic equation for the metric $g = e^{|x|^2}|dx|^2$ on $\mathbb{R}^2$.

The $t = 1/2$ slice of a regular expanding self-similar network consists of a union of curves, each an arc or half-ray solution of (2.6), which together constitute a regular network. Such a network is a stable critical point of the length functional in $(\mathbb{R}^2, g)$ (where, as usual, it suffices to look at the length of networks in any large ball $B_R(0)$). The condition that expanding self-similar solution be a flowout of the initial fan $\mathcal{F}$ means that for each external edge, the image of $x \mapsto \sqrt{2t}\eta^{(j)}(x/\sqrt{2t})$ converges to $\ell^{(j)}$ as a set as $t \searrow 0$.

The individual geodesic arcs and rays for the metric g are qualitatively similar to geodesics in hyperbolic space, as expected since the curvature of g is everywhere negative. For example, if ℓ and ℓ' are any two rays emanating from the origin, then there is a unique complete geodesic for this metric which is asymptotic to these rays along its two ends. A convenient way to picture this is to realize the geodesic compactification of $(\mathbb{R}^2, g)$ as a closed ball B . A limiting direction, i.e., the asymptotic limit of any ray ℓ , corresponds to a point $q \in \partial B$. Thus the specification of the k rays in the fan $\mathcal{F}$ is the same as choosing k distinct points $q_1, \dots, q_k \in \partial B$. A solution of this problem, i.e., the configuration at $t = 1/2$ of an expanding self-similar solution emanating from $\mathcal{F}$ is thus a geodesic network for this metric. If there are no loops, then it is a Steiner tree for $(\mathbb{R}^2, g)$.

Proposition 2.5. [20, Main Theorem] *Let $\mathcal{F} = \{\ell^{(1)}, \dots, \ell^{(k)}\}$ be a fan of rays in $\mathbb{R}^2$ emanating from the origin, and $q_1, \dots, q_k$ the corresponding points (listed in cyclic order) around ∂B . Then the set of expanding self-similar solutions of the network flow with initial datum $\mathcal{F}$ is in bijective correspondence with the set of (possibly disconnected) regular networks on B , each arc of which is a geodesic for the metric g , with boundary $\{q_1, \dots, q_k\}$. For each decomposition of the set of boundary points into subsets J_i of collections of adjacent points, there is at least one geodesic Steiner tree whose components are characterized by having boundary in the points q_j , $j \in J_i$.*

We ‘normalize’ by only considering solutions of (2.4) rather than (2.1).

Definition 2.6. *An expanding soliton is a solution $\Gamma(t) = \{\gamma^{(j)}(t, x)\}$ of the network flow with initial network Γ_0 , where for each j ,*

$$\gamma^{(j)}(t, x) = \sqrt{2t}\eta^{(j)}(x/\sqrt{2t}),$$

where, if we set $s = x/\sqrt{2t}$, then each $\eta^{(j)}$ solves

$$\frac{\partial_s^2 \eta^{(j)}}{|\partial_s \eta^{(j)}|^2} + (s \partial_s \eta^{(j)} - \eta^{(j)}) = 0. \quad (2.7)$$

This soliton is called *regular* if the curves $\eta^{(j)}(s)$ are arcs of rays of a regular geodesic network in $(\mathbb{R}^2, g)$, and *minimizing* if it is regular and a tree.

The parametrization of each $\gamma^{(j)}$ is singular at $t = 0$; we explain how to think about this in the next section.

To derive (2.7), define $s = x/\sqrt{2t}$ as above, and $\tau = \sqrt{2t}$. Thus $\gamma^{(j)} = \tau \eta^{(j)}(s)$. We then compute that

$$\partial_t = \frac{1}{\sqrt{2t}} \partial_\tau - \frac{1}{2\sqrt{2}} \frac{x}{\sqrt{t^3}} \partial_s = \frac{1}{\tau} \partial_\tau - \frac{s}{\tau^2} \partial_s \quad \partial_x = \frac{1}{\sqrt{2t}} \partial_s = \frac{1}{\tau} \partial_s \quad \partial_x^2 = \frac{1}{2t} \partial_s^2 = \frac{1}{\tau^2} \partial_s^2,$$

and hence the flow equation reduces to

$$\left(\frac{1}{\tau} \partial_\tau - \frac{s}{\tau^2} \partial_s \right) (\tau \eta(s)) = \frac{1}{\tau} \frac{\partial_s^2 \eta(s)}{|\partial_s \eta(s)|^2}.$$

This is the same as (2.7).

Later we shall require precise asymptotic information about the solutions η to this equation.

Lemma 2.7. *Let η be a solution to (2.7). Then as $s \rightarrow \infty$,*

$$\eta(s) = as + \mathcal{O}(e^{-Cs^2})$$

for some $a \in \mathbb{R}^2$ and $C > 0$.

Proof. Define the quantities

$$\mu = s \partial_s \eta - \eta, \quad \sigma = |\mu|^2, \quad \text{and} \quad \nu = |\partial_s \eta|^2;$$

using $'$ for ∂_s for simplicity, we compute

$$\mu' = s \eta'' = -s \nu \mu, \quad \sigma' = -2s \nu \sigma, \quad \text{and} \quad \nu' = 2 \langle \eta', \eta'' \rangle = -2 \nu \langle \eta', \mu \rangle \geq -2 \nu^{3/2} \sigma^{1/2}.$$

The equation for σ gives $(\log \sigma)' = -2s \nu \leq 0$, hence $\sigma(s) \leq \sigma(0)$ for $s > 0$, while

$$\nu^{-3/2} \nu' \geq -2 \sigma^{1/2} \Rightarrow \nu(s) \geq \left(C_1 + \int_1^s \sigma^{1/2}(s') ds' \right)^{-2}. \quad (2.8)$$

Our proximate goal is to show that $\nu(s) \geq C > 0$ as $s \rightarrow \infty$. Indeed, if that is the case, then $(\log \sigma)' \leq -Cs$, hence $\sigma \leq C_1 e^{-C_2 s^2}$.

To show this positive lower bound on ν , insert the initial estimate $\sigma \leq C$ into (2.8) to get $\nu \geq C/s^2$, hence $-s \nu \leq -C/s$. Using this in the equation for $(\log \sigma)'$ yields $\sigma(s) \leq C s^{-\alpha}$ for some $\alpha > 0$. We now repeat this argument, taking advantage of this improved bound. There are three possible estimates for $\int_1^s \sigma^{1/2}(s') ds'$, depending on whether $\alpha < 2$, $\alpha = 2$ or $\alpha > 2$. In the first case, we derive that

$$\int_1^s \sigma^{1/2} \leq C s^{1-\alpha/2} \Rightarrow \nu \geq C s^{\alpha-2} \Rightarrow (\log \sigma)' \leq -C s^{\alpha-1} \Rightarrow \sigma \leq C e^{-C s^\alpha} \Rightarrow \nu \geq C > 0.$$

In the third case, $\int_1^s \sigma^{1/2} \leq C$, which gives directly the positive lower bound for ν . The borderline case between these two produces an exponential decay rate for σ which implies as before that $\nu \geq C > 0$.

Returning to the original equation for σ , a positive lower bound for ν shows that $\sigma \leq C_1 e^{-C_2 s^2}$. With slightly more care, one can deduce that $C_2 = 1$, but this is not important for our purposes.

Now observe that $(s^{-1}\eta)' = s^{-1}\eta' - s^{-2}\eta = s^{-2}\mu$ decays like e^{-Cs^2} , so we conclude finally that

$$\eta(s) = as + \mathcal{O}(e^{-Cs^2})$$

for some $a \in \mathbb{R}^2$. □

Lemma 2.8. *The solutions of (2.6) and (2.7) are in bijective correspondence.*

Proof. Any solution of the network flow is a solution of the motion by curvature, so it suffices to consider only the normal component of the velocity. If $\gamma^{(j)}(t, x) = \sqrt{2t}\eta^{(j)}(x/\sqrt{2t})$ is a solution of the network flow, then $\eta^{(j)}$ is a solution of (2.6). On the other hand, fix a solution of (2.6). This is the value at $t = 1/2$ of a self-similar solution of the network flow, and the collection of $\eta^{(j)}$ determines a regular network. By [17, Theorem 4.14], regular solutions of this flow are unique up to a reparametrization. The change of variable $s = x/\sqrt{2t}$, $\tau = \sqrt{2t}$ yields a solution of (2.7). □

Combining Proposition 2.5 and Lemma 2.8, we see that for every $\mathcal{F}$, there exists at least one expanding soliton that flows out from $\mathcal{F}$, and that these flowouts are in one-to-one correspondence with geodesic networks with asymptotic boundary values determined by $\mathcal{F}$.

3 Parabolic blowups and the lifted flow

We now come to the main conceptual step of this paper. This involves recasting the flow equations by introducing parabolic blowups of the interior vertices of the network at $t = 0$, both in the domain and the range. This process exhibits the true homogeneities of the problem and leads to the precise regularity theory below. It also makes the appearance of solitons particularly natural.

Consider for the moment the simplest situation, where $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^3$ is a “smooth triod”, i.e., a network with exactly three curves meeting at a regular vertex, the other three endpoints are three distinct external vertices. Even in this case a solution Γ_t of the network flow is not necessarily smooth at the interior vertex at $t = 0$. In such a situation, smoothness at this ‘corner’ only holds if the initial triod $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^3$ satisfies an infinite set of compatibility conditions at this vertex. This is explained for the nonlinear flow in [5, 18]. These compatibility conditions are obtained by taking successive derivatives of the equation and boundary conditions and using these to equate mixed partials of the various $\gamma^{(j)}$ at the corner $x = t = 0$. Equality of mixed partials of course presupposes smoothness at this corner. Thus, for example, the first compatibility condition comes from evaluating (2.3) at this corner. The next condition arises by combining the t derivative of the first equation in (2.3) with the equation of motion (2.4), again evaluated at the corner, to get

$$\frac{\partial_x^2 \gamma_0^{(i)}(0)}{|\partial_x \gamma_0^{(i)}(0)|^2} = \frac{\partial_x^2 \gamma_0^{(j)}(0)}{|\partial_x \gamma_0^{(j)}(0)|^2}$$

for every $i \neq j$. The full sequence of compatibility conditions is obtained similarly.

This issue is not special to these systems, of course. Even for the most basic linear parabolic problem on a manifold with boundary Ω , a solution of a mixed Cauchy–Dirichlet boundary problem

$$(\partial_t - \Delta)u = 0, \quad u(0, \cdot) = \phi \in \mathcal{C}^\infty, \quad u|_{\partial\Omega \times \mathbb{R}^+} = \psi \in \mathcal{C}^\infty$$

is not smooth at $\partial\Omega \times \{0\}$ unless infinitely many compatibility conditions relating the Taylor series of ϕ at $\partial\Omega$ and ψ at $t = 0$ are satisfied. While certainly known as folklore for a century, this was investigated in some detail by Smale [22]. Our point is that it is not just a minor curiosity, but has some important consequences. For example, if the compatibility conditions are only satisfied up to some order K , then the mixed boundary problem can only be solved in $\mathcal{C}^{k+\delta, (k+\delta)/2}([0, T) \times \Omega)$ when $k \leq K$. In particular, if $\phi \equiv 0$, the Dirichlet heat semigroup is not strongly continuous on $\mathcal{C}^{k+\delta}(\Omega)$.

It is not a standard perspective, but also not hard to prove, that for this linear problem, u becomes smooth if we pass to the space obtained by parabolically blowing up this corner. This means the following. Choose local coordinates (x, y) on Ω with x a defining function for the boundary. Now introduce “parabolic polar coordinates” in (x, t) , namely $x = r \cos \theta$, $t = r^2 \sin \theta$ (or equivalently, $r = \sqrt{x^2 + t}$, $(x/r, t/r^2) = (\cos \theta, \sin \theta)$). The parabolic blowup

$$\Omega_h := [\Omega \times \mathbb{R}^+; \partial\Omega \times \{0\}; dt]$$

is the space on which r, θ, y become smooth coordinates. This is called the heat single space and has corners up to codimension 2; it has the two original boundary hypersurfaces, $\partial\Omega \times \mathbb{R}^+$ and $\Omega \times \{0\}$, as well as the new hypersurface $r = 0$, which we call the front face and denote ff . It is often more convenient to use projective coordinates on this space. One such coordinate system, for example, is

$$s = x/\sqrt{t}, \tau = t, y,$$

while another is

$$\sigma = t/x^2, \eta = x, y.$$

Note that (s, τ, y) is only a nonsingular coordinate system on ff and the lift of $\partial\Omega \times \mathbb{R}^+$, while (σ, η, y) is only nonsingular on ff and $\Omega \times \{0\}$. The regularity theorem stated above says, more precisely, that if u is a solution to the Cauchy–Dirichlet problem above, with ϕ, ψ both smooth, then the lift of u lies in $\mathcal{C}^\infty(\Omega_h)$. The new front face intermediates between ϕ and ψ and regularizes the solution.

3.1 Blowup constructions

Let us now return to the problem at hand. Suppose that Γ_0 is a (possibly irregular) network in a compact smoothly bounded domain $\Omega \subset \mathbb{R}^2$. We shall introduce two separate blowups, one of the domain and the other of the range, which ‘regularize’ the network flow problem.

Recalling the notation of §2, consider an initial network $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^n$ where each $\gamma_0^{(j)}$ is a smooth map $[0, 1] \rightarrow \mathbb{R}^2$. The images of these maps constitute the set of initial edges $\mathcal{E}_0$. This network has a set of external vertices $q_1, \dots, q_r \in \partial\Omega$ and interior vertices $p_1, \dots, p_s$. For simplicity, we require that the compatibility conditions are satisfied to all orders at every exterior vertex; these conditions state that $\partial_x^i \gamma_0^j(1) = 0$ for all $i \geq 2$ and indices j corresponding to exterior edges. If these conditions are all satisfied, then classical methods are sufficient to prove estimates near these boundary points. It is straightforward to modify the constructions and arguments of this paper to include this analysis, and in fact to extend this to the case where these compatibility conditions are not satisfied. We leave this to the interested reader.

The eventual solution will be an evolving network $\Gamma(t) = \{\gamma^{(j)}\}$ with $m \geq n$ curves. Extra curves are generated at every interior vertex where the valence is greater than 3. We may regard this entire network as a collection of mappings from a disjoint union of regions, each in the (t, x) plane. These regions are labelled Q_j , $j = 1, \dots, n$, for the evolutions of each of the initial curves $\gamma_0^{(j)} \in \Gamma_0$ and P_j , $j = n+1, \dots, m$, for the new families of curves generating out of the vertices. We shall often think of each curve $\gamma^{(j)}$ simultaneously as a 1-parameter family of maps $x \mapsto \gamma^{(j)}(t, x) \in \Omega$ but also as a map $(t, x) \mapsto (t, \gamma^{(j)}(t, x)) \in \mathbb{R}^+ \times \Omega$. For simplicity, no separate notation is introduced for the latter; this conflation should not lead to any confusion.

Blowup of the domain: For each $j = 1, \dots, n$, let $Q^{(j)} = \{(t, x) \in \mathbb{R}^2 : 0 \leq t, 0 \leq x \leq 1\}$ be the domain parametrizing the evolution of $\gamma_0^{(j)}$. If this is an interior edge, we define the parabolic blowup

$$Q_h^{(j)} = [Q^{(j)}, (0, 0) \cup (0, 1); dt];$$

this is, by definition, the set obtained by replacing each of the two points $(0, 0)$ and $(0, 1)$ by their inward-pointing unit parabolic normal bundles, and endowing the resulting space with the natural topology and smooth structure for which the lifts of smooth functions on $Q^{(j)}$ and parabolic polar coordinates around these two corner points are both smooth. These parabolic polar coordinates are defined near $(0, 0)$ as

$$\rho = \sqrt{t + x^2} \geq 0, \quad \omega = \arcsin(t/\rho^2) = \arccos(x/\rho) \in [0, \pi/2] \Leftrightarrow (t, x) = (\rho \cos \omega, \rho^2 \sin \omega),$$

and similarly near $(0, 1)$. The space $Q_h^{(j)}$ has two new boundary faces, called the front faces of $Q_h^{(j)}$, and denoted ff_0 and ff_1 (or just ff_0 if we do not need to specify). If $\gamma_0^{(j)}$ is an exterior edge, then $Q_h^{(j)}$ is defined by only blowing up the corner at $(0, 0)$ (see Figure 1).

There is another useful way to think about parabolic blowup: points of ff_0 , for example, are identified as equivalence classes of smooth paths $\tilde{\zeta}(\xi) = (\zeta_1(\xi), \zeta_2(\xi))$ in $Q^{(j)}$, $\xi \geq 0$, with $\tilde{\zeta}(0) = (0, 0)$. Two paths $\zeta, \tilde{\zeta}$ are equivalent, $\zeta \sim \tilde{\zeta}$, if there exists some $c > 0$ such that $|\zeta_1(s) - \zeta_1(cs)| \leq Cs^2$, $|\zeta_2(s) - \zeta_2(cs)| \leq Cs^3$. Thus this construction separates out the endpoints of the paths $(\alpha s^2, s)$, $0 < \alpha < \infty$; each such path determines a unique point in the interior of ff_0 . Paths $\zeta(\xi)$ which do not approach the corner parabolically limit to one or the other boundary of ff_0 .

Define

$$Q_h = \sqcup_{j=1}^n Q_h^{(j)}, \quad Q = \sqcup_{j=1}^n Q^{(j)}.$$

There is a natural blowdown map $\beta : Q_h \rightarrow Q$ which is the identity away from ff and which collapses each front face to $(0, 0)$ and $(0, 1)$, respectively. As above, $\mathcal{C}^\infty(Q_h)$ is generated by $\beta^*(\mathcal{C}^\infty(Q))$ and the functions ρ and ω .

