THE INTEGRAL CHOW RINGS OF MODULI OF WEIERSTRASS FIBRATIONS

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Abstract. We compute the Chow rings with integral coefficients of moduli stacks of minimal Weierstrass fibrations over the projective line. For each integer $N \geq 1$, there is a moduli stack $\mathcal{W}_N^{\min}$ parametrizing minimal Weierstrass fibrations with fundamental invariant N. Following work of Miranda and Park–Schmitt, we give a quotient stack presentation for each $\mathcal{W}_N^{\min}$. Using these presentations and equivariant intersection theory, we determine a complete set of generators and relations for each of the Chow rings. For the cases $N = 1$ (respectively, $N = 2$), parametrizing rational (respectively, K3) elliptic surfaces, we give a more explicit computation of the relations.

1. Introduction

The study of the Chow rings of moduli spaces has played a central role in algebraic geometry ever since Mumford's introduction of an intersection product for the moduli space of curves $\mathcal{M}_g$ and its compactification by stable curves. Mumford's intersection product requires the use of *rational coefficients*, but Totaro [Tot99] and Edidin–Graham [EG98a] developed an intersection theory for quotient stacks that works with *integral coefficients*. Many moduli stacks of interest in algebraic geometry, including the moduli stacks of curves, are quotient stacks.

Chow rings with integral coefficients are often quite difficult to compute, but in turn they have a much richer structure than their rational counterparts: for instance, rational Chow rings of moduli of hyperelliptic curves are trivial, but the integral ones are not (see [Vis98, EF09, DL21]).

Only a few examples have been computed in full for moduli stacks of curves $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ with g and n small (see [DLFV21, Lar21, DLV21, DLPV21, Inc22]), and even less is known for moduli stacks parametrizing higher dimensional varieties.

In this paper, we study *integral* Chow rings of certain moduli stacks of surfaces that we denote $\mathcal{W}_N^{\min}$, indexed by an integer $N \geq 1$. The stacks $\mathcal{W}_N^{\min}$ parametrize surfaces called *minimal Weierstrass fibrations* over $\mathbb{P}^1$. The arithmetic and geometry of moduli spaces of minimal Weierstrass fibrations over $\mathbb{P}^1$ has already been the subject of investigation of several works (see for instance [Mir81, HP19, PS21, CK23]). Moreover, the stack $\mathcal{W}_2^{\min}$ is of particular interest, as it can be regarded as the moduli stack of elliptic K3 surfaces with a section (equivalently, K3 surfaces polarized by a hyperbolic lattice).

Minimal Weierstrass fibrations over $\mathbb{P}^1$ are flat, proper morphisms $p : X \to \mathbb{P}^1$ together with a section $s : \mathbb{P}^1 \to X$ satisfying the following conditions:

(1) X is normal, irreducible, with at most ADE singularities;
(2) every fiber of p is isomorphic to an elliptic curve, a rational curve with a node, or a rational curve with a cusp;
(3) the section does not intersect the singular point of any of the fibers.

These fibrations arise naturally from contracting the components of the fibers of a smooth elliptic surface over $\mathbb{P}^1$ that do not meet the section. Associated to a minimal Weierstrass fibration is a fundamental

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invariant $N = \deg(R^1 p_* \mathcal{O}_X)^\vee \geq 0$. For each $N \geq 1$, we consider moduli stacks $\mathcal{W}_N^{\min}$ parametrizing minimal Weierstrass fibrations with fundamental invariant N.

Our main result is the following. For a more precise formulation, see Theorem 5.5.

Theorem 1.1. *Suppose that the ground field has characteristic $\neq 2, 3$ and let $N \geq 1$ be an integer. Then*

(1) for N odd, we have

$$\mathrm{CH}^*(\mathcal{W}_N^{\min}) \simeq \mathbb{Z}[c_1, c_2]/I_N$$

where the generators are Chern classes of a certain rank two vector bundle $\mathcal{E}_N$ and the ideal of relations I_N is generated by $\binom{N+2}{2}$ relations, of which one has degree $8N+1$ and the others have degree $9k+m$ for $1 \leq k \leq N$ and $0 \leq m \leq k$. Explicit formulas for these relations are given in (14).

(2) for N even, we have

$$\mathrm{CH}^*(\mathcal{W}_N^{\min}) \simeq \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, I_N)$$

where the generators are Chern classes of certain vector bundles $\mathcal{L}_N$ and $\mathcal{E}_N$ and the ideal of relations I_N is generated by $\binom{N+2}{2}$ relations, of which one has degree $8N+1$ and the others have degree $9k+m$ for $1 \leq k \leq N$ and $0 \leq m \leq k$. Explicit formulas for these relations are given in (15) and (16).

Perhaps the most interesting cases are when $N = 1$ and $N = 2$. Minimal Weierstrass fibrations with fundamental invariant 1 are rational. They arise from the elliptic fibrations obtained by blowing up the base points of a pencil of cubics in $\mathbb{P}^2$. Moduli spaces of rational elliptic fibrations are well studied, and in particular are closely related to several other interesting moduli problems [Vak01].

