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Some remarks on the history of Ricci's absolute differential calculus

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Abstract

The article offers a general account of the genesis of the absolute differential calculus (ADC), paying special attention to its links with the history of differential geometry. In relatively recent times, several historians have described the development of the ADC as a direct outgrowth either of the theory of algebraic and differential invariants or as a product of analytical investigations, thus minimizing the role of Riemann's geometry in the process leading to its discovery. Our principal aim consists in challenging this historiographical tenet and analyzing the intimate connection between the development of Riemannian geometry and the birth of tensor calculus.

1 Introduction

The importance of the absolute differential calculus (ADC) in the history of mathematics and physics cannot be overestimated. Besides the well-known role of the ADC in the formulation of Einstein's theory of gravitation, its discovery also had a great impact on the historical development of pure mathematics, for example by providing a theoretical framework within which Riemannian geometry and generalizations thereof could find a proper and effective formulation.

In view of its significance, the ADC has been made the object of several studies to determine its historical roots and describe its evolution by analyzing the work of mathematicians such as Riemann, Christoffel, Lipschitz and, Ricci-Curbastro. Far from being the discovery of a single man, the history of the ADC has properly been described as the outcome of a collective enterprise and the final result of a long process that marked the convergence of distinct research lines of late 19th century mathematics: Riemannian geometry, algebraic invariant theory, the theory of differential invariants and differential parameters.

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The origin of ADC can be traced back to Riemann's *Habilitationsvortrag* where the notions of multi-dimensional manifolds and Riemannian metrics were first introduced. Evidently inspired by the reading of this work and by the problems that it aroused, Christoffel and Lipschitz gave the first thorough analysis of the equivalence problem for quadratic differential forms. The need for analytical tools specifically designed to treat this and similar problems gradually evolved into a homogeneous corpus of methods and techniques (including the notion of covariant derivative) that found their first systematic treatment in some of Ricci-Curbastro's papers dating back to the late 1880s.

In relatively recent times, a different view has been proposed that the ADC had its "primary root" within the theory of algebraic invariants. On this interpretation, infinitesimal geometry and Riemannian geometry, in particular, would have played a secondary, almost marginal role by merely representing one of many research areas in which the ADC could find application. Only after the advent of General Relativity would Ricci's calculus be recognized by its own founders as a fundamental tool in the field of geometric research.

An example of this view is offered by the following remarks (Reich 1994, p. 213):

> The absolute differential calculus does not have its main roots (*Hauptwurzeln*) in differential geometry, as has often been assumed previously, but rather in the theory of invariants. Christoffel and Ricci, its most important representatives, come from this tradition. Their goal is not the extension of differential geometry or non-Euclidean geometry, but rather the transfer of the methods of the theory of forms and invariants to the theory of differential forms and differential invariants. Thus, differential geometry played, at best, a secondary role in Christoffel, and it was just one area of application among others for Ricci. Consequently, it can be said that absolute differential calculus lacks geometrical characteristics (*Aspekte*).

The idea that the development of the ADC should be separated from that of infinitesimal geometry was also advanced in Farwell and Knee (1990), with particular reference to Riemann's *Commentatio Mathematica*.[1] The content of the latter is interpreted as separate from the geometric research of the *Habilitationsvortrag*. Supposedly, the view that the *Commentatio* represents a turning point for the developments of differential geometry is actually a misunderstanding resulting from the anachronism imposed by Einstein's theory, after which the tensor calculus was regarded as the privileged language of Riemannian geometry:

> Tensor analysis developed independently of geometry for decades. Indeed there was no significant development in metric geometry per se until it linked with tensor analysis. What provided the impetus for the linking between tensor analysis and metric geometry was the theory of general relativity. Tensors may have continued to develop independently of geometry, and metric geometry may have remained dormant, but for the theory of relativity. (Farwell and Knee 1990, pp. 239–240)

[1] Riemann (1876). For a critique of this interpretation, see Cogliati (2014) and Darrigol (2015).

A less drastic standpoint can be found in Dell'Aglio (1997). Here the role of differential geometry in the genesis of the ADC is described in terms of three distinct phases which gradually led Ricci to appreciate the geometrical significance of the notions of tensor and covariant derivative. The first phase, which was characterized as "interpretative" in character, concerns Ricci's early works on differential invariants, where the geometrical interpretation of his results was allegedly employed as a simple illustration of the meaning of purely algebraic and analytical notions. Dell'Aglio (1997, §6) singled out a subsequent phase called "applicative," in which research topics from differential geometry, though still marginal, were regarded as providing specific fields of application of Ricci's emerging ADC. Only at a later stage, would the connection between the ADC and differential geometry become essential, especially with regard to Ricci's theory of congruences and the reformulation of classical surface theory. Dell'Aglio regarded the genesis of the ADC as a multifaceted process to which various research lines (algebraic, analytical, and geometrical) contributed in different ways. However, he emphasized the role of analytical research by asserting, for example, that "the concept of tensor emerges in Ricci-Curbastro's work in view of strictly analytical purposes" (Dell'Aglio 1997, p. 14).[2]

In many respects, these remarks do not appear completely convincing. Let us start from Farwell and Knee's interpretation of Riemann's *Commentatio*. The idea according to which it does not include any explicit reference to the content of the *Habilitionsvortrag* is simply unfounded and contradicted by textual evidence. Indeed, after providing his own deduction of the conditions for a differential quadratic form to be transformable into a differential form with constant coefficients—in terms of the vanishing of a four-index symbol—Riemann conveyed a geometric discussion about the possibility of interpreting such symbols in light of the Gaussian curvature of suitable 2-dimensional (geodesic) submanifolds of the n-dimensional manifold of which the quadratic form represents the metric.

Concerning Reich's remarks on Christoffel's paper, it cannot be denied that Christoffel heavily relied upon notions and methods that come from the algebraic theory of invariants (and are somehow extraneous to the context of geometry) with the explicit objective of reducing the analytical-differential problem of equivalence to the determination of purely algebraic conditions. Nonetheless, it is important to observe that Christoffel himself was quite clear in acknowledging that his work on the subject had originated from a geometrical setting when he stated that

> the current investigations were originally prompted by the extension of the problem of surfaces that can be developed on one another to n-dimensional regions (*Gebiete*). (Christoffel 1869, p. 47)

Incidentally, one can notice that the connection of Lipschitz's contributions to the problem of equivalence with infinitesimal geometry is even more transparent. Besides explicitly mentioning Riemann's *Habilitationsvortrag* and its role in promoting an increased interest in the theory of differential quadratic forms, Lipschitz deduced his criteria for flatness by employing a special type of coordinates, now known as a normal

[2] "Il concetto di tensore emerge nell'opera di Ricci-Curbastro in virtù di finalità di carattere strettamente analitico."

coordinate system, which Riemann had introduced—albeit only discursively—in the course of his treatment of the curvature of n-dimensional manifolds.

Concerning Ricci's contributions, Reich's view according to which Ricci regarded differential geometry merely as a field of application of his ADC appears problematic, if only for the fact that in the introduction to Ricci-Curbastro and Levi-Civita (1901) one can find an explicit statement of a different (almost opposite) standpoint:

> L'algorithme du Calcul différentiel absolu, c'est a dire l'instrument matériel des méthodes, sur lesquelles nous allons entretenir les lecteurs des *Mathematische Annalen*, se trouve tout entière dans une remarque due a M. Christoffel. Mais les méthodes même et les avantages, qui'ils présentent, ont leur raison d'être et leur source dans les rapports intimes, qui les lient à la notion de variété à n dimensions, que nous devons aux génies de Gauss et de Riemann. (Ricci-Curbastro and Levi-Civita 1901, p. 127)

Clearly, one might object that these judgements can be regarded as the product of a later (and thus biased) reconstruction made by authors who are reflecting on events that occurred many years earlier and, therefore, attributing characteristics to them that are extraneous to the actual discovery process. Actually, as will be explained, there are reasons to believe that this is not the case: These remarks provide us with a sufficiently sound, though incomplete, picture of the "primary source" of the ADC.

Dell'Aglio's remarks concerning the role of geometry in the development of the ADC are certainly more accurate. However, his emphasis on "strictly analytical" motivations still runs the risk of overshadowing important connections with the development of Riemannian geometry and underestimating its role in an *early* phase of the discovery process. Indeed, it is certainly true that the ADC was the fruit of a complex interaction between multiple research traditions, such as the algebraic theory of invariants—from which it inherited a goodly number of methods and notions—and the theory of differential invariants. Nevertheless, we believe that differential geometry contributed to the creation of the ADC in a decisive and essential way.

Our principal aim in this paper is to describe the emergence of the ADC by paying special attention to its connection with infinitesimal geometry. As our analysis will make clear, this connection involved different aspects of the discovery process. Indeed, as will be shown, differential geometry played a prominent role in the genesis of the ADC, not only by identifying problems that were regarded as significant but also by suggesting solution strategies and offering useful techniques that it is very unlikely purely algebraic and analytical approaches would have made available. In this respect, it is interesting to draw attention to the following circumstances:

(a) the use by Lipschitz of the notion of geodesic coordinates in his characterization of differential quadratic forms with constant coefficients;
(b) Ricci's recourse to the $n-$dimensional generalization of the notion of the second fundamental form for hypersurfaces in his classification of quadratic differential forms (Ricci-Curbastro 1884); and, finally,
(c) the emergence in Ricci-Curbastro (1886) of the concept of covariant differentiation in the context of Ricci's generalization of the notion of n-fold orthogonal systems of hypersurfaces immersed in a Riemannian manifold.

The present paper is organized as follows: Sect. 2 is devoted to an analysis of Christoffel (1869) in which special attention is paid to discussing the range of validity of his *Reduktionsatz*. In doing so, we will see that Christoffel himself provided an explicit geometrical interpretation of his results. Section 3 covers Lipschitz's contributions to the equivalence problem. Its connections with Riemann's works and differential geometry are generally well established and unquestioned. Still, some aspects of the treatment Lipschitz offered, such as his use of normal geodesic coordinates, are worth considering and reveal the intimate connection between his work and Riemann 's. Section 4 discusses the genesis of the ADC in Ricci's work. Specifically, it focuses on those aspects of the discovery process that are better suited to highlight the link between Ricci's work and geometrical notions and geometrical methods. Special attention is given to the role of the emerging theory of hypersurfaces in prompting the technical choices Ricci made, which gradually led him to the creation of his calculus. Finally, section 5 is devoted to outlining Ricci's rather extensive contributions to surface theory and the reformulation of the latter in terms of the language of the ADC. This represented, for some time, Ricci's main research priority and, certainly, the main field of application through which he intended to disseminate the use of his calculus.

2 Christoffel and his *Reduktionsatz*

The year after Riemann's *Habilitationsvortrag* was published at the initiative of Dedekind, Erwin Christoffel, a professor at the polytechnic school in Zurich, completed a fundamental work in which he tackled the problem of the equivalence of two quadratic differential forms. The problem consists in finding necessary and sufficient conditions for the existence of a change of coordinates $x = f(\bar{x})$ that transforms a given differential quadratic form, say, $\mathbf{F} = \sum_{i,j=1}^{n} a_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j$, into another one, $\bar{\mathbf{F}} = \sum_{i,j=1}^{n} \bar{a}_{ij}(\bar{x})\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j$.

Christoffel succeeded in producing a partial solution to the problem of providing necessary and sufficient conditions (a rigorous and general result was reached only about fifty years later by Élie Cartan) for the existence of such a transformation $x = f(\bar{x})$. To this end, he employed methods and techniques that proved to be extraordinarily fruitful in light of future developments of tensor calculus. Although Christoffel never formalized the notion of a tensor, he made use of the transformation properties of certain sets of multi-index functions, retrospectively identifiable with the components of a covariant tensor. Furthermore, he introduced an operation, later known as the covariant derivative, which allowed him to produce a (possibly infinite) sequence of linear forms associated with a given quadratic differential expression, in terms of which the aforementioned conditions could be expressed in a purely algebraic way.

Christoffel (1869) has been properly described as an attempt to apply standard algebraic techniques stemming from algebraic invariant theory to the, as yet unexplored, field of differential quadratic forms. This, however, does not imply that the development of Riemannian geometry had no influence on this work. On the contrary, it played a significant role in stimulating Christoffel's interest in the subject of quadratic differential forms (QDFs) and their equivalence and also helped him to gain a clearer understanding of his results.

The proof of Christoffel's main theorem is rather technical and long. Still, it is necessary to examine the theorem in some detail in order to understand the limitations imposed upon its validity and the geometrical interpretation discussed in Christoffel (1869, §11.)

To this end, let us suppose that $\mathbf{F}$, $\bar{\mathbf{F}}$ are equivalent differential quadratic forms. This means that there exists a (sufficiently regular) map $\bar{x}^i \mapsto x^i(\bar{x})$ (to be defined upon an appropriate open domain of $\mathbb{R}^n$) such that $\mathbf{F} := a(x, \mathrm{d}x) = \bar{a}(\bar{x}, \mathrm{d}\bar{x}) =: \bar{\mathbf{F}}$. This condition implies that

$$\bar{a}_{kr}(\bar{x}) = \sum_{i,j=1}^{n} a_{ij}(x(\bar{x}))\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^r}, \qquad k, r = 1, \ldots, n, \tag{1}$$

which, upon differentiation with respect to $\bar{x}_s$, produce the transformation equations for the first derivatives of the coefficients a_{ij} that are induced by the coordinate transformation $\bar{x} = \bar{x}(x)$:

$$\frac{\partial \bar{a}_{kr}}{\partial \bar{x}^s} = \sum_{i,j,t} \frac{\partial a_{ij}}{\partial x^t}\frac{\partial x^t}{\partial \bar{x}^s}\frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^r} + \sum_{i,j=1}^{n} a_{ij}\left[\frac{\partial^2 x^i}{\partial \bar{x}^s \partial \bar{x}^k}\frac{\partial x^j}{\partial \bar{x}^r} + \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial^2 x^j}{\partial \bar{x}^s \partial \bar{x}^r}\right],$$
$$k, r, s = 1 \ldots, n. \tag{2}$$

By introducing the 3-index symbols $\Gamma_{ij,k}$ (later to be named Christoffel symbols of the first kind),

$$\Gamma_{ij,k} = \frac{1}{2}\left[\frac{\partial a_{ik}}{\partial x^j} + \frac{\partial a_{jk}}{\partial x^i} - \frac{\partial a_{ij}}{\partial x^k}\right] \tag{3}$$

and $\Gamma^k_{ij} = \sum_{r=1}^{n} a^{kr}\Gamma_{ij,r}$ (later to be named Christoffel symbols of the second kind), Christoffel wrote the conditions in (2) as follows:

$$\frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} + \sum_{r,s=1}^{n} \Gamma^k_{rs}\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j} = \sum_{l=1}^{n} \bar{\Gamma}^l_{ij}\frac{\partial x^k}{\partial \bar{x}^l}, \qquad i, j, k = 1, \ldots, n. \tag{4}$$

The set of these relations defines a second-order system of $n\dfrac{n(n+1)}{2}$ partial differential equations (PDEs) equations in the unknowns $x^i(\bar{x})$, $i = 1, \ldots, n$.

Clearly, the equations in (4) represent a set of conditions that are necessary to guarantee the existence of a transformation $\bar{x}^i \mapsto x^i(\bar{x})$ satisfying (1). But are they also sufficient? That is, does a solution to (4), $x^i = x^i(\bar{x})$, transform $\mathbf{F}$ into $\bar{\mathbf{F}}$, thus assuring that $\mathbf{F}$ and $\bar{\mathbf{F}}$ are equivalent? Christoffel argued as follows.

When the function $x^i = x^i(\bar{x})$, $i = 1, \ldots, n$, supposed to be a solution to (4), is substituted in $\mathbf{F}$ in place of x^i, $i = 1, \ldots, n$, one obtains a new differential quadratic form, to be denoted by $\tilde{\mathbf{F}}$:

$$\sum_{i,j=1}^{n} a_{ij}(x)\mathrm{d}x^i \mathrm{d}x^j = \sum_{i,j=1}^{n} \tilde{a}_{ij}(\bar{x})\mathrm{d}\bar{x}^i \mathrm{d}\bar{x}^j =: \tilde{\mathbf{F}}. \tag{5}$$

Since $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are equivalent by construction, we will have[3]

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 x^i}{\partial \bar{x}^r \partial \bar{x}^s} \frac{\partial x^j}{\partial \bar{x}^t} + \sum_{i,j,k=1}^{n} \Gamma_{ij,k} \frac{\partial x^i}{\partial \bar{x}^r} \frac{\partial x^j}{\partial \bar{x}^s} \frac{\partial x^k}{\partial \bar{x}^t} = \tilde{\Gamma}_{rs,t}, \qquad r, s, t = 1, \ldots, n. \tag{6}$$

Furthermore, from (4), one has

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 x^i}{\partial \bar{x}^r \partial \bar{x}^s} \frac{\partial x^j}{\partial \bar{x}^t} + \sum_{i,j,k=1}^{n} \Gamma_{ij,k} \frac{\partial x^i}{\partial \bar{x}^r} \frac{\partial x^j}{\partial \bar{x}^s} \frac{\partial x^k}{\partial \bar{x}^t} = \bar{\Gamma}_{rs,t}, \qquad r, s, t = 1, \ldots, n. \tag{7}$$

A comparison between (4) and (7) indicates that $\tilde{\Gamma}_{rs,t} = \bar{\Gamma}_{rs,t}, r, s, t = 1, \ldots, n$, which, in turn, imply the following:

$$\frac{\partial \tilde{a}_{ij}}{\partial \bar{x}^k} = \frac{\partial \bar{a}_{ij}}{\partial \bar{x}^k}, \qquad i, j, k = 1, \ldots, n. \tag{8}$$

This means that the difference $\tilde{a}_{ij} - \bar{a}_{ij}$ between the coefficients of $\bar{\mathbf{F}}$, $\tilde{\mathbf{F}}$ is a constant. Thus, if we suppose that there is a value $\bar{x} = \bar{x}^0$ such that

$$\sum_{k,r=1}^{n} a_{kr}(x(\bar{x}^0)) \frac{\partial x^k}{\partial \bar{x}^i}\bigg|_{\bar{x}=\bar{x}^0} \frac{\partial x^r}{\partial \bar{x}^j}\bigg|_{\bar{x}=\bar{x}^0} = \tilde{a}_{ij}(\bar{x}^0) = \bar{a}_{ij}(\bar{x}^0), \tag{9}$$

then, in view of (8), we obtain $\mathbf{F} = \bar{\mathbf{F}}$. Thus, the existence of solution to (1) assures the equivalence between $\mathbf{F}$ and $\bar{\mathbf{F}}$, provided that the conditions in (9) hold true.