Each component of Q_h has either three or four boundary faces: the front faces ff_0 and ff_1 , two side faces lf and rf and the bottom face bf . Each ff_j is given by $\rho = 0$ in local parabolic polar coordinates; the left and right faces are the vertical sides ‘above’ ff where $\omega = \pi/2$, and the bottom face is the initial face $t = 0$, between the two front faces, and at $\omega = 0$. The eventual map $\gamma^{(j)}(t, x)$ determined by the evolution will be defined on $Q_h^{(j)}$ rather than $Q^{(j)}$. It has initial condition $\gamma^{(j)}(x)$ on bf and satisfies the matching (Herring) conditions along the left and right faces. Its behavior on the front face(s) is one of the key issues addressed below.

As in §2, the evolving network $\Gamma(t)$ will also include a set of new internal edges $\gamma^{(j)}$, $j = n+1, \dots, m$. These are defined on the set $P^{(j)}$ defined by $t \geq 0$ and $0 \leq x \leq \sqrt{t}$. The fact that this region shrinks to a point at $t = 0$ corresponds to the fact that these curves $\gamma^{(j)}$ disappear

into the interior vertex points p_i . We also blow these up parabolically at $(0, 0)$, obtaining regions

$$P_h^{(j)} = [P^{(j)}, (0, 0); dt].$$

In parabolic polar coordinates defined exactly as above, this space has coordinates (ρ, ω) with $\rho \geq 0$ and $0 \leq \omega \leq \arcsin(1/2) = \pi/6$. We write

$$P_h = \sqcup_{j=n+1}^m P_h^{(j)}.$$

The space $Q_h \sqcup P_h$ is the entire desingularized domain of the evolving network $\Gamma(t)$.

In computations it is usually simpler to work in coordinate systems different than the parabolic polar coordinates above. In particular, we introduce two sets of projective coordinates near each front face. In the following, for simplicity, we describe these only near the front face over $(0, 0)$, but these are obviously defined near the front face over $(0, 1)$.

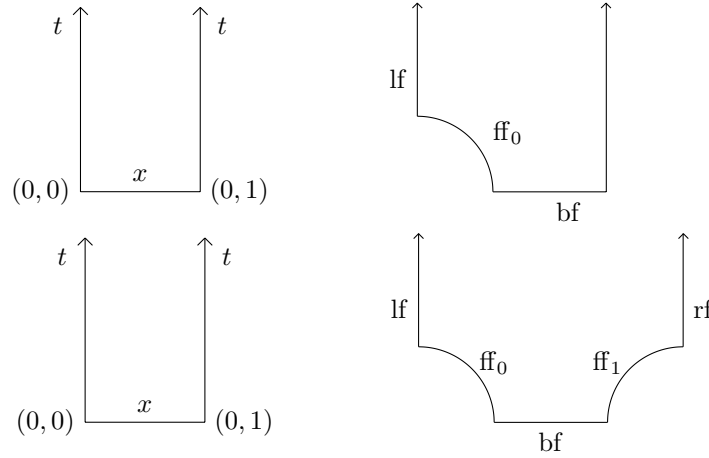


Figure 1: Above: The spaces Q and Q_h in the case $\gamma^{(j)}(0)$ is an interior irregular vertex and $\gamma^{(j)}(1)$ is an exterior vertex. Below: The spaces Q and Q_h in the case both $\gamma^{(j)}(0)$ and $\gamma^{(j)}(1)$ are interior irregular vertices.

- **Coordinates (τ, s) .**

Define $\tau = \sqrt{2t}$ and $s = \frac{x}{\sqrt{2t}}$. Then (τ, s) is a nondegenerate coordinate system on $Q_h^{(j)}$ near ff_0 away from bf ; there is a corresponding set of projective coordinates near ff_1 , of course. In addition, we may also use these coordinates in $P_h^{(j)}$ near its front face where $\rho = 0$. Note that in each $Q_h^{(j)}$, $s \in [0, \infty)$, while in $P_h^{(j)}$, $0 \leq s \leq 1$. The variable τ is a defining function for the front face in each of these cases in the sense that it vanishes simply on ff and nowhere else, and is commensurable with ρ on any compact set in $Q_h^{(j)}$ which does not intersect rf , and on the entirety of $P_h^{(j)}$. The variable s is a defining function for lf or rf , and identifies int ff with $\mathbb{R}^+$.

We compute

$$\partial_t = \frac{1}{\tau} \partial_\tau - \frac{s}{\tau^2} \partial_s, \quad \partial_x = \frac{1}{\tau} \partial_s, \quad \partial_x^2 = \frac{1}{\tau^2} \partial_s^2,$$

and hence the linear heat equation becomes

$$\partial_t - \partial_x^2 = \frac{1}{\tau^2} (\tau \partial_\tau u - s \partial_s u - \partial_s^2 u) = 0. \quad (3.1)$$

The singular factor on the right is inevitable because the two sides of this equality must both have homogeneity -2 with respect to the scaling $(t, x) \mapsto (\lambda^2 t, \lambda x)$, and s and τ are homogeneous of degree 0. In any case, it is sufficient to study the operator

$$\tau \partial_\tau - s \partial_s - \partial_s^2.$$

- **Coordinates** (T, y) .

The (τ, s) coordinates are not valid near the bottom face where $t = 0$ and in particular near the intersection of ff and bf. Near this corner we introduce an alternate set of projective coordinates $y = x$ and $T = \frac{t}{x^2}$. These are singular along the positive t -axis; the variable y is now the defining function for ff and T is the defining function for bf.

The expressions for ∂_t and ∂_x in these new coordinates are

$$\partial_t = \frac{1}{y^2} \partial_T, \quad \partial_x = \partial_y - \frac{2T}{y} \partial_T, \quad \partial_x^2 = \partial_y^2 - \frac{4T}{y} \partial_y \partial_T + \frac{4T^2}{y^2} \partial_T^2 + \frac{6T}{y^2} \partial_T,$$

and now the linear heat operator takes the much less tractable form

$$\frac{1}{y^2} (\partial_T u - 6T \partial_T u - 4T^2 \partial_T^2 u + 4Ty \partial_T \partial_y u - y^2 \partial_y^2 u) = 0.$$

As before, it suffices to consider the slightly nicer looking operator

$$\partial_T - 6T \partial_T - 4T^2 \partial_T^2 + 4Ty \partial_T \partial_y - y^2 \partial_y^2. \quad (3.2)$$

We also record that

$$T = \frac{1}{2s^2}, \quad y = s\tau.$$

This change of variables leads to sufficiently ungainly expressions that we do not use this coordinate system for any serious calculations.

Blowup of the range: The dilation properties of expanding solitons suggest that the homogeneity in the range should also be emphasized. In other words, the ansatz that solitons take the form $\gamma(t, x) = \sqrt{2t}\eta(x/\sqrt{2t})$ already indicates the relevance of the projective coordinate $s = x/\sqrt{2t}$, but also suggests the change of variable $z \in \mathbb{R}^2$ to $w = z/\sqrt{2t}$ in the range. We formalize this as follows.

Define $Z = \mathbb{R}_t^+ \times \mathbb{R}_z^2$. Following the remark just before §3.1, in the following we regard each $\gamma^{(j)}$ as a map into $\mathbb{R}^+ \times \Omega$ via $(t, x) \mapsto (t, \gamma^{(j)}(t, x))$. As suggested by the preceding paragraph, we lift this map by blowing up both the domain and the range. In other words, each lift should be regarded as a map

$$\gamma^{(j)}(t, x) : Q_h^{(j)} \longrightarrow Z_h = [Z, \{(0, p_1), \dots, (0, p_s)\}; dt].$$

This latter space is obtained from Z by taking the parabolic blowup of Z at each of the interior vertices p_i at $t = 0$. This parabolic blowup is defined exactly as before, by replacing each $(0, p_i)$ with the inward-pointing spherical parabolic normal bundle. As before, this becomes more tangible in locally defined parabolic polar coordinates. Fixing one interior vertex p_i , suppose that $p_i = 0$ and define

$$R = \sqrt{t + |z|^2}, \quad \Theta = (t/R^2, z/R) = (\Theta_0, \Theta').$$

Then Z_h has a front face $\text{ff} = \{R = 0\}$ and a bottom face $\text{bf} = \{\Theta_0 = 0\}$. There is a codimension two corner where these two faces intersect.

There are useful projective coordinates here too, namely

$$\tau = \sqrt{2t}, \quad w = z/\sqrt{2t}.$$

Thus $\tau = 0$ is a defining function for ff , while w is a projectively natural linear coordinate for ff . These coordinates are valid away from bf . Thus $(t, \gamma^{(j)}(t, x))$ lifts to $(\frac{1}{2}\tau^2, \tau\eta^{(j)}(\tau, s))$.

The lifted equation: We now consider the evolution equation in terms of these blowups, i.e., we lift the maps $\gamma^{(j)}$ to maps between $Q_h^{(j)}$ and Z_h . This lifting is effected simply by writing

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2} \quad (3.3)$$

using the coordinate systems (τ, s) on $Q_h^{(j)}$ and (τ, w) on Z_h .

Set $\gamma^{(j)} = \tau\eta^{(j)}$, which corresponds to the introduction of the projective coordinate on Z_h , and for simplicity, drop the superscript (j) for the time being. As noted earlier, if γ is an arc in an expanding soliton, then η depends only on s . Recalling that $\partial_t = \tau^{-2}(\tau\partial_\tau - s\partial_s)$ and $\partial_x = \tau^{-1}\partial_s$, hence (3.3) becomes

$$\frac{1}{\tau^2}(\tau\partial_\tau - s\partial_s)(\tau\eta) = \frac{\tau^{-2}\partial_s^2(\tau\eta)}{|\tau^{-1}\partial_s(\tau\eta)|^2},$$

or finally,

$$(\tau\partial_\tau + 1 - s\partial_s)\eta = \frac{\partial_s^2 \eta}{|\partial_s \eta|^2}. \quad (3.4)$$

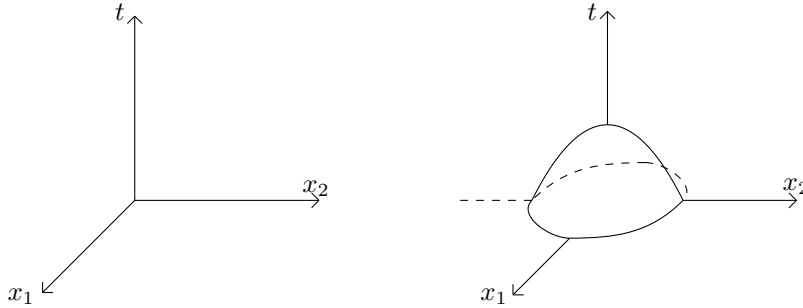


Figure 2: From $Z = \mathbb{R}_t^+ \times \mathbb{R}_z^2$ to Z_h

In particular, if γ is an expander, so $\eta = \eta(s)$, this yields the dimensionally reduced soliton equation

$$\frac{\partial_s^2 \eta}{|\partial_s \eta|^2} + (s\partial_s - 1)\eta = 0. \quad (3.5)$$

Observe that if $\eta(s)$ solves this last equation, then so does $\eta(cs)$ for any constant $c > 0$. We are therefore free to normalize by assuming that

$$|\partial_s \eta| \rightarrow 1 \quad \text{as } s \rightarrow \infty. \quad (3.6)$$

Boundary conditions: We now explain the various boundary conditions and how they should be formulated along each of the boundary faces of $Q_h \sqcup P_h$. Clearly we must demand that the restriction of each $\gamma^{(j)}$, $j \leq n$, to the bottom face of $Q_h^{(j)}$ is the given parametrization $\gamma_0^{(j)}$ of that arc of the initial network. Next, fix any j and consider either the left or right face of the appropriate domain region $Q_h^{(j)}$ or $P_h^{(j)}$. If the left face of this domain does not correspond to an exterior vertex, then it is paired to exactly two other faces (either left or right) of some other $Q_h^{(i)}$ or $P_h^{(i)}$, corresponding to the evolving trivalent vertex of the regular network $\Gamma(t)$. Let us denote this ‘cluster’ by $N(\ell)$, where ℓ is a vertex of $\Gamma(t)$, and assume that the matching boundaries are the left faces of each. Then in terms of the (τ, s) coordinate system, the boundary conditions at these faces become

$$\eta^{(i)}(\tau, 0) = \eta^{(j)}(\tau, 0) \quad \forall i, j \in N(\ell), \quad \sum_{i \in N(\ell)} \frac{\partial_s \eta^{(i)}(\tau, 0)}{|\partial_s \eta^{(i)}(\tau, 0)|} = 0.$$

Every left face corresponding to an interior vertex of $\Gamma(t)$ arises exactly once in one of these clusters. This specifies the boundary conditions along all of the side faces.

Finally, it is perhaps not obvious how to specify an ‘initial’ condition along any one of the front faces. To determine this, note that we expect the lifted map $\eta^{(j)}(\tau, s)$ to be bounded as $\tau \rightarrow 0$. We make the assumption that $\eta^{(j)}$ is in fact smooth up to the front face. This means that it has a boundary value $\eta_0^{(j)}(s)$. Noting that $\tau \partial_\tau \eta^{(j)}|_{\tau=0} = 0$, we deduce that

$$\frac{\partial_s^2 \eta_0}{|\partial_s \eta_0|^2} + (s \partial_s - 1) \eta_0 = 0,$$

which is precisely the soliton equation! In other words, solitons arise naturally as the initial conditions for the flow along these new front faces.

As a word of extra motivation, notice that the lifted flow equation (3.4) is characteristic along the front face. Change variables, setting $\bar{\tau} = \log \tau$. Then (3.4) becomes

$$(\partial_{\bar{\tau}} + 1 - s \partial_s) \eta = \frac{\partial_s^2 \eta}{|\partial_s \eta|^2},$$

which appears to be nondegenerate, but now of course, $\bar{\tau} \rightarrow -\infty$ along ff. In order for there to be a reasonable short-time solution for the original problem, it is necessary that this solution restrict to an ancient solution near the front face. As is well-known in evolution problems, the choice for limiting ‘initial’ data as $\bar{\tau} \rightarrow -\infty$ must be extremely rigid in order for the flow to exist on this semi-infinite time interval.

The final point is to recall that we have chosen a particular expanding self-similar solution $\Gamma_j^{\text{ss},(i)}$ for each interior vertex p_i of Γ_0 . The various arcs of this soliton correspond precisely to the various front faces of the domains $Q_h^{(j)}$ and $P_h^{(j)}$ which have left faces limiting to p_i . The $\tau = 1$ slice of the soliton itself should be regarded as the geometric limit of these maps along the corresponding front face of Z_h over $(0, p_i)$.

4 A model: the scalar heat equation

As a further step toward explaining our main theorem, we now carry out in full detail the method we are espousing for the scalar heat equation on the half-line, where a number of simplifications occur. We describe briefly at the end of this section the minor changes that need to be made to prove the corresponding result for higher dimensional domains.

Theorem 4.1. *Suppose that u is the (unique moderate-growth) solution of the heat equation*

$$u_t - u_{xx} = 0$$

on $Q := [0, \infty)_t \times [0, \infty)_x$ with boundary conditions

$$u(0, x) = \phi(x), \quad u(t, 0) = \psi(t),$$

where both ϕ and ψ are $\mathcal{C}^\infty$ on the closed half-line. for simplicity we assume that each is supported in $[0, 1)$. Then u is smooth on the space $Q_h = [[0, \infty)^2; (0, 0); \{dt\}]$ obtained by parabolically blowing up the quarter-space at the origin. This solution is $\mathcal{C}^\infty$ on the closed quarter-space only when infinitely many ‘matching conditions’ are satisfied.

The last assertion is classical, but the fact that u is always “smooth” on the blown up space if the matching conditions are not satisfied is the novel part of this.

The fact that the spatial domain is $\mathbb{R}^+$ rather than a closed interval $[0, 1]$ is immaterial, and the proof below could be extended to that case with only minimal change of notation. It is easy to adapt the proof below to the higher dimensional case, i.e., to solutions of the scalar heat equation on $\mathbb{R}_t^+ \times \Omega_x$, where Ω is a domain in $\mathbb{R}^n$ with smooth boundary. We comment further on this extension at the end of this section.

Proof of Theorem 4.1 There are two steps. In the first we show that the Taylor series of the boundary data at $x = 0$ and $t = 0$ respectively determine all terms in the expansion of u on Q_h . In other words, we produce a smooth function v on Q_h such that $(\partial_t - \partial_x^2)v = f$ vanishes to all orders as $t \searrow 0$ in ∂Q_h , and is supported in $\{(t, x) : 0 \leq t, x \leq 2\}$. In the second step we use the heat kernel to produce a function w which also vanishes to all orders at the boundary of Q and which solves $(\partial_t - \partial_x^2)w = f$ with $w(t, 0) = w(0, x) = 0$. The solution $u = v - w$ then has the desired properties. There is one non-local aspect in this procedure which only becomes apparent once we pass to the blowup.

Construction of the approximate solution v : We now construct an approximate solution v to the homogeneous linear heat equation. For simplicity, assume that $v(0, x) = \phi(x) \in \mathcal{C}^\infty(\mathbb{R}^+)$ and $v(t, 0) = 0$. We comment on the more general case where $v(t, 0) = \psi(t) \in \mathcal{C}^\infty(\mathbb{R}^+)$ later.

We shall *assume* that the lift of v to Q_h is smooth, and show how the heat equation and boundary conditions determine its Taylor series at each boundary.

First observe that the equation $(\partial_t - \partial_x^2)v \equiv 0$ (in Taylor series as $t \rightarrow 0$) determines the full Taylor series of v along $t = 0$ for $x > 0$ in terms of the initial data ϕ and its derivatives.

Now consider this equation near the front face ff of Q_h . Using the (τ, s) coordinates, write

$$v \sim \sum_{j=0}^{\infty} v_j(s) \tau^j.$$

Each $v_j(s)$ is a function on ff. The boundary condition $v(t, 0) = 0$ forces each $v_j(0) = 0$. Now substitute this expansion into (3.1), and cancel the extra factor of τ^{-2} , to get

$$(\tau \partial_\tau - s \partial_s - \partial_s^2)v \sim \sum_{j=0}^{\infty} (j v_j - s v_j' - v_j'') \tau^j \equiv 0,$$

where the prime denotes a derivative with respect to s . We conclude that

$$L_j v_j := v_j''(s) + s v_j'(s) - j v_j(s) = 0, \quad j = 0, 1, 2, \dots, \quad (4.1)$$

We bring in the boundary condition at $t = 0$, i.e., as $s \rightarrow \infty$, momentarily.