Minimal Weierstrass fibrations with fundamental invariant 2 come from elliptic K3 surfaces. The intersection theory with rational coefficients of moduli spaces of K3 surfaces has been the subject of much recent research and is expected to behave analogously to that of the moduli space of curves [CK23, MOP17, PY20].

Specializing our Theorem 1.1 to $N = 1, 2$, we obtain the following completely explicit result.

Theorem 1.2. *Suppose that the ground field has characteristic $\neq 2, 3$. Then*

(1) the integral Chow ring of the moduli stack $\mathcal{W}_1^{\min}$ of rational elliptic surfaces with a section is

$$\mathbb{Z}[c_1, c_2]/(6c_1c_2r_6, c_1^3 r_6, c_1^2 c_2 r_6)$$

where the generators are Chern classes of a certain rank two vector bundle $\mathcal{E}_1$ and

$$r_6 = 576(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3);$$

(2) the integral Chow ring of the moduli stack $\mathcal{W}_2^{\min}$ of elliptic K3 surfaces with a section is

$$\mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, r_9, r_{10}, r_{18}, r_{19})$$

where the generators are Chern classes of certain vector bundles $\mathcal{L}_2$ and $\mathcal{E}_2$ and

$$\begin{aligned}
r_9 =&1152(691c_2^4\tau_1 - 38005c_2^3\tau_1^3 + 309568c_2^2\tau_1^5 - 497520c_2\tau_1^7 + 124416\tau_1^9),\\
r_{10} =&1152(30c_2^5 - 6811c_2^4\tau_1^2 + 133495c_2^3\tau_1^4 - 481528c_2^2\tau_1^6 + 327600c_2\tau_1^8 - 20736\tau_1^{10}),\\
r_{18} =&1152c_2^5(108314154642930c_2^4 + 1045672c_2^3\tau_1^2 - 89483c_2^2\tau_1^4 + 35c_2\tau_1^6 - 4\tau_1^8),\\
r_{19} =&2304c_2^6\tau_1(118203201c_2^3 + 180502c_2^2\tau_1^2 - 7c_2\tau_1^4 + 4\tau_1^6).
\end{aligned}$$

When $N \geq 2$, the rational Chow rings $\mathrm{CH}^*(\mathcal{W}_N^{\min}) \otimes \mathbb{Q}$ have been computed by the first author and Kong [CK23]: in particular, they proved that only r_9 and r_{10} are needed in order to generate the ideal of relations *with rational coefficients*. This implies that there are many torsion classes in $\mathrm{CH}^*(\mathcal{W}_2^{\min})$, so the theory with integral coefficients is genuinely different from that with rational coefficients and contains much more information.

Structure of the paper. In Section 2, we construct the moduli stacks $\mathcal{W}_N^{\min}$ as quotient stacks, following the work of Miranda [Mir81] who constructed coarse spaces for $\mathcal{W}_N^{\min}$ and Park–Schmitt [PS21], who constructed $\mathcal{W}_N^{\min}$ as quotients of a weighted projective stack by PGL_2. Our approach is slightly different from that of Park–Schmitt, but we will show that the two approaches coincide.

In Section 3, we discuss the equivariant intersection theory of projective spaces, which is a key tool in the proof of Theorem 1.1. In particular, we obtain generators for $\mathrm{CH}^*(\mathcal{W}_N^{\min})$.

In Sections 4 and 5, we compute relations among the generators of the Chow ring that result from excising the locus of non-minimal Weierstrass fibrations, finishing the proof of Theorem 1.1.

Finally, in Section 6 we make explicit calculations of the relations in the cases $N = 1$ and $N = 2$, proving Theorem 1.2.

Notation. In what follows, we denote $V_m := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m))$. Moreover, we denote

$$V_{m_1,m_2} := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_1)) \oplus \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m_2)).$$

The symbol G will be used to denote the group $\mathrm{GL}_2 / \boldsymbol{\mu}_N$ for some N. Observe that for N odd we have $G \simeq \mathrm{GL}_2$ and for N even we have $G \simeq \mathrm{PGL}_2 \times \mathbb{G}_m$ (see [AV04, Proposition 4.4]). We will consider the following G-actions

- when $G = \mathrm{GL}_2$, we identify V_m with $\mathrm{Sym}^m E^\vee$, where E is the standard GL_2-representation. In other terms, we let GL_2 acts on V_m via the formula $A \cdot f(x,y) := f(A^{-1}(x,y))$;
- when $G = \mathrm{PGL}_2 \times \mathbb{G}_m$, we regard V_{2m} as the representation on which PGL_2 acts via the formula $A \cdot f(x,y) := \det(A)^m f(A^{-1}(x,y))$ and $\mathbb{G}_m$ acts trivially. This definition does not depend on the choice of a representative A for the element in PGL_2.

The symbol V'_{2dm} will stand for the following G-representations:

- when $G = \mathrm{GL}_2$ we set $V'_{2dm} := V_{2dm} \otimes \det(E)^{\otimes d(m-1)}$.
- when $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ we set $V'_{2dm} := V_{2dm} \otimes L^{\otimes(-2d)}$, where L is the standard representation of $\mathbb{G}_m$.