In light of these remarks, Christoffel obtained the following criterion for the equivalence of two forms $\mathbf{F}$, $\bar{\mathbf{F}}$ (Christoffel 1869, pp. 50–51):

Theorem 2.1 (Christoffel 1869) *Let $\boldsymbol{F}$, $\bar{\boldsymbol{F}}$ be two differential forms in the variables x^i and $\bar{x}^i$, $i = 1, \ldots, n$, respectively. If one can determine functions x^i, $i = 1, \ldots, n$ of*

[3] This is a consequence of (4). Here, $\tilde{\Gamma}_{rs,t}$ are defined by the following:

$$\tilde{\Gamma}_{rs,t} = \frac{1}{2}\left[\frac{\partial \tilde{a}_{rt}}{\partial x^s} + \frac{\partial \tilde{a}_{st}}{\partial x^r} - \frac{\partial \tilde{a}_{rs}}{\partial x^t}\right], \quad r, s, t = 1, \ldots, n.$$

$\bar{x}^i$, $i = 1, \ldots, n$ *in such a way that the equations in* (4) *hold true, and furthermore, if there exists a value* $\bar{x} = \bar{x}^0$ *for which the conditions in* (9) *are satisfied, then the differential forms* $\boldsymbol{F}, \overline{\boldsymbol{F}}$ *are equivalent.*

Accordingly, the whole problem boils down to the study of the solutions to (4). A first set of conditions to be taken into account is represented by the "integrability conditions" that are obtained by considering equality among mixed (third order) derivatives:

$$\frac{\partial^3 x^r}{\partial \bar{x}^k \partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial^3 x^r}{\partial \bar{x}^j \partial \bar{x}^i \partial \bar{x}^k}, \quad r, i, j, k = 1, \ldots, n. \tag{10}$$

Christoffel proved that as a consequence of (4), these integrability conditions can be written as follows

$$\overline{R}_{j_1 j_2 j_3 j_4} = \sum_{i_1, i_2, i_3, i_4 = 1}^{n} X_{j_1}^{i_1} X_{j_2}^{i_2} X_{j_3}^{i_3} X_{j_4}^{i_4} R_{i_1 i_2 i_3 i_4}, \tag{$G_4 = \overline{G}_4$}$$

where $X_j^i := \dfrac{\partial x^i}{\partial \bar{x}^j}$ and the quantities $R_{i_1 i_2 i_3 i_4}$ are defined by the following:

$$\begin{aligned} R_{i_1 i_2 i_3 i_4} = \frac{1}{2} \left(\frac{\partial^2 a_{i_1 i_4}}{\partial x^{i_2} \partial x^{i_3}} + \frac{\partial^2 a_{i_2 i_3}}{\partial x^{i_1} \partial x^{i_4}} - \frac{\partial^2 a_{i_1 i_3}}{\partial x^{i_2} \partial x^{i_4}} - \frac{\partial^2 a_{i_4 i_2}}{\partial x^{i_1} \partial x^{i_3}} \right) \\ + \sum_{j=1}^{n} \Gamma_{i_1 i_4}^{j} \Gamma_{i_3 i_2, j} - \sum_{j=1}^{n} \Gamma_{i_1 i_2}^{j} \Gamma_{i_4 i_3, j}. \end{aligned} \tag{11}$$

Unbeknownst to Christoffel, these four-index symbols had made their first appearance in Riemann's *Commentatio Mathematica*, a paper that Riemann submitted to the French Academy of Sciences in 1861 on the occasion of a prize competition relating to heat conduction. They were introduced by Riemann in *Pars Secunda* of his essay in the course of his search for conditions guaranteeing that a given differential quadratic expression, say, $\mathbf{F} = \sum_{ik} a_{ik} \mathrm{d}x^i \mathrm{d}x^k$, can be transformed into another one of the form $\mathbf{G} = \sum_i (\mathrm{d}y^i)^2$. Riemann limited himself to proving that the vanishing of the functions constitutes a set of necessary conditions for the existence of such a coordinate transformation, leaving completely unexplored the problem consisting in also providing sufficient conditions. As is clear from our previous discussion, Christoffel considered the more general situation when the nature of the form $\mathbf{G}$ is not restricted by any hypothesis concerning its coefficients. It is interesting to observe that relations $G_4 = \overline{G}_4$, which Christoffel simply interpreted as a set of conditions that are necessary to guarantee the equivalence of two DQFs, can now be regarded as the equations expressing the tensorial character of the Riemann curvature tensor. Fully aware of the relevance of this set of functions, Christoffel provided a thorough analysis of the symmetry properties of the symbols $R_{i_1 i_2 i_3 i_4}$ by proving the existence of algebraic relations among them, thus demonstrating that the number of independent components is $n^2(n^2 - 1)/12$.

A key aspect of Christoffel's solution to the equivalence problem was the possibility of regarding equations ($G_4 = \overline{G}_4$) as "trasformation relations" (*Transformationsrelationen*[4]) of the quadrilinear form:

$$\mathbf{G}_4 = \sum_{i,j,k,r=1}^{n} R_{ijkr} \mathrm{d}^{(1)}x^i \mathrm{d}^{(2)}x^j \mathrm{d}^{(3)}x^k \mathrm{d}^{(4)}x^r, \tag{12}$$

where the differentials $\mathrm{d}^{(k)}x^i$, $k = 1, \ldots, 4, i = 1, \ldots, n$ are treated as a new set variables that are subject to the linear transformations

$$\mathrm{d}^{(k)}x^j = \sum_{j=1}^{n} X^i_j \mathrm{d}^{(k)}\bar{x}^j,$$

induced by a coordinate transformation $x = x(\bar{x})$. Furthermore, by introducing a new algorithm, which Ricci later dubbed a *covariant derivative*, he deduced a (possibly infinite) series of algebraic forms in the differentials $\mathrm{d}^{(k)}x^i$, $i = 1, \ldots, n$, $k = 1, \ldots$, in terms of which necessary and sufficient conditions for the equivalence of two QDFs could be given.

Indeed, consider a differential form of mth order (*Ordnung*):

$$\mathbf{\Phi}_m = \sum_{j_1,\ldots,j_m=1}^{n} \Phi_{j_1 j_2 \ldots j_m} \mathrm{d}^{(1)}x^{j_1} \mathrm{d}^{(2)}x^{j_2} \ldots \mathrm{d}^{(m)}x^{j_m}. \tag{13}$$

If, under an arbitrary change of coordinates $x = x(\bar{x})$, the components of $\mathbf{\Phi}_m$, $\Phi_{j_1 j_2 \ldots j_m}$ transform according to

$$\overline{\Phi}_{j_1 j_2 \ldots j_m} = \sum_{i_1,\ldots,i_m=1}^{n} X^{i_1}_{j_1} X^{i_2}_{j_2} \ldots X^{i_m}_{j_m} \Phi_{i_1 i_2 \ldots i_m}, \tag{14}$$

as Christoffel indicated, one can deduce from $\mathbf{\Phi}$ a set of differential forms of increasing order such that their components satisfy transformation relations of type ($G_4 = \overline{G}_4$). Indeed, upon differentiation of ($G_4 = \overline{G}_4$) with respect to $\bar{x}^t$, and by also using (4) to obtain the second-order derivatives of x with respect to $\bar{x}$ in terms of the symbols Γ^i_{jk}, $\overline{\Gamma}^i_{jk}$, we easily obtain the following set of conditions:

$$\overline{\Phi}_{j_1 j_2 \ldots j_m t} = \sum_{i_1,\ldots,i_m,s=1}^{n} X^{i_1}_{j_1} X^{i_2}_{j_2} \ldots X^{i_m}_{j_m} X^s_t \Phi_{i_1 i_2 \ldots i_m s}, \tag{15}$$

where the functions $\Phi_{i_1 i_2 \ldots i_m k}$, which were interpreted by Christoffel as the components of a new differential form $\mathbf{\Phi}_{m+1}$ of order $m + 1$, are defined by the relations:

[4] On the use of this word, see (Dell'Aglio 1996, p. 225).

$$\Phi_{i_1 i_2 \ldots i_m s} = \frac{\partial \Phi_{i_1 i_2 \ldots i_m}}{\partial x^s} - \sum_{i=1}^{n} \Gamma^i_{i_1 s} \Phi_{i i_2 \ldots i_m} - \sum_{i=1}^{n} \Gamma^i_{i_2 s} \Phi_{i_1 i \ldots i_m} - \ldots - \sum_{i=1}^{n} \Gamma^i_{i_m s} \Phi_{i_1 i_2 \ldots i} \tag{16}$$

that we can *nowadays* regard as the components of the covariant derivative with respect to x^s of the covariant tensor $\mathbf{\Phi}$.

Repeated application of this algorithm produces a (possibly infinite) series of differential forms (DFs), $\mathbf{G}_5, \mathbf{G}_6, \ldots$ that are generated from $\mathbf{G}_4$, their coefficients being given by the successive covariant derivatives of the components of the curvature tensor $R_{i_1 i_2 i_3 i_4}$. By means of this series of DFs, Christoffel could deduce the existence of *purely algebraic conditions* that provide criteria for the equivalence of two given QDFs.

In order to better understand the nature of these conditions, we will recast the problem by closely following the treatment provided in Christoffel (1869, §4). The existence of a coordinate transformation $x = x(\bar{x})$ such that $\sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j = \sum_{k,l=1}^{n} \bar{a}_{kl} \mathrm{d}\bar{x}^k \mathrm{d}\bar{x}^l$ can be translated into the requirement that there exists a function $\Xi : U \subset \mathbb{R}^n \to \mathbb{R}^{n+n^2}$, $\bar{x} \mapsto \Xi(\bar{x}) = (x^i(\bar{x}), X^i_\gamma(\bar{x}))$ such that

$$\bar{a}_{kr} = \sum_{i,j=1}^{n} X^i_k X^j_r a_{ij}, \qquad k, r = 1, \ldots, n; \tag{A}$$

$$X^i_k = \frac{\partial x^i}{\partial \bar{x}^k}, \qquad i, k = 1, \ldots, n. \tag{B}$$

If the differentials $\mathrm{d}^{(k)} x^i$, $i = 1, \ldots, n$ are now regarded as independent variables, the DFs

$$\mathbf{G}_m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} R_{i_1 i_2 \ldots i_m} \mathrm{d}^{(1)} x^{i_1} \mathrm{d}^{(2)} x^{i_2} \ldots \mathrm{d}^{(m)} x^{i_m}, \qquad m \geq 5 \tag{$\mathbf{G}_m$}$$

become algebraic forms in the differentials $\mathrm{d}^{(k)} x^i$, $i = 1, \ldots, m$. Furthermore, if $\mathrm{d}^{(k)} x^i$ are supposed to be transformed according to the following linear relations:

$$\mathrm{d}^{(k)} x^i = \sum_{j=1}^{n} X^i_j \mathrm{d}^{(k)} \bar{x}^j, \qquad i = 1, 2, \ldots, n, \ k \text{ being fixed,} \tag{17}$$

the conditions for the equivalence of F, $\overline{F}$,

$$\overline{R}_{i_1 i_2 \ldots i_m} = \sum_{j_1, \ldots, j_m = 1}^{n} X^{j_1}_{i_1} \ldots X^{j_m}_{i_m} R_{j_1 j_2 \ldots j_m} \tag{$\overline{G}_m = G_m$}$$

can be considered as a set of relations between the covariants R, $\overline{R}$ of the algebraic forms $\mathbf{G}_m$ e $\overline{\mathbf{G}}_m$. It should be stressed that the word "covariant," which Christoffel himself employed, was taken directly from the theory of algebraic forms (quantics) where it was introduced by Sylvester in the early 1850s.

We are now in the position to state Christoffel's main result, by means of which, *under appropriate hypotheses*, one can find algebraic conditions that are sufficient to guarantee the equivalence of two given QDFs. Quite appropriately, to emphasize the fact that the criterion that intervenes in the theorem is of a purely algebraic nature, Klein called it *Reduktionssatz*[5]:

Theorem 2.2 (Christoffel 1869) *Let $\boldsymbol{F}, \overline{\boldsymbol{F}}$ be QDFs in the variables x and $\bar{x}$, respectively. Suppose that from*[6]

$$F = \overline{F}, \quad G_4 = \overline{G}_4, \quad G_5 = \overline{G}_5, \quad \ldots, \quad G_q = \overline{G}_q,$$

the quantities x^i, X^i_j, which are functions of $\bar{x}$, can be completely determined (vollständig bestimmt) *in such a way that the conditions $G_{q+1} = \overline{G}_{q+1}$ are identically satisfied. Then,*

$$X^i_j = \frac{\partial x^i}{\partial \bar{x}^j}, \qquad i, j = 1, \ldots, n.$$

As a consequence of this, $F = \sum_{i,j} a_{ij} dx^i dx^j$ and $\overline{F} = \sum_{i,j=1}^{n} \bar{a}_{ij} d\bar{x}^i d\bar{x}^j$ are equivalent.

Before providing a general proof of this result, Christoffel (1869, §9) discussed a special case that is worth considering, in order to better understand how the demonstration works and how the algebraic reduction is performed.

To this end, let us consider the following assumptions: (1) The $n + n^2$ functions $x^i, X^j_k, i, j, k = 1, \ldots, n$ of $\bar{x}$ are univocally determined from ($G_4 = \overline{G}_4$); (2) these functions are such that the conditions $G_5 = \overline{G}_5$ are identically verified. In light of these assumptions, one can suppose that from the set of conditions ($G_4 = \overline{G}_4$), $n + n^2$ equations,

$$F_k(x, X, \bar{x}) = 0, \qquad k = 1, \ldots, n + n^2,$$

can be chosen such that the Jacobian determinant $\det \left[\frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial X^i_r} \right]$ is not equal to zero. Furthermore, we set:

$$\binom{i}{j} := \frac{\partial x^i}{\partial \bar{x}^j} - X^i_j,$$

5 See Christoffel (Christoffel 1869, §10) and Klein (1927, II, p. 169).

6 The relation $F = \overline{F}$ coincides with (A). $\mathbf{F}$ and $\overline{\mathbf{F}}$ are here regarded as algebraic forms in the differentials dx.

$$\binom{k}{ij} := \frac{\partial X^k_j}{\partial \bar{x}^i} + \sum_{r,s=1}^{n} \Gamma^k_{rs} X^r_i X^s_j - \sum_{s=1}^{n} \overline{\Gamma}^s_{ij} X^k_s,$$

with $i, j, k = 1, \ldots, n$. It is clear that the equations $F_k(x, X, \bar{x}) = 0$ constitute a set of $n + n^2$ conditions of the form $\overline{R}_{i_1 i_2 i_3 i_4} - \sum_{j_1, j_2, j_3, j_4=1}^{n} X^{j_1}_{i_1} X^{j_2}_{i_2} X^{j_3}_{i_3} X^{j_4}_{i_4} R_{j_1 j_2 j_3 j_4} = 0$. If we differentiate them with respect to $\bar{x}^i$ (i being a fixed index $i = 1, \ldots, n$), we obtain the $n^2 + n$ relations:

$$\frac{\partial \overline{R}_{i_1 i_2 i_3 i_4}}{\partial \bar{x}^i} - \sum_{j_1, j_2, j_3, j_4, q=1}^{n} \frac{\partial R_{j_1 j_2 j_3 j_4}}{\partial x^q} \frac{\partial x^q}{\partial \bar{x}^i} X^{j_1}_{i_1} X^{j_2}_{i_2} X^{j_3}_{i_3} X^{j_4}_{i_4}$$
$$- \sum_{l, j_2, j_3, j_4=1}^{n} R_{l j_2 j_3 j_4} \frac{\partial X^l_{i_1}}{\partial \bar{x}^i} X^{j_2}_{i_2} X^{j_3}_{i_3} X^{j_4}_{i_4} - \ldots = 0.$$

By replacing $\dfrac{\partial x^q}{\partial \bar{x}^i}$ and $\dfrac{\partial X^l_{i_k}}{\partial \bar{x}^\alpha}$ with the following expressions,

$$\frac{\partial x^q}{\partial \bar{x}^i} = \binom{q}{i} + X^q_i,$$
$$\frac{\partial X^l_{i_k}}{\partial \bar{x}^i} = - \sum_{r,s=1}^{n} \Gamma^l_{rs} X^r_i X^s_{i_k} + \sum_{r=1}^{n} \overline{\Gamma}^r_{i i_k} X^l_r + \binom{l}{i i_k},$$

and also by using $\overline{G}_5 = G_5$, that is, $\overline{R}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5} = \sum_{j_1, j_2, j_3, j_4, j_5=1}^{n} X^{j_1}_{\alpha_1} X^{j_2}_{\alpha_2} X^{j_3}_{\alpha_3} X^{j_4}_{\alpha_4} X^{j_5}_{\alpha_5} R_{j_1 j_2 j_3 j_4 j_5}$, one obtains:

$$\sum_{i=1}^{n} \frac{\partial F_k}{\partial x^i} \binom{i}{j} + \sum_{r,i=1}^{n} \binom{i}{jr} \frac{\partial F_k}{\partial X^i_r} = 0, \qquad j = 1, \ldots, n; k = 1, \ldots, n^2 + n.$$

Since we supposed that $\det \left[\dfrac{\partial F_k}{\partial x^i}, \dfrac{\partial F_k}{\partial X^i_r} \right] \neq 0$, we easily obtain:

$$\binom{i}{j} = 0, \qquad \binom{i}{jr} = 0, \qquad i, j, r = 1, \ldots, n, \tag{18}$$

which proves the equivalence of the two QDFs $\mathbf{F}$ and $\bar{\mathbf{F}}$, by virtue of Theorem 2.1.