The equation $L_0 v_0 = 0$ is explicitly solvable. Indeed, setting $z = v'_0$, then the equation becomes $z' = -sz$, which means $z(s) = ce^{-s^2/2}$, and hence

$$v_0(s) = c \int_0^s e^{-\sigma^2/2} d\sigma$$

for some $c \in \mathbb{R}$. Observe that $v_0(s)$ converges to a limit $\alpha_0 > 0$ as $s \rightarrow \infty$. In fact, $v_0(s) = \alpha_0 + \mathcal{O}(R(s)e^{-s^2/2})$ for some polynomial $R(s)$. We do not need to keep careful track of these exponential remainders, so in all the following, we write simply $v_0(s) = \alpha_0 + E$ where $E = \mathcal{O}(e^{-\delta s^2})$ for some $\delta > 0$. We use similar expressions below when this leading part is more complicated than just a constant.

A fortuitous identity allows us to solve the other equations in this sequence for $j > 0$. To do this, observe that differentiating (4.1) with respect to s gives

$$(L_j v_j)' = L_{j-1} v_j' = (v_j')'' + s(v_j')' - (j-1)v_j' = 0.$$

Iterating $j-1$ times shows that $L_0 v_j^{(j)} = 0$. As shown above, this yields that $v_j^{(j)} = \text{const.} + E$, and hence

$$v_j(s) = P_j(s) + E$$

where P_j is a polynomial of degree j in s . Writing this out explicitly, we have

$$v_j(s) = c_{j,j}s^j + c_{j,j-1}s^{j-1} + \dots + c_{j,1}s + c_{j,0} + E. \quad (4.2)$$

We now use the heat equation to determine recursion relationships between these coefficients. First recall that $c_{0,0} \neq 0$ unless $v_0 \equiv 0$, which we assume is not the case. Inserting (4.2) into (4.1), and reindexing, we obtain

$$-c_{j,j-1}s^{j-1} + \sum_{\ell=0}^{j-2} ((\ell+2)(\ell+1)c_{j,\ell+2} + (\ell-j)c_{j,\ell})s^\ell = 0,$$

hence

$$(\ell+2)(\ell+1)c_{j,\ell+2} + (\ell-j)c_{j,\ell} = 0, \quad \ell = 0, \dots, j. \quad (4.3)$$

Since $c_{j,\ell} = 0$ for $\ell > j$, these recursion relationships show that $c_{j,j}$ may be chosen freely, while $c_{j,j-1} = 0$. They also recursively determine each $c_{j,j-2k}$ as a nonzero multiple of $c_{j,j}$, and show that each $c_{j,j-2k-1} = 0$ since it is a multiple of $c_{j,j-1}$.

The only point to check is that each $c_{j,j} \neq 0$, or more accurately, that $c_{j,j} = 0$ implies that v_j vanishes identically. To see this, first note that $c_{j,j} = 0$ implies that all $c_{j,\ell} = 0$, so that v_j decays exponentially. In addition, $v_j(0) = 0$. Now observe that

$$\int_0^\infty -(L_j v_j)v_j ds = \int_0^\infty (v_j')^2 + (j + \frac{1}{2})(v_j)^2 ds = 0$$

since

$$\int_0^\infty s v_j v_j' ds = \frac{1}{2} \int_0^\infty s (v_j^2)' ds = -\frac{1}{2} \int_0^\infty (v_j)^2 ds,$$

which would show that $v_j \equiv 0$ as claimed.

To determine v_j completely, it suffices to determine the coefficient $c_{j,j}$, and this is where the initial condition at $t = 0$ is required. It is now convenient to switch to the other coordinate system (T, y) , since this is nonsingular near the corner $\text{ff} \cap \text{bf}$ of Q_h .

Using that $T = \frac{1}{2s^2}$, $y = s\tau$, or equivalently $s = (2T)^{-1/2}$, $\tau = y(2T)^{1/2}$, we see that $s^\ell \tau^j = y^j (2T)^{\frac{\ell-j}{2}}$. Altogether, modulo the exponentially decreasing error term,

$$v \sim \sum_{j=0}^{\infty} \sum_{\ell=0}^j c_{j,\ell} s^\ell \tau^j = \sum_{j=0}^{\infty} \sum_{\ell=0}^j 2^{(\frac{j-\ell}{2})} c_{j,\ell} y^j T^{(\frac{j-\ell}{2})}, \quad (4.4)$$

where the $c_{j,\ell}$ are the same coefficients as were discussed above. Now suppose that

$$v \sim \sum_{j,p=0}^{\infty} A_{jp} y^j T^p, \quad \text{as } y, T \rightarrow 0.$$

Inserted into (3.2), this leads to the recursion relation

$$(p+1)A_{j,p+1} = [j(j-1) + 2p(2p+1-2j)]A_{jp},$$

or more simply

$$A_{j,p+1} = \frac{1}{p+1} (j-2p)(j-2p-1)A_{jp}. \quad (4.5)$$

To compare with (4.4), set $p = (j-\ell)/2$, i.e., $\ell = j-2p$, so

$$2^p c_{j,j-2p} = A_{jp}. \quad (4.6)$$

For this to be consistent, it is necessary that $A_{jp} = 0$ for $p > j/2$, but this is immediate from (4.5). Comparing the recursion formulas (4.3) and (4.5) more carefully, one can check that they are actually the same, so that (4.6) is true for all indices. Furthermore, the coefficient

$$c_{j,j} = A_{j0} = \partial_y^j v(T, y)|_{y=T=0}$$

is nothing other than the j^{th} derivative of $\phi(x)$ at $x = 0$. This shows that the initial condition $u_t|_{t=0} = \phi$ determines all of the $c_{j,j}$, and hence the entire set of coefficients $v_j(s)$, $j \geq 0$.

We have now shown that the initial data ϕ and the heat equation itself determine the full set of Taylor coefficients of the solution near the front and bottom faces of Q_h .

To determine the Taylor series of v at $x = 0$ for $t > 0$, note that the heat equation shows that all even derivatives $\partial_x^{2j} v(t, 0)$ must vanish since we have specified that $v(t, 0) \equiv 0$. The odd Taylor coefficients $\partial_x^{2j+1} v(t, 0)$ are all determined by $\partial_x v(t, 0)$. We can choose this normal derivative to be any smooth function, but for simplicity, let us assume that $\partial_x v(t, 0)$ vanishes identically. This shows that v vanishes to infinite order as $x \rightarrow 0$. It is perhaps slightly more reassuring to repeat this calculation in the (τ, s) coordinates in order to be sure that it really can be done uniformly up to the front face, but this is indeed the case.

When the boundary condition $u(t, 0) = \psi(t)$ is smooth, but not identically zero, we see that all even Taylor coefficients of v are determined recursively in terms of derivatives of ψ ; as before, assuming that $\partial_x v(t, 0) = 0$, all the odd coefficients vanish. In this case where $\psi(t) \neq 0$, we must also alter the prior step in determining the solutions to $L_j v_j = 0$ along the front face, since now we must prescribe the initial condition $v_j(0)$ to equal the recursively determined value of $(1/j!) \partial_s^j v(0, 0)$.

Having determined all of these Taylor coefficients at all boundaries of Q_h , we now invoke Borel's Lemma. This classical result states that given any prescribed functions on the boundary components of a manifold with corners, there exists a smooth function for which these functions are precisely the coefficients of the Taylor series with respect to some fixed choice of boundary

defining function at that face. We apply this to construct a smooth function v on Q_h , the Taylor coefficients of which at all boundaries of Q_h are the quantities we have determined above. By construction, $(\partial_t - \partial_x^2)v = f$ vanishes to infinite order as $t \searrow 0$ in Q_h .

Construction of the correction term w : The final step is to define the function

$$w(t, x) = \int_0^\infty \int_0^\infty H_D(t - \tilde{t}, x, \tilde{x}) f(\tilde{t}, \tilde{x}) d\tilde{t} d\tilde{x},$$

where $H_D(t, x, \tilde{x})$ is the Dirichlet heat kernel on $\mathbb{R}^+$, i.e., the solution operator for $(\partial_t - \Delta)u = 0$, $u(0, x) = \phi(x)$ and $u(t, 0) = 0$. This heat kernel is known explicitly, of course, by the method of reflections. Using the rapid vanishing of f , it is easy to see that $w(t, x)$ vanishes to all orders as $t \rightarrow 0$, uniformly for $x \geq 0$. In particular, the lift of w to Q_h vanishes to all orders on the front face. In addition, $w(t, 0) = 0$ since $H_D(t - \tilde{t}, 0, \tilde{x})$ vanishes. On the other hand, $\partial_x H_D(t, 0, \tilde{x})$ is a smooth nonvanishing function, as are each of the odd derivatives $\partial_x^{2j+1} H_D(t, 0, \tilde{x})$. Hence $\partial_x^{2j+1} w$ is a smooth function on the left face; as remarked earlier, all of these functions vanish rapidly as $t \searrow 0$. This proves that $w(t, x)$ is $\mathcal{C}^\infty$ on $[0, \infty)_t \times [0, \infty)_x$ and vanishes to all orders at $t = 0$, and all its even Taylor coefficients at $x = 0$ also vanish. Lifted to Q_h , it vanishes rapidly on the front and bottom faces, and all even terms in its series vanish at the left face.

The true solution $u(t, x) = v(t, x) - w(t, x)$ to the heat equation with specified initial and boundary conditions is now smooth on Q_h as claimed. Its series expansion at the front face of Q_h is determined only ‘semilocally’ on this space, since the coefficients in the expansion here are solutions to the induced ordinary differential equations on that face. Note that the arbitrary choice of $\partial_x v(t, 0)$ is compensated by the nonvanishing of $\partial_x w(t, 0)$. As expected, the normal derivative in x of the final solution u is determined globally by ϕ and ψ . $\square$

We conclude this section by remarking on the higher dimensional case. Suppose that $\Omega \subset \mathbb{R}^n$ is a smoothly bounded region, or more generally, a smooth manifold with boundary (not necessarily compact). We assume that there is a well-behaved Dirichlet heat kernel H_D for Ω , which is the case if Ω is compact or has boundary having reasonable asymptotics at infinity. Consider the problem

$$(\partial_t - \Delta_\Omega)u = 0, \quad u(0, z) = \phi(z) \in \mathcal{C}^\infty(\overline{\Omega}), \quad u(t, y) = \psi(t, y) \in \mathcal{C}^\infty([0, \infty) \times \partial\Omega),$$

where $y \in \partial\Omega$. An almost identical result to the one proved above remains true. Namely, define $Q = [0, \infty) \times \overline{\Omega}$ and Q_h the parabolic blowup of Q at $\{0\} \times \partial\Omega$.

Theorem 4.2. *Suppose that the boundary data ϕ and ψ are smooth up to the boundaries of their respective domains of definition. Then the unique (moderate growth) solution u to the mixed initial/Dirichlet problem with this prescribed boundary data is smooth on Q_h .*

The proof is essentially identical to the one above. The extra y -dependence is straightforward to include in the recursion formulas above, and the sequence of equations that must be solved on the front face is still a sequence of ordinary differential equations in which y appears as a parameter. The recursion relationships now involve derivatives in y of the previously determined coefficients.

5 Approximate solutions for the nonlinear flow

We now begin the proof of our short-term existence result for the network flow. As earlier in the paper, assume that the initial network Γ_0 comprises n parametrized arcs $\gamma_0^{(j)}(x)$, $j = 1, \dots, n$,

meeting pairwise nontangentially at the origin. Following the logic and strategy explained in the previous section for the linear heat equation, we first construct an approximate solution to the problem, i.e., a collection of families of parametrized arcs $\hat{\gamma}^{(j)}(t, x)$, $j = 1, \dots, m$, or more properly the lifts of these curves to the blown up spaces, which satisfy

$$(\tau \partial_\tau + 1 - s \partial_s) \hat{\gamma}^{(j)} = \frac{\partial_s^2 \hat{\gamma}^{(j)}}{|\partial_s \hat{\gamma}^{(j)}|^2} + f, \quad (5.1)$$

along with the appropriate initial conditions on the bottom and front faces and matching conditions along the left faces. Here f is smooth on Q_h or P_h and vanishes to all orders at every boundary component. This step will be similar in spirit, but unfortunately rather more intricate, than in the linear case.

In the next sections we take up the problem of correcting this approximate solution to an exact solution. To do that, we will need a detour to explain the structure of the heat kernel for networks. For simplicity, drop both the $\hat{}$ and the superscript (j) from the notation and consider the lift of a single curve as a map from Q_h to Z_h . (We focus for brevity on these outer arcs, rather than the inner ones which have domain P_h .) As in §3, write $\gamma(t, x)$ in terms of the (τ, s) and (τ, w) coordinate systems on Q_h and Z_h , respectively. Thus $\gamma = \tau \eta(\tau, s)$. We make the ansatz that η is smooth,

$$\eta \sim \sum_{j=0}^{\infty} \tau^j \eta_j(s)$$

near ff. Inserting this series in (5.1) and collecting like powers of τ gives the sequence of equations

$$\left[\frac{\partial_s^2 \eta}{|\partial_s \eta|^2} \right]_j + s \partial_s \eta_j - (j+1) \eta_j = 0, \quad j = 0, 1, 2, \dots \quad (5.2)$$

The initial expression on the left, $[\dots]_j$, denotes the coefficient of τ^j in the series expansion of the term inside the brackets.

When $j = 0$, (5.2) reduces to the soliton equation (3.5); hence, as already explained in §3, once we have chosen a particular expanding soliton, then the arcs of this soliton appear as the leading asymptotic terms of each arc of this approximate solution at ff.

The study of (5.2) for $j \geq 1$ is conducted in a series of steps. In the first, we study the sequence of operators

$$\mathcal{L}_j \eta_j = \partial_s^2 \eta_j + (s \partial_s - (j+1)) \eta_j.$$

These operators appear as the principal parts of (5.2), and their precise role in this analysis will be explained below.

In the following we write $A \sim B$ if $A - B$ decays exponentially as $s \rightarrow \infty$. (In fact, the ‘error terms’ encountered below all decay like e^{-cs^2} for some $c > 0$.) We also say that a function u is of degree ℓ if $u \sim P$ where P is a polynomial of degree ℓ . All functions below are assumed to be smooth.

Lemma 5.1. *The space of solutions to $\mathcal{L}_j u = 0$ on $\mathbb{R}^+$ is spanned by two functions, U_j, V_j , where U_j is of degree $j+1$ and $V_j \sim 0$. If R has degree $j-1$, and a is any constant, then there exists a unique solution u to $\mathcal{L}_j u = R$ with u of degree $j+1$ and $u(0) = a$.*

The analogous statement is true for any operator of the form $\mathcal{L}_j + E_1 \partial_s^2 + E_2 \partial_s + E_3$, where each $E_j \sim 0$.

Proof. Observe first that $\mathcal{L}_{-1}u = u'' + su' = 0$ has the explicit solutions 1 and $G(s) := -\int_s^\infty e^{-\sigma^2/2} d\sigma$, so a general solution has the form $A + BG(s)$ for some constants A, B .

Next, applying the intertwining formula

$$\partial_s \mathcal{L}_j = \mathcal{L}_{j-1} \partial_s$$

$(j+1)$ times to $\mathcal{L}_j u = 0$ gives $\mathcal{L}_{-1} u^{(j+1)} = 0$, so $u^{(j+1)} = A + BG(s)$, and hence u has degree $j+1$ as claimed. Note that if $U_j \not\equiv 0$, then its leading coefficient must be nonzero.

To obtain a solution of the inhomogeneous equation $\mathcal{L}_j u = R$, we can use the method of undetermined coefficients to solve away the polynomial terms of R , and reduce to the case where $R \sim 0$. Assuming this has been done for the moment, then the explicit formula

$$w = U(s) \int_s^\infty U(s')^{-2} \int_{s'}^\infty U(s'') R(s'') ds'' ds'$$

produces a solution $w \sim 0$.

Now suppose that $u(s) \sim a_{j+1}s^{j+1} + a_j s^j + \dots + a_0 s^0$ and write $R \sim b_{j-1}s^{j-1} + \dots + b_0 s^0$. Then

$$\mathcal{L}_j u \sim \sum_{i=0}^{j-1} a_{i+2}(i+2)(i+1)s^i + \sum_{i=0}^{j+1} (i-(j+1))a_i s^i \sim \sum_{i=0}^{j-1} b_i s^i,$$

hence

$$(j+1-(j+1))a_{j+1} = 0, \quad (j-(j+1))a_j = 0, \quad (i+2)(i+1)a_{i+2} + (i-j-1)a_i = b_i, \quad i = 0, \dots, j-1.$$

The first equation leaves a_{j+1} free, and the second equation gives that $a_j = 0$. The remaining equations show that each a_i , $i \leq j-1$, is determined by b_i and $a_{j+1}, \dots, a_{i+1}$. Notice that if we had assumed that the polynomial part of this solution has leading term different than s^{j+1} , there would have been no cancellation and this ansatz would not have worked.

To attain the correct boundary value, simply add on the appropriate homogeneous solution, which by the above has degree at most $j+1$.

Finally, $(1+E_1)u'' + s(1+E_2)u' + (-(j+1)+E_3)u = R$ as $\mathcal{L}_j u = f$ where $f = R - E_1 u'' - E_2 s u' - E_3 u$. The same explicit integral formula, and standard perturbation arguments, gives the existence of solutions with the specified asymptotics. $\square$

The relevance of the operators $\mathcal{L}_j$ is explained by the following

Lemma 5.2. *For $j \geq 1$, equation (5.2) can be written in the form*

$$\mathcal{L}_j \eta_j + E_1 \partial_s^2 \eta_j + E_2 \partial_s \eta_j + Q_j(\eta_0, \dots, \eta_{j-1}) = 0,$$

where Q_j is a polynomial in $\partial_s \eta_i$ and $\partial_s^2 \eta_i$, $0 \leq i \leq j-1$ and each $E_j \sim 0$ depends only on η_0 .