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2. Moduli of minimal Weierstrass fibrations

In this Section, after recalling some basic definitions, we give in Theorem 2.8 a presentation as a quotient stack of $\mathcal{W}_N^{\min}$, the moduli stack of minimal Weierstrass fibrations with fundamental invariant N. We also introduce some vector bundles $\mathcal{L}_N$ and $\mathcal{E}_N$ (see Definition 2.10 and Definition 2.12) which will be relevant for our computations.

2.1. Stacks of conics with sections.

We start by introducing a stack, denoted $\mathcal{F}_N^{\min}$, which admits a natural presentation as a quotient stack (Proposition 2.3). The reason for introducing this stack is that it will turn out to be isomorphic to $\mathcal{W}_N^{\min}$, the moduli stack of minimal Weierstrass fibrations.

Definition 2.1. We denote by $\mathcal{F}_N^{\min}$ the following fibered category over $\mathfrak{Sch}/\mathrm{Spec}(k)$, the category of quasi-separated schemes of finite type over a field k:

Objects: The objects over a scheme T are tuples $(f : \mathcal{P} \to T, \mathcal{L}, A, B)$ consisting of a flat and proper morphism $\mathcal{P} \to T$ with geometric fibers isomorphic to $\mathbb{P}^1$, a line bundle $\mathcal{L}$ over $\mathcal{P}$ of degree N along each fiber of f, and two sections A, B of $\mathrm{H}^0(\mathcal{P}, \mathcal{L}^{\otimes 4})$ and $\mathrm{H}^0(\mathcal{P}, \mathcal{L}^{\otimes 6})$ respectively. We require the sections A and B to satisfy the following two conditions:

(1) the global section $4A^3 + 27B^2$ of $\mathcal{L}^{\otimes 12}$ is not the zero section, and
(2) for each geometric point s of T, there is no point p of $\mathbb{P}^1 \simeq \mathcal{P}_s$ such that A_s (resp. B_s) vanishes in p with order ≥ 4 (resp. with order ≥ 6).

Morphisms: A morphism $(f : \mathcal{P} \to T, \mathcal{L}, A, B) \to (f' : \mathcal{P}' \to T', \mathcal{L}', A', B')$ consists of a morphism $T \to T'$, together with two isomorphisms $\phi : \mathcal{P} \to \mathcal{P}' \times_{T'} T$ and $\psi : \phi^*\mathcal{L}' \to \mathcal{L}$, such that ψ sends A' (resp. B') to A (resp. B).

Recall from the Notation section that V'_{2dN} is the representation of $G := \mathrm{GL}_2 / \boldsymbol{\mu}_N$ where the action is defined as follows:

- for N odd, V'_{2dN} is the GL_2-representation $\mathrm{Sym}^{2dN} E^\vee \otimes \det(E)^{d(N-1)}$, where E is the standard rank two representation of GL_2;
- for N even, V'_{2dN} is the $\mathrm{PGL}_2 \times \mathbb{G}_m$-representation $V_{2dN} \otimes L^{\otimes(-2d)}$, where L is the $\mathbb{G}_m$-representation of weight one and $V_{2dN} = \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2dN))$ is the PGL_2-representation with action defined by the formula $A \cdot f(X_0, X_1) := \det(A)^{dN} f(A^{-1}(X_0, X_1))$.

Without using the isomorphisms of G with GL_2 (respectively, $\mathrm{PGL}_2 \times \mathbb{G}_m$), the representation V'_{2dN} can be defined as follows: V'_{2dN} is the vector space $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2dN))$, and the group GL_2 acts on $\mathcal{O}_{\mathbb{P}^1}(N)$ by the formula $A \cdot f(x, y) = f(A^{-1}(x, y))$, so that the subgroup of diagonal matrices of the form

$$\begin{bmatrix} \xi & 0 \\ 0 & \xi \end{bmatrix},$$

where $\xi \in \boldsymbol{\mu}_N$, acts trivially; this induces an action of G on $\mathcal{O}_{\mathbb{P}^1}(N)$ and consequently an action of G on the global sections of $\mathcal{O}_{\mathbb{P}^1}(2dN)$. In this Section, we will be mostly interested in the representation $V'_{4N,6N} := V'_{4N} \oplus V'_{6N}$.

Definition 2.2. We define the G-invariant closed subscheme Δ_N in $V'_{4N,6N}$ as the union of Δ^1_N and Δ^2_N, where

- the subscheme Δ^1_N is the locus of pairs (A, B) such that $4A^3 + 27B^2 = 0$, and
- the subscheme Δ^2_N is the locus of pairs (A, B) such that there exists a point $p \in \mathbb{P}^1$ such that A (resp. B) vanishes in p with order ≥ 4 (resp. with order ≥ 6).

The following Proposition gives a presentation of $\mathcal{F}^{\min}_N$ as a quotient stack.