It is important to observe that this argument requires that the quantities $(x^i(\bar{x}), X^i_j)(\bar{x})$ are completely determined as functions of $\bar{x}$, since one somehow wants to assume that $\det \left[\dfrac{\partial F_k}{\partial x^i}, \dfrac{\partial F_k}{\partial X^i_r} \right] \neq 0$. This is also true for the proof of Theorem 2.2. Hence, the range of validity of the *Reduktionssatz* is limited to those QDFs for which some sort of maximality condition for the rank of the Jacobian matrix $\mathbf{J}(F) := \left[\frac{\partial F_k}{\partial x^i}, \frac{\partial F_k}{\partial X^i_r}\right]$ is verified.

To summarize, one can say that Christoffel's approach to the equivalence problem consisted of a combination of analytical and algebraic tools that led him to a set of algebraic relations to which the equivalence of the differential forms can be reduced. Clearly, the main difficulty boils down to the question of which algebraic conditions are sufficient to ensure the validity of the following equations:

$$\binom{i}{j} := \frac{\partial x^i}{\partial \bar{x}^j} - X^i_j = 0, \qquad i, j = 1, \ldots, n.$$

In the course of his treatment, Christoffel did not provide any extensive discussion of the geometrical implications of his results. Rather, he limited himself to offering some brief remarks on the equivalence of surfaces and a cursory reference to Riemann's *Habilitationsvortrag* at the end of the paper.

Possibly, this very circumstance has led certain historians to deny the existence of a link, either direct or indirect, between his paper and the development of differential geometry and to interpret the achievements of Christoffel as a mere outcome of his interest in the theory of invariants of algebraic forms. In her monograph on the history of tensor calculus, Reich vigorously defended this viewpoint. For example, she asserted that:

> the theory of algebraic invariants and the works of Aronhold represent the basis for Christoffel's first contributions to the theory of quadratic differential forms: *Ueber die Transformation der homogenen Differentialaudrücke* […]"[7] (Reich 1994, p. 59);

She also commented as follows:

> Only in the last sentence of his paper does Christoffel mention Riemann's Hypothesen, and only in passing, without following up on them […]. In contrast to Lipschitz, the publication of Riemann's Hypothesen had no influence on Christoffel's investigation; he would have published his treatise in this way even without Riemann.[8] (Reich 1994, p. 65)

Undoubtedly, Christoffel greatly profited from his knowledge of invariant theory, to which he had actively contributed with publications such as Christoffel (1868) in which he acknowledged his debt to the work of Aronhold on the invariant theory of algebraic forms of any degree. Nonetheless, the claim that Riemann's work had played no role in prompting Christoffel's research on the equivalence problem can be

[7] "Die algebraische Invariantentheorie und wiederum die Arbeiten Aronholds sind die Grundlage für Christoffels ersten Beitrag zur Theorie der quadratischen Differentialformen: *Ueber die Transformation der homogenen Differentialaudrücke* […].

[8] "Erst im letzten Satz seiner Abhandlung erwähnt Christoffel Riemanns Hypothesen und dies nur beiläufig, ohne daran anzuknüpfen […]. Für Christoffels Untersuchung hat im Gegensatz zu Lipschitz die Veröffentlichung von Riemanns Hypothesen keinen Einfluß ausgeübt, er hätte seine Abhandlung auch ohne Riemann in dieser Weise publiziert."

safely excluded, if only because Christoffel himself disproved this possibility in the introduction to Christoffel (1869). Indeed, after several brief remarks concerning the equivalence problem of homogeneous forms of degree (*Ordnung*) $p > 2$, he pointed out:

> I will, therefore, use the following to explain the means by which this exceptional case [i.e. the case of 2nd degree QDFs], which is so important in a wide variety of fields, can be dealt with, and I will only note that in the case where $F = \partial x_1^2 + \partial x_2^2 + \partial x_3^2$, a detailed work by Mr. Lamé exists (Théorie des cordonnées curvilignes), while the current investigations were originally prompted by the extension of the problem of applicability of surfaces [*des Problems der aufeinander abwicklbaren Flächen*] to regions of n dimensions.[9] (Christoffel 1869, p. 47)

Although we cannot find here an explicit mention of either Riemann or his work, there can be little doubt that Christoffel was, in this instance, referring to those passages of Riemann's *Habilitationsvortrag* in which the author discussed the problem of deciding whether two distinct QDFs can be interpreted as different expressions (in different coordinates) of the same (Riemannian) metric.

Quite recently, Darrigol (2015, §3.2) advanced the hypothesis that before the final redaction of his paper Christoffel might have had access to manuscripts by Dedekind and, as a result, became acquainted with some of the content of Riemann's *Commentatio Mathematica*. Indeed, before the publication of *Commentatio Mathematica*, in 1876, Dedekind had worked with handwritten notes from Riemann's *Nachlass* containing some material that Riemann had written down in preparation for his Paris essay. This eventuality cannot be excluded, although it appears to be unlikely. Indeed, Christoffel never mentioned the existence of a manuscript by Dedekind. His only reference to the latter was to the footnote in Riemann (1868) where Dedekind promised to provide the analytical details that are relevant to understanding the content of Riemann's prolusion.

The existence of a connection between the solution to the equivalence problem and infinitesimal (differential) geometry appears to be confirmed by a passage in Christoffel (Christoffel 1869, §11) where he clarified the limits of the validity of his *Reduktionsatz*. Notably, he observed that the conditions required by the hypotheses of this theorem cannot occur, for example, when one deals with the question of applicability of two surfaces, and the surfaces admit of a continuous motion. Indeed, in this case, the algebraic relations $F = \overline{F}, G_4 = \overline{G}_4$, etc., do not allow a complete determination of the function $\Xi : U \subset \mathbb{R}^n \to \mathbb{R}^{n+n^2}, \bar{x} \mapsto \Xi(\bar{x}) = (x^i(\bar{x}), X^i_r(\bar{x}))$.

> In order to understand the content of the theorem established in the previous paragraph [i.e. Christoffel's *Reduktionsatz*], it must be emphasized that these conditions cannot be fulfilled under all circumstances. If, for example,

[9] "Ich werde daher im Folgenden die Hülfsmittel auseinandersetzen, mittelst deren dieser auf den verschiedensten Gebieten so wichtige Ausnahmefall sich erledigen lässt, und bemerke nur noch, dass über den Fall, wo $F = \partial x_1^2 + \partial x_2^2 + \partial x_3^2$ ist, ein ausführliches Werke von Herrn *Lamé* existirt (Théorie des cordonnées curvilignes), während die gegenwärtigen Untersuchungen ursprünglich durch die Ausdehnung des Problems der aufeinander abwickelbaren Flächen auf Gebiete von n Dimensionen veranlasst worden sind."

> the question is whether two given surfaces can be developed one onto the other without distortion, one has to examine an equation of type $F = F'$ for $n = 2$. If the development of one surface onto the other, i.e., the above equation, is possible, then from the transformation relations contained in the equations $F = F', G_4 = G'_4, G_5 = G'_5, \ldots$, however this series may be continued, one cannot determine the values of x and u [read X, in our notation] in all those cases where the given surfaces admit a continuous motion. (Christoffel 1869, §11).

In addition to the situation described here, all those cases in which the differential forms F, F' admit a continuous group of automorphisms (i.e., of local isometries) were excluded from the criterion established by Christoffel's *Reduktionsatz*. A fortiori, this theorem cannot find application when the differential forms admit a group of isometries of maximal dimension, that is, the cases in which there are manifolds with constant curvature—also including locally Euclidean manifolds. The problem of equivalence for Euclidean manifolds, which Riemann had addressed in the *Habilitationsvortrag* and more explicitly in the *Commentatio*, therefore remained unsolved and still open.[10]

3 Rudolf Lipschitz and his variational approach

In the same volume of *Crelle's Journal* in which Christoffel (1869) was published, we can find an equally remarkable memoir by Rudolf Lipschitz where, albeit with different methods and intentions, the problem of equivalence investigated by Christoffel was tackled. As was the case for Christoffel, the motivation for Lipschitz to perform his research on the subject came from the recent publication of Riemann's *Habilitationsvortrag*. Indeed, he explicitly remarked that his investigations could be seen as a quite natural extension of Gauss's theory of surfaces. As he explained,

> The main purpose of this work is to provide a direct answer to the question posed concerning second-degree forms of n differentials, the determinant of which does not vanish identically, according to the restriction made, and in this sense to extend the theory of Gaussian curvature.[11] (Lipschitz 1869, p. 73)

Unlike Christoffel, however, Lipschitz sought to investigate the transformation properties of homogeneous forms in the differentials of the variables $x^1, \ldots, x^n$ of *any* degree p. Possibly, this extension was suggested to Lipschitz by a paragraph in the *Habilitationsvortrag*, where Riemann had envisioned the possibility of considering homogeneous forms of order four in the differentials of the variables.[12] Despite his desire to achieve a greater generality, Lipschitz mostly concentrated upon the problem, excluded by Christoffel's treatment, of establishing necessary and sufficient conditions for a given differential form to have constant coefficients.

[10] In this respect, see Ehlers (1981).

[11] Die gegenwärtige Arbeit verfolgt vornehmlich den Zweck, die gestellte Frage in Betreff der Formen zweiten Grades von n Differentialen, deren Determinante zufolge der getroffenen Einschränkung nicht identisch verschwindet, direct zu beantworten, und in diesem Sinne die Theorie des *Gauss*ischen Krümmungsmasses auszudehnen.

[12] In this respect, see Tazzioli (Tazzioli 1994, p. 115).

On a more technical level, we can observe an interesting connection with Riemann's *Habilitationsvortrag* in the extensive use of special coordinate systems that Lipschitz introduced in order to solve his equivalence problem. As will be seen, these special coordinate systems represent a generalization of the notion of normal coordinates that Riemann had employed while providing a geometrical meaning of the notion of curvature of a manifold.[13] As a consequence of this technical choice, while Christoffel's treatment was characterized by the use of algebraic and analytical tools with the aim of tracing back the solution of the equivalence problem of two differential forms to the equivalence of algebraic forms, Lipschitz opted for techniques drawn from the calculus of variations which allowed him to investigate geometric and mechanical aspects of the theory of differential forms.

Lipschitz started by considering homogeneous differential forms of degree p in the variables $x^1, \ldots, x^n$ and by proposing to study their invariance properties with respect to changes of coordinates $(y^1, \ldots, y^n) = (y^1(x^1, \ldots, x^n) \ldots, y^n(x^1, \ldots, x^n))$. By a homogeneous differential form (*ganze homogene Function*), Lipschitz meant an expression of the following type:

$$\mathbf{F}(\mathrm{d}x) = \sum_{i_1,\ldots,i_p=1}^{n} F_{i_1,\ldots,i_p}(x^1, \ldots, x^n)\mathrm{d}x^{i_1} \ldots \mathrm{d}x^{i_p},$$

which he interpreted as a function in the $2n$ variables $x^1, \ldots, x^n, \mathrm{d}x^1, \ldots, \mathrm{d}x^n$. A change of coordinates $y = y(x)$ transforms the form $\mathbf{F}(\mathrm{d}x)$ into $\mathbf{G}(\mathrm{d}y)$, which is obtained from the former upon substitution of $x^i(y)$ in place of x^i and of $\mathrm{d}x^i = \sum_{k=1}^{n} \frac{\partial x^i}{\partial y^k}\mathrm{d}y^k$ in place of $\mathrm{d}x^i$. By assuming that $\mathbf{G}(\mathrm{d}y)$ is obtained from $\mathbf{F}(\mathrm{d}x)$ precisely in this way, one evidently has $\mathbf{G}(\mathrm{d}y) = \mathbf{F}(\mathrm{d}x)$. From this relation, Lipschitz deduced the following equation:

$$\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial^2\mathbf{F}(\mathrm{d}x)}{\partial \mathrm{d}x^k \partial \mathrm{d}x^j}\mathrm{d}^2x^j + \mathbf{F}_k(\mathrm{d}x)\right)\delta x^k = \sum_{k=1}^{n}\left(\sum_{r=1}^{n}\frac{\partial^2\mathbf{G}(\mathrm{d}y)}{\partial \mathrm{d}y^k \mathrm{d}y^r}\mathrm{d}^2y^r + \mathbf{G}_k(\mathrm{d}y)\right)\delta y^k, \tag{19}$$

where

$$\begin{cases} \mathbf{F}_k(dx) := \sum_{s=1}^{n} \dfrac{\partial^2\mathbf{F}(dx)}{\partial dx^k \partial x^s}dx^s - \dfrac{\partial \mathbf{F}(dx)}{\partial x^k}, \\ \mathbf{G}_r(dy) := \sum_{s=1}^{n} \dfrac{\partial^2\mathbf{G}(dy)}{\partial dy^r \partial y^s}dy^s - \dfrac{\partial \mathbf{G}(dy)}{\partial y^r} \end{cases} \quad k, r = 1, \ldots, n. \tag{20}$$

[13] See Riemann (1868, §2.2).