Proof. Recalling again that $'$ is the same as ∂_s , write

$$\eta''/|\eta'|^2 = F(|\eta'|^2)\eta'' \quad \text{with} \quad F(w) = 1/w.$$

By (3.6), $|\eta'_0|^2 \sim 1$, hence

$$|\eta'|^2 \sim 1 + \sum_{\ell=1}^{\infty} \tau^\ell \left(2\eta'_0 \cdot \eta'_\ell + \sum_{i=1}^{\ell-1} \eta'_i \cdot \eta'_{\ell-i} \right).$$

Expanding F about $w = 1$ yields

$$F(|\eta'|^2) \sim \sum_{p=0}^{\infty} (-1)^p \left(\sum_{\ell=1}^{\infty} (2\eta'_0 \cdot \eta'_\ell + \sum_{i=1}^{\ell-1} \eta'_i \cdot \eta'_{\ell-i}) \tau^\ell \right)^p \sim 1 + \sum_{\ell=1}^{\infty} (-2\eta'_0 \cdot \eta'_\ell + B_\ell) \tau^\ell$$

where each B_ℓ is a sum of products, each summand having at least two factors, of terms $\eta'_i \cdot \eta'_j$. In fact, a moment's reflection reveals that each monomial in B_ℓ is a constant multiple of $(\eta'_{r_1} \cdot \eta'_{r_2}) \dots (\eta'_{r_{2i+1}} \cdot \eta'_{r_{2i+2}})$ where $r_1 + \dots + r_{2i+2} = \ell$ and $i \geq 0$. Note that because of these last two conditions, η'_ℓ does not appear in B_ℓ , so we can write $B_\ell = B_\ell(\eta'_0, \dots, \eta'_{\ell-1})$.

Next,

$$\begin{aligned} F(|\eta'|^2) \eta'' &\sim \left(1 + \sum_{\ell=1}^{\infty} (-2\eta'_0 \cdot \eta'_\ell + B_\ell) \tau^\ell \right) (\tau \eta''_1 + \tau^2 \eta''_2 + \dots) \\ &\sim \sum_{j=1}^{\infty} \tau^j \left(\eta''_j + \sum_{i=1}^{j-1} \eta''_i (B_{j-i} - 2\eta'_0 \cdot \eta'_{j-i}) \right), \end{aligned}$$

where η''_0 has been omitted since it decays exponentially. We read off from this that

$$\left[\frac{\eta''}{|\eta'|^2} \right]_j = \eta''_j + Q_j,$$

where Q_j is a polynomial in η'_i , $0 \leq i \leq j-1$ and η''_i , $1 \leq i \leq j-1$. □

Lemma 5.3. *For every $j \geq 0$, η_j has degree $j+1$.*

Proof. We shall prove this by induction, making the hypothesis that

$$\text{For each } i < j, \eta_i \text{ has degree } i+1.$$

We have already shown that $\eta_0(s) \sim as$, where a is a constant vector, so this hypothesis is true for $j = 0$.

We first show that $Q_j(\eta_0, \dots, \eta_{j-1})$ has degree $j-1$. We have already shown that

$$Q_j = \sum_{i=1}^{j-1} \eta''_i (B_{j-i} - 2\eta'_0 \cdot \eta'_{j-i}).$$

Each term in B_{j-i} is a product $(\eta'_{r_1} \cdot \eta'_{r_2}) \dots (\eta'_{r_{2\ell+1}} \cdot \eta'_{r_{2\ell+2}})$ with $r_1 + \dots + r_{2\ell+2} = j-i$ and $\ell \geq 0$. By induction, each η'_s has degree s , $s \leq j-1$, so the total degree of B_{j-i} is $j-i$; the same is true for the other term, $\eta'_0 \cdot \eta'_{j-i}$. Therefore, the product with η''_i has degree $i-1 + j-i = j-1$.

Next, using the intertwining formula, we see that

$$(\mathcal{L}_j \eta_j)^{(j+1)} = \mathcal{L}_{-1} \eta_j^{(j+1)} \sim -(Q_j)^{(j+1)} \sim 0,$$

which implies that $\eta_j^{(j+1)} \sim A$, hence $\eta_j \sim P_{j+1}(s)$, as claimed. This completes the inductive step. □

Collecting and summarizing these results, we have now proved the following

Proposition 5.4. *Given any constant vectors $a, b \in \mathbb{R}^2$ and an $\mathbb{R}^2$ -valued function R_{j-1} of degree $j-1$, there exists a unique solution to $\mathcal{L}_j \eta_j = R_{j-1}$ such that $\eta_j(0) = b$ and $\eta_j(s) \sim as^{j+1} + \mathcal{O}(s^j)$ as $s \rightarrow \infty$.*

The last in this sequence of results concerns the polynomial structure of η_j . We say that a polynomial is even if it only has terms with even powers of s , and odd if it only has terms with odd powers of s .

Lemma 5.5. *If j is even, then the polynomial part P_{j+1} of η_j is odd, while if j is odd, then P_{j+1} is even.*

Proof. We prove this again by induction, assuming the statement is true for all $i < j$. Following the construction of the solution above, it suffices to show that each R_{j-1} (the polynomial part of Q_j) is even or odd, depending on the parity of j . Indeed, suppose j is even and we have shown that R_{j-1} is odd. Then the recursion formula for the coefficients of P_{j+1} above, and the fact that the coefficients b_i vanish when i is even, show that it too involves only odd powers of s ; similarly when j is odd, then P_{j+1} can only involve even powers.

To prove the assertion about Q_j , we analyze each summand $\eta_i'' B_{j-i}$ and $\eta_i'' \eta_0' \cdot \eta_{j-i}'$. Suppose j is even. If i is also even, then η_i is odd, so the same is true for η_i'' . Furthermore, each monomial in B_{j-i} is composed of factors η_{r_k}' , which are polynomials of degree r_k . The sum of these degrees adds up to $j - i$, which is even, and so there must be an even number of factors which are odd polynomials. Hence each summand in B_{j-i} is even, so the product with η_i'' is odd. On the other hand, if i is odd, then η_i'' is even, while B_{j-i} is a sum of products of polynomials, each either even or odd, and since the overall product is odd, there must be an odd number of odd factors, so B_{j-i} is odd and the product with η_i'' is odd. Similar considerations apply to the other term $\eta_i'' \eta_0' \cdot \eta_{j-i}'$. This proves the parity assertion when j is even. The corresponding assertion when j is odd is proved similarly. $\square$

Here is what we have accomplished so far. Proposition 5.4 shows that once we have specified the coefficients b_j and a_j giving the boundary value at $s = 0$ and the leading term of the polynomial asymptotic as $s \rightarrow \infty$, then there exists an infinite sequence of functions $\eta_j(s)$ such that the asymptotic sum $\sum \tau^j \eta_j(s)$ is a formal solution of the flow equation near ff. Said slightly differently, if we choose a smooth function $\tilde{\eta}(\tau, s)$ with this Taylor series as $\tau \rightarrow 0$, then $N(\tilde{\eta})$ vanishes to infinite order at ff. To complete this construction of the approximate solution, we must show how to choose these values a_j and b_j . As already seen in our treatment of the linear heat equation, the values a_j will be determined by the Taylor coefficients of the arcs $\gamma^{(\ell)}$ at $x = 0$. On the other hand, the Herring conditions only specify half of the necessary Taylor coefficients along the left face of each Q_h (or P), and hence we will need to specify the other parts of the Taylor series (which we can do arbitrarily).

Consider first the issue of matching coefficients at the intersection of the front and bottom faces. In this region we use the coordinates $T = t/x^2 = 1/2s^2$ and $y = x = s\tau$, and also revert to $\gamma = \tau\eta$. Then

$$\begin{aligned} \gamma(\tau, s) &\sim \sum_{j=0}^{\infty} \tau^{j+1} \eta_j(s) \sim \sum_{j=0}^{\infty} \sum_{i=0}^{j+1} a_{ij} s^{j+1-i} \tau^{j+1} \\ &\sim \sum_{j=0}^{\infty} \sum_{i=0}^{[(j+1)/2]} a_{ij} (2T)^{-(j+1-2i)/2} (y2T^{1/2})^{j+1} \sim \sum_{j=0}^{\infty} \sum_{i=0}^{[(j+1)/2]} 2^i a_{ij} T^i y^{j+1}. \end{aligned}$$

Now dropping the superscript, each initial curve $\gamma_0(x) = \gamma_0^{(\ell)}(x)$ is smooth for $x \geq 0$, so in the (y, T) coordinates,

$$\gamma_0(x) \sim \sum_{j=0}^{\infty} A_j y^j.$$

It is thus necessary that a_{0j} , the coefficient of s^{j+1} in η_j , equals A_j . As we have showed earlier, computing in the (τ, s) , the rest of the terms a_{ij} , $i \geq 0$, are determined by this leading coefficient. Establishing the equivalent recursion in the (T, y) coordinates is more difficult, but is not necessary simply because the recursion is determined by the underlying equation, and hence must come out the same in either coordinate system.

Now let us return to the expansion along the left face, where $s = 0$ and $\tau \geq 0$. In the following, recall that along this face we impose the boundary conditions that three curves meet at equal angles. For simplicity, call these three curves $\eta^{(1)}$, $\eta^{(2)}$ and $\eta^{(3)}$. The relevant endpoint of each $\eta^{(j)}$ is specified by four real numbers (the coordinates of the position and tangent vector at that point), so the full space of parameters for these endpoints for the three curves is 12-dimensional. The Herring conditions impose 6 (nonlinear) constraints, namely that the endpoints of these curves coincide (four real conditions) and that the unit tangents at this point of coincidence sum to 0 (two further conditions). We now describe the remaining six free parameters. Two are the coordinates of the vertex where the three curves come together, and three are the lengths of the tangents $(\eta^{(j)})'(0)$ at this point. For the remaining coordinate, observe that, up to reflection, the normalized (unit length) triod is completely determined by specifying the direction of $(\eta^{(1)})'(0)$, and this in turn is determined by the angle it makes with some fixed direction (e.g. the positive x_1 axis). This angle might be called the phase of the triod.

To formalize this, consider a collection of six vectors $v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}^2$, and write $\mathbf{v}$ and $\mathbf{w}$ for the two triples of vectors respectively. Now define

$$F : \{(\mathbf{v}, \mathbf{w}) : |w_i| \neq 0, i = 1, 2, 3\} \longrightarrow \mathbb{R}^6 \times \mathbb{R}^2 \times (\mathbb{R}^+)^3 \times S^1$$

$$F(\mathbf{v}, \mathbf{w}) = (v_1 - v_2, v_2 - v_3, \sum w_i/|w_i|, v_1, |w_1|, |w_2|, |w_3|, \arccos(w_1 \cdot e_1/|w_1|)).$$

The coordinate θ is only defined locally, where $\cos \theta = w_1 \cdot e_1/|w_1|$, but there is a local branch of this map which suffices for our purposes. A straightforward calculation shows that this map is a local diffeomorphism.

The Herring conditions are equivalent to

$$F((\eta^{(1)}(0), \eta^{(2)}(0), \eta^{(3)}(0), (\eta^{(1)})'(0), (\eta^{(2)})'(0), (\eta^{(3)})'(0))) \in \{0\} \times (\mathbb{R}^+)^3 \times S^1 \times \mathbb{R}^2$$

Now choose any smooth mappings $\beta(\tau), \zeta_1(\tau), \zeta_2(\tau), \zeta_3(\tau), \theta(\tau)$ where $\zeta_i(0) = |(\eta_0^{(i)})'(0)|$ and the initial angle $\theta(0)$ that $(\eta_0^{(1)})'(0)$ makes with the e_1 direction are determined by the fixed soliton solution at time $\tau = 0$, and $\beta(\tau)$ represents the common vertex at time τ . Then $F^{-1}(0, 0, 0, \beta(\tau), \zeta_1(\tau), \zeta_2(\tau), \zeta_3(\tau), \theta(\tau))$ specifies the full Cauchy data of the three curves $\eta^{(i)}(s)$ at $s = 0$.

Having made these choices of complementary data, it is now easy to construct the entire Taylor series of each $\eta^{(i)}(\tau, s)$ at $s = 0$ for $0 \leq \tau < \tau_0$ using (3.4). In particular, this determines all of the constant initial values b_j for $\eta_j(0)$.

Altogether, we have determined the full Taylor series at all faces of each Q_h , and of each P , for any putative solution to the full nonlinear equation with coupled to the Herring boundary conditions, and also imposing the extra complementary boundary conditions as above. We now use Borel's lemma to choose functions $\hat{\gamma}^{(i)}$ which have precisely these Taylor series at all boundaries. These constitute the evolving network $\hat{\Gamma}(t)$. By construction, $N(\hat{\Gamma}) := f$ is $\mathcal{C}^\infty$ and vanishes to all orders at all boundary faces.

6 Linearization

We next study the linearization of (2.4) around the approximate solution $\widehat{\Gamma}$ constructed in the last section. By the naturality of the process of computing linearizations, we may either compute the linearization of the equation on the unblown up spaces and then lift to the blowups, or else first lift the nonlinear equation and then compute blowups. We do the former.

First suppose that $\Gamma_\kappa = \{\gamma_\kappa^{(j)}\}$ is a family of networks, i.e., for each κ , Γ_κ is a time-dependent network which does not necessarily satisfy the flow equations. Differentiating (2.4) with respect to κ leads to the operator

$$\partial_t u^{(j)} = \frac{\partial_x^2 u^{(j)}}{|\partial_x \gamma^{(j)}|^2} - 2 \frac{\langle \partial_x \gamma^{(j)}, \partial_x u^{(j)} \rangle}{|\partial_x \gamma^{(j)}|^4} \partial_x^2 \gamma^{(j)},$$

where $u^{(j)} = \partial_\kappa \gamma^{(j)}|_{\kappa=0}$. The linearized boundary conditions are

$$u^{(i)}(t, 0) = u^{(j)}(t, 0) \text{ for all } i \neq j, \quad \sum \left(\frac{\partial_x u^{(j)}}{|\partial_x \gamma^{(j)}|} - \frac{\langle \partial_x \gamma^{(j)}, \partial_x u^{(j)} \rangle}{|\partial_x \gamma^{(j)}|^3} \partial_x \gamma^{(j)} \right) (t, 0) = 0,$$

where the sum is over the triples associated to any interior vertex. The latter boundary condition can be rewritten in the more revealing form

$$\sum |\partial_x \gamma^{(j)}(t, 0)|^{-1} P^{(j)}(\partial_x u^{(j)}(t, 0)) = 0, \quad \text{or more simply} \quad \sum c_j(t) P^{(j)}(\partial_x u^{(j)}(t, 0)) = 0$$

where $c_j(t) = |\partial_x \gamma^{(j)}(t, 0)|^{-1}$ and $P^{(j)}$ is the orthogonal projection onto the unit normal to $\gamma^{(j)}$. The interior and boundary operators will be written as $\partial_t - \mathcal{L}$ and β , respectively. It will be useful to separate the boundary operator into two separate terms, $\beta(u, \partial_x u) = (\beta_0(u), \beta_1(\partial_x u))$, where the first only depends on $u(t, 0)$ and the second only depends on $\partial_x u(t, 0)$. We give a brief derivation of the ellipticity of these boundary conditions later. We will consider the lifts of these operators, or rather the operators $t(\partial_t - \mathcal{L})$ and $x\beta$ to each Q_h and P .

Our goal is to solve the nonlinear parabolic equation $\mathbb{M}(\widehat{\Gamma} + U) = 0$ and $B(\widehat{\Gamma} + U) = 0$. The operators $\mathbb{M}$ and B are simply the full nonlinear operators corresponding to the parametrized flow and the Herring boundary conditions, respectively. Expanding around the approximate solution, this becomes

$$(t\partial_t - t\mathcal{L})U = f + Q_i(U), \quad \beta(U) = S_b(U),$$

where $f = -\mathbb{M}(\widehat{\Gamma})$ is the error term and Q_i and S_b are the quadratically vanishing remainders. By construction, f vanishes to all orders at all boundaries of Q_h, P_h , and similarly, $B(\widehat{\Gamma}) = 0$. We shall prove the existence and uniqueness of a solution U which vanishes rapidly as $t \rightarrow 0^+$, and by the usual contraction mapping arguments, to do so it suffices to solve the linear system

$$(t\partial_t - \mathcal{L})U = H, \quad \beta(U) = K \tag{6.1}$$

with good estimates, where $H(t, x)$ and $K(t)$ both vanish to all orders in t , or equivalently, along the front and bottom faces. Notice that we may as well assume that the correction term U satisfies the matching condition $U^{(i)}(t, 0) = U^{(j)}(t, 0)$, and the nonlinear part of the boundary term $S_b(U)$ only involves the second component of the nonlinear boundary operator $B(U)$. Therefore, it suffices to consider boundary conditions of the form $\beta(U) = (\beta_0(U), \beta_1(\partial_x U)) = (0, k)$, but since it is no harder, we consider the full system where $\beta_0(U) \neq 0$ as well.

Our aim is to show that if the inhomogeneous terms H and K in (6.1) are smooth and decay to some high order as $t \rightarrow 0$, then there exists a solution U to this system with these same properties. We focus on obtaining solutions with appropriate decay first; the higher regularity is standard and will be discussed at the end. The main step is to recast this problem by showing that

$$t(\partial_t - \mathcal{L}) : t^N \mathcal{X} \rightarrow t^N \mathcal{Y} + t^N \mathcal{Z} \quad (6.2)$$

is an isomorphism for all $N \gg 1$, where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are appropriately defined weighted L^2 -based Sobolev spaces (thus $U \in t^N \mathcal{X}$, $H \in t^N \mathcal{Y}$ and $K \in t^N \mathcal{Z}$). The uniqueness part of this shows that if H and K vanishes to all orders, then so does U . Writing $H = t^N h$, $K = t^N k$, and $U = t^N u$, then the problem is equivalent to showing that

$$(t\partial_t + N - t\mathcal{L})u = h, \quad \beta(u) = k.$$

is an isomorphism for N large.