Proposition 2.3. *There is an isomorphism* $\mathcal{F}^{\min}_N \cong [(V'_{4N,6N} \smallsetminus \Delta_N)/G]$.

Proof. Our argument follows [AV04, Theorem 4.1]. It suffices to construct a map $(V'_{4N,6N} \smallsetminus \Delta_N) \to \mathcal{F}^{\min}_N$ which is a G-torsor.

The data of a map $T \to V'_{4N,6N}$ is equivalent to a section of the projection $\pi_2 : V'_{4N,6N} \times T \to T$. Let $p : \mathbb{P}^1 \times T \to T$ be the second projection. Since π_2 is affine, a section of π_2 induces a morphism

$$\mathrm{Sym}^\bullet_{\mathcal{O}_T}(p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N))^\vee) \to \mathcal{O}_T.$$

This is the same as a map $p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N))^\vee \to \mathcal{O}_T$ that in turn is equivalent to a map $\mathcal{O}_T \to p_*(\mathcal{O}_{\mathbb{P}^1 \times T}(4N) \oplus \mathcal{O}_{\mathbb{P}^1 \times T}(6N))$, namely a choice of a pair of sections $A \in \mathrm{H}^0(T, p_*\mathcal{O}_{\mathbb{P}^1 \times T}(4N))$ and $B \in \mathrm{H}^0(T, p_*\mathcal{O}_{\mathbb{P}^1 \times T}(6N))$, equivalently a pair of sections of $\mathrm{H}^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(4N))$ and $\mathrm{H}^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(6N))$.

In particular, the data of a morphism $T \to V'_{4N,6N} \smallsetminus \Delta_N$ is equivalent to the data of two sections (A, B) of $\mathrm{H}^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(4N))$ and $\mathrm{H}^0(\mathbb{P}^1 \times T, \mathcal{O}_{\mathbb{P}^1 \times T}(6N))$ such that the two conditions given in Definition 2.2 are both not verified.

There is a natural transformation $\Phi : (V'_{4N,6N} \smallsetminus \Delta_N) \to \mathcal{F}_N^{\min}$, that on objects is defined as follows. Given a map $T \to V'_{4N,6N} \smallsetminus \Delta_N$, which corresponds to two sections A, B as above, we can associate the object of $\mathcal{F}_N^{\min}$ given by $(\mathbb{P}^1 \times T \to T, \mathcal{O}_{\mathbb{P}^1 \times T}(N), A, B)$.

Let $\sigma : G \times (V'_{4N,6N} \smallsetminus \Delta_N) \to (V'_{4N,6N} \smallsetminus \Delta_N)$ be the map that defines the action of G on $V'_{4N,6N} \smallsetminus \Delta_N$, and denote $\mathrm{pr}_2 : G \times (V'_{4N,6N} \smallsetminus \Delta_N) \to (V'_{4N,6N} \smallsetminus \Delta_N)$ the projection on the second factor.

We claim that Φ is a G-torsor. We need to show that:

(1) The two arrows $\Phi \circ \sigma$ and $\Phi \circ \mathrm{pr}_2$ are isomorphic,
(2) For every scheme T and every object $(f : \mathcal{P} \to T, \mathcal{L}, A, B)$ of $\mathcal{F}_N^{\min}(T)$, there is an étale cover $T' \to T$ such that the pull-back $(f' : \mathcal{P}' \to T', \mathcal{L}', A', B')$ is isomorphic to an object of $(V'_{4N,6N} \smallsetminus \Delta_N)(T)$ (i.e. it is in the essential image of Φ), and
(3) If $\alpha := (f' : \mathcal{P}' \to T', \mathcal{L}', A', B')$ is in the essential image of Φ, the action of G on its essential fiber (i.e. the pairs (β, γ) consisting of an element $\beta \in (V'_{4N,6N} \smallsetminus \Delta_N)(T)$ and an isomorphism $\Phi(\beta) \to \alpha$) is simply transitive.

To check point (1) it suffices to observe that the action of G on $U_N = V'_{4N,6N} \smallsetminus \Delta_N$ extends to an action of G on $(\mathcal{O}_{\mathbb{P}^1 \times U_N}(N), A, B)$. Then by definition there is an isomorphism $(\mathcal{O}_{\mathbb{P}^1 \times U_N \times G}(N), \sigma^* A, \sigma^* B) \cong (\mathcal{O}_{\mathbb{P}^1 \times U_N \times G}(N), \pi_2^* A, \pi_2^* B)$.

To check (2), recall that as $f : \mathcal{P} \to T$ is a Severi-Brauer variety, there is an étale cover $T' \to T$ and an isomorphism $\mathcal{P} \times_T T' \cong \mathbb{P}^1 \times T'$ over T'. Then if we denote by $\mathcal{L}'$ the pull-back of $\mathcal{L}$ to $\mathbb{P}^1 \times T'$ we have two line bundles, $\mathcal{L}'$ and $\mathcal{O}_{\mathbb{P}^1 \times T'}(N)$ that are isomorphic along each fiber. Therefore there is a line bundle $\mathcal{G}$ on T' such that $\mathcal{G} \otimes \mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1 \times T'}(N)$ (see [Har13, Exercise III.12.4]). Then up to replacing T' with a covering that trivializes $\mathcal{G}$, we can assume that $\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^1 \times T'}(N)$. This proves point (2).