When $p = 1$, one obtains a well-known property of the so-called *bilinear covariant* that is associated with a differential 1-form $\mathbf{F}(\mathrm{d}x) = a_1(x)\mathrm{d}x^1 + \ldots + a_n\mathrm{d}x^n$.[14]

Lipschitz's main goal was to exploit these relations in order to obtain invariant differential expressions that are built using the coefficients of the form $\mathbf{F}(\mathrm{d}x)$ and their derivatives. As he explained,

> In the study of the form $\mathbf{F}(\mathrm{d}x)$, it seems to me above all desirable to know those functions which, when transformed together with $\mathbf{F}(\mathrm{d}x)$, are modified in such a way that their relationship remains unchanged.[15]

In accordance with this remark, he proposed a thorough study of the case $p = 2$ which was based on the introduction of the following quadrilinear form that he discovered by considering the explicit expression for the curvature of surfaces that Gauss had provided in his *Disquisitiones*:

$$\mathbf{\Psi} = \sum_{s,l=1}^{n} \left[\delta \frac{\partial \mathbf{F}_s(\mathrm{d}x)}{\partial \mathrm{d}x^l} - \mathrm{d}\frac{\partial \mathbf{F}_s(\delta x)}{\partial \delta x^l} + \sum_{i,j=1}^{n} \frac{1}{2} a^{ij} \left(\frac{\mathbf{F}_i(\mathrm{d}x)}{\partial \mathrm{d}x^s} \frac{\mathbf{F}_j(\delta x)}{\partial \delta x^l} - \frac{\mathbf{F}_i(\mathrm{d}x)}{\partial \delta x^s} \frac{\mathbf{F}_j(\delta x)}{\partial \mathrm{d}x^l} \right) \right] \mathrm{d}u^s \delta u^l. \tag{21}$$

Using (19) Lispchitz proved the covariance of $\mathbf{\Psi}$ by showing that $\mathbf{\Psi} = \mathbf{\Phi}$, where $\mathbf{\Phi}$ denotes the differential form that is obtained by $\mathbf{\Psi}$ on introduction of the variables $y = y(x)$ (it is intended that both $\mathbf{F}(\mathrm{d}x)$ and $\mathbf{F}_\alpha(\mathrm{d}x)$ should be replaced by $\mathbf{G}(\mathrm{d}y)$ and $\mathbf{G}_\alpha(\mathrm{d}y)$, respectively). Finally, an explicit formula for $\mathbf{\Psi}$—derived in Lipschitz (1869, p. 84)—permits the expression of $\mathbf{\Psi}$ in terms of the covariant components of the Riemann curvature tensor, precisely as Christoffel had done,

$$\mathbf{\Psi} = \sum_{ijkr} R_{ijkr} \mathrm{d}x^i \delta x^j \bar{\mathrm{d}} x^k \bar{\delta} x^r. \tag{22}$$

After providing a proof of Gauss's *Theorema Egregium*, which consisted in deriving the invariance of the curvature starting from the relation $\mathbf{\Psi} = \mathbf{\Phi}$, Lipschitz proceeded to deduce necessary and sufficient conditions in order that a given QDF, $\mathbf{F}(\mathrm{d}x)$, can be transformed into a QDF with constant coefficients. For this purpose, certainly inspired by Riemann (1868) who had introduced an extension of the notion of curvature through the use of a new coordinate system, Lipschitz offered a detailed analytical treatment

[14] The bilinear covariant of $\mathbf{F}(\mathrm{d}x) = a_1(x)\mathrm{d}x^1 + \ldots + a_n(x)\mathrm{d}x^n$ is defined as $\sum_{ik} \left(\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i} \right) \mathrm{d}x^k \delta x^i$. If $\mathbf{G}(\mathrm{d}y)$ denotes the result of a change of coordinates on $f(\mathrm{d}x)$ by means of $y = y(x)$, $\mathbf{G}(\mathrm{d}y) = b_1(y)\mathrm{d}y^1 + \ldots + b_n(y)\mathrm{d}y^n$, one has:

$$\sum_{ik} \left(\frac{\partial a_i}{\partial x^k} - \frac{\partial a_k}{\partial x^i} \right) dx^k \delta x^i = \sum_{ik} \left(\frac{\partial b_i}{\partial y^k} - \frac{\partial b_k}{\partial y^i} \right) dy^k \delta y^i.$$

[15] Lipschitz (1869, p. 75).

of normal coordinates and of the transformations induced by them on the forms $\mathbf{F}(\mathrm{d}x)$ and $\mathbf{G}(\mathrm{d}y)$. He did so by obtaining a criterion that establishes necessary and sufficient conditions guaranteeing that a given form $\mathbf{F}(\mathrm{d}x)$ is equivalent to a flat (to use Riemann's terminology) QDF.

If $p \geq 2$, as was remarked in Lipschitz (1869, §5), the equation in (19) admits an interpretation in the light of the invariance properties of the isoperimetric system of equations that can be associated to the differential form $\mathbf{F}(\mathrm{d}x) = \sum_{i_1 \dots i_p=1}^{n} F_{i_1,\dots i_p} \mathrm{d}x^{i_1} \dots \mathrm{d}x^{i_p}$. Indeed, if we consider the functional

$$\mathrm{E}[x(t)] = \int_{t_0}^{t} \left(\sum_{i_1 \dots i_p=1}^{n} F_{i_1,\dots i_p} \frac{\mathrm{d}x^{i_1}}{\mathrm{d}t} \dots \frac{\mathrm{d}x^{i_p}}{dt} \right) \mathrm{d}t,$$

and the associated Euler–Lagrange equations,

$$\sum_j \frac{\partial^2 \mathbf{F}(\dot{x})}{\partial \dot{x}^\alpha \partial \dot{x}^j} \ddot{x}^j + \mathbf{F}_\alpha(\dot{x}) = 0, \qquad \alpha = 1, \dots, n, \tag{23}$$

we easily recognize that, as a consequence of (19), any change of coordinates $y = y(x)$ transforms a solution to (24) into a solution to the transformed equation:

$$\sum_j \frac{\partial^2 \mathbf{G}(\dot{y})}{\partial \dot{y}^\alpha \partial \dot{y}^j} \ddot{y}^j + \mathbf{G}_\alpha(\dot{y}) = 0, \qquad \alpha = 1, \dots, n. \tag{24}$$

On the basis of these premises, Lipschitz turned to investigating the conditions under which a given differential form $\mathbf{F}(\mathrm{d}x)$ can be transformed into another one, say, $\mathbf{G}(\mathrm{d}y) = \sum_{i_1 \dots i_p=1}^{n} G_{i_1,\dots i_p} \mathrm{d}y^{i_1} \dots \mathrm{d}y^{i_p}$, that has constant coefficients, that is $G_{i_1,\dots i_p} = \text{const.}$ We will limit ourselves to stating the criterion and discussing its proof in the case of quadratic ($p = 2$) differential forms. However, it is important to observe that Lipschitz's criterion holds true for homogenous differential forms of any degree, $p \geq 2$.[16]

To this end, let us consider the QDF, $\mathbf{F}(\mathrm{d}x) = \sum a_{ij} \mathrm{d}x^i \mathrm{d}x^j$, and let us suppose that the associated system of isoperimetric equations have been integrated. This means that the Euler-Lagrange equations for the functional

$$E[x(t)] = \int_{t_0}^{t} \left(\sum_{i,k} a_{ik} \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t} \right) \mathrm{d}t,$$

are solved by the set of functions $x(t) = (x^1(t), \dots, x^n(t))$ corresponding to given initial values $x(t_0)$ and $\dot{x}(t_0)$. It can be proved that the solutions $x(t)$ depend upon t through the combinations $\dot{x}^i(t_0)(t - t_0)$, $i = 1, \dots, n$, only. Indeed, $x(t) = (x^1(t), \dots, x^n(t))$

[16] See Lipschitz (1869, p. 92).

can be developable in power series, which are convergent in an appropriate (sufficiently small) neighborhood of $x(t_0)$, as follows:

$$x^k(t) = x^k(t_0) + \dot{x}^k(t_0)(t - t_0) - \frac{1}{2}\sum_{i,j} \Gamma^k_{ij} \dot{x}^i(t_0)\dot{x}^j(t_0)(t - t_0)^2 - \dots.$$

By introducing the variables $u^i := \dot{x}^i(t_0)(t - t_0)$, we obtain a coordinate transformation $x^k = F^k(u^1, \dots, u^n)$, which is invertible in a neighborhood of $x(t_0)$, as a consequence of the fact that $\det\left[\dfrac{\partial x^k}{\partial u^j}\right]_{u=0} = \det\left[\delta^k_j\right] = 1 \neq 0$. The coordinate system $u = (u^1, \dots, u^n)$ is easily recognized as what is now known as the geodesic coordinate system with centre in $x(t_0)$. Lipschitz did not explicitly mention the relevant passage of Riemann's *Habilitationvortrag* in which, albeit discursively, these new coordinates were first introduced. Nonetheless, it is reasonable to conjecture that Riemann represented—in this context too—the main source of inspiration for Lipschitz's treatment. Actually, it can even be said that Lipschitz's discussion of geodesic coordinate systems was an attempt to clarify Riemann's obscure treatment of this topic.

If the coordinates $u = (u^1, \dots, u^n)$ are introduced into the QDF $\mathbf{F}(\mathrm{d}x)$, a new QDF, $\Phi(\mathrm{d}u)$, is obtained by means of which a criterion for flatness could finally be expressed

Theorem 3.1 *A necessary and sufficient condition for the form $\boldsymbol{F}(dx) = \sum_{ik} a_{ik} dx^i dx^k$ to be transformable into a QDF with constant coefficients is that the coefficients of $\boldsymbol{\Phi}(du)$ be independent from u^i, $i = 1, \dots, n$.*[17]

Clearly, the condition expressed by the theorem is sufficient. The difficult part consists in proving that it is necessary as well. The proof given by Lipschitz was based upon this simple remark: If $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$ denotes an arbitrary coordinate transformation then the geodesic coordinates $(u^1, \dots, u^n)$ and $(z^1, \dots, z^n)$ corresponding to $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$, respectively, satisfy a linear relation, namely

$$z^k = \sum_{j=1} \left.\frac{\partial y^k}{\partial x^j}\right|_{x(t_0)} u^j, \qquad k = 1, \dots, n. \tag{25}$$

Indeed, suppose that there is a change of coordinates that transforms $\mathbf{F}(\mathrm{d}x)$ into $\mathbf{G}(\mathrm{d}y) = \sum_{hs} b_{hs} \mathrm{d}y^h \mathrm{d}y^s$, with $b_{hs} = \text{const.}$, $h, s = 1, \dots, n$. Let u and z be the geodesic coordinates corresponding to x, y, respectively, and consider the following definitions: $\mathbf{F}_0(u) = \sum_{ik} a_{ik}(x(t_0)) u^i u^k$, $\mathbf{F}_0(\mathrm{d}u) = \sum_{ik} a_{ik}(x(t_0)) \mathrm{d}u^i \mathrm{d}u^k$; $\mathbf{G}_0(z) = \sum_{ik} b_{ik} z^i z^k$, $\mathbf{G}_0(\mathrm{d}z) = \sum_{ik} b_{ik} \mathrm{d}z^i \mathrm{d}z^k$. Since $\mathbf{G}(dy)$ has constant coefficients, the corresponding equations for the geodesic curves admit the following solutions: $y^k(t) = y^k(t_0) - (t - t_0)\dot{y}^k(t_0)$. As a consequence of $z^k = (t - t_0)\dot{y}^k(t_0)$—this is the definition of geodesic coordinates—we obtain $\mathrm{d}y^k = \mathrm{d}z^k$, $\quad k = 1, \dots, n$.

[17] See Lipschitz (1869, p. 92).

Furthermore, in view of (25), the following relations hold true:

$$\mathbf{F}_0(u) = \mathbf{G}_0(z), \quad \mathbf{F}_0(\mathrm{d}u) = \mathbf{G}_0(\mathrm{d}z).$$

By denoting with $\mathbf{\Phi}(\mathrm{d}u)$ and $\mathbf{X}(\mathrm{d}z)$ the expressions in geodetic coordinates corresponding to x and y of $\mathbf{F}(\mathrm{d}x)$ and $\mathbf{G}(\mathrm{d}y)$, respectively, we obtain:

$$\mathbf{\Phi}(\mathrm{d}u) - \mathbf{F}_0(\mathrm{d}u) = \mathbf{X}(\mathrm{d}z) - \mathbf{G}_0(\mathrm{d}z).$$

Finally, since $\mathbf{X}(\mathrm{d}z) = \mathbf{G}(\mathrm{d}y) = \mathbf{G}_0(\mathrm{d}z)$ (the last equality being a consequence of the fact that $\mathbf{G}(\mathrm{d}y)$ has constant coefficients), one arrives at[18]

$$\mathbf{\Phi}(\mathrm{d}u) = \mathbf{F}_0(\mathrm{d}u),$$

which establishes the condition of Theorem (3.1).

Lipschitz (1869, §7) explicitly admitted that the criterion sanctioned by his theorem offered only an indirect method for deciding the possibility of transforming the differential form $\mathbf{F}(\mathrm{d}x)$ into one with constant coefficients. Indeed, the application of his criterion requires that the equations of the isoperimetric problem associated with $\mathbf{F}(\mathrm{d}x)$ are integrated to ensure that the transformed form $\mathbf{\Phi}(\mathrm{d}u)$ has constant coefficients. Because of this drawback, Lipschitz took care to obtain a more direct criterion that consisted in the vanishing of the covariant form $\mathbf{\Psi}$. In modern terms, the criterion consisted in the fact that the components of the Riemann curvature tensor are identically zero. Lipschitz's treatment of these conditions can be seen as the first rigorous demonstration of the fact that the vanishing of the Riemann symbols is also a sufficient condition to guarantee that a given form can be transformed into a flat metric.

While Christoffel mainly operated within the field of invariant theory, Lipschitz's research on the equivalence problem was mainly focused towards geometrical and mechanical applications. Indeed, to these applications Lipschitz dedicated a substantial number of works. In particular, Lipschitz (1870) is worth mentioning, in which he provided a detailed verification (probably the first truly complete one) of several assertions in Riemann's *Habilitationsvortrag*. In this respect, it is important to recall the explicit computation of the second-order term in the development of the metric in a neighborhood of the center of normal coordinates that, as Riemann had stated, could be interpreted in terms of the curvature Gaussian of a suitable geodesic surface.

The interest in Riemann's ideas aroused by the publication of *Habilitationsvortrag* had been decisive in prompting Lipschitz's research in the field of geometry and mechanics in curved spaces. A further stimulus for reflection followed the publication of Riemann's *Collected Works*, which included his *Commentatio Mathematica*. Both the nature and the extent of the influence that Riemann's geometrical papers had exerted upon Lipschitz and his works can be directly grasped through a series of interesting letters that he sent to Dedekind, one of the editors of Riemann's *Werke*, between the summer and autumn of 1876. The following passage, taken from a letter to Dedekind on September 22, 1876, is particularly revealing:

[18] This is formula (56) of Lipschitz (1869).

> Soon after I last wrote to you, it happened that I went to my homeland, and I spent this break from work on the book [Riemann's *Werke*] to study Riemann's Latin essay that you published, including your addition to it. It gave me the greatest interest to explore the connection between this investigation and my own work. In particular, a concept here comes to light, which has been known to me for several years, and of which I have, so far, made only a few public mentions. To this end, I intend to write my observations on Riemann's Latin essay and have them printed by Borchardt. But I do not want to do it without informing you, so I am addressing you now. My gratitude for what I have learned through Riemann's lecture on the "Hypotheses of Geometry" is so great that I have a keen desire to make some observations on the Latin essay, which is such an essential complement to that lecture.[19] (Lipschitz 1986, pp. 82–83)

The works of Christoffel and Lipschitz on the equivalence problem between QDFs constitute only a very first step in the reception process of Riemann's geometrical ideas. Nonetheless, it was an essential one, above all on the grounds of the attention that both authors accorded the study of the invariance properties of certain functions associated with the coefficients of the metric and for the introduction (by Christoffel) of an algorithm substantially equivalent to the covariant derivative.

4 The birth of the absolute differential calculus

The ADC was the outcome of a slow and gradual discovery process that took place over the course of about a decade (1884–1892), during which Ricci gradually devised and refined the key notions of his algorithm. A crucial contribution to the emergence of the ADC was a paper entitled *Principi di una teoria delle forme quadratiche differenziali* (Ricci-Curbastro 1884), in which Ricci provided his own treatment of the equivalence problem for QDFs. In actual fact, the scope of this work was much broader. Ricci's aim was twofold: to offer a treatment of QDFs by means of purely analytical and algebraic tools and to introduce a classification criterion that might open the way to more systematic research on the subject.

The introduction to Ricci-Curbastro (1884) is interesting in this respect. Here, Ricci expressed the desire to achieve an ideal of uniformity and generality of methods that, in his opinion, could not be found in studies published earlier. After recalling Riemann's pioneering works, notably, the *Habilitationsvortrag* and the *Commentatio*

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[19] Bald nachdem ich das letzte mal an Sie schrieb, hat es sich so gefügt, dass ich hierher in meine Heimat gegangen bin, und ich habe diese Unterbrechung der Arbeit an dem Buche benutzt, um den von Ihnen publicierten lateinischen Aufsatz Riemanns samt Ihrem Zusatz zu demselben zu studieren. Es gewährte mir das grösseste Interesse, den Zusammenhang dieser Untersuchung mit meinen eigenen Arbeiten aufzusuchen. Namentlich kommt hierbei ein Begriff zu Tage, den ich schon seit einer Reihe von Jahren kenne, von dem ich aber bisher nur beiläufig etwas publicirt habe. Um dieses Begriffs willen beabsichtige ich, meine Bemerkungen in Betreff des lateinischen Aufsatzes von Riemann niederzuschreiben und bei Borchardt drucken zu lassen. Ich möchte dies aber nicht thun, ohne dieselben Ihnen vorher mitzutheilen, und daher wende ich mich jetzt an Sie. Meine Dankbarkeit für das, was ich durch Riemanns Vortrag über die Hypothesen der Geometrie empfangen habe, ist so gross, dass ich den lebhaften Wunsch habe, über den hinterlassenen lateinischen Aufsatz, der eine so wesentliche Ergänzung jenes Vortrages bildet, […] solche Äusserungen zu publicieren […].

Mathematica, Ricci briefly described the contributions of his predecessors (Christoffel, Lipschitz, Voss, and Beez) by observing that, in his view, (almost) all of them present common methodological defects:

> Almost all the geometers who have, so far, dealt with this topic [i.e. with the theory of differential quadratic forms], [...] regarded the theories that are valid for our space and for ordinary 2-dimensional surfaces as a model for their research. And while the results they achieved were notable, their methods do not always appear clear and do not sufficiently account for the results themselves.[20] (Ricci-Curbastro 1884, pp. 139–140)

Furthermore, Ricci intended to abandon any explicit reference to analogies of a mechanical and geometric nature in order to favor a purely analytical treatment that was free from any involvement in “idle" discussions about the nature of multi-dimensional spaces. As he explained

> The object of this paper is to begin a series of research on quadratic differential forms, which, because they are conducted on purely analytical notion, will lead to a better knowledge of their nature, and thus also will allow to avoid those rather idle discussions on the existence and nature of spaces with more than three dimensions. The interpretations dictated by the geometric or mechanical analogies that can be given to those results will only be illustrations of such a theory.[21] (Ricci-Curbastro 1884, pp. 139–140)

Unsurprisingly, these remarks have attracted a great deal of attention.[22] They have been interpreted as a programmatic statement to the effect that the theory of QDFs and, more generally, the emerging ADC, should be developed independently from any recourse to geometrical insight, thus confining differential geometry to playing only a secondary role in the development of the calculus. Certainly, Ricci's words indicate a preference for analytical and algebraic methods as opposed, for example, to those employed by Lipschitz, who had frequently had recourse to arguments taken from mechanics and the calculus of variations. At the same time, however, they should not be seen as a sign that Ricci intended to belittle the role of geometry or even to disregard its importance for his research on the theory of QDFs. His claim according that he intended to avoid any discussion on the nature of multidimensional spaces has specific historical reasons, which should be considered when examining Ricci's programmatic statements and their actual impact on the process leading to the creation of the ADC.