The space $\mathcal{Z}$ is simply $L^2(Q_h; \frac{dt}{t} dx)$. Away from the boundaries, elements of $\mathcal{X}$ lie in the parabolic Sobolev space $H^{1,2}$, but the precise weight factors at each boundary face of Q_h are not obvious at the outset. We also impose fractional Sobolev regularity on k_0 and k_1 . To show that (6.2) is an isomorphism, we prove the a priori estimate

$$\|u\|_{\mathcal{X}} \leq C(\|h\|_{\mathcal{Y}} + \|k\|_{\mathcal{Z}}). \quad (6.3)$$

Concealed is the fact that the norms depend on N , again as spelled out carefully below. Given (6.3), a standard continuity argument implies the unique solvability of the original problem, and hence that (6.2) is an isomorphism. We defer a precise statement of the estimate until the end of the proof.

This estimate is not standard because of the behavior of the coefficients of $\mathcal{L}$ at the front faces of Q_h . To handle this, we consider the lift of this operator to the blowup. We follow in broad outline a proof of a priori estimates given by Krylov [15], adapted to the present situation. By standard localization and approximation arguments, it is enough to prove a priori estimates for each of the following simpler model operators near the different faces of each Q_h . In the following, we use either the original coordinates (t, x) or else the projective coordinates (τ, s) .

- a) $\tau\partial_\tau + 2N - \partial_s^2 - s\partial_s$ on functions supported on the region $0 < \tau < T$, $s_0 < s < \infty$, i.e., near $\text{ff} \cap \text{bf}$;
- b) $t\partial_t + N - t\partial_x^2$ on smooth functions compactly supported on $0 < t < T$, $0 < c_1 < x < c_2$, i.e., near bf but away from ff ;
- c) $\tau\partial_\tau + 2N - \partial_s^2$ on smooth functions on three copies of Q_h joined along lf where $0 < \tau < T$, $0 \leq s_i < s_0$, $i = 1, 2, 3$, and satisfying the boundary conditions $\beta(u) = k$, i.e., near $\text{ff} \cap \text{lf}$.

We call these regions I, II and III, respectively. These are models in the sense that the actual lifted operator in each of these regions can be considered as a lower order perturbation of these models, and those lower order terms are readily absorbed into the estimates below. The appearance of $2N$ rather than N in a) and c) is because we are conjugating by $t^N = \tau^{2N}$, but we can obviously replace $2N$ by N here. In each of these cases, by density arguments, it suffices to prove the estimate for smooth compactly supported functions whose support does not intersect $t = 0$ or $\tau = 0$.

Region I: First suppose that u is supported near the intersection of the front and bottom faces. Although the nondegenerate coordinate system here are $T = t/x^2$ and $y = x$, it is simpler

to derive estimates using the (τ, s) coordinates. An auxiliary argument, which gives the same estimate but in the coordinates (t, x) , is included here to show how to join this estimate to the one in region II.

Suppose then that

$$\tau \partial_\tau u - \partial_s^2 u - s \partial_s u + Nu = \tau^2 h. \quad (6.4)$$

First multiply both sides of (6.4) by Nu and integrate in both variables with respect to the measure $d\tau ds$. Given the support properties of u , integrations by parts leave no boundary terms. We obtain the identities:

$$N \int \tau (\partial_\tau u) u \, d\tau ds = \frac{N}{2} \int \tau \partial_\tau u^2 \, d\tau ds = -\frac{N}{2} \int u^2 \, d\tau ds,$$

$$-N \int u \partial_s^2 u \, d\tau ds = N \int (\partial_s u)^2 \, d\tau ds,$$

and

$$-N \int s u \partial_s u \, d\tau ds = -\frac{N}{2} \int s \partial_s u^2 \, d\tau ds = \frac{N}{2} \int u^2 \, d\tau ds.$$

Using these, and applying the Cauchy-Schwarz and arithmetic/geometric mean inequality on the right hand side, we arrive at

$$N \int (\partial_s u)^2 \, d\tau ds + N^2 \int u^2 \, d\tau ds \leq C \int \tau^4 h^2 \, d\tau ds. \quad (6.5)$$

Next, multiply (6.4) by $s^2 u$ and integrate with respect to the same measure to get

$$\int \tau \partial_\tau u (s^2 u) - \partial_s^2 u (s^2 u) - s \partial_s u (s^2 u) + N s^2 u^2 \, d\tau ds = \int \tau^2 h s^2 u \, d\tau ds.$$

Here we observe that

$$\int (\tau \partial_\tau u) s^2 u \, d\tau ds = \frac{1}{2} \int \tau \partial_\tau (u^2) s^2 \, d\tau ds = -\frac{1}{2} \int s^2 u^2 \, d\tau ds,$$

$$- \int (\partial_s^2 u) (s^2 u) \, d\tau ds = \int (\partial_s u) (s^2 \partial_s u + 2su) \, d\tau ds = \int s^2 (\partial_s u)^2 \, d\tau ds - \int u^2 \, d\tau ds,$$

and

$$- \int s^3 u \partial_s u \, d\tau ds = -\frac{1}{2} \int s^3 \partial_s (u^2) \, d\tau ds = \frac{3}{2} \int s^2 u^2 \, d\tau ds.$$

Putting these together and estimating the right hand side the same way gives

$$N \int s^2 u^2 \, d\tau ds + \int s^2 (\partial_s u)^2 \, d\tau ds \leq C \int s^2 \tau^4 h^2 \, d\tau ds + \int u^2 \, d\tau ds \leq C \int s^2 \tau^4 h^2 \, d\tau ds \quad (6.6)$$

since $s \geq s_0 \geq 1$.

Now rewrite the equation as

$$(\tau \partial_\tau - \partial_s^2) u = (s \partial_s u - Nu + \tau^2 h).$$

Squaring both sides and integrating, we get

$$\int \tau^2 (\partial_\tau u)^2 + (\partial_s^2 u)^2 - 2\tau \partial_\tau u \partial_s^2 u \, d\tau ds \leq C \int s^2 (\partial_s u)^2 + N^2 u^2 + \tau^4 h^2 \, d\tau ds.$$

The cross-term equals

$$-2 \int \tau \partial_\tau u \partial_s^2 u \, d\tau ds = 2 \int \tau \partial_\tau (\partial_s u) \partial_s u \, d\tau ds = \int \tau \partial_\tau (\partial_s u)^2 \, d\tau ds = - \int (\partial_s u)^2 \, d\tau ds.$$

Thus using (6.5) and (6.6), we have

$$\int \tau^2 (\partial_\tau u)^2 + (\partial_s^2 u)^2 \, d\tau ds \leq C \int s^2 \tau^4 h^2 \, d\tau ds. \quad (6.7)$$

Finally, (6.5), (6.6) and (6.7) together prove that

$$\int (\tau \partial_\tau u)^2 + (\partial_s^2 u)^2 \, d\tau ds + \int s^2 (\partial_s u)^2 + N^2 u^2 \, d\tau ds + N \int s^2 u^2 \, d\tau ds \leq C \int s^2 \tau^4 h^2 \, d\tau ds. \quad (6.8)$$

This completes the main estimate in this region.

We now translate this estimate to the (t, x) coordinate system. As explained earlier, this is necessary to show how to join this estimate to the one for region II. We have

$$d\tau ds = \det \begin{pmatrix} 1/\sqrt{2t} & 0 \\ -x/2\sqrt{2}t^{3/2} & 1/\sqrt{2t} \end{pmatrix} dt dx = \frac{1}{2t} dt dx \quad \text{and} \\ \tau \partial_\tau = 2t \partial_t + x \partial_x, \quad s \partial_s = x \partial_x, \quad \partial_s^2 = 2t \partial_x^2,$$

and therefore, (6.8) is the same as

$$\int [((2t \partial_t + x \partial_x)u)^2 + 4t^2 (\partial_x^2 u)^2 + (x \partial_x u)^2 + N(x^2/2t)u^2 + N^2 u^2] \frac{dt dx}{2t} \leq C \int x^2 h^2 \, dt dx,$$

or equivalently,

$$\int \frac{1}{t} (2t \partial_t u + x \partial_x u)^2 + t (\partial_x^2 u)^2 + \frac{x^2}{t} (\partial_x u)^2 + (N \frac{x^2}{t^2} + N^2 \frac{1}{t}) u^2 \, dt dx \leq C \int x^2 h^2 \, dt dx. \quad (6.9)$$

We now prove this latter estimate by similar means. It suffices now, by approximation, to consider the simple model equation $(t \partial_t + N - t \partial_x^2)u = th$. For simplicity, we suppose that u is supported in the region where $t/x^2 \leq 1$.

First multiply the equation by $x^2 u/t^2$ and integrate to get

$$\int \frac{x^2}{t} u \partial_t u + N \frac{x^2}{t^2} u^2 - (\partial_x^2 u) \left(\frac{x^2}{t} u \right) \, dt dx = \int \left(\frac{x^2}{t} \right) h u \, dt dx.$$

The first term on the left equals

$$\frac{1}{2} \int \frac{x^2}{t} \partial_t (u^2) \, dt dx = \frac{1}{2} \int \frac{x^2}{t^2} u^2 \, dt dx.$$

The third term on the left becomes

$$\int \frac{1}{t} \partial_x u \partial_x (x^2 u) \, dt dx = \int \frac{x^2}{t} (\partial_x u)^2 + \frac{2x}{t} u \partial_x u \, dt dx = \int \frac{x^2}{t} (\partial_x u)^2 - \frac{1}{t} u^2 \, dt dx..$$

The right hand side of this equality is estimated by

$$\frac{1}{2} \int \frac{x^2}{t^2} u^2 \, dt dx + \int x^2 h^2 \, dt dx.$$

Altogether, these prove that

$$\int \frac{x^2}{t} (\partial_x u)^2 + N \frac{x^2}{t^2} u^2 dt dx \leq \int x^2 h^2 dt dx + \int \frac{1}{t} u^2 dt dx.$$

Now multiply the equation by Nu/t and integrate, to obtain

$$\int Nu \partial_t u + N^2 \frac{1}{t} u^2 + N (\partial_x u)^2 dt dx = \int Nhu dt dx \leq C \int x^2 h^2 + \frac{1}{4} \int N^2 \frac{1}{x^2} u^2 dt dx.$$

The first term integrates to 0. Furthermore, in this region, $(1/x^2) = (1/t)(t/x^2) \leq 1/t$, so we can move the second term on the right over to the left, leading to

$$\int N (\partial_x u)^2 + N^2 \frac{1}{t} u^2 dt dx \leq C \int x^2 h^2 dt dx. \quad (6.10)$$

For the next step, rewrite the equation as $\partial_t u - \partial_x^2 u = h - (N/t)u$, square both sides and multiply by t . This gives

$$\int t (\partial_t u)^2 + t (\partial_x^2 u)^2 - 2t \partial_t u \partial_x^2 u dt dx = \int t h^2 - 2Nhu + \frac{N^2}{t} u^2 dt dx.$$

The third term on the left can be transformed as

$$2 \int t \partial_t u \partial_x^2 u dt dx = \int t \partial_t (\partial_x u)^2 dt dx = - \int (\partial_x u)^2 dt dx.$$

For the first term on the right, use $t \leq x^2$. The middle term on the right can be estimated by

$$\int N^2 \frac{1}{x^2} u^2 + \int x^2 h^2 = \int N^2 (1/t) (t/x^2) u^2 + \int x^2 h^2 \leq \int N^2 \frac{1}{t} u^2 + \int x^2 h^2 \leq C \int x^2 h^2,$$

using (6.10).

Altogether, this step shows that

$$\int t (\partial_t u)^2 + t (\partial_x^2 u)^2 dt dx \leq C \int x^2 f^2 + \int (\partial_x u)^2 dt dx \leq C' \int x^2 h^2 dt dx$$

the last inequality following from (6.10).

We have now estimated almost all the terms on the left in (6.9). To complete the proof, note that

$$\int \frac{1}{2t} (2t \partial_t u + x \partial_x u)^2 dt dx = \int 2t (\partial_t u)^2 + \frac{x^2}{2t} (\partial_x u)^2 + 2 \partial_t u x \partial_x u dt dx.$$

The first and second terms have already been estimated, while

$$\int \partial_t u (x \partial_x u) dt dx \leq \int t (\partial_t u)^2 + \frac{x^2}{t} (\partial_x u)^2 dt dx$$

and both of these terms have been estimated. This proves

$$\int \left[t (\partial_t u)^2 + t (\partial_x^2 u)^2 + (N + \frac{x^2}{t}) (\partial_x u)^2 + (N \frac{x^2}{t^2} + N^2 \frac{1}{t}) u^2 \right] dt dx \leq \int x^2 h^2 dt dx. \quad (6.11)$$

Region II: The previous calculations have been written out carefully, but these all adapt readily when $0 < c \leq x \leq C$. Indeed, in such a region, (6.11) becomes

$$\int \left(t(\partial_t u)^2 + t(\partial_x^2 u)^2 + (N + \frac{1}{t})(\partial_x u)^2 + (N/t)(N + 1/t)u^2 \right) dt dx \leq C \int h^2 dt dx.$$

This is not a particularly familiar estimate, but is easy enough to derive. One proceeds through essentially the same steps as above for the equation $(\partial_t + (N/t) - \partial_x^2)u = h$: first multiply by Nu and integrate, then multiply by u/t , and finally, square the equation $(\partial_t - \partial_x^2)u = h - (N/t)u$, multiply by t and integrate; in each case one has to apply Cauchy-Schwarz judiciously.

Region III: Finally consider a neighbourhood of the intersection of the left and front faces. We use coordinates (τ, s) but now must take the linearized Herring boundary conditions and the interaction between the different evolving arcs into account. So consider three separate functions $U = (u^{(j)})$, one for each of the three intersecting arcs, and each supported in $s \leq 2$. (This use of U is different than the notation earlier in this section.) The model operator in this region is $\tau\partial_\tau + N - \partial_s^2$, and we write $(\tau\partial_\tau + N - \partial_s^2)u^{(j)} = \tau^2 h^{(j)}$ and $\beta(U) = K = (k_0, k_1)$, where each $u^{(j)}$ (and hence each $h^{(j)}$ and K as well) vanishes to all orders as $\tau \rightarrow 0$, but is nonvanishing at $x = 0$.

We next choose a decomposition $u^{(j)} = v^{(j)} + w^{(j)}$, or $U = V + W$, as follows. The functions $v^{(j)}$ are chosen to satisfy $(\tau\partial_\tau + N - \partial_s^2)v^{(j)} = \tau^2 h^{(j)}$, $v^{(j)}(\tau, 0) = 0$ (and, say, $v^{(j)}(\tau, s) = 0$ for $s \geq 2$ as well). The $w^{(j)}$ are then chosen so that $(\tau\partial_\tau + N - \partial_s^2)w^{(j)} = 0$ and $\beta(W) = K - \beta(V)$. Although W is not supported in $s \leq 1$, we can cut both it and V off to have support in $s \leq 3/2$, and estimate the terms coming from the cutoff functions using the earlier estimates for Region I. The estimates for V and W are handled differently.

We assume for simplicity that the coefficients $c_j(\tau) = |\partial_s \gamma^{(j)}(\tau, 0)|^{-1}$ of the projectors in the definition of β are constant; the general case can be handled by approximation. The steps in the estimates for each $v^{(j)}$ are essentially the same as those in Region I, but slightly simpler. First multiply the equation by $Nv^{(j)}$ and integrate to get the estimate (6.5). The boundary terms in the various integrations by parts all vanish since each $v^{(j)}(\tau, 0) = 0$. Next write the equation as $(\tau\partial_\tau - \partial_s^2)v^{(j)} = (\tau^2 h^{(j)} - Nv^{(j)})$ square both sides and integrate, just as before. We handle the cross-term just as in region I. This proves that

$$\int (\tau\partial_\tau v^{(j)})^2 + (\partial_s^2 v^{(j)})^2 + N(\partial_s v^{(j)})^2 + N^2(v^{(j)})^2 d\tau ds \leq C \int \tau^4 |h^{(j)}|^2 d\tau ds. \quad (6.12)$$

Finally, let us turn to the estimates for $W = (w^{(j)})$. Recall the Mellin transform and its inverse

$$\eta(\tau) \mapsto \mathcal{M}(\eta)(\mu) := \tilde{\eta}(\mu) = \int_0^\infty \tau^{i\mu} \eta(\tau) \frac{d\tau}{\tau}, \quad \eta(\tau) = \frac{1}{2\pi} \int \tilde{\eta}(\mu) d\mu.$$

Taking the Mellin transform of the equation gives

$$(-\partial_s^2 + (N + i\mu))\tilde{w}^{(j)}(\mu, s) = 0,$$

so, throwing out the exponentially growing solution, we find that

$$\tilde{w}^{(j)}(\mu, s) = \tilde{b}_j e^{-\sqrt{N+i\mu}s}$$

for some $\tilde{b}_j(\mu)$ to be determined. Note that $\tilde{w}^{(j)}(\mu, 0) = \tilde{b}_j$ and $\partial_s \tilde{w}^{(j)}(\mu, 0) = -\sqrt{N+i\mu}\tilde{b}_j$, so if we set $b_j(\tau) = \mathcal{M}^{-1}(\tilde{b}_j)$, then

$$w^{(j)}(\tau, 0) = b_j, \quad \partial_s w^{(j)}(\tau, 0) = S_N b_j,$$

where $S_N b := \mathcal{M}^{-1}(\sqrt{N + i\mu} \tilde{b}(\mu))$ is an invertible elliptic pseudodifferential operator of order $1/2$.