To check point (3) it suffices to recall that the functor sending a scheme T to $\underline{\mathrm{Aut}}_T(\mathbb{P}^1_T, \mathcal{O}_{\mathbb{P}^1_T}(N))$ is represented by G (see [AV04, Proof of Theorem 4.1]). Indeed, we need to check that:

- The action of G is transitive on the fibers of $(V'_{4N,6N} \smallsetminus \Delta_N) \to \mathcal{F}_N^{\min}$, and
- The action is simply transitive (this is analogous to the representability of $[(V'_{4N,6N} \smallsetminus \Delta_N)/G] \to \mathcal{F}_N^{\min}$).

To check the first bullet point, we need to check that if two objects of $\mathcal{F}_N^{\min}(T)$ that belong to the image of $\Phi(T)$ are isomorphic, then there is an element of $G(T)$ which sends the first one to the second one. To check the second bullet point, we need to check that such an element is unique. Both bullet points follow since $\underline{\mathrm{Aut}}_T(\mathbb{P}^1_T, \mathcal{O}_{\mathbb{P}^1_T}(N))$ is represented by G. □

2.2. **Moduli of Weierstrass fibrations.** Let $\mathcal{W}_N^{\min}$ be the moduli stack of minimal Weierstrass fibrations, as defined in [PS21, Section 4.2]. We will prove in Proposition 2.7 that $\mathcal{W}_N^{\min}$ is isomorphic to the stack $\mathcal{F}_N^{\min}$ that we introduced before. We start by recalling the relevant definitions from *loc. cit.*

Definition 2.4. A *family of minimal Weierstrass fibrations* of degree N over a scheme T is the data of:

(1) a flat and proper morphism $\mathcal{P} \to T$ with geometric fibers isomorphic to $\mathbb{P}^1$, and
(2) a flat and proper morphism $f : \mathcal{X} \to \mathcal{P}$ with a section $\mathcal{S} \subseteq \mathcal{X}$.

We require that for every geometric point $p \in T$, the fiber $(\mathcal{X}_p, \mathcal{S}_p) \to \mathcal{P}_p$ is a minimal Weierstrass fibration of degree N, and we refer the reader to [Mir89] for a more detailed exposition on Weierstrass fibrations.

Given two families $((\mathcal{X}, \mathcal{S}) \to \mathcal{P} \to T)$ and $((\mathcal{X}', \mathcal{S}') \to \mathcal{P}' \to T')$, a morphism from the latter to the former consists of a morphism $g : T' \to T$ and isomorphisms $\mathcal{X}' \cong \mathcal{X} \times_T T'$ and $\mathcal{P}' \cong \mathcal{P} \times_T T'$ which preserve the section and make the obvious square commutative.

It is shown in [PS21, Theorem 1.2] that there is an algebraic stack, which we denote by $\mathcal{W}_N^{\min}$, that parametrizes families of minimal Weierstrass fibrations. Our goal is to prove that $\mathcal{W}_N^{\min} \cong \mathcal{F}_N^{\min}$. We need the following preparatory Lemma, that can be understood as a generalization of some of the results in [Mir89] to families. See also [PS21, Subsection 4.1] for the case where $\mathcal{P} \cong \mathbb{P}^1 \times B$.

Lemma 2.5. *Let* $((\mathcal{X}, \mathcal{S}) \xrightarrow{f} \mathcal{P} \to T)$ *be a family of minimal Weierstrass fibrations over* T*. Then:*

- $R^1 f_* \mathcal{O}_{\mathcal{X}}$ *is a line bundle, which we denote by* $\mathcal{L}^{-1}$*,*
- $f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \cong f_* \mathcal{O}_{\mathcal{X}}$*, and*
- *For every* $n \geq 2$*, we have a* split *exact sequence*

$$0 \to f_* \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S}) \to f_* \mathcal{O}_{\mathcal{X}}(n\mathcal{S}) \to f_* \mathcal{O}_{\mathcal{S}}(n\mathcal{S}) \to 0.$$

In particular, $f_* \mathcal{O}_{\mathcal{X}}(n\mathcal{S}) = \mathcal{O}_{\mathcal{P}} \oplus \mathcal{L}^{\otimes -2} \oplus \ldots \oplus \mathcal{L}^{\otimes -n}$.