Not long before the publication of Ricci-Curbastro (1884), the question regarding the legitimacy of multidimensional spaces had been central to the interests of several

[20] quasi tutti i Geometri che si sono fino ad ora occupati di questo argomento [cioè della teoria delle forme quadratiche differenziali], [...] chiesero alle teorie che valgono pel nostro spazio e per le superficie ordinarie a due dimensioni, norma alle loro ricerche. E se i risultati a cui giunsero furono notevoli, i metodi non appaiono sempre chiari e non rendono dei risultati stessi sufficiente ragione [...].

[21] Oggetto di questo scritto è di iniziare sulle forme differenziali quadratiche una serie di ricerche, le quali, condotte su concetti puramente analitici, meglio ci addentrino nella conoscenza della loro natura, e sfuggano anche così alle discussioni, a dir vero, alquanto oziose sulla esistenza e sulla natura degli spazi a più di tre dimensioni. Le interpretazioni dettate dalle analogie geometriche o meccaniche, che a quei risultati si potranno dare, non saranno che illustrazioni di una tale teoria.

[22] See Reich (1994, p. 67) and Darrigol (2015, §3.4).

investigations of the German mathematician Richard Beez.[23] Since late 1870s, after the publication of Riemann's *Werke*, the author had pursued a wide research program for providing an extension of Gauss's theory of surfaces to multidimensional geometry and for developing a general theory of hypersurfaces embedded into multidimensional Euclidean spaces $\mathbb{R}^n$, with $n \geq 4$. In the course of these investigations, Beez had encountered several unexpected results that he interpreted as a manifestation of the logical inconsistency of multidimensional geometry. The most important one was the following: In contrast to the 2-dimensional case, higher dimensional hypersurfaces—that is, $n-1$-dimensional hypersurfaces in n-dimensional Euclidean space with $n \geq 4$—cannot be isometrically deformed. Indeed, hypersurfaces in Euclidean space are completely determined by their first fundamental form, thus exhibiting a rigidity property which finds no counterpart in classical surface theory. In his view, this very behavior appeared to be inconceivable in view of its counterintuitive character.

As was soon made clear by Killing (1885) and Schur (1887), this result proved to be incorrect or, at best, imprecisely formulated. Indeed, as was first proved by Killing in his monograph (Killing 1885), Beez's rigidity result can be considered correct only under specific conditions concerning the rank of the matrix corresponding to the second fundamental form of hypersurfaces.

These limitations notwithstanding, Beez and his papers appear to have exerted an important influence on Ricci-Curbastro (1884). Indeed, as will be seen, Ricci's treatment of QDFs of the first class can be interpreted as a purely analytical reformulation of theory of hypersurfaces in higher dimensional Euclidean space. Through this, Ricci intended to provide alternative proofs of some of Beez's results, including the (incorrect) theorem according to which the determinant of the second fundamental form that is associated to n-dimensional hypersurfaces, with $n > 2$, is different from zero[24].

Undoubtedly, by offering heuristic tools that proved to be essential in selecting technical choices and classification criteria, Beez's contributions to multidimensional geometry were an important inspiration for Ricci's theory. At the same time, however, Ricci clearly thought it wise to distance himself from Beez's geometrical treatment, probably in order to avoid becoming involved in a delicate debate concerning the legitimacy of multi-dimensional geometry, by focusing on a purely analytical discussion of the subject.

As a consequence, it appears that the above quotation should be understood on the basis of precise historical circumstances rather than as a consequence of a clear and thoughtful methodological viewpoint on Ricci's part. A review of later papers seems to provide confirmation of this interpretation. Indeed, it suffices to consider for example the introduction to Ricci-Curbastro and Levi-Civita (1901) where the authors explicitly emphasized the intimate connection between the ADC and the geometry of Riemannian multidimensional manifolds, which are regarded as the main source of the theory itself.

[23] See Beez (1874, 1875, 1876).

[24] See (Ricci-Curbastro 1884, §3)

4.1 Ricci's *Principi*

The starting point of Ricci-Curbastro (1884) was an observation made by Ludwig Schläfli[25] according to whom if a QDF $\mathbf{\Phi} = \sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j$ in the n variables $x^1, \ldots, x^n$ is given, then there exist integers h, $0 \le h \le n(n-1)/2$ and functions $u^r = u^r(x^1, \ldots, x^n)$, $r = 1, \ldots, n+h$, such that $\mathbf{\Phi}$ can be transformed into a QDF of type $\sum_{r=1}^{n+h} (\mathrm{d}u^r)^2$. In light of this result, Ricci sought to provide a classification of QDF's based upon the minimal integer $\tilde{h}$ that satisfies this condition, which he called the class of the QDF.

He decided to concentrate his efforts on the classification of QDFs of the zeroth and first classes. Possibly, his choice was dictated by a simplicity criterion as much as by the fact that this type of QFD was relevant to the study of geometry of flat manifolds and hypersurfaces, respectively.

As for QDFs of the zeroth-class, Ricci offered a new demonstration of the criterion, already established by Lipschitz, according to which the vanishing of the curvature tensor is a condition both necessary and sufficient for a QDF to be transformable into a flat metric.

Ricci's solution to the problem of providing necessary and sufficient conditions for a QDF to be of the first-class is particularly interesting, since it was clearly modeled upon well-known properties of hypersurfaces in Euclidean space, as established by Beez's research. Indeed, Ricci's treatment of QDFs of the first-class $\mathbf{\Phi} = \sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j = \sum_{r=1}^{n+1} (\mathrm{d}u^r)^2$ relied upon the introduction of a QDF, covariant to $\mathbf{\Phi}$, that can be interpreted geometrically as a generalization to higher dimensions of the second fundamental form employed in classical surface theory.

Inspired by a definition introduced in Beez (Beez 1875, p. 428), Ricci considered the following set of functions[26]:

$$\sqrt{a} \cdot b_{ik} = \det \begin{bmatrix} \frac{\partial^2 u^1}{\partial x^i \partial x^k} & \frac{\partial^2 u^2}{\partial x^i \partial x^k} & \cdots & \frac{\partial^2 u^{n+1}}{\partial x^i \partial x^k} \\ \frac{\partial u^1}{\partial x^1} & \frac{\partial u^2}{\partial x^1} & \cdots & \frac{\partial u^{n+1}}{\partial x^1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u^1}{\partial x^n} & \frac{\partial u^2}{\partial x^n} & \cdots & \frac{\partial u^{n+1}}{\partial x^n} \end{bmatrix}, \qquad \text{where } a := \sqrt{\det(a_{ij})}, \tag{26}$$

and demonstrated that the Riemann symbols associated to $\mathbf{\Phi}$ can be expressed through the second-order minors of the symmetric matrix of coefficients b_{lq}. Fur-

[25] Ricci referred to Schläfli (1873).

[26] Ricci's formula differs from Beez's simply by the factor $-\sqrt{a}$. It is interesting to note that the factor $\sqrt{a}$ is essential in order for b_{rs} to transform covariantly with respect to a_{rs}. On this circumstance, Ricci did not mention Beez's work on hypersurfaces. However, he repeatedly referred to it when he discussed Beez's rigidity theorem in light of his results (Ricci-Curbastro 1884, pp. 167–170).

thermore, Ricci could prove that these coefficients constitute a differential quadratic form $\sum_{i,j=1}^{n} b_{ij} \mathrm{d}x^i \mathrm{d}x^j$, to be denoted with $\mathbf{K}$ and called the derivative form (*forma derivata*) of $\mathbf{\Phi}$, which exhibits the notable property of being "covariant" (see Ricci-Curbastro 1884, p. 168) to $\mathbf{\Phi}$. This means that the transformation laws for the coefficients of $\mathbf{K}$ are identical to those for the coefficients of $\mathbf{\Phi}$. Indeed if $\bar{b}_{ij}$, $i, j = 1, \ldots, n$ denote the components $\mathbf{K}$ with respect to a new local coordinate system $(y_1, \ldots, y_n)$, we have

$$\bar{b}_{ij} = \sum_{ij} b_{rs} \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^j}, \quad i, j = 1, \ldots, n.$$

The introduction of the QDF $\mathbf{K}$ played a decisive role in orienting Ricci's subsequent research on invariant properties of QDFs. Indeed, the consideration of the couple $(\mathbf{\Phi}, \mathbf{K})$ enabled Ricci to present, for the first time, a general approach to the determination of the differential invariants associated to $\mathbf{\Phi}$.

The main idea consisted in regarding both $\mathbf{\Phi}$ and $\mathbf{K}$ as algebraic forms (*forme algebriche quadratiche*) in the variables $\mathrm{d}x^i$, $i = 1, \ldots, n$ (precisely as Christoffel had done), to which standard techniques from algebraic invariant theory could be applied in order to find absolute algebraic invariants. In this expression, Ricci referred to rational functions $\mathfrak{F}(a_{ij}, b_{ij})$ of the coefficients of the forms $\mathbf{\Phi}$ and $\mathbf{K}$ that are left invariant by the following transformations:

$$\bar{a}_{lq} = \sum_{ij} a_{ij} \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^q}, \qquad \bar{b}_{lq} = \sum_{ij} b_{ij} \frac{\partial x^i}{\partial y^l} \frac{\partial x^j}{\partial y^q}, \qquad l, q = 1, \ldots, n.$$

Such functions were then interpreted as differential invariants of $\mathbf{\Phi}$ on the grounds of the *incorrect* result that the coefficients of $\mathbf{K}$ can be expressed in terms of the coefficients of $\mathbf{\Phi}$ and their derivatives, only.

The procedure adopted by Ricci can be described as follows[27]: In order to compute the invariant functions $\mathfrak{F}(a_{ij}, b_{ij})$, he considered the characteristic polynomial of the matrix $\mathbf{BA}^{-1}$ (where $\mathbf{A} = [a_{ij}], \mathbf{B} = [b_{ij}]$) and remarked that its n (non-constant) coefficients represent as many absolute invariants, depending upon the coefficients of $\mathbf{\Phi}$ and their derivatives. These invariance properties are a consequence of a simple observation: if $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ denote the matrices whose elements are given by $\bar{a}_{ij}$ and $\bar{b}_{ij}$, respectively, then the matrices $\mathbf{BA}^{-1}$ and $\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1}$ have the same characteristic polynomial. Indeed, if we introduce the matrix $\mathbf{J} := [J_{ki}] = \dfrac{\partial x^k}{\partial y^i}$, it is easy to see that $\bar{\mathbf{A}} = [\bar{a}_{ij}] = \mathbf{J}^t\mathbf{AJ}$, $\tilde{\mathbf{B}} = [\bar{b}_{ij}] = \mathbf{J}^t\mathbf{BJ}$. As a consequence, $\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1} = \mathbf{J}^t\mathbf{BJ}(\mathbf{J}^t\mathbf{AJ})^{-1} = \mathbf{J}^t\mathbf{BA}^{-1}(\mathbf{J}^t)^{-1}$ and thus, $\det(\bar{\mathbf{B}}\bar{\mathbf{A}}^{-1} - \lambda\mathbb{I}) = \det(\mathbf{BA}^{-1} - \lambda\mathbb{I})$.

Ricci was particularly interested in studying the geometrical significance of this set of invariants and in reconnecting it with Beez's theory of hypersurfaces. To this end, Ricci proved that the n invariants thus determined admit expressions in terms of the principal curvatures of the hypersurface that is associated to $\mathbf{\Phi}$.

27 Here and in what follows matrix notation will be employed for the sake of brevity. Ricci did not use it.

4.2 Differential invariants and parameters

His study of QDFs of the first-class, though biased by the incorrect belief that the derived form $\mathbf{K}$ is completely determined by $\mathbf{\Phi}$, prompted Ricci to develop a general approach to the determination of differential invariants and differential parameters. In 1886, he published a brief memoir *Sui parametri e gli invarianti delle forme quadratiche differenziali*, in which the procedure already adopted in Ricci-Curbastro (1884) could find general application.

The central idea to this approach consisted in producing QDFs that are covariant to the given fundamental form and in applying standard techniques from algebraic invariant theory in order to obtain from them invariant functions. It is precisely in this context that Christoffel's algorithm (corresponding to the operation of covariant differentiation) acquired prominence as an essential technical tool of Ricci's investigations.

These contributions to the theory of differential invariants and parameters have sometimes been described as a result of Ricci's concern for methodological issues and desire to offer a systematic treatment of the subject that is free from the use of variational techniques, such as those employed by Lipschitz and Beltrami. Ricci evidently regarded these techniques as spurious and somehow extraneous to the type of problems to which they were applied. It is certainly true that Ricci was quite sensitive to concerns of a formal and methodological character. Nonetheless, the role of his previous research on the QDFs that are associated to hypersurfaces embedded in Euclidean space was, by far, more decisive in orienting his technical choices and in leading him to the notion of a covariant derivative. In the introduction to his 1886 work, Ricci himself was quite explicit in describing the heuristic pathway he had followed. Indeed, after recalling several previous works on differential invariants by Casorati and Beltrami, namely (Casorati 1861) and (Beltrami 1904), Ricci described in detail Casorati's algebraic approach, which was based upon elimination techniques—he remarked:

> The results relating to the research that is contained in my *Principles of a theory of quadratic differential forms*, according to which the invariants of the linear elements of surfaces of any dimensions presented themselves as algebraic invariants of a system of two covariant quadratic forms, led me to look for a similar way to obtain second-order differential parameters.[28] (Ricci-Curbastro 1886, pp. 177–178)

Before analyzing the content of the paper, it is useful to recall the definition of differential parameters and differential invariants provided in Ricci (1886, §1). To this end, let $\mathbf{\Phi}$ denote a non-singular QDF in the variables $x^1, \ldots, x^n$, $\mathbf{\Phi} = \sum_{r,s} a_{rs} \mathrm{d}x^r \mathrm{d}x^s$; a differential parameter of $\mathbf{\Phi}$ is an expression containing the coefficients of $\mathbf{\Phi}$ and arbitrary functions, say, $U(x^1, \ldots, x^n)$, $V(x^1, \ldots, x^n)$, etc., along with their derivatives

[28] I risultati relativi alla ricerca medesima, contenuti nei miei *Principi di una teoria delle forme differenziali quadratiche*, pei quali gli invarianti degli elementi lineari delle superficie di quante si vogliano dimensioni si sono presentati come veri e propri invarianti algebrici di un sistema di due forme quadratiche covarianti, mi hanno indotto a cercare una via simile per giungere ai parametri differenziali di 2° ordine.

($a_{ik,r} := \dfrac{\partial a_{ik}}{\partial x^r}$), such that it preserves its form whenever the coordinates $(x^1, \ldots, x^n)$ are replaced by new coordinates, say $(y^1, \ldots, y^n)$. This means that if $\bar{a}_{ik}, \bar{a}_{ik,r}, \ldots$, $\overline{U}, \overline{U}_r$, etc., are defined by:

$$\begin{cases} \bar{a}_{ik} = \sum_{r,s=1}^{n} \dfrac{\partial x^r}{\partial y^i} \dfrac{\partial x^s}{\partial y^k} a_{rs}, \\ \bar{a}_{ik,r} = \dfrac{\partial \bar{a}_{ik}}{\partial y^r}, \\ \vdots \\ \overline{U}(y^1, \ldots, y^n) = U(x^1(y), \ldots, x^n(y)), \\ \overline{U}_r(y^1, \ldots, y^n) = \dfrac{\partial \overline{U}}{\partial y^r}, \\ \vdots \end{cases} \tag{27}$$

then the function $\mathfrak{F}(a_{ik}, a_{ik,r}, \ldots; U, U_r, \ldots)$ is a differential parameter if, and only if, one has:

$$\mathfrak{F}(a_{ik}, a_{ik,r}, \ldots; U, U_r, \ldots) = \mathfrak{F}(\bar{a}_{ik}, \bar{a}_{ik,r}, \ldots; \overline{U}, \overline{U}_r, \ldots). \tag{28}$$

As he had done in the case of QDFs of the first-class, Ricci applied the method of algebraic invariant theory to obtain differential parameters in terms of what he called "absolute algebraic invariants" of two covariant forms. His strategy consisted in associating to $\mathbf{\Phi}$ a QDF that depends upon the function U and its derivatives up to a certain order.