Replacing $K - \beta(V)$ by K for simplicity, the boundary conditions are:

$$\begin{aligned} \beta_0(W) &= (w^{(2)}(\tau, 0) - w^{(1)}(\tau, 0), w^{(3)}(\tau, 0) - w^{(1)}(\tau, 0)) = k_0 = (k'_0, k''_0), \quad \text{and,} \\ \beta_1(W) &= \sum_{j=1}^3 P^{(j)} \partial_s w^{(j)}(\tau, 0) = k_1. \end{aligned}$$

Using these, we can determine the b_j from K as follows. Write $w^{(1)}(\tau, 0) = b_1$ and $k_0 = (k'_0, k''_0)$ (these primes do not denote differentiation) so that $w^{(2)}(\tau, 0) = b_2 = k'_0 + b_1$, and $w^{(3)}(\tau, 0) = b_3 = k''_0 + b_1$. On the other hand, by a short calculation, the second boundary condition determines the unknown coefficient b_1 by the relationship

$$\mathbb{P}(b_1) = S_N^{-1} k_1 - P_2 k'_0 - P_3 k''_0,$$

where $\mathbb{P} = \sum c_j P^{(j)}$. (Recall that $c_j = |\partial_s \gamma^{(j)}(0, 0)|^{-1}$.) To solve the problem completely, we need to show that $\mathbb{P}$ is invertible. (This invertibility is of course precisely the Lopatinski-Shapiro condition for the linearized Herring boundary condition.) In the special case where all $c_j = 1$, an explicit calculation using that the $P^{(j)}$ project onto three unit vectors making a mutual angle $2\pi/3$ yields

$$\mathbb{P} = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}.$$

For the general case, if some or all of the $c_j \neq 1$, we note that $\mathbb{P} = \sum c_j (P^{(j)})^2$, so $\mathbb{P}b_1 = 0$ implies

$$\langle \mathbb{P}b_1, b_1 \rangle = \sum c_j |P^{(j)}b_1|^2 = 0,$$

hence $b_1 = 0$.

Remark 6.1. *The invertibility of the matrix $\mathbb{P}$ is precisely the Lopatinski-Shapiro condition for this boundary problem.*

It remains to estimate the L^2 norm of W and its derivatives. We first carry this out assuming that $N = 1$, and after that obtain the final estimates by a rescaling argument. Let us focus on any one of the components $w^{(j)}$, and drop the superscript j for the moment. In the following we shall use the measures $d\tau/\tau$ and $(d\tau/\tau)ds$, which are more natural when using the Mellin transform. We can convert back to the measures $d\tau$ and $d\tau ds$ at the end of the proof if we replace w by $\tau^{1/2}w$, N by $N + \frac{1}{2}$, etc.

By the Plancherel theorem,

$$\|w\|_{L^2}^2 = \int_{-\infty}^{\infty} \int_0^{\infty} |\tilde{w}(\mu, s)|^2 d\mu ds \leq C \int_{-\infty}^{\infty} |\tilde{b}(\mu)|^2 \left(\int_0^{\infty} \left| e^{-s\sqrt{1+i\mu}} \right|^2 ds \right) d\mu.$$

Now $1 + i\mu = Re^{i\theta}$, where $R = \sqrt{1 + \mu^2}$ and $\tan \theta = \mu$, so

$$\left| e^{-s\sqrt{1+i\mu}} \right|^2 = \exp \left(-2s(1 + \mu^2)^{1/4} \cos(\theta/2) \right) \leq C \exp \left(-Cs(1 + \mu^2)^{1/4} \right),$$

since $\cos(\theta/2) \rightarrow 1/\sqrt{2}$ as $|\mu| \rightarrow \infty$. Integrating in s , we conclude that

$$\|w\|_{L^2}^2 \leq C \int_{-\infty}^{\infty} |\tilde{b}(\mu)|^2 (1 + \mu^2)^{-1/4} d\mu = \|b\|_{-1/4}^2,$$

where $\|\cdot\|_s$ denotes the H_b^s norm of the function $b(\tau)$. For completeness, recall that

$$\|b\|_s^2 = \int |\tilde{b}(\mu)|^2 (1 + \mu^2)^s d\mu.$$

In the special case $s = \ell \in \mathbb{N}$,

$$\|b\|_\ell^2 = \sum_{i=0}^{\ell} \int |(\tau \partial_\tau)^i b|^2 \frac{d\tau}{\tau}.$$

Since b is a linear combination of $S_1^{-1}k_1$ and k_0 , we conclude finally that

$$\|w\|_{L^2}^2 \leq C(\|k_0\|_{-1/4}^2 + \|k_1\|_{-3/4}^2).$$

A similar calculation can be done to estimate the L^2 norms of $\partial_s w$, $\partial_s^2 w$ and $\tau \partial_\tau w$, leading to the estimate

$$\int |\tau \partial_\tau w|^2 + |\partial_s^2 w|^2 + |\partial_s w|^2 + |w|^2 \leq C(\|k_0\|_{3/4}^2 + \|k_1\|_{1/4}^2).$$

(The estimates for w and $\partial_s w$ require norms of k_0 and k_1 which are 1 and 1/2 orders weaker, respectively.)

Now consider how these estimates depend on N . Changing variables by $\hat{\tau} = \tau^N$ and $\hat{s} = \sqrt{N}s$ gives $\tau \partial_\tau - \partial_s^2 = N(\hat{\tau} \partial_{\hat{\tau}} + 1 - \partial_{\hat{s}}^2)$. For want of better notation, write $\mathbf{w}(\hat{\tau}, \hat{s}) = w(\hat{\tau}^{1/N}, N^{-1/2}\hat{s})$. Then

$$\hat{\mathcal{L}}\mathbf{w}(\hat{\tau}, \hat{s}) = (\mathcal{L}w)(\hat{\tau}^{1/N}, N^{-1/2}\hat{s}) = 0,$$

and in addition

$$\mathbf{w}(\hat{\tau}, 0) = b(\hat{\tau}), \quad \partial_{\hat{s}}\mathbf{w}(\hat{\tau}, \hat{s}) = N^{-1/2}(\partial_s w)(\hat{\tau}^{1/N}, N^{-1/2}\hat{s}).$$

Hence the boundary data appropriate for the problem in the $(\hat{\tau}, \hat{s})$ coordinate system is:

$$\beta(\mathbf{W}) = (\mathbf{k}_0(\hat{\tau}^{1/N}), \mathbf{k}_1(\hat{\tau}^{1/N})) = (k_0(\hat{\tau}^{1/N}), N^{-1/2}k_1(\hat{\tau}^{1/N})).$$

It remains to compute how the various norms change under this rescaling. For this we first note that

$$\begin{aligned} \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} &= N^{3/2} \frac{d\tau}{\tau} ds, \quad \partial_{\hat{s}}\mathbf{w}(\hat{\tau}, \hat{s}) = N^{-1/2}(\partial_s w)(\hat{\tau}^{1/N}, N^{-1/2}\hat{s}), \\ \partial_{\hat{s}}^2\mathbf{w}(\hat{\tau}, \hat{s}) &= N^{-1}(\partial_s^2 w)(\hat{\tau}^{1/N}, N^{-1/2}\hat{s}), \quad \hat{\tau} \partial_{\hat{\tau}}\mathbf{w}(\hat{\tau}, \hat{s}) = N^{-1}(\tau \partial_\tau w)(\hat{\tau}^{1/N}, N^{-1/2}\hat{s}). \end{aligned}$$

Thus

$$\begin{aligned} \int |\mathbf{w}(\hat{\tau}, \hat{s})|^2 \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} &= \int |w(\hat{\tau}^{1/N}, N^{-1/2}\hat{s})|^2 \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} = N^{3/2} \int |w(\tau, s)|^2 \frac{d\tau}{\tau} ds, \\ \int |\partial_{\hat{s}}\mathbf{w}(\hat{\tau}, \hat{s})|^2 \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} &= N^{1/2} \int |\partial_s w(\tau, s)|^2 \frac{d\tau}{\tau} ds, \\ \int |\partial_{\hat{s}}^2\mathbf{w}(\hat{\tau}, \hat{s})|^2 \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} &= N^{-1/2} \int |\partial_s^2 w(\tau, s)|^2 \frac{d\tau}{\tau} ds, \end{aligned}$$

and

$$\int |\hat{\tau} \partial_{\hat{\tau}}\mathbf{w}(\hat{\tau}, \hat{s})|^2 \frac{d\hat{\tau}}{\hat{\tau}} d\hat{s} = N^{-1/2} \int |\tau \partial_\tau w(\tau, s)|^2 \frac{d\tau}{\tau} ds.$$

As for the boundary values, first note that if $\mathbf{k}(\widehat{\tau}) = k(\widehat{\tau}^{1/N})$, then

$$\int_0^\infty k(\widehat{\tau}^{1/N}) \widehat{\tau}^{i\mu} \frac{d\widehat{\tau}}{\widehat{\tau}} = N \int_0^\infty k(\tau) \tau^{iN\mu} \frac{d\tau}{\tau},$$

or in other words,

$$\mathcal{M}(\mathbf{k})(\mu) = N\mathcal{M}(k)(N\mu).$$

Therefore,

$$\begin{aligned} \|\mathbf{k}_0\|_{3/4}^2 &= \int |\widetilde{\mathbf{k}}_0|^2 (1 + \mu^2)^{3/4} d\mu = N^2 \int |\widetilde{k}_0(N\mu)|^2 (1 + \mu^2)^{3/4} d\mu \\ &= N \int |\widetilde{k}_0(\nu)|^2 (1 + N^{-2}\nu^2)^{3/4} d\nu \leq N \int |\widetilde{k}_0(\nu)|^2 (1 + \nu^2)^{3/4} d\nu, \end{aligned}$$

and

$$\|\mathbf{k}_1\|_{1/4}^2 = N \int |\widetilde{k}_1(N\mu)|^2 (1 + \mu^2)^{1/4} d\mu \leq \int |\widetilde{k}_1(\nu)|^2 (1 + \nu^2)^{1/4} d\nu.$$

We used here that if $a > 0$, then

$$(1 + N^{-2}\nu^2)^a = N^{-2a}(N^2 + \nu^2)^a = N^{-2a}(1 + \nu^2)^a \left(\frac{N^2 + \nu^2}{1 + \nu^2} \right)^a \leq (1 + \nu^2)^a.$$

Multiplying all terms by $N^{1/2}$, we derive, at long last, the final estimate

$$\int (\tau \partial_\tau w)^2 + (\partial_s^2 w)^2 + N(\partial_s w)^2 + N^2 |w|^2 \leq C \left(N^{3/2} \|k_0\|_{3/4}^2 + N^{1/2} \|k_1\|_{1/4}^2 \right).$$

Recall that we had replaced $k_1 - \beta_1(V)$ by k_1 . This means that the correct full estimate in region III is

$$\int (\tau \partial_\tau u)^2 + (\partial_s^2 u)^2 + N(\partial_s u)^2 + N^2 |u|^2 \leq C \left(N^{3/2} \|k_0\|_{3/4}^2 + N^{1/2} \|k_1\|_{1/4}^2 + \|\tau^2 h\|_{L^2}^2 \right). \quad (6.13)$$

Again we remind that this is with respect to the measures $d\tau ds/\tau$ and $d\tau/\tau$, but we can obtain an identical-looking estimate by replacing u by $\sqrt{\tau}u$.

We are finally in a position to define the spaces $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$. First, the measures on the right hand sides of (6.8), (6.9), (6.12) and (6.13) are

$$s^2 \tau^4 d\tau ds, \quad x^2 dt dx, \quad \text{and} \quad \tau^4 d\tau ds;$$

the first of these holds in the region where $s \geq s_0$ and the latter when $s \leq s_0$. After changing coordinates, these measures are all equivalent to

$$d\mu_{\mathfrak{Y}} = (1 + s^2) \tau^4 d\tau ds \cong (t + x^2) dt dx;$$

either of these expressions localize to the correct measure in each of the regions.

Next, the space $\mathfrak{X}$ is defined by the (N -dependent) conditions that

$$\begin{aligned} \|u\|_{\mathfrak{X}}^2 &:= \int (\tau \partial_\tau u)^2 + (\partial_s^2 u)^2 + (N + s^2)(\partial_s u)^2 + (Ns^2 + N^2)u^2 d\tau ds \\ &\cong \int t(\partial_t u)^2 + t(\partial_x^2 u)^2 + (N + \frac{x^2}{t})(\partial_x u)^2 + (N\frac{x^2}{t^2} + N^2\frac{1}{t})u^2 dt dx < \infty. \end{aligned}$$

Finally, $\mathfrak{Z} = \mathfrak{Z}_0 \oplus \mathfrak{Z}_1$ is defined as $H_b^{3/4}$ and $H_b^{1/4}$, the fractional Sobolev spaces on $\mathbb{R}^+$ defined in terms of the vector field $\tau \partial_\tau$, and with respect to the measures $N^{3/2} d\tau$ and $N^{1/2} d\tau$, respectively.

With these very extensive calculations, we can now have the

Theorem 6.2. *The mapping*

$$\begin{aligned} (t\partial_t - t\mathcal{L}, \beta) : t^N \mathcal{X} &\longrightarrow t^N \mathcal{Y} \oplus t^N \mathcal{Z} \\ U &\longmapsto ((t\partial_t - t\mathcal{L})U, \beta(U)) \end{aligned}$$

is an isomorphism when N is sufficiently large.

We sketch the proof briefly. First, recall that the mapping here is equivalent to

$$(t\partial_t + N - t\mathcal{L}, \beta) : \mathcal{X} \longrightarrow \mathcal{Y} \oplus \mathcal{Z},$$

so we consider this latter operator instead. Next, if N is sufficiently large, we may use the model estimates proved above to obtain the corresponding estimate for this operator on $\widehat{\Gamma}$. This is done by the usual process of approximation, passing from constant coefficient to variable coefficient operators, and then pasting together these local a priori estimates to get

$$\|U\|_x \leq C(\|t\partial_t + N - t\mathcal{L}U\|_y + \|\beta(U)\|_z)$$

on the entire network. The constants in this estimate depend only on $\mathcal{C}^0$ norms of the coefficients of the operator, and in particular are invariant under continuous deformation of the approximate solution and the network.

This estimate implies, first of all, that any solution to $(t\partial_t + N - t\mathcal{L})U = H$, $\beta(U) = K$, if it exists, must be unique. In addition, the range of this boundary problem is closed, so the theorem will be proved if we can show that its range is dense. However, this is straightforward: the spaces here are all L^2 -based, and by standard parabolic theory, the range of this operator in any region $\{t \geq \epsilon\}$ is onto.

We next define spaces suitable for the higher regularity version. If X is any manifold with corners, we define $\mathcal{V}_b(X)$ to be the space of all smooth vector fields on X which are tangent to all boundary faces. Thus, for example, $\mathcal{V}_b(Q_h)$ is spanned over $\mathcal{C}^\infty(Q_h)$ by $\tau\partial_\tau$ and $s\partial_s$ near $\text{lf} \cap \text{ff}$ and by $y\partial_y$ and $T\partial_T$ near $\text{bf} \cap \text{ff}$; similarly, $\mathcal{V}_b(\mathbb{R}^+)$ is spanned over $\mathcal{C}^\infty(\mathbb{R}^+)$ by $t\partial_t$. These spaces of vector fields are closed under Lie bracket, and are the natural starting point in the analysis of many classes of degenerate differential operators on such spaces. In particular, note that $t\partial_t - t\mathcal{L}$ is a sum of products of elements of $\mathcal{V}_b(Q_h \cup P_h)$. Their main use here is based on the fact that if $V \in \mathcal{V}_b$, then the commutator $[V, t\partial_t - t\mathcal{L}]$ is of the form $t\partial_t - t\mathcal{L}'$ where $\mathcal{L}'$ is an operator of exactly the same type as $\mathcal{L}$. Thus

$$V(t\partial_t - t\mathcal{L})U = (t\partial_t - t\mathcal{L}')(VU) = V(tH) = t(VH + [V, t]H) = tH',$$

with a similar expression for the boundary values.

Now for any $\ell \in \mathbb{N}$, define

$$\mathcal{X}^k = \{U \in \mathcal{X} : V_1 \dots V_j U \in \mathcal{X} \ \forall j \leq k \text{ and } V_i \in \mathcal{V}_b\},$$

with similar definitions for $\mathcal{Y}^k$ and $\mathcal{Z}^k$. Applying a standard commutator argument and the same conjugation trick as above, we can reduce the analysis of $(t\partial_t - t\mathcal{L}, \beta) : t^N \mathcal{X}^\ell \longrightarrow t^N \mathcal{Y}^\ell \oplus t^N \mathcal{Z}^\ell$ to that of a similar operator $(t\partial_t + N - t\mathcal{L}', \beta) : \mathcal{X} \longrightarrow \mathcal{Y} \oplus \mathcal{Z}$, and for this we obtain estimates and an existence and uniqueness theorem exactly as above.

Corollary 6.3. *For every $\ell \geq 0$ and $N \gg 1$,*

$$(t\partial_t - t\mathcal{L}, \beta) : t^N \mathcal{X}^\ell \longrightarrow t^N \mathcal{Y}^\ell \oplus t^N \mathcal{Z}^\ell$$

is an isomorphism.

In particular, if H and K are smooth and vanish to infinite order, then there exists a solution U to this problem which is also smooth and vanishes to all orders. This solution is unique.

7 Short-time existence

We now have all the tools in place to prove our main result:

Theorem 7.1. *Fix an initial network $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^n$ which consists of n smooth curves $\gamma_0^{(j)} \in C^\infty([0, 1])$, $j = 1, \dots, n$. Suppose that at each interior vertex p , there are m_p curves which come together there, and that these intersections are all nontangential. For each interior vertex p , choose an expanding self-similar solution S_p . Then there exists a family of networks $\Gamma(t)$ which are regular for each $t > 0$, and which vary smoothly in the sense that the worldsheet $\{(t, \Gamma(t)) : 0 \leq t < \epsilon\}$ is the image of a smooth function with domain Q_h . The topology of $\Gamma(t)$ for $t > 0$ is obtained by excising a small ball around each interior vertex p and inserting a truncated copy of S_p . For a given topology, the solution is unique.*

Proof. Construct an approximate solution $\widehat{\Gamma}(t) = \{\widehat{\gamma}^{(i)}\}_{i=1}^m$; as per our earlier convention, suppose that each $\widehat{\gamma}^{(j)}$, $j \leq n$, limits to $\gamma_0^{(j)}$, whereas the $\widehat{\gamma}^{(j)}$ with $j > n$ are the curves which disappear into the vertices as $t \rightarrow 0$. By abuse of language, we consider each $\widehat{\gamma}^{(j)}$ as a map from Q_h into Z_h .