Proof. To check the first and second bullet point, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}} \longrightarrow \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \longrightarrow \mathcal{O}_{\mathcal{S}}(\mathcal{S}) \longrightarrow 0.$$

First observe that $R^1 f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) = 0$ from cohomology and base change applied to f. Then if we push it forward the exact sequence above, we get

$$0 \to f_* \mathcal{O}_{\mathcal{X}} \to f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \to f_* \mathcal{O}_{\mathcal{S}}(\mathcal{S}) \to R^1 f_* \mathcal{O}_{\mathcal{X}} \to 0. \qquad (*)$$

Now, $f_* \mathcal{O}_{\mathcal{X}} \to f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S})$ is surjective. It suffices to check for each point b, $f_* \mathcal{O}_{\mathcal{X}} \otimes k(b) \to f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \otimes k(b)$ is surjective. But $\dim_{k(b)}(\mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})) = \dim_{k(b)}(\mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(\mathcal{S}_b))) = 1$, so the two canonical maps $f_* \mathcal{O}_{\mathcal{X}} \otimes k(b) \to \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})$ and $f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \otimes k(b) \to \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(\mathcal{S}_b))$ are surjective (the image of the constant function 1 does not get sent to 0). Then from cohomology and base change, we have two isomorphisms $f_* \mathcal{O}_{\mathcal{X}} \otimes k(b) \cong \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b})$ and $f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \otimes k(b) \cong \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(\mathcal{S}_b))$, which make the following diagram commutative:

$$\begin{array}{ccc} f_* \mathcal{O}_{\mathcal{X}} \otimes k(b) & \longrightarrow & f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \otimes k(b) \\ {\scriptstyle \cong \text{ cohomology and b.c.}} \downarrow & & \downarrow {\scriptstyle \cong \text{ cohomology and b.c.}} \\ \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) & \xrightarrow[\cong]{} & \mathrm{H}^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(\mathcal{S}_b)). \end{array}$$

In particular, the top map is also an isomorphism, so it is surjective.

Then from the exactness of $(*)$, we have two isomorphisms $f_* \mathcal{O}_{\mathcal{S}}(\mathcal{S}) \cong R^1 f_* \mathcal{O}_{\mathcal{X}}$ and $f_* \mathcal{O}_{\mathcal{X}}(\mathcal{S}) \cong f_* \mathcal{O}_{\mathcal{X}}$. The first two bullet points follow since $f_* \mathcal{O}_{\mathcal{S}}(\mathcal{S})$ is a line bundle as $\mathcal{S}$ is a section.

We prove the third bullet point by induction. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S}) \longrightarrow \mathcal{O}_{\mathcal{X}}(n\mathcal{S}) \longrightarrow \mathcal{O}_{\mathcal{S}}(n\mathcal{S}) \longrightarrow 0.$$

We push it forward via f, observing that from cohomology and base change it remains exact, and we get

$$0 \longrightarrow f_* \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S}) \longrightarrow f_* \mathcal{O}_{\mathcal{X}}(n\mathcal{S}) \longrightarrow f_* \mathcal{O}_{\mathcal{S}}(n\mathcal{S}) \longrightarrow 0.$$

Now, $f_* \mathcal{O}_{\mathcal{S}}(n\mathcal{S}) \cong \mathcal{L}^{\otimes -n}$ as $\mathcal{S}$ is a section, and

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{L}^{\otimes -n}, f_* \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S})) &= \mathrm{Ext}^1(\mathcal{O}_{\mathcal{P}}, \mathcal{L}^{\otimes n} \otimes f_* \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S})) \\ &= R^1 f_*(\mathcal{L}^{\otimes n}) \oplus R^1 f_*(\mathcal{L}^{\otimes n-2}) \oplus \ldots \oplus R^1 f_*(\mathcal{L}). \end{aligned}$$

However, $\mathcal{L}$ has non-negative degree along each fiber of $\mathcal{P} \to T$ ([Mir89, Lemma II.5.6]), so from cohomology and base change $R^1 f_*(\mathcal{L}^{\otimes k}) = 0$ for every $k > 0$. Then $\mathrm{Ext}^1(\mathcal{L}^{\otimes -n}, f_* \mathcal{O}_{\mathcal{X}}((n-1)\mathcal{S})) = 0$ and the desired sequence splits. □

The following is just a relative version of the arguments in [Mir89, II.5]. We report them below for convenience of the reader.

Consider $((\mathcal{X},\mathcal{S}) \xrightarrow{\phi} \mathcal{P} \to T)$ be a family of minimal Weierstrass fibrations over T. First, we choose a covering of $\mathcal{P}$ which trivializes $\mathcal{L}$, and we choose a generator e_1 for $\mathcal{L}^{-1}$. In particular, $e_n := e_1^{\otimes n}$ will be a generator for $\mathcal{L}^{-n}$. From Lemma 2.5 this covering also trivializes $\phi_*\mathcal{O}_{\mathcal{X}}(n\mathcal{S})$, and we choose an element f of $\phi_*\mathcal{O}_{\mathcal{X}}(2\mathcal{S})$ (resp. g of $\phi_*\mathcal{O}_{\mathcal{X}}(3\mathcal{S})$) that via the projection to $\phi_*\mathcal{O}_{\mathcal{S}}(2\mathcal{S})$ (resp. $\phi_*\mathcal{O}_{\mathcal{S}}(3\mathcal{S})$) of Lemma 2.5 maps to e_2 (resp. e_3). Then g^2 and f^3 are sections of $\phi_*\mathcal{O}_{\mathcal{X}}(6\mathcal{S})$, and $g^2 = f^3 + h$ where h maps to 0 via the projection $\phi_*\mathcal{O}_{\mathcal{X}}(6\mathcal{S}) \to \phi_*\mathcal{O}_{\mathcal{S}}(6\mathcal{S})$.