In this very circumstance, Christoffel's procedure, which Ricci later dubbed "derivazione covariante," made its first appearance in Ricci's work. Indeed, Ricci remarked that if $U(x^1, \ldots, x^n)$ denotes an arbitrary function, then its first order derivatives $U_r = \dfrac{\partial U}{\partial x^r}$, $r = 1, \ldots, n$ transform "contragrediently" with respect to dx^r, $r = 1, \ldots, n$. In modern terms, this can be translated by stating that the U_r are the components of a first-order covariant tensor while the $\mathrm{d}x^r$, $r = 1, \ldots, n$ behave like the components of a contravariant first-order tensor. By considering the set of functions $U^r = \sum_{k=1}^{n} a^{rk} U_k$, Ricci introduced the quantities

$$U_{rs} = \frac{\partial^2 U}{\partial x^r \partial x^s} - \sum_{k=1}^{n} \Gamma_{rs,k} U^k, \qquad r, s = 1, \ldots, n,$$

and proved that they transform covariantly with respect to the coefficients of $\mathbf{\Phi} = \sum_{rs} a_{rs} \mathrm{d}x^r \mathrm{d}x^s$, that is

$$\overline{U}_{ik} = \sum_{r,s=1}^{n} \frac{\partial x^r}{\partial y^i} \frac{\partial x^s}{\partial y^k} U_{rs}, \qquad i, k = 1, \ldots, n.$$

Precisely this covariance property can be exploited in order to compute the differential parameters that are associated to the function U. Notably, by introducing the matrices $\mathbf{U} = [U_{rs}]$ and $\mathbf{A} = [a_{rs}]$, Ricci determined the differential parameters in terms of the coefficients of the characteristic polynomial of the matrix $\mathbf{U}\mathbf{A}^{-1}$. This is due to the fact that a change of coordinates $x^k = x^k(y^1, \ldots, y^n)$ conjugates the matrix $\mathbf{U}\mathbf{A}^{-1}$. Indeed, if $\mathbf{J}$ denotes the matrix $J_{ik} = \dfrac{\partial x^i}{\partial y^k}$, one has $\overline{\mathbf{U}} = \mathbf{J}^t\mathbf{U}\mathbf{J}$, $\overline{\mathbf{A}} = \mathbf{J}^t\mathbf{A}\mathbf{J}$ and thus, $\overline{\mathbf{U}}(\overline{\mathbf{A}})^{-1} = \mathbf{J}^t\mathbf{U}\mathbf{A}^{-1}(\mathbf{J}^t)^{-1}$. As a consequence, the coefficients of the characteristic polynomial of $\mathbf{U}\mathbf{A}^{-1}$ are invariants under any arbitrary change of coordinates. They are in fact, second-order differential parameters of the function $U(x^1, \ldots, x^n)$, the dependence on second-order derivatives being given by the coefficients of the matrix $\mathbf{U}$.

In this way, besides other new differential parameters, Ricci obtained the second-order differential parameter introduced in Beltrami (1904, p. 98). Indeed, if c_r denotes the coefficient of the r-th order term r of the characteristic polynomial of $\mathbf{U}\mathbf{A}^{-1}$, then it can be easily verified that the following equation holds true:

$$(\Delta_2)U = c_{n-1}(-1)^{n-1} = \mathrm{Tr}(\mathbf{U}\mathbf{A}^{-1}) = \sum_{h,k=1}^{n} a^{hk} U_{hk} = \frac{1}{\sqrt{a}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(a^{ij} \sqrt{a} \frac{\partial U}{\partial x^j} \right), \tag{29}$$

where it is intended that $a^{ij} = [\mathbf{A}^{-1}]_{ij}$ and that $a = \det \mathbf{A}$.

Ricci was particularly explicit in stressing the advantages of his new approach in the search for differential invariants and parameters. In his view, the methods he had pursued allowed for a unitary treatment of the subject that required no appeal to indirect techniques such as those of a variational kind, which had been employed by Jacobi and Beltrami. In actual fact, the importance of Ricci's new approach went far beyond that of providing a new strategy for determining differential invariants and differential parameters. Indeed, his work on the subject quite naturally led Ricci to the notion of covariant derivative.

4.3 Covariant derivative

The first explicit occurrence of the notion of covariant derivative (*derivata covariante*) can be found in Ricci-Curbastro, a brief memoir that was published in the *Rendiconti dell'Accademia dei Lincei*. In the introduction to this paper, Ricci expressed his desire to provide a systematic treatment of a general procedure (*operazione* is the word he employed) that he had recently utilized in a number of different circumstances. He mentioned his research on differential invariants and differential parameters and his work on the notion of n-uple orthogonal systems in a Riemannian manifold. In his own words

> In my latest research, I was naturally led to associate to functions of n variables a quadratic form ϕ^2 as an expression of the square of the linear element of a manifold, of which those variables represent the coordinates. By indicating with U an arbitrary function of these variables, I came across with expressions with two or three indices, whose consideration can replace that of second or third derivatives of U, and which have the advantage over these of being the coefficients of covariant forms with respect to ϕ^2. I also mentioned the possibility of a similar substitution for the derivatives of any order. The usefulness of this substitution is evident especially in those investigations which are essentially independent of the nature of the manifold or of the choice of coordinates in a given manifold. Thus, for example, these expressions (which I will call covariant derivatives in the manifold of linear element ϕ) necessarily give a simpler and clearer form to all expression that enjoy the characteristic property of differential parameters, and allowed me to derive the equations that the parameter of a family of hypersurfaces with $n-1$ dimensions in any n-dimensional manifold must satisfy, independently of the chosen coordinate system, in order for it to be part of an n-ple orthogonal system, a form as simple as that given by Darboux in the case in which the proposed manifold is flat or Euclidean and the coordinates are Cartesian orthogonal coordinates.[29] (Ricci-Curbastro 1887, p. 199)

The explicit mention of the problem of determining n-uple orthogonal systems is particularly revealing of the role that differential geometry had in promoting the genesis of the ADC and, notably, the emergence of the notion of covariant derivative. In order to understand the significance of Ricci's remarks it is necessary to offer an extensive discussion of this technical point. We will provide it in the next paragraph, before we briefly describe the content of Ricci-Curbastro (1887).

The importance of this memoir is twofold. Indeed, in addition to introducing the notion of covariant differentiation, Ricci here presented for the first time an explicit definition of what he called "quantità a p indici," which we can identify, in modern terms, as the components of a fully covariant $p-$th order tensor:

[29] Nelle mie ultime ricerche sono stato naturalmente condotto ad associare alle funzioni di n variabili una forma quadratica ϕ^2 come espressione del quadrato dell'elemento lineare di una varietà, di cui quelle variabili rappresentano le coordinate. Indicando con U una funzione arbitraria di queste, mi sono così presentate delle espressioni a due o tre indici, la cui considerazione si può sostituire a quella delle derivate seconde o terze di U, e che hanno su queste il vantaggio di essere coefficienti di forme covarianti a ϕ^2. Ho pure accennato alla possibilità di una simile sostituzione per le derivate di un ordine qualunque. La utilità della sostituzione stessa in ispecie nelle ricerche, che sono per loro essenza indipendenti dalla natura della varietà o dalla scelta delle coordinate in una varietà data è evidente. Così, per esempio, queste espressioni (che chiamerò *derivate covarianti nella varietà di elemento lineare ϕ*) danno necessariamente forma più semplice e perspicua a tutte le espressioni, che godono della proprietà caratteristica dei parametri differenziali e mi hanno permesso di dare alle equazioni, cui deve soddisfare il parametro di una famiglia di luoghi ad $n-1$ dimensioni in una varietà qual si voglia ad n dimensioni e qualunque sia il sistema delle coordinate per poter far parte di un sistema n-plo ortogonale, una forma tanto semplice quanto quella data dal Darboux nel caso in cui la varietà proposta sia piana od euclidea e le coordinate siano cartesiane ortogonali.

Definition 4.1 Let $U_{r_1 \ldots r_p}$ be p functions that are connected to $\mathbf{\Phi} = \sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j$ in such a way that when one goes from $\sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j$ to $\sum_{i,j=1}^{n} \bar{a}_{ij} \mathrm{d}u^i \mathrm{d}u^j$, they transform according to the following relations:

$$\overline{U}_{r_1 \ldots r_p} = \sum_{s_1, s_2, \ldots, s_p = 1}^{n} \frac{\partial x^{s_1}}{\partial u^{r_1}} \cdots \frac{\partial x^{s_p}}{\partial u^{r_p}} U_{s_1 \ldots s_p}; \tag{30}$$

we say that the functions $U_{r_1 \ldots r_p}$ are coefficients with p indices of a form that is covariant with $\mathbf{\Phi}$.

Concerning the operation of covariant differentiation as introduced in Ricci-Curbastro, we can regard it as an analytical reinterpretation of Christoffel's procedure consisting in generating covariant differential forms of increasing order that are associated to the quadrilinear form G_4.

In order to introduce this notion, Ricci considered a set of coefficients $U_{r_1 \ldots r_p}$ that transform covariantly with respect to $\mathbf{\Phi}$ (according to the definition above) and defined the quantities as follows

$$U_{r_1 \ldots r_p r_{p+1}} := \frac{\partial U_{r_1 r_2 \ldots r_p}}{\partial x^{r_p+1}} - \sum_{q,s=1}^{n} a^{qs} \sum_{h=1}^{p} \Gamma_{r_h r_{p+1}, s} U_{r_1 \ldots r_{h-1} q r_{h+1} r_p}.$$

By making recourse to the relations

$$\frac{\partial^2 x^h}{\partial u^r \partial u^s} = \bar{a}^{pq} \overline{\Gamma}_{rs,q} \frac{\partial x^h}{\partial u^p} - a^{hp} \Gamma_{qt,p} \frac{\partial x^s}{\partial u^t},$$

Ricci proved that the set of quantities $U_{r_1 \ldots r_p r_{p+1}}$ so defined transforms as follows with respect to a change of coordinates $(x^1, \ldots, x^n) \mapsto (u^1(x), \ldots, u^n(x))$

$$\overline{U}_{r_1 \ldots r_p r_{p+1}} = \sum_{s_1, s_2, \ldots, s_p, s_{p+1} = 1}^{n} \frac{\partial x^{s_1}}{\partial u^{r_1}} \cdots \frac{\partial x^{s_p}}{\partial u^{r_p}} \frac{\partial x^{s_{p+1}}}{\partial u^{r_{p+1}}} U_{s_1 \ldots s_p s_{p+1}},$$

and thus behaves like "the coefficients with $p + 1$ indices" of a form that is covariant with $\mathbf{\Phi}$.

Ricci then turned to examining some general properties of the operation that he had just introduced, by observing that it is distributive with respect to the sum. He also investigated the validity of the commutative property, establishing that "in any manifold, covariant derivation enjoys the commutative property up to the second order and beyond the second order in Euclidean manifolds only." In this way, a direct link between the Riemannian curvature and the commutative property of the covariant differentiation was established for the first time. More precisely, if $\mathfrak{F}$ is an arbitrary function of the variables $x_1, \ldots, x_n$ and if $\mathfrak{F}_r$ denotes $\frac{\partial \mathfrak{F}}{\partial x^r}$, $r = 1, \ldots, n$, Ricci proved that $\mathfrak{F}_{rs} = \mathfrak{F}_{sr}$, $r, s = 1, \ldots, n$ and that, for third order derivatives, one has the following[30]:

$$\mathfrak{F}_{rst} - \mathfrak{F}_{rts} = \sum_{i,k=1}^{n} a^{ik} R_{krst} \mathfrak{F}_i. \tag{31}$$

A more general definition of tensor (Ricci did not use this denomination), either $p-$time covariant or contravariant, was given in terms of a system of np functions of the variables $(x_1, \ldots, x_n)$ subject to transformation laws of the type (30). The definition was further discussed in a review article aimed at presenting his methods to a wider audience, which Ricci published in the *Bulletin de Sciences Mathématiques* in 1892: Here, tensor quantities were referred to as invariant systems of functions. Furthermore, it is in this work that Ricci introduced a specific symbol to indicate the operation of covariant (and contravariant) differentiation with respect to a given differential quadratic form $\mathbf{\Phi}$, denoted by $D_{\mathbf{\Phi}}$ (and by $D^{\mathbf{\Phi}}$).

4.4 Orthogonal systems in a Riemannian manifold

The notion of covariant differentiation acquired prominence within Ricci's research priorities quite gradually. As he explained in the above quotation, he soon became convinced of the fruitfulness of this tool also as a consequence of his geometrical investigations on $n-$fold orthogonal systems of hypersurfaces in a general Riemannian manifold. The problem of determining such systems, which had originated in the work of Lamé on differential parameters and triply orthogonal systems, had recently been addressed in Darboux (1878), an extensive memoir divided into three parts. In it, Darboux provided conditions for the existence of mutually orthogonal families of hypersurfaces embedded in n-dimensional Euclidean spaces.

For his part Ricci intended to convey a generalization of the theory proposed by Darboux by obtaining an extension of his results to Riemannian manifolds.[31] The problem that he addressed can be expressed as follows:

Problem 4.1 (Problem A) *To find conditions (both necessary and sufficient) such that to a given family of hypersurfaces $\rho(x^1, \ldots, x^n) = c$ in a Riemannian manifold, whose metric is $\mathbf{\Phi} = \sum_{i,k=1}^{n} a_{ik} dx^i dx^k$, one can associate other families of hypersurfaces, say, $\rho_1 = c_1, \rho_2 = c_2, \ldots, \rho_{n-1} = c_{n-1}$, which, together with $\rho_n := \rho = c$, constitute an n-fold orthogonal system in the sense that, the following relations hold true:*

$$\sum_{r,s=1}^{n} a^{rs} \frac{\partial \rho_h}{\partial x^r} \frac{\partial \rho_k}{\partial x^s} = 0, \qquad h \neq k. \tag{32}$$

[30] If D_k denote the covariant differentiation with respect to x^k, then $\mathfrak{F}_{rs} := D_s(\mathfrak{F}_r)$ and $\mathfrak{F}_{rst} := D_t(D_s(\mathfrak{F}_r))$.

[31] An analysis of Ricci-Curbastro (1887) can also be found in Tonolo (1961).

In order to solve the problem, Ricci introduced an auxiliary problem, which we will call Problem B, to which Problem A could be traced back. He considered a first-order linear, homogeneous differential equation,

$$\mathbf{Y}(F) = \sum_{i=1}^{n} Y^i \frac{\partial F}{\partial x^i} = 0, \tag{33}$$

and searched for conditions (both necessary and sufficient) that for any integral $\rho = \rho(x^1, \ldots, x^n)$ of (33), there are independent solutions $\rho_2, \rho_3, \ldots, \rho_{n-1}$ of the same equation such that:

$$\sum_{r,s=1}^{n} a^{rs} \frac{\partial \rho}{\partial x^r} \frac{\partial \rho_k}{\partial x^s} = 0, \qquad k = 2, \ldots, n-1. \tag{34}$$

Upon introduction of the quantities $H^r := \sum_{r=1}^{n} a^{rs} \frac{\partial \rho}{\partial x^s}$, the existence of $n-2$ independent integrals is tantamount to supposing that the system of equations

$$\mathbf{Y}(F) = 0, \qquad \mathbf{H}(F) = \sum_{r=1}^{n} H^r \frac{\partial F}{\partial x^r} = 0, \tag{35}$$

is complete, that is that there are functions of $(x^1, \ldots, x^n)$, say, ξ_1 and ξ_2, such that

$$(\mathbf{Y}, \mathbf{H})F := \mathbf{Y}(\mathbf{H}(F)) - \mathbf{H}(\mathbf{Y}(F)) = \xi_1 \mathbf{Y}(F) + \xi_2 \mathbf{H}(F). \tag{36}$$

This implies that there exist functions Ξ_0 and ω (it is sufficient to define them as $\xi_1/2$ and $\xi_2/2$, respectively) such that the following n equations hold true:

$$\frac{1}{2} \sum_{r=1}^{n} \left(Y^r \frac{\partial H^s}{\partial x^r} - H^r \frac{\partial Y^s}{\partial x^r} \right) = \Xi_0 Y^s + \omega H^s, \quad s = 1, \ldots, n. \tag{37}$$

By introducing the functions $Y^{rs}, r, s = 1, \ldots, n$ as defined by

$$Y^{rs} := \sum_{k=1}^{n} \left(a^{sk} \frac{\partial Y^r}{\partial x^k} + a^{rk} \frac{\partial Y^s}{\partial x^k} - Y^k \frac{\partial a^{rs}}{\partial x^k} \right),$$

after some straightforward manipulation, one easily proves that, as a consequence of (35), (37), the derivatives of ρ, $\frac{\partial \rho}{\partial x^r}, = 1, \ldots, n$, satisfy to the following system of $n+1$ linear equations:

$$\begin{cases} \sum_r Y^r \frac{\partial \rho}{\partial x^r} = 0, \\ Y^s \Xi_0 + \sum_{r=1}^{n} (Y^{rs} + \omega a^{rs}) \frac{\partial \rho}{\partial x^r} = 0, \qquad s = 1, \ldots, n. \end{cases} \tag{38}$$