We seek a family of perturbations $U = \{u^{(j)}\}$, where each $u^{(j)}$ is a function on $Q^{(j)}$ or $P^{(j)}$, as appropriate, and this collection of functions is chosen so that the flow equations and boundary conditions hold, i.e.,

$$\mathbb{M}(\widehat{\Gamma} + U) = 0, \quad B(\widehat{\Gamma} + U) = 0.$$

We approach this in the standard way. Expand the nonlinear operators $\mathcal{M}$ and B around the approximate solution, and denote their linearizations by $\partial_t - \mathcal{L}$ and β , respectively, as we have done in the first part of §6. Now set $H = -\mathcal{M}(\widehat{\Gamma})$, $J = -B(\widehat{\Gamma})$. We can then write this nonlinear initial-boundary problem as

$$(\partial_t - \mathcal{L})U = H + Q(U), \quad \beta(U) = J + S(U), \quad (7.1)$$

where Q and S are the quadratically vanishing remainder terms in these two Taylor approximations.

Our careful choice of approximate solution ensures that H and J are smooth and vanish to all orders as $t \rightarrow 0$. We therefore seek to find a solution U which is also smooth and vanishing to all orders. Choose ℓ large enough so that multiplication $t^N \mathcal{X}^\ell \times t^N \mathcal{X}^\ell \rightarrow t^{2N} \mathcal{X}^\ell$ is bounded. Next, suppose that we are solving the problem on some interval $0 \leq t < t_0$. If Q is any quadratically vanishing expression in the components of U and $N \geq 2$, then

$$\|Q(U)\|_{t^N \mathcal{X}^\ell} \leq t_0^N \|U\|_{t^N \mathcal{X}^\ell} \quad (7.2)$$

for any $U \in t^N \mathcal{X}^\ell$.

Now let $\mathbb{H}$ denote the solution operator for $(t\partial_t - t\mathcal{L}, \beta)$, as in Theorem 6.2, and rewrite (7.1) as the integral equation

$$U = \mathbb{H}(H + Q(U), J + S(U)). \quad (7.3)$$

It is now quite standard, using Corollary 6.3 and taking advantage of (7.2), to check that (7.3) is a contraction mapping. Hence there is a unique solution U , and clearly U is smooth and vanishes to all orders as $t \rightarrow 0$.

This completes the proof of short-time existence.

As for uniqueness, notice that if two solutions $\Gamma(t)$ and $\Gamma'(t)$ are two smooth flowouts from the same initial network Γ_0 and both have the same topology, then all Taylor coefficients of the constituent curves $\gamma^{(j)}(t, x)$ and $\gamma^{(j)'}(t, x)$ must agree. This means that the differences $\gamma^{(j)'} - \gamma^{(j)}$ necessarily vanish to all orders as $t \rightarrow 0$. By the uniqueness inherent in the contraction mapping step of the argument, this implies that $\gamma^{(j)}(t, x) \equiv \gamma^{(j)'}(t, x)$. $\square$

Proposition 7.2. *Theorem 7.1 also holds for $\mathcal{C}^2$ irregular networks $\Gamma_0 = \{\gamma_0^{(j)}\}_{j=1}^n$.*

Proof. We prove this by taking a $\mathcal{C}^2$ approximation of Γ_0 by a sequence of $\mathcal{C}^\infty$ networks $\Gamma_{0,\ell}$, and showing that the local existence theorem around any of these is valid in a fixed-sized neighborhood in the space $\mathcal{X}^0$. For ℓ large enough, this neighborhood includes an evolving network with initial condition Γ_0 .

First consider the approximate solution construction, where to each smooth initial network $\Gamma_{0,\ell}$ we assign a flowout $\widehat{\Gamma}_\ell$ of some fixed topology. It is possible to choose these approximate solutions to depend continuously in the $\mathcal{C}^\infty$ topology on the initial network. Indeed, the coefficient functions in the series expansion at each front face clearly depend continuously on the derivatives of each $\gamma_{0,\ell}^{(j)}$ at $x = 0$ or $x = 1$. We then use the Seeley extension theorem, which is a quantitative form of the Borel lemma. It states specifically that the extension map, carrying the coefficients to all orders of these expansions at each boundary face of $Q_h \sqcup P_h$ to the approximate solution $\widehat{\Gamma}_\ell$, can be chosen to depend continuously on these coefficients.

Now turn to the contraction argument. We may as well work in some fixed space $t^N \mathcal{X}^0$. We seek a solution U_ℓ to $\mathbb{M}(\widehat{\Gamma}_\ell + U_\ell) = 0$, $B(\widehat{\Gamma}_\ell + U_\ell) = 0$, which we set up as the equivalent integral equation (7.3). The usual argument shows that this is a contraction on a ball of size R_0 in the space $t^N \mathcal{X}^0$, where the radius R_0 depends on the norm of the solution operator $\mathbb{H}_\ell$ on $[0, t_0)$. In particular, this proof also shows that t_0 and hence R_0 can be chosen to be uniform in ℓ .

Recalling that each arc in $\Gamma_{0,\ell}$ is $\mathcal{C}^2$, we see that the coefficients in the linearized operators $\mathcal{L}_\ell$ are convergent in the $\mathcal{C}^0$ topology. We have remarked that the constants in the main a priori estimate leading to the fact that $\mathcal{L}_\ell : t^N \mathcal{X} \rightarrow t^N \mathcal{Y} \oplus t^N \mathcal{Z}$ depend only on the size in $\mathcal{C}^0$ of the coefficients of $\mathcal{L}_\ell$, and these are uniformly controlled. Thus for this particular sequence of smooth initial networks $\Gamma_{0,\ell}$, the solution $\widehat{\Gamma}_\ell + U_\ell$ exists and is unique in a ball around $\widehat{\Gamma}_\ell$ of fixed radius R_0 . Furthermore, the $t^N \mathcal{X}^0$ norm of U_ℓ is uniformly controlled.

We now take a weak limit of these solutions U_ℓ to some $U \in t^N \mathcal{X}^0$. By classical parabolic regularity theory, this convergence can be assumed to be strong in the $\mathcal{C}^\infty$ topology in any region $t > \epsilon$. This convergence is also in $\mathcal{C}^2$ on the bottom face, i.e., as $t \searrow 0$ but $x \neq 0, 1$, and in $\mathcal{C}^\infty$ on the front face in any region where $s \leq C$. The one place where classical methods do not apply is near the corner where the front and bottom faces meet. We obtain in this way a nonzero solution U to the flow equations on some fixed time-interval $[0, t_0)$ which have initial $\mathcal{C}^2$ network Γ_0 . This proves existence of a flowout for any admissible $\mathcal{C}^2$ network Γ_0 . $\square$

Remark 7.3. *We do not assert uniqueness in this last result, but note that it is highly likely to be true.*

8 Final comments and further directions

In this last section we present a number of results which round out and provide further context to our main theorem.

8.1 The unparametrized evolution equation

The analysis in this paper has focused on the parametrized network flow, i.e., the specific parabolic equation

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2}$$

with the Herring boundary conditions. As explained in §2, this formulation is equivalent to imposing a specific tangential term in the velocity vector. We study here the question of whether

the geometric objects, i.e. network of curves, corresponding to the solution we have found, which of course depends on the choice of expanding soliton at each interior vertex, also depends on this tangential term. To do so, we introduce a “geometric” formulation of this curvature flow that only involves the evolution of the set, but does not fix a choice of parametrization. In the following we denote networks by $\mathcal{N}$ rather than Γ to emphasize the difference.

Definition 8.1. *A network of n curves $\mathcal{N}_0 = \{n_0^{(j)}\}$ is an admissible initial network if $n_0^{(j)}$ admits a regular $\mathcal{C}^2$ admissible parametrization $\gamma_0^{(j)}$.*

Definition 8.2 (Motion by curvature of networks). *Let $\mathcal{N}_0 = \{n_0^{(j)}\}$, $j = 1, \dots, n$ be an admissible initial network, and $\gamma^{(j)}$ a regular parametrization of $n_0^{(j)}$. A time dependent family of networks $\mathcal{N}(t)$ with $m \geq n$ constituent curves for each $t \in (0, T_{\max})$ is a maximal geometric solution of the geometric motion by curvature in $[0, T_{\max})$ with Dirichlet boundary conditions and initial condition $\mathcal{N}_0$ if each curve $n^{(j)}$ in $\mathcal{N}$ has a $\mathcal{C}^2$ parametrization $\gamma^{(j)}$ and the following is true. First, for each j ,*

$$\left\langle \partial_t \gamma^{(j)}(t, x), \nu^{(j)}(t, x) \right\rangle = \kappa^{(j)}(t, x),$$

where $\nu^{(j)}$ is the unit normal to $\gamma^{(j)}$; second, $\mathcal{N}(t)$ is a regular network for each $t > 0$, i.e., has only regular triple junctions; third, its exterior vertices are fixed points on $\partial\Omega$; finally, for $j \leq n$, the curves $n^{(j)}$ converge uniformly to $n_0^{(j)}$ as $t \rightarrow 0$, while for $j > n$, the curves $n^{(j)}$ converge to one of the interior vertices of $\mathcal{N}_0$. We say that $T_{\max}$ is the maximal time of existence for this regular evolution if this solution exists for $0 \leq t < T_{\max}$, but there exists no other regular network on a longer time interval.

Proposition 8.3. *For any admissible initial network $\mathcal{N}_0$, there exists a geometric solution $\mathcal{N}(t)$, $0 \leq t \leq T$, to the network flow with initial datum $\mathcal{N}_0$.*

Proof. If $\{\gamma_0^{(j)}\}$ is a $\mathcal{C}^2$ parametrization of the curves in $\mathcal{N}_0$, then by Theorem 7.1 there exists a solution $\Gamma(t) = \{\gamma^{(j)}\}$ of the parametrized flow with initial datum $\Gamma_0 := \{\gamma_0^{(j)}\}$. Clearly this is a solution of the unparametrized geometric flow as well. $\square$

Definition 8.4. *Two admissible networks $\mathcal{N}$ and $\mathcal{N}'$ in the smoothly bounded set Ω with the same outer vertices are said to have the same topology if there is a diffeomorphism Φ of Ω which fixes the outer vertices and is such that $\Phi(\mathcal{N}) = \mathcal{N}'$ and $\Phi(\Omega \setminus \mathcal{N}) = \Omega \setminus \mathcal{N}'$. In particular, two such networks have the same number of interior and exterior curves, vertices, and overall combinatorial type.*

Proposition 8.5. *Let $\mathcal{N}_0$ be a smooth admissible irregular initial network, and $\mathcal{N}(t)$, $0 \leq t \leq T$ a smooth solution of the unparametrized geometric flow starting from $\mathcal{N}_0$. Choose any smooth parametrization $\Gamma(t) = \{\gamma^{(j)}\}$ of $\mathcal{N}(t)$. Now suppose that $\tilde{\Gamma}(t) = \{\tilde{\gamma}^{(j)}\}$, $0 \leq t \leq \tilde{T}$ is any other solution of the parametrized flow which limits geometrically to $\mathcal{N}_0$ and which has the same topology as $\Gamma(t)$ for any $t \in (0, \min\{T, \tilde{T}\})$. Then the curves in $\tilde{\Gamma}(t)$ are reparametrizations of those in $\Gamma(t)$.*

Remark 8.6. *If $\mathcal{N}_0$ is a regular network containing precisely 3 curves, then this result reduces to [9, Theorem 3.16].*

Proof. We must prove that there exists a collection of functions $\psi^{(j)} : [0, T] \times [0, 1] \rightarrow [0, 1]$ such that $\tilde{\gamma}^{(j)} = \gamma^{(j)}(t, \psi^{(j)}(t, x))$. We shall derive evolution equations for the $\psi^{(j)}$, and in fact also for their inverses, $\xi^{(j)} : [0, \tilde{T}] \times [0, 1] \rightarrow [0, 1]$, $\psi^{(j)}(t, \xi^{(j)}(t, x)) = x$, $\xi^{(j)}(t, \psi^{(j)}(t, x)) = x$. Not unexpectedly, it is necessary to pass to the blowups $Q_h \cup P_h$ to write these equations.

Suppose for the moment that $t > 0$, and we derive the equations away from the front faces. Obviously, reparametrizations do not affect the normal velocities, so the normal components of $\partial_x \tilde{\gamma}^{(j)}$ and $\partial_x \gamma^{(j)}$ are the same at corresponding points. The reparametrization must be chosen so that the tangential components of these two tangent vectors also correspond. This will be the case if

$$\left\{ \begin{array}{lcl} \psi_t^{(j)}(t, x) & = & \frac{\psi_{xx}^{(j)}(t, x)}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|^2} \frac{\psi_x^{(j)}(t, x)^2}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|^2} - \frac{\langle \tilde{\gamma}_t^{(j)} \circ \psi^{(j)}(t, x), \tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x) \rangle}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|^2} \\ & + & \frac{1}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|} \left\langle \frac{\tilde{\gamma}_{xx}^{(j)} \circ \psi^{(j)}(t, x)}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|^2}, \frac{\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)}{|\tilde{\gamma}_x^{(j)} \circ \psi^{(j)}(t, x)|} \right\rangle \\ \psi^{(j)}(t, 0) & = & 0 \\ \psi^{(j)}(t, 1) & = & 1 \\ \tilde{\gamma}(0, \psi^{(j)}(0, x)) & = & \gamma_0^{(j)}(x) \end{array} \right.$$

here we have used the notation $\tilde{\gamma}^{(j)} \circ \psi^{(j)}(t, x)$ to stand for $\tilde{\gamma}^{(j)}(t, \psi^{(j)}(t, x))$. Unfortunately, the coefficients of this system depend on the $\psi^{(j)}(t, x)$. To remedy this, we consider instead the evolution problem for the functions $\xi^{(j)}$ instead. These are

$$\begin{aligned} \partial_t \xi^{(j)}(t, y) &= -\partial_t \psi^{(j)}(t, \xi^{(j)}(t, y)) \partial_y \xi^{(j)}(t, y) \\ &= -\frac{\psi_{xx}^{(j)}(t, x)}{|\tilde{\gamma}_x^{(j)}(t, y)|^2} \partial_y \xi(t, y)^3 + \frac{\langle \tilde{\gamma}_t^{(j)}(t, y), \tilde{\gamma}_x^{(j)}(t, y) \rangle}{|\tilde{\gamma}_x^{(j)}(t, y)|^2} \partial_y \xi(t, y) \\ &\quad - \frac{\partial_y \xi(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|} \left\langle \frac{\tilde{\gamma}_{xx}^{(j)}(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|^2}, \frac{\tilde{\gamma}_x^{(j)}(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|} \right\rangle, \end{aligned}$$

and

$$\begin{aligned} \partial_y \xi^{(j)}(t, y) &= 1 / \partial_y \psi^{(j)}(t, \xi^{(j)}(t, y)), \\ \partial_y^2 \xi^{(j)}(t, y) &= -\frac{\partial_y^2 \psi^{(j)}(t, \xi^{(j)}(t, y))}{\partial_y \psi^{(j)}(t, \xi^{(j)}(t, y))^3} = -\partial_y \xi^{(j)}(t, y)^3 \partial_y^2 \psi^{(j)}(t, \xi^{(j)}(t, y)). \end{aligned}$$

this leads to the *linear* system

$$\left\{ \begin{array}{lcl} \partial_t \xi^{(j)}(t, y) & = & \frac{\partial_y^2 \xi(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|^2} + \frac{\langle \tilde{\gamma}_t^{(j)}(t, y), \tilde{\gamma}_x^{(j)}(t, y) \rangle}{|\tilde{\gamma}_x^{(j)}(t, y)|^2} \partial_y \xi(t, y) \\ & - & \frac{\partial_y \xi(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|} \left\langle \frac{\tilde{\gamma}_{xx}^{(j)}(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|^2}, \frac{\tilde{\gamma}_x^{(j)}(t, y)}{|\tilde{\gamma}_x^{(j)}(t, y)|} \right\rangle \\ \xi^{(j)}(t, 0) & = & 0 \\ \xi^{(j)}(t, 1) & = & 1 \\ \xi^{(j)}(x) & = & \tilde{\gamma}^{-1}(y). \end{array} \right.$$

The coefficients here are at least continuous since $\Gamma(t)$ solves the network flow and its curves are of class $\mathcal{C}^{1,2}$. We write this main equation as

$$\partial_t \xi^{(j)}(t, x) = a^{(j)}(t, x) \partial_x^2 \xi^{(j)}(t, x) + b^{(j)}(t, x) \partial_x \xi^{(j)}(t, x); \quad (8.1)$$

here $a^{(j)}(t, x), b^{(j)}(t, x) \in \mathcal{C}^0$.

Now a bit of calculation shows that in the (τ, s) coordinates, (8.1) takes the form

$$\tau \partial_\tau \xi^{(j)} - s \partial_s \xi^{(j)} - \tilde{a} \partial_s^2 \xi^{(j)} - \tilde{b} \tau \partial_s \xi^{(j)} = 0.$$

Taking into account the explicit formulæ for a and b , and writing $\xi = \tau\varphi$ and $\gamma = \tau\eta$, we obtain finally that

$$(\tau \partial_\tau + 1 - 2 \partial_s) \varphi^{(j)} = \frac{\partial_s^2 \varphi^{(j)}}{|\partial_s \eta^{(j)}|^2} + \frac{\partial_s \varphi^{(j)}}{|\partial_s \eta^{(j)}|} \left\langle (\tau \partial_\tau + 1 - 2 \partial_s) \eta^{(j)}, \frac{\partial_s \eta^{(j)}}{|\partial_s \eta^{(j)}|} \right\rangle - \frac{\partial_s \varphi^{(j)}}{|\partial_s \eta^{(j)}|} \left\langle \frac{\partial_s^2 \eta^{(j)}}{|\partial_s \eta^{(j)}|^2}, \frac{\partial_s \eta^{(j)}}{|\partial_s \eta^{(j)}|} \right\rangle.$$

This system has the same structure as the main one we have studied earlier in this paper, and following the steps of that argument, we can build first an approximate solution to infinite order, then correct this to an exact solution. We omit the details. $\square$

Corollary 8.7. *The number of geometric flowouts of an initial geometric network $\mathcal{N}_0$, up to reparametrization, is precisely the same as the product of the number of expanders at each internal vertex.*

Proof. The claim is that for every possible topology for a flowout, there exists exactly one unparametrized geometric solution starting from $\mathcal{N}_0$. First, choose a compatible expander at each internal vertex of $\mathcal{N}_0$. This determines the topology of an approximate solution to the parametrized flow equations, and thus an exact solution of these equations. On the other hand, we have proved that any two geometric solutions with the same topology coincide up to reparametrization. This establishes the bijection. $\square$

This answers the two Open Problems, 11.19 and 11.20, as stated in [17].