Proceeding as in [Mir89, II.5] (i.e. completing the square and the cube), we can choose f and g appropriately such that there are two sections $A \in \mathrm{H}^0(\mathcal{P},\mathcal{L}^{\otimes 4})$ and $B \in \mathrm{H}^0(\mathcal{P},\mathcal{L}^{\otimes 6})$ such that $g^2 = f^3 + Af + B$.

Using [Mir89, II.5.3] we see that the choice of A and B is not unique (for example, it depends on the choice of e_1). However, given two different choices (A,B) and (A',B') there is an invertible function $\lambda \in \mathrm{H}^0(\mathcal{P},\mathcal{O}_{\mathcal{P}}^*)$ such that $A = \lambda^{-4}A'$ and $b = \lambda^{-6}B'$. Moreover, from [Mir81, Corollary 2.5], there is no point p in a fiber of $\mathcal{P} \to T$ where the order of vanishing of A, at p is greater than 4 and the order of vanishing of B is greater than 6 (as ϕ is a family of *minimal* Weierstrass fibrations). Note that here, for A and B we intend the restriction of the sections to the fiber of $\mathcal{P} \to T$ containing p.

Combining the previous paragraph with Lemma 2.5, we have

Corollary 2.6. *Consider $((\mathcal{X},\mathcal{S}) \xrightarrow{\phi} \mathcal{P} \to T)$ a family of minimal Weierstrass fibrations over T. Then:*

1. *The sheaf $\mathcal{L} = (R^1\phi_*\mathcal{O}_{\mathcal{X}})^\vee$ is a line bundle,*
2. *From the data above we can construct two sections A, B of $\mathrm{H}^0(\mathcal{P},\mathcal{L}^{\otimes 4})$ and $\mathrm{H}^0(\mathcal{P},\mathcal{L}^{\otimes 6})$,*
3. *Given two different choices (A,B) and (A',B') there is an invertible function $\lambda \in \mathrm{H}^0(\mathcal{P},\mathcal{O}_{\mathcal{P}}^*)$ such that $A = \lambda^{-4}A'$ and $B = \lambda^{-6}B'$, and*
4. *For every $x \in T$, there is no point $y \in \mathcal{P}_x$ such that the sections A_x and B_x of $\mathrm{H}^0(\mathcal{P}_x,\mathcal{L}^{\otimes 4}_{|\mathcal{P}_x})$ and $\mathrm{H}^0(\mathcal{P}_x,\mathcal{L}^{\otimes 6}_{|\mathcal{P}_x})$ vanish at y to order ≥ 4 and ≥ 6, respectively.*

Corollary 2.6 gives a map $\mathcal{W}_N^{\min} \to \mathfrak{F}_N^{\min}$. We show that this map is an isomorphism, by producing an inverse.

Given a family $(f : \mathcal{P} \to T, \mathcal{L}, A, B)$ we can construct a family of Weierstrass fibrations by taking a closed subscheme of $\mathrm{Proj}_{\mathcal{O}_{\mathcal{P}}}(\mathcal{O}_{\mathcal{P}} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3})$ as follows. First consider a covering $\mathcal{U} \to \mathcal{P}$ which trivializes $\mathcal{L}$. Then consider the closed subscheme of $\mathrm{Proj}_{\mathcal{O}_{\mathcal{U}}}(\mathcal{O}_{\mathcal{U}} \oplus \mathcal{L}^{-2}_{|\mathcal{U}} \oplus \mathcal{L}^{-3}_{|\mathcal{U}})$ given by $y^2z = x^3 + Axz^2 + Bz^3$ where x (resp. y) is a generator for $\mathcal{L}^{-3}_{|\mathcal{U}}$ (resp. $\mathcal{L}^{-2}_{|\mathcal{U}}$).

One can check that these closed subschemes descend to a closed subscheme of $\mathcal{X} \subseteq \mathrm{Proj}_{\mathcal{O}_{\mathcal{P}}}(\mathcal{O}_{\mathcal{P}} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3})$. The map $\mathcal{X} \to \mathcal{P}$ has a section $\mathcal{S} \subseteq \mathcal{X}$, that over $\mathcal{U}$ is given by $z = x = 0$. To check that this is a family in $\mathcal{W}_N^{\min}$ we need to check that when $T = \mathrm{Spec}(k)$ for an algebraically closed field k, the resulting surface $\mathcal{X}$ with section $\mathcal{S}$ is a minimal Weierstrass fibration. The fact that it is a Weierstrass fibration follows from [Mir89], whereas minimality follows from [Mir81, Corollary 2.5]. We have proven the following.

Proposition 2.7. *We have an isomorphism $\mathcal{W}_N^{\min} \simeq \mathfrak{F}_N^{\min}$.*

Combining the result above with Proposition 2.3, we obtain the main result of this Section.