The system in (38) admits non-trivial solutions, that is non-zero solutions corresponding to the integral $\rho = \rho(x^1, \dots, x^n)$, if and only if ω is a solution to the $(n-1)$-th degree polynomial equation:

$$P(\omega) := \det(\mathbf{A}(\omega)) = \begin{vmatrix} 0 & Y^1 & Y^2 & \dots & Y^n \\ Y^1 & Y^{11} + \omega a^{11} & Y^{12} + \omega a^{12} & \dots & Y^{1n} + \omega a^{1n} \\ Y^2 & Y^{21} + \omega a^{21} & Y^{22} + \omega a^{22} & \dots & Y^{2n} + \omega a^{2n} \\ \dots & \dots & \dots & \dots & \dots \\ Y^n & Y^{n1} + \omega a^{n1} & Y^{n2} + \omega a^{n2} & \dots & Y^{nn} + \omega a^{nn} \end{vmatrix} = 0. \tag{39}$$

To each root, say ω_h, $h = 1 \dots, n-1$, of $P(\omega) = 0$, there exist quantities $\Xi_{0,h}$, $\Xi_{r,h}$, $r = 1, \dots, n$ such that the following equations hold:

$$\begin{cases} \sum_r Y^r \, \Xi_{r,h} = 0, \\ Y^s \, \Xi_{0,h} + \sum_{r=1}^n (Y^{rs} + \omega_h a^{rs}) \, \Xi_{r,h} = 0, \qquad s = 1, \dots, n. \end{cases} \tag{40}$$

Furthermore, as Ricci easily proved, if ω_k denotes another root of $P(\omega) = 0$, then the corresponding quantities $\mathbf{\Xi}_k = (\Xi_{0,k}, \Xi_{1,k}, \dots, \Xi_{n,k})$ satisfy the relation

$$(\omega_h - \omega_k) \sum_{rs} a^{rs} \, \Xi_{r,k} \, \Xi_{s,h} = 0, \qquad h, k = 1, \dots, n, h \neq k, \tag{41}$$

which can be expressed by stating that the contravariant vectors $H_k^s = \sum_{i=1}^n a^{si} \, \Xi_{i,k}$, $H_h^r = \sum_{i=1}^n a^{ri} \, \Xi_{i,h}, s, r = 1, \dots, n, k \neq h, h, k = 1, \dots, n-1$, are orthogonal with respect to the metric $\mathbf{\Phi} = \sum_{r,s=1}^n a_{rs} \mathrm{d}x^r \mathrm{d}x^s$. Finally, it can be proved that if $\omega = \overline{\omega}$ is a root of $P(\omega)$=0 of multiplicity m then $\dim \ker(\mathbf{A}(\overline{\omega})) = m$, and consequently, the equation $\mathbf{A}(\overline{\omega})(\Xi_0, \Xi_1, \dots, \Xi_n)^t = 0$ admits m linearly independent solutions.

In the light of these remarks, and with the aim of solving problem A, Ricci turned to considering the following auxiliary task.

Problem 4.2 (Problem B) *To provide necessary and sufficient conditions for the existence of* $n-1$ *independent solutions* $\rho_1, \dots, \rho_{n-1}$ *to the equation*

$$\sum_{k=1}^n Y^r(x^1, \dots, x^n) \frac{\partial F}{\partial x^r} = 0$$

that are mutually orthogonal, *that is,*

$$\sum_{r,s=1}^n a^{rs} \frac{\partial \rho_h}{\partial x^r} \frac{\partial \rho_k}{\partial x^s} = 0, \qquad h, k = 1, \dots, n-1. \tag{42}$$

First, Ricci supposed that the equation $P(\omega) = 0$ does not admit any multiple root. As a consequence, the system of equations that is associated to the simple root ω_k

admits one "system of independent solutions," that is $\dim\ker(\mathbf{A}(\omega_k)) = 1$. Furthermore, by posing:

$$Y(\omega) = \det(Y^{rs} + \omega a^{rs}), \qquad \Xi_r(\omega) = \sum_{s=1} \frac{\partial Y(\omega)}{\partial(Y^{rs} + \omega a^{rs})} Y^s, \quad r = 1, \ldots, n,$$

he easily recognized that the quantities

$$\Xi_{0,k} := -\det(a_{rs})Y(\omega_k), \qquad \Xi_{r,h} = \det(a_{rs})\Xi_r(\omega_k)$$

are solutions to the system (40). Finally, he observed that if integrals $\rho_1, \ldots, \rho_{n-1}$ exist, then each of them, say ρ_k, is such that its derivatives $\dfrac{\partial \rho_k}{\partial x^r}$, $r = 1, \ldots, n$ satisfy the conditions imposed by the system (40). As a result, the derivatives $\dfrac{\partial \rho_k}{\partial x^r}$ must be proportional to $\Xi_{r,k}$ (this is due to the fact that $\dim\ker(\mathbf{A}(\omega_k)) = 1$), that is

$$\frac{\partial \rho_k}{\partial x^r} = \mu_k \cdot \Xi_{r,k}. \tag{43}$$

Conversely, if for any $k = 1, \ldots, n-1$ the solutions $\Xi_{r,k}$ are proportional to the derivatives of a function ρ_k, then as a consequence of the first equation of (40), we easily obtain that ρ_k are integrals of (33), too. Furthermore, the equations in (41) assure that these integrals are mutually orthogonal, as required by Problem B.

Ricci proposed various reformulations of the conditions (43) that provide a solution to Problem B. In doing so, he emphasized the need to express these conditions in an invariant way, which made manifest their independence from any particular choice of coordinate system, as is required by the geometrical setting of the problem. Indeed, it was precisely with this idea in mind that Ricci came to appreciate the advantages offered by the notion of covariant differentiation.

The first step in this process consisted in providing an equivalent set of conditions that give a solution to Problem B. Ricci gave the following argument. If a system of orthogonal integrals $\rho_1, \ldots, \rho_{n-1}$ exists, then each of them, say, ρ_k, is a solution to the set of $n-1$ linear PDEs:

$$\mathbf{Y}(\rho_k) = \sum_r Y^r \frac{\partial \rho_k}{\partial x^r} = 0, \quad \mathbf{H}_h(\rho_k) = \sum_r H_h^r \frac{\partial \rho_k}{\partial x^r} = 0, \; h \neq k, h = 1, \ldots, n-1. \tag{44}$$

As a consequence, one also has $(\mathbf{Y}, \mathbf{H}_h)(\rho_k) := \mathbf{Y}(\mathbf{H}_h(\rho_k)) - \mathbf{H}_h(\mathbf{Y}(\rho_k)) = 0$, which, in view of (43), implies the following:

$$\sum_{r,s=1}^{n} \left(Y^s \frac{\partial H_h^r}{\partial x^s} - H_h^s \frac{\partial Y^r}{\partial x^s} \right) \Xi_{r,k} = 0, \quad h \neq k, \quad h, k = 1, \ldots, n-1. \tag{Ω_{hk}}$$

Similarly, by observing that the equations $\mathbf{H}_h(\mathbf{F}) = 0$, $\mathbf{H}_k(\mathbf{F}) = 0$, with $h \neq k$, admit ρ_i, $i \in \{1, \ldots, n\}$, $i \neq h, k$ as a solution, Ricci easily obtained

$$\sum_{r,s=1}^{n} \left(H_h^s \frac{\partial H_k^r}{\partial x^s} - H_k^s \frac{\partial H_h^r}{\partial x^s} \right) \Xi_{r,i} = 0, \qquad i \neq h, i \neq k, h \neq k. \qquad (\Omega_{hki})$$

Conversely, he could prove that conditions (Ω_{hk}) and (Ω_{hki}) are also sufficient to guarantee the existence of $n - 1$ mutually orthogonal integrals, thus providing a complete solution to Problem B.

At this point, an important observation concerning the invariance property of conditions (Ω_{hk}) and (Ω_{hki}) came into effect. Ricci emphasized that these conditions are, in fact, independent of any choice of coordinate system; as he explained, the lefthand sides of conditions (Ω_{hk}), (Ω_{hki}) are *invariabili*,[32] that is, invariant with respect to any change of coordinates.

Guided by this insight, Ricci explored the possibility of providing a full answer to Problem A, by formulating conditions that are manifestly invariant. To this end, he first observed that a solution to Problem A can be obtained from conditions (Ω_{hk}) and (Ω_{hki}) simply by posing $Y^i := \sum_{k=1}^{n} a^{ik} \dfrac{\partial \rho}{\partial x^k}$, $i = 1, \ldots, n$, and then sought to express them in terms of the function ρ and its derivatives. His final result was a set of conditions, denoted by (P_{hk}) and (P_{hki}), in which he recognized the components of the covariant derivatives of ρ as introduced in Ricci-Curbastro (1886):

$$\sum_{r,s,q=1}^{n} \rho_{rsq} \rho^q H_h^r H_k^s = 2 \sum_{p,q,r,s=1}^{n} \rho_{qr} \rho_{qs} H_h^r H_k^s, \qquad (P_{hk})$$

$$\sum_{r,s,q=1}^{n} \rho_{rsq} H_i^q H_h^r H_k^s = 0, \qquad (P_{hki})$$

where $\rho^q := \sum_{k=1}^{n} a^{qk} \partial_k \rho$ and $\rho_{qr} := D_r(\partial_q \rho)$, $\rho_{rsq} := D_q((D_s)(\rho_r))$, $r, s, q = 1, \ldots, n$.[33]

4.5 Ricci's calculus and surface theory

The influence of differential geometry on the works of Ricci was not confined to an early phase of his production. Indeed, besides representing one major factor in the discovery process leading to the discovery of the ADC, differential geometry, and surface theory in particular, soon became the main source of applications within which the efficiency of the new tools could be tested and appreciated.

The review article written for Darboux's *Bulletin* contained a brief discussion of the applications of the ADC to surface theory. Indeed, together with the advantages offered by the new techniques for the study of elasticity theory and heat diffusion, Ricci

32 See Ricci (Ricci-Curbastro 1887, p. 220) and Ricci (Ricci-Curbastro 1887, p. 223).

33 It is intended that D_q denotes the covariant differential operator.

briefly discussed how the equations for the asymptotic lines and the lines of curvature of a surface could be obtained from the principles of the ADC. One year later, in 1893, the applications of his calculus to the theory of surfaces were the subject of a detailed study. The purpose of the study was twofold: first, to demonstrate the fruitfulness of the new methods in a field that, "precisely because thoroughly explored by geometers," was "better suited to highlighting the advantages [of absolute differential calculus] through comparison" and secondly, to present some preliminary results necessary for the understanding of further developments.

The basic idea at the basis of Ricci's reformulation of classical surface theory consisted of a clever combination of the methods of the ADC and of techniques traceable to some works of Beltrami, notably (Beltrami 1865), where the author had investigated systems of curves drawn upon a surface, especially isothermal curves. A fruitful insight that Ricci gained from Beltrami's research was the possibility of characterizing curves on a surface through their "analytical representation," which is obtained by specifying the vector field of which the curve is an integral curve, rather than by providing their finite equations. More specifically, Ricci proposed to study the notion of congruence of curves and that of a bundle of congruences with the specific aim of translating key concepts of surface theory into the language of his new calculus. The definition of congruence was given by Ricci in the following manner. He considered a surface Σ (apparently an abstract one) whose Gaussian coordinates are x^1, x^2 and whose linear element (i.e., first fundamental form) is $\mathbf{\Phi} = \sum_{i,j=1}^{2} a_{ij} \mathrm{d}x^i \mathrm{d}x^j$; a congruence on Σ is defined as a simply infinite system of curves on Σ (*sistema semplicemente infinito*) such that through every point of Σ, there exists a single curve belonging to the system. In analytical terms, giving a congruence is tantamount to assigning the vector field that at each point is tangent to a curve of the system. This means that a congruence can be given by assigning the components of this vector field $\lambda^r(x)$, $r = 1, 2$ with respect to a certain coordinate system x^1, x^2. If the vector field λ is normalized, that is, the condition $\sum_{r=1}^{2} \lambda^r \lambda^s a_{rs} = \sum_{r=1}^{2} \lambda^r \lambda_r = 1$ holds true, the differential equations of the congruence are

$$\lambda^r = \frac{\mathrm{d}x^r}{\mathrm{d}s}, \qquad r = 1, 2; \tag{45}$$

where s denotes the arc length of the integral curves. The so-called finite equations of the congruence can be obtained by integrating the linear first-order equation $\sum_{r=1}^{2} \lambda^r \frac{\partial F}{\partial x^r} = 0$ and equating its solution F to an arbitrary constant. The equation $F(x^1, x^2) = c$, where c is now fixed, singles out a curve of the congruence.

A key notion of Ricci's reformulation of surface theory was that of "orthogonal canonical system" (*sistema canonico ortogonale*, OCS), which is associated to a given congruence $\lambda = (\lambda^1, \lambda^2)$. We can find a clear definition in Ricci (1893, p. 313): Let $\lambda = (\lambda^1, \lambda^2)$ be the vector field of a congruence, the OCS associated to λ (to be indicated either with $\bar{\lambda}$ or, alternatively, with $N_\phi(\lambda)$) is the unitary vector field that is orthogonal to λ, defined by the relations:

$$\bar{\lambda} = \left(\bar{\lambda}^1, \bar{\lambda}^2\right) = \left(-\frac{1}{\sqrt{a}}\lambda_2, \frac{1}{\sqrt{a}}\lambda_1\right), \tag{46}$$

where a denotes the determinant of the matrix $\mathbf{A} = [a_{rs}]$, and λ_r, $r = 1, 2$ are the covariant components that are associated to λ, $\lambda_r = \sum_{s=1}^{2} a_{rs}\lambda^s$, $r = 1, 2$. As Ricci easily proved, this definition is well-posed. Indeed, one has to prove that the components of $\bar{\lambda}$ transform as the components of a first-order contravariant tensor (Ricci referred to it as "sistema semplice controvariante"). Furthermore, the vector field $\bar{\lambda}$ is easily seen to have unitary norm and to be everywhere orthogonal to the curves defined by λ. In moderately anachronistic terms, we can restate his definition by stating that the vectors λ, $\bar{\lambda}$ constitute an orthonormal moving frame, which is locally defined in a coordinate chart of a surface whose metric admits the local expression (with respect to the same local coordinate system) $\boldsymbol{\Phi} = \sum_{r,s=1}^{2} a_{rs}\mathrm{d}x^r\mathrm{d}x^s$. Finally, since at each point of the surface, the vectors $\lambda(x)$, $\bar{\lambda}(x)$ span a basis of the tangent space, Ricci could express the components of a_{rs} by means of the components λ_r, $\bar{\lambda}_s$ of the 1-forms (*sistemi semplici covarianti*) associated to the vectors λ, $\bar{\lambda}$:.[34]

$$a_{rs} = \lambda_r\lambda_s + \bar{\lambda}_r\bar{\lambda}_s, \qquad r, s = 1, 2 \tag{47}$$

From the condition $\sum_{r=1}^{2} \lambda^r\lambda_r = 1$, by means of covariant derivation (with respect to the metric a_{rs}), Ricci easily deduced that: $\sum_{r=1}^{2} \lambda^r D_s\lambda_r = 0$, which, in turn, implies the existence of a covariant tensor ϕ_s, $s = 1, 2$ such that:

$$D_s\lambda_r = \bar{\lambda}_r\phi_s, \qquad r, s = 1, 2. \tag{48}$$

By further differentiating (48) and by using (31) rewritten for the 2-dimensional case, Ricci could deduce the following relation involving the Gaussian curvature G of the surface:

$$D_2\phi_1 - D_1\phi_2 = \sqrt{a}G. \tag{49}$$

In light of these remarks, Ricci introduced the notion of a *bundle* of congruences upon a surface. This is defined to be a set of congruences on a surface such that any two curves belonging to two distinct congruences of the set intersect at a constant angle. It can be proved that two congruences of curves, which are integral curves of the vectors λ^r e ξ^r, $r = 1, 2$, belong to the same bundle if and only if the covariant tensors that are associated to λ^r e ξ^r by means of (48), to be denoted by ϕ_s and ψ_s, $s = 1, 2$, respectively, coincide. Furthermore, Ricci demonstrated that to any covariant tensor ϕ_s, $s = 1, 2$ that satisfies equation (49) there corresponds a bundle F. For this reason,

[34] The metric tensor is a symmetric tensor of order two; the set, $S^k(V)$, of all symmetric tensors of order k on a N-dimensional vector space V is a vector space of dimension $\binom{N+k-1}{k}$. As a consequence, the set of metric tensors on a surface has dimension equal to 3. Thus the metric $a_{ij}\mathrm{d}x^i\mathrm{d}x^j$ can be decomposed with respect to the base $\{\theta \otimes \theta, \theta \otimes \bar{\theta} + \bar{\theta} \otimes \theta, \bar{\theta} \otimes \bar{\theta}\}$, where $\theta = \sum_{r=1}^{2} \lambda_r\mathrm{d}x^r$, $\bar{\theta} = \sum_{r=1}^{2} \bar{\lambda}_r\mathrm{d}x^r$. Indeed, one has: $a_{rs} = \alpha_1\lambda_r\lambda_s + \alpha_2(\lambda_r\bar{\lambda}_s + \bar{\lambda}_r\lambda_s) + \alpha_3\bar{\lambda}_r\bar{\lambda}_s$. Finally, by recalling that $\sum\lambda_s\lambda^s = \sum\bar{\lambda}_s\bar{\lambda}^s = 1$ and, $\sum\lambda_s\bar{\lambda}^s = \bar{\lambda}_s\lambda^s = 0$, we obtain $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = 0$.