Corollary 8.8. *If the initial network has precisely one triple junction and no other internal vertices, then there is a unique solution of the network flow. If the initial datum has one quadruple junction forming angles of $\pi/3$ and $2\pi/3$, then there is a unique solution of the network flow.*

8.2 Disconnected and unstable solutions

The solutions we have found in this paper are regular networks when $t > 0$; in particular, every vertex of the original network resolves into an expanding cluster of vertices, connected by a system of new curves, in such a way that the evolving vertices are all regular, i.e., points where precisely three curves meet at $2\pi/3$. In this section we describe briefly two other possible evolutions, both of which are immediately accessible by our methods. The first is where the network instantaneously disconnects (locally) at one or more of the interior vertices, and the second is where the network evolves but maintaining irregular vertices modeled on unstable expanding solutions. The fact that this network flow is the gradient flow of the overall length of the network indicates that the ‘physically relevant’ solutions are those which are locally and globally stable. This suggests that the second of these ‘new’ flows is an oddity. On the other hand, there are situations where the physics suggest that the network should disconnect. We explain all of this now.

Regarding the first of these issues, we recall from [20] that disconnected self-similar expanders exist. Indeed, as described in that paper, such expanders are in bijective correspondence with geodesic Steiner trees in the negatively curved space $(\mathbb{R}^2, g)$, where $g = e^{|x|^2} |dx|^2$. The initial

condition that the expander limits to a particular union of n rays meeting at the origin corresponds to the fact that these Steiner trees have n exterior edges limiting to points $q_1, \dots, q_n$ lying on the boundary of the geodesic compactification of this space. By virtue of the negative curvature of g , this geodesic compactification is naturally identified with the closed unit disk, so we may consider these points as arranged in cyclic order around the circle.

Now suppose that we decompose the set $\{1, \dots, n\}$ into disjoint subsets $S_1 \sqcup S_2 \sqcup \dots \sqcup S_k$, where each S_j consists of a contiguous set of elements (mod n) and $|S_j| \geq 2$ for all j . In other words, each S_i is of the form $\{\ell, \ell + 1, \dots, \ell'\}$. The general methods there (which are based on minimization of 1-rectifiable currents with coefficients in $\mathbb{Z}_r$ (for various values of r) show that there is a connected geodesic Steiner tree $\mathcal{T}_j$ with asymptotic boundary S_j for each j . It follows by the regularity theory for these varifold minimizers that $\mathcal{T}_i \cap \mathcal{T}_j = \emptyset$ for $i \neq j$. In other words, this produces a geodesic Steiner tree with precisely k components.

Now let Γ_0 be any irregular geodesic network such that at least one vertex has valence strictly greater than 3. According to our main theorem, the choice of a self-similar expander at each interior vertex of Γ_0 leads to a unique evolution $\Gamma(t)$, $0 < t < T$. If the expander is disconnected, the corresponding flowout $\Gamma(t)$ becomes immediately locally disconnected near that vertex. Depending on the overall topology, $\Gamma(t)$ itself may become disconnected. In some situations this may be the more physically appropriate situation.

In each of the pictures below, the initial irregular network is the same. In the first picture it is the boundary of the blue set, in the second one the boundary of the yellow set, and in the third, it is the boundary for four different phases. In the first two cases, the disconnected flowouts are physically realistic, whereas in the last case, the four phases must remain separate so the evolving network should also remain connected, but there are two possible combinatorial arrangements.

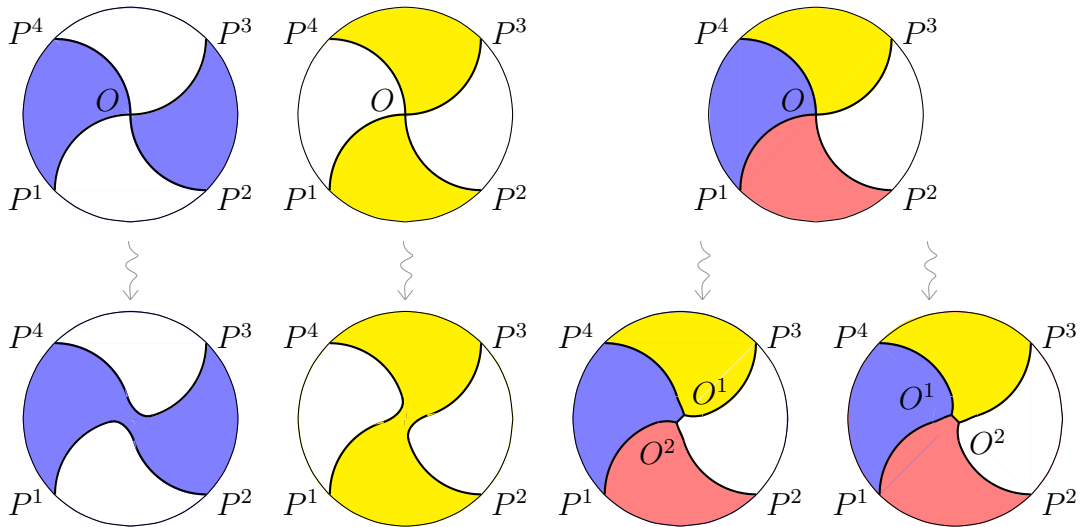


Figure 3: Up: initial datum. Down: expected evolution.

Now let us turn to the second problem, where a network evolves in such a way that some of its vertices remain irregular but stable. To explain this, we return to the description of self-similar expanders as geodesic Steiner trees with given asymptotic boundary on the circle at infinity. For many (and probably all) arrangements of n points on this circle, there exist Steiner networks where some or all of the interior vertices are not trivalent, but do satisfy the criticality condition

that the sum of the unit tangents of all curves meeting at each vertex sum to 0. Steiner networks with vertices satisfying this weaker condition are no longer minimizing, or even stable, critical points for the length functional; instead, they are critical points for length.

As an example, suppose that the arrangement of four points on the circle is invariant with respect to reflection across the origin (thus $q_3 = -q_1$ and $q_4 = -q_2$). In this case, the two diameters of the circle are geodesics for g and the ‘ray-balancing’ condition that the unit tangents sum to 0 holds at the origin, where these lines intersect.

We assert that it is possible to carry out the entirety of our proof of short-time existence in this paper using such irregular self-similar expanders. The steps are all clear: we may construct the approximate solution in precisely the same way, and then derive estimates for the linear equation precisely as in §6 to solve away the rapidly vanishing error term, just as in §7. The only point that needs to be made is that even at an evolving irregular vertex satisfying this balancing condition the Lopatinski-Shapiro condition still holds.

8.3 Theorem 1.1 as a restarting theorem

Given any initial network $\mathcal{N}_0$, Theorem 1.1 guarantees the existence of a flowout $\mathcal{N}(t)$, i.e., a solution of the network flow, for some possibly short time-interval $(0, T)$ which is a regular network for all such positive times. It is quite conceivable, however, that at some later time the network $\mathcal{N}(t)$ degenerates and becomes irregular at some first positive time. There are various ways this might happen. For example, some curve in $\mathcal{N}(t)$ might shrink to zero length at time t_0 so that the limit of $\mathcal{N}(t)$ as $t \nearrow t_0$ is an irregular network in which the vertex which is the limit of that edge has valence four. Another type of degeneration occurs if $\mathcal{N}(t)$ has a closed region bounded by fewer than six curves. One can prove that the area of the enclosed region decreases monotonically and shrinks to zero in finite time. At that time the lengths of some or all of the boundary curves of that region have decreased to zero, so the limiting network will have a vertex of valence four or five. In such limits, the curvature of the curves around this region is unbounded above, cf. [17]. We refer to the well-known numerical simulations of network flow that appear on Ken Brakke’s webpage; these phenomena all can be seen there.

Suppose that $\mathcal{N}(t)$ does degenerate as $t \nearrow t_0$, but in such a way that none of the curvatures of any constituent curve blow up. This happens if two triple-junctions coalesce into a vertex of valence four, where the incoming edges meet at angles of either $\pi/3$ or $2\pi/3$. At least in this case, it is natural to ask whether the network flow can be continued past this singular time. We refer to this type of singularity as a ‘standard transition’ and we emphasize that if we assume that if $\mathcal{N}(t)$ is a tree, and if none of the curves in the limit $\mathcal{N}(t_0)$ has multiplicity greater than one, then this is the only possible singularity type.

We now interpret Theorem 1.1 as a ‘restarting’ theorem for a suitably defined network flow.

Definition 8.9. *Let Ω be a smoothly bounded open subset of $\mathbb{R}^2$ and $\mathcal{N}_0$ an admissible initial network. A time dependent family of networks $\mathcal{N}(t)$, $0 \leq t < T_{\max}$ is said to be a geometric solution of network flow if it has fixed outer vertices, converges to the initial network $\mathcal{N}_0$ as $t \searrow 0$, and if $[0, T_{\max})$ decomposes as a finite union of subintervals $[0, a_1) \cup [a_1, a_2) \cup \dots, [a_\ell, T_{\max})$ so that for each interval (a_j, a_{j+1}) there is a regular network $\mathcal{N}^{(j)}(t)$ which evolves by network flow, and so that*

$$\lim_{t \nearrow a_j} \mathcal{N}^{(j-1)}(t) = \lim_{t \searrow a_j} \mathcal{N}^{(j)}(t) = \mathcal{N}_j.$$

In each of these limits, $\mathcal{N}^{(j-1)}(t)$ has some arcs whose length decreases to 0, while $\mathcal{N}^{(j)}$ has a collection of new arcs emanating from all vertices in $\mathcal{N}_j$ which have valence greater than 3.

Proposition 8.10. *Suppose $N(t)$ is a solution to the network flow in $[0, T)$ and that as $t \rightarrow T$ the lengths of one or more curves in $N(t)$ converge to 0, but that the curvatures of all curves in $N(t)$ remain bounded in this limit. Then the solution can be extended to a longer time-interval $[0, T')$ for some $T' > T$.*

Proof. By the hypotheses, the limit as $t \rightarrow T$ is a network with bounded curvature and irregular vertices. It can thus be used as an initial network for a subsequent flow. The juxtaposition of this ‘later’ flow with the original one produces a solution to this flow, in the sense described in Definition 8.9 on the union of the two time intervals. $\square$

It is worth stating explicitly that one may control the topological complexity of the network as it passes through each singular time in terms of the choices of expanders at each irregular vertex. Considering only connected expanders without loops, then the total number of enclosed regions is nonincreasing for the duration of this flow.

Notice that when a ‘standard transition’ occurs, differently from [11], our Theorem 1.1 produces a unique network flow.

8.4 Other approaches

In this final section we discuss several approaches that have been used to analyze mean curvature flow in settings which allow for nonsmooth initial data. We do not include here method based on the maximum principle, i.e. viscosity solutions.

Let Σ_t denote a time-dependent smooth submanifold in a Riemannian manifold which evolves with normal velocity equal to its mean curvature at any point and time. This can be formulated as a parabolic PDE for as long as Σ_t remains smooth and embedded. Smoothness, in particular, is guaranteed by parabolic regularity theory so long as certain curvatures do not blow up, or if certain components (or regions) collapse. However, at some time T the flow ceases to exist in a classical sense. It is then natural to ask whether there is any natural way (or perhaps multiple ways) to continue the flow past this singular time.

In a different direction, it has long been of interest to define the evolution by mean curvature of more complicated objects, for example clusters of surfaces, as necessitated by the evolution of grain boundaries.

One of the first successful approaches to this was contained in the thesis of Brakke [4]. In it, he introduced a weak formulation, phrased in the geometric measure theoretic notion of varifolds. This is a broad enough class to contain highly irregular objects, and mean curvature is defined in a distributional sense. More precisely, a family of n -dimensional varifolds $\{V_t\}_{t \in [0, T)}$ is a solution of the Brakke flow if, for almost every time, V_t is an integral n -varifold with locally bounded first variation and generalized curvature h in L^2 such that for any compact set K in $\mathbb{R}^n$ and $t \in (0, T)$ we have $\sup_{\tau \in [0, t)} \|V_\tau\|(K) < \infty$. The flow is dictated by the requirement that for every function $\phi \in C^0(\mathbb{R}^N \times [0, T); \mathbb{R}^+)$, we have

$$\frac{d}{dt} \int \phi(x, t) d\|V_t\| \leq \int (-\phi(x, t)|h(V_t, x)|^2 + \langle \nabla \phi, h(V_t, x) \rangle + (\partial_t \phi)(x, t)) d\|V_t\|. \quad (8.2)$$

Brakke was able to prove global existence of solutions for any initial data which is an integral varifold. This flow has many advantages, but also some deficiencies. For example, this definition allows the instantaneous loss of mass, so even the empty set is a solution. Solutions are not unique (see also [23]), and in addition, it is difficult to control the multiplicity of the evolving varifolds.

These difficulties motivate the continuing search for formulations which allow for singular configurations but which provide better geometric control of the evolution.

An important bit of progress was made almost forty years after Brakke's original work by Kim and Tonegawa [12], at least for codimension one varifolds. They enhanced the definition of Brakke flow by coupling the time dependent family of varifolds with a finite number of time-dependent mutually disjoint open sets. Roughly speaking the varifold is the boundary of the union of these sets, and represents the evolving surfaces of a cluster (the open sets are the grains and the varifold is the grain boundary). The sets evolve continuously in time with respect to the Lebesgue measure and the varifold satisfies (8.2). Kim and Tonegawa proved the existence and could also exclude trivial solutions. Furthermore, when the initial datum is a closed 1-rectifiable set in $\mathbb{R}^2$ with (locally) finite measure (thus, a generalization of the networks we consider here), they obtain regularity [13]: for almost every time the support of the evolving varifolds consists locally of embedded $W^{2,2}$ curves whose endpoints meet at junctions with either 0° , 60° or 120° at almost every time. It is expected that these angles are (almost always) 120° and thus these solutions resemble the network flow.

On the other hand, as explained in [17], solutions to the network flow are solutions of the Brakke flow, and there is *equality* in (8.2) when the network is *regular*.

An important point is that none of these formulations describe what happens on sets of time of measure zero, so cannot give precise information on the types of instantaneous jumps we study in this paper.

We next recall the theory of minimizing movements; this is a time discrete approximation to gradient flow, which was considered in the setting of geometric flows by a number of authors, including De Giorgi, Almgren, Taylor and Wang, Luckhaus and Sturzenhecker. This method involves solving a sequence of variational problems at every one of the discrete time steps. To pass from one time-step to the next, one minimizes the sum of the area functional and an additional term which is added to make sure that we do not move too far in a single time step. There is a recent result by Fischer, Hensel, Laux and Simone [6] concerning BV solutions to planar multiphase mean curvature flow. They prove that in the absence of topological changes, solutions are unique.

As for methods which rely specifically on PDE, the theory started with the paper of Bronsard and Reitich [5], who proved local existence and uniqueness provided the initial network is regular and of class $\mathcal{C}^{2+\alpha}$, with the sum of the curvatures at each triple junctions equal to zero. This last condition is tantamount to the first compatibility condition. Their proof is based on a standard linearization procedure and a fixed point argument. If the initial datum is a regular network of class $\mathcal{C}^2$, but without any restriction on the curvatures at the junctions, then [17] proves existence but not uniqueness. Uniqueness was proved in [9] for initial networks in $W_p^{2-2/p}$ for any $p \in (3, \infty)$. A fortiori, this gives uniqueness for regular networks of class $\mathcal{C}^2$. Noting that $W_p^{2-2/p} \hookrightarrow \mathcal{C}^{1+\alpha}$, we see that the space $W_p^1((0, T); L_p((0, 1); \mathbb{R}^2)) \cap L_p((0, T); W_p^2((0, 1); \mathbb{R}^2))$ where solutions lie embeds into the space of bounded, uniformly continuous functions from $[0, T]$ to $\mathcal{C}^{1+\alpha}$. This degree of regularity is needed in order to get the contraction estimates in the fixed point argument and to be able to invoke classical parabolic theory for the associated linear system.

None of these approaches allows one to treat initial networks which contain triple junctions where the curves do not meet at 120° , or which contain vertices with valence greater than three. Clearly, in either of these cases one needs to work in a slightly broader class which does not impose continuity as $t \searrow 0$.

As already mention in the introduction, there is already a proof of short-time existence due to

Ilmanen, Neves and Schulze [11] if the initial network is an irregular tree with bounded curvature. We describe this in a bit more detail. As we do here, the main issue is to desingularize the higher valence vertices using expanding solitons. They glue a truncation of a connected regular expanding soliton in place of a higher valence vertex, in a ball of radius comparable to $\sqrt{t}$ to obtain an approximation of the flow. They then argue using an expander monotonicity formula which implies that self-similar expanding flows are “dynamically stable”. The existence of the approximating flow for a positive time is guaranteed by this monotonicity, which shows that this approximating flows stays close to the self-similar expanding flows in a suitable neighborhood. The authors establish that the flow converges to the initial datum only in the sense of varifolds. This makes it difficult to address questions of uniqueness or multiplicity, as is possible in our approach.

This brief tour of various methods hopefully makes clear that the methods espoused in this paper do provide some useful additional information regarding multiplicity of solutions that may not be so easy to obtain by other ways.

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