Ricci dubbed either ϕ_s or its contravariant counterpart $\phi^s = \sum_{r=1}^{2} a^{sr}\phi_r$ covariant (or contravariant) coordinate system of the bundle F.

After having introduced such an elaborate system of definitions and notions, Ricci quite naturally turned to discussing the absolute invariants associated to the forms $\lambda_r, \bar{\lambda}_r, \phi_r$, carefully distinguishing between invariants that are characteristic of a congruence from those that are instead associated to a bundle,[35] Among the invariants of a congruence, Ricci mentioned $\gamma = \sum_{r=1}^{2} \lambda^r \phi_r$ and $\gamma' = \sum_{r=1}^{2} \bar{\lambda}^r \phi_r$, as for the latter, the so-called curvature of the bundle (*curvatura del fascio*) that he defined as $\sum_{r=1}^{2} \phi^r \phi_r$.

Although the intrinsic character of these quantities is manifest, their geometrical meaning is not. For this reason, Ricci devoted a great deal of attention to investigating their geometrical interpretation. Thus, for instance, the quantities γ and γ' were easily recognized to be the geodesic curvatures of the integral curves associated to the vector fields λ^r and $\bar{\lambda}^r$, $r = 1, 2$, respectively. As he promptly remarked in the introduction to Ricci-Curbastro, the notion of geodesic curvature, which had been introduced for the first time by Minding in 1830 in a purely geometrical way, here "spontaneously presents itself under an analytic garb, as an absolute invariant" associated to the quadratic differential form $\mathbf{\Phi} = \sum_{r,s=1}^{2} a_{rs} \mathrm{d}x^r \mathrm{d}x^s$ and other covariant forms such as $\lambda_r, \bar{\lambda}_r$.

By a simple application of covariant derivative operations, Ricci discovered a characterization of those covariant systems μ_r, $r = 1, 2$ that are the coordinate systems of a bundle. Such systems μ_r are precisely those for which the covariant derivatives of the associated canonical orthogonal system $\bar{\mu}_{rs} := D_s\bar{\mu}_r$ satisfy:

$$\sum_{r,s=1}^{2} a^{rs}\bar{\mu}_{rs} = G, \tag{50}$$

G being the Gaussian curvature corresponding to a_{rs}.

Ricci employed this result in order to propose a more direct and expeditious deduction of well-known formulas of classical surface theory. A noteworthy example is offered by a formula, which was first found in Liouville (1851), that connects the Gaussian curvature to the geodesic curvatures of two systems of curves drawn upon a surface. Indeed, consider the integral curves corresponding to the 1-form λ_r, and suppose that they belong to the bundle F whose associated system is μ_r; by denoting with γ e γ' the geodesic curvatures of the integral lines of λ_r and $\bar{\lambda}_r$, respectively, and with ρ the curvature of the bundle, the condition (50) can be rewritten as follows:

$$\sum_{r=1}^{2} \gamma'^r \lambda_r - \sum_{r=1}^{2} \gamma^r \bar{\lambda}_r + \rho^2 + G = 0, \tag{51}$$

which is indeed equivalent to Liouville's relation.

Several years later, Ricci devoted a long paper Ricci-Curbastro (1895) to a discussion of the theory of surfaces embedded in Euclidean space in the light of the tools of

[35] See Ricci (1893, p. 321).

his ADC. Of particular interest is the first chapter, in which Ricci offered a new proof of the fundamental theorem of surface theory (i.e., the result establishing that the first and second fundamental forms uniquely determine an embedded surface, except for its position in space), which was based upon the possibility of writing the so-called Mainardi–Codazzi equations in tensorial form. In Ricci's words,

> In the first paragraph I employ the methods of the absolute differential calculus to deduce the well-known fundamental relations between the coefficients of the two quadratic differential forms, through which surfaces are determined in shape, regardless of their position in space. The first of these quadratic forms, which I take as fundamental, represents the square of the linear element of the surface. Thus, while I directly introduce the notations of those methods, I give a new simple and direct demonstration, which seems to me to highlight the nature and advantages of the methods themselves.[36] (Ricci-Curbastro 1895, p. 445)

Following Ricci-Curbastro (1895), we consider a surface embedded in $\mathbb{R}^3$. Let y^1, y^2, y^3 denote the Cartesian orthogonal coordinates of a point in $\mathbb{R}^3$, and let x^1, x^2 indicate the Gaussian coordinates of S, whose local parametrization is consequently given by $y^k = y^k(x^1, x^2)$, $k = 1, 2, 3$.

In light of these suppositions, the coefficients of the first fundamental form can be expressed as follows: $a_{rs} = \sum_{k=1}^{3} \frac{\partial y^k}{\partial x^r}\frac{\partial y^k}{\partial x^s}$. Regarding the coefficients of the second fundamental form, Ricci first introduced the components of the normal vector to the surface:

$$\begin{aligned} \zeta^1 &= \frac{1}{\sqrt{a}}\left(\frac{\partial y^2}{\partial x^1}\frac{\partial y^3}{\partial x^2} - \frac{\partial y^2}{\partial x^2}\frac{\partial y^3}{\partial x^1}\right), \\ \zeta^2 &= \frac{1}{\sqrt{a}}\left(\frac{\partial y^3}{\partial x^1}\frac{\partial y^1}{\partial x^2} - \frac{\partial y^3}{\partial x^2}\frac{\partial y^1}{\partial x^1}\right), \\ \zeta^3 &= \frac{1}{\sqrt{a}}\left(\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1}\right). \end{aligned} \tag{52}$$

Then, he defined the following quantities

$$b_{rs} := \sum_{h=1}^{3} \zeta^h D_s \frac{\partial y^h}{\partial x^r}, \qquad r, s = 1, 2, \tag{53}$$

where the operator D_s denotes the covariant differentiation with respect to x^s (it is important to observe that the functions y^h, $h = 1, 2, 3$, are to be treated as scalar functions with regard to Gaussian coordinates).

[36] Nel primo paragrafo stabilisco direttamente coi metodi del calcolo differenziale assoluto le note relazioni fondamentali tra i coefficienti delle due forme differenziali quadratiche, mediante le quali le superficie risultano determinate di forma, prescindendo dalla loro posizione nello spazio; e di cui la prima, che assumo come fondamentale, rappresenta il quadrato dell'elemento lineare della superficie. Così, mentre introduco direttamente in quelle relazioni le notazioni proprie di quei metodi, ne do una nuova dimostrazione semplice e diretta, che mi sembra mettere in evidenza la natura ed i vantaggi dei metodi stessi.

By turning his attention to his previous work on the classification of QDFs, Ricci had no difficulty in recognizing that the coefficients b_{rs} so defined coincide with those that he had introduced in Ricci-Curbastro (1884) when investigating necessary and sufficient conditions for a QDF to be of the first-class. Indeed, it is clear that the QDFs that are susceptible of being interpreted as first fundamental forms of a surface (and more generally of a hypersurface) in Euclidean space are precisely those QDFs that Ricci had named QDFs of the first-class (*forme quadratiche differenziali di prima classe*). Furthermore, it is not difficult to prove that the covariant tensor introduced in (39) coincides, in the two-dimensional case, with the $b_r s$ tensor as defined in (53).

As early as Ricci-Curbastro (1892), Ricci had explicitly observed that this set of conditions can be given a geometrical interpretation in the light of Gauss's equation for the curvature of a surface and the Mainardi–Codazzi equations, which establish a relation among the coefficients of the first and second fundamental forms. Indeed, as Ricci had proved in Ricci-Curbastro (1884), every QDF $\Phi = \sum_{i,j=1}^{n} a_{ij} \mathrm{d}x^i \mathrm{d}x^j$ of the first class satisfies the following conditions:

i) if $R_{ij,kl}, i, j, k, l = 1, \ldots, n$ denote the coefficients of the Riemann curvature that are associated to Φ, there exists a symmetric covariant tensor of the second order, say b_{ij}, such that $R_{ij,kl} = b_{ik}b_{jl} - b_{il}b_{kj}$;
ii) furthermore, the tensor b_{rst} defined by $b_{rst} := D_t(b_{rs})$ is fully symmetric, in particular: $b_{rts} = b_{rst}$. When $n = 2$, conditions (i) yield Gauss's equation since in this case the Gaussian curvature can be expressed in terms of one coefficient only of $R_{ij,kl}$, e.g. $R_{12,12}$: $G = \dfrac{R_{12,12}}{a}$. Finally, the symmetry property of the tensor b_{rts} provides two independent equations that are easily recognized to coincide with the Mainardi–Codazzi equations.

Furthermore, it was easy to prove the reciprocal result—known as fundamental theorem of surface theory—that if two symmetric tensors a_{rs} and b_{rs} (a_{rs} being a positive definite quadratic form) are given that satisfy the so-called fundamental (*fondamentali*) equations[37]

$$G = \frac{\det(b_{rs})}{\det(a_{rs})}, \qquad b_{rst} = b_{rts}, \quad r, s, t = 1, 2; \tag{54}$$

then there exists a surface in $\mathbb{R}^3$ that is defined by the following intrinsic (*intrinseche*) equations

$$\begin{cases} \sum_{h=1}^{3} \dfrac{\partial y^h}{\partial x^r} \dfrac{\partial y^h}{\partial x^s} = a_{rs}, \qquad r, s = 1, 2; \\ D_s\left(\dfrac{\partial y^h}{\partial x^r}\right) = \zeta^h b_{rs}, \qquad h = 1, 2, 3; \quad r, s = 1, 2. \end{cases} \tag{55}$$

The system of equations in (55) constitutes a system of a particular type that was called complete, due to the fact that all higher order derivatives (in this case, second-order derivatives) can be expressed in terms of the lower order derivatives. It was

[37] Ricci (Ricci-Curbastro 1895, pp. 398–399).

already known that such systems admit solutions that depend only on a finite number of arbitrary parameters. More precisely, if the system is of order s and contains q independent equations in the derivatives of order $s-1$, denoted by ϵ_k the number of derivatives of the unknown functions of order k, then the number of such constants is $N = \epsilon_{s-1} + \ldots + \epsilon_1 + \epsilon_0 - q$. In the present case, $N = 6$, since $\epsilon_1 = 6, \epsilon_0 = 3$ and $q = 3$. In fact, it can be easily demonstrated that if $y_k(x_1, x_2), k = 1, 2, 3$ is the solution to (55), then also $\tilde{y}^k = \sum_{h=1}^{3} C_{kh} y^h + C^k$ is a solution to (55), where $\mathbf{C} = [C_{kh}]$ is an orthogonal matrix and $C_k, k = 1, 2, 3$ are arbitrary constants. On the basis of this approach, characterized by a clear analytical orientation, Ricci addressed the resolution of some classic problems of the theory of surfaces.

Since the early 1890s Ricci had mainly concentrated on a number of applications of his ADC within the scope of surface theory, and its n-dimensional generalization (hypersurface theory). Indeed, for some time the desire to provide a reformulation of surface theory by means of the tools of the ADC was at the top of his research priorities, as the publication of his ponderous volume *Lezioni sulla teoria delle superficie* (1898) attests. However, Ricci's commitment to this task and concern with classical problems somehow came at the expense of the discovery of new results, so much so that the new achievements that he attained by means of his ADC were sometimes regarded as few in number and of limited importance. For example, in his report on the 1901 Royal Prize Competition,[38] Luigi Bianchi, at that time a leading figure in the field of differential geometry in Italy and abroad, observed the following:

> But it seems to us that in the field of the infinitesimal geometry of surfaces the advantages produced by the new algorithms are more formal than anything else, or limited to secondary issues; and this judgment is confirmed by the observation that the new methods have not contributed significantly to the most recent and important advances made in this field.[39] (Bianchi 1904, p. 148)

The fact that the ADC was regarded as essentially useless in promoting new achievements within the scope of the applications investigated by Ricci himself—the theory of surfaces and hypersurfaces—was later singled out as a main reason for the delay in the ADC finding recognition among mathematicians. In this respect, the following remarks of Angelo Tonolo, a student of Ricci in Padua, are particularly explicit:

> The most decisive cause [of the lack of scientific recognition of his *Lectures*[40]] was that Ricci tested his algorithm in a field not suitable for appreciating its usefulness since, in ordinary surface theory, the absolute calculus appears to be a complication of things that are simple in themselves.[41] (Tonolo 1954, p. 10).

38 On the history of the 1901 Royal Prize Competion, see Bottazzini (1999).

39 Ma per verità ci sembra che, nel campo della geometria infinitesimale delle superficie, i vantaggi inerenti ai nuovi algoritmi siano più che altro formali, o limitati a questioni secondarie; ed in questo giudizio ci conferma l'osservare che ai più recenti ed importanti progressi compiuti nel detto campo non hanno contribuito in modo significante i nuovi metodi.

40 Tonolo referred to Ricci Curbastro (2019).

41 la causa più decisiva [del mancato riconoscimento scientifico delle *Lezioni*] fu che il Ricci saggiò il Suo algoritmo in un campo non adatto a farne apprezzare l'utilità in quanto che, nell'ordinaria teoria delle superfici, il Calcolo assoluto appare una complicazione di cose di per sè semplici.

What Tonolo interpreted as a manifestation of poor strategic foresight on Ricci's part, was the result of a choice that led Ricci to prove the effectiveness of his calculus in a field that, precisely because already widely explored, was better suited to highlighting its advantages. It is certainly true that the most relevant and numerous results of the new algorithm were obtained in the field of multidimensional geometry, as Bianchi himself was willing to admit. Nonetheless, Ricci's contributions to the theory of surfaces went well beyond a translation of already known geometric results into an analytical guise. In addition to new results, such as the characterization of minimal immersions of an abstract surface in Euclidean space, Ricci's contributions in the field of surface theory marked a mature reflection on methods and techniques, thus offering a common conceptual framework within which different approaches to the subject could find a common theoretical basis.

5 Final remarks

The birth of the ADC was the result of a complex process that has been interpreted as the result of the interactions of different research traditions, such as the theory of algebraic and differential invariants, the theory of differential parameters, and differential geometry. Historians, such as Reich and Dell'Aglio, acknowledged this syncretistic origin, for example, by describing the emergence of the notion of covariant differentiation "as marking the convergence" of two research traditions: an algorithmic one of a "purely algebraic" character, which found in Christoffel's work its most explicit expression and an analytical one consisting in the investigations on differential parameters by Lamé, Beltrami, and others.

This historiographical approach, certainly, allows for a full appreciation of the complexity involved in the genesis of the ADC. However, at the same time it presents several drawbacks, the main one being that it assumed as unproblematic the possibility of drawing a clear and unambiguous distinction between disciplines, such as the theory of differential invariants and the theory of differential parameters on one side and differential geometry on the other. Indeed, a closer inspection of the historical context in which the works by Christoffel, Lipschitz, and Ricci were produced and on their mathematical content indicates that the existence of clear-cut boundaries between these research areas is entirely questionable.

As we have seen, geometrical considerations also played a part in Christoffel's purely algebraic treatment of the equivalence problem, when he discussed the limitations of his *Reduktionssatz*.

Concerning the contributions of Ricci, the case of his theory of differential parameters is particularly interesting in this respect. To qualify Ricci's achievements in this subject as purely analytical results can overshadow noteworthy connections between the discovery process of the ADC and the development of Riemannian geometry. It is certainly true that Ricci often expressed his preference for algebraic tools over other methods, namely variational methods, and that in many circumstances, for example in Ricci-Curbastro (1886), he was not particularly explicit in explaining the geometrical motivation at the basis of his investigations. Nonetheless, as was seen, considerations of a geometrical nature and especially his interest in Beez's work on hypersurface theory proved to be essential in orienting Ricci's proof strategies. Indeed, his treatment of differential invariants and differential parameters can be seen as an attempt at generalizing to any QDF the methods that he had employed in his study of QDFs of the first class, which Ricci interpreted as the metric (*elemento lineare*) of a hypersurface embedded in Euclidean space. Finally, the attempt at generalization of Lamé and Darboux's theory of $n-$uple orthogonal systems, as pursued in Ricci-Curbastro (1887) clearly indicates that the notion of covariant derivative (together with an increasing emphasis on invariant properties of tensorial relations) acquired prominence in Ricci's work, precisely in a geometrical context.

Hence, the genesis of the ADC is more conveniently described as a more or less direct outgrowth of the investigations that followed the publication of Riemann's *Habilitationsvortag* and as a response to the problems it posed while disclosing new research fields: the equivalence problem for QDFs in any number of variables, the generalizations of Gauss surface theory to higher dimensions, and the search for differential invariants and differential parameters in a Riemannian setting.

Accordingly, Ricci's (and Levi–Civita's) claim to the effect that:

> [the methods of the ADC] ont leur raison d'être et leur source dans les rapports intimes, qui les lient à la notion de variété à n dimensions, que nous devons aux génies de Gauss et de Riemann [. . .]

seems to find a substantial confirmation. Indeed, we can regard it as a sufficiently accurate synthesis of the historical and conceptual roots at the origin of their work.

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