Theorem 2.8. *Suppose that the ground field k has characteristic $\neq 2, 3$. Then we have*

$$\mathcal{W}_N^{\min} \simeq [(V'_{4N,6N} \smallsetminus \Delta_N)/(\mathrm{GL}_2/\boldsymbol{\mu}_N)].$$

Remark 2.9. The presentation above specializes to the two following cases depending on the parity of N, that is:

- if N is odd, then $\mathcal{W}_N^{\min} \simeq [(V'_{4N,6N} \smallsetminus \Delta_N)/\mathrm{GL}_2]$;

- if N is even, then $\mathcal{W}_N^{\min} \simeq [(V'_{4N,6N} \smallsetminus \Delta_N)/\operatorname{PGL}_2 \times \mathbb{G}_m]$.

The actions of these two groups are the ones explained in the Notation section.

2.3. **Vector bundles on $\mathcal{W}_N^{\min}$ when N is odd.** Let us suppose N odd. As oberved in Remark 2.9, the stack $\mathcal{W}_N^{\min}$ has a presentation as a quotient by the action of GL_2. In particular, the GL_2-equivariant morphism $V'_{4N,6N} \smallsetminus \Delta_N \to \operatorname{Spec} k$ induces a morphism of quotient stacks $\mathcal{W}_N^{\min} \to \mathcal{B}\operatorname{GL}_2$. This should correspond to a rank two vector bundle on $\mathcal{W}_N^{\min}$.

Definition 2.10. For N odd, we define the rank two vector bundle $\mathcal{E}_N$ on $\mathcal{W}_N^{\min}$ as follows:

$$\mathcal{E}_N((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) := p_*((R^1 f_*\mathcal{O})^\vee \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N-1}{2}}).$$

Proposition 2.11. *The map $\mathcal{W}_N^{\min} \to \mathcal{B}\operatorname{GL}_2$ is given by*

$$((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) \longmapsto \mathcal{E}_N((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T).$$

Proof. The stack $\mathcal{B}\operatorname{GL}_2$ can be regarded as the stack whose objects are pairs $(\mathcal{P} \to T, \mathcal{N})$, where $\mathcal{P} \to T$ is a Severi-Brauer variety whose geometric fibers have dimension one, and $\mathcal{N}$ is a line bundle on $\mathcal{P}$ that on the geometric fibers has degree one. Therefore, to give a map to $\mathcal{B}\operatorname{GL}_2$ is equivalent to giving such a pair $(\mathcal{P} \to T, \mathcal{N})$.

On the other hand, the stack $\mathcal{B}\operatorname{GL}_2$ can also be regarded as the stack of rank two vector bundles. The equivalence between these two descriptions is given in one direction by sending a pair $(p : \mathcal{P} \to T, \mathcal{N})$ to the rank two vector bundle $p_*\mathcal{N}$, and in the other direction by sending the vector bundle $\mathcal{E}$ on T to $(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathcal{P}(\mathcal{E}^\vee)}(1))$.

By construction, the map $\mathcal{W}^N_{\min} \to \mathcal{B}\operatorname{GL}_2$ is as follows:

$$((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) \longmapsto (\mathcal{P} \to B, (R^1 f_*\mathcal{O})^\vee \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N-1}{2}}).$$

The corresponding rank two vector bundle is then given by $p_*((R^1 f_*\mathcal{O})^\vee \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N-1}{2}})$. □

2.4. **Vector bundles on $\mathcal{W}_N^{\min}$ when N is even.** In this case, the presentation of the stack $\mathcal{W}_N^{\min}$ given in Theorem 2.8 can be recasted in terms of the group $\operatorname{PGL}_2 \times \mathbb{G}_m$. In particular, this shows that there is a map $\mathcal{W}_N^{\min} \to \mathcal{B}\operatorname{PGL}_2 \times \mathcal{B}\mathbb{G}_m$, which then must be induced by a Severi-Brauer stack on $\mathcal{W}_N^{\min}$ together with a line bundle.

Definition 2.12. For N even, we define the rank three vector bundle $\mathcal{E}_N$ and the line bundle $\mathcal{L}_N$ on $\mathcal{W}_N^{\min}$ as follows:

$$\mathcal{E}_N((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) := p_*(\omega_{\mathcal{P}/T}^\vee),$$
$$\mathcal{L}_N((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) := p_*((R^1 f_*\mathcal{O}_\mathcal{X})^\vee \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N}{2}}).$$

The vector bundle $\mathcal{E}_N$ actually plays no role here, but it will be relevant later on.

Proposition 2.13. *The map $\mathcal{W}_N^{\min} \to \mathcal{B}\operatorname{PGL}_2 \times \mathcal{B}\mathbb{G}_m$ is given by*

$$((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T) \longmapsto (\mathcal{P} \xrightarrow{p} T, \mathcal{L}_N((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T))$$

Proof. The stack $\mathcal{B}(\operatorname{GL}_2/\boldsymbol{\mu}_N)$ classifies pairs $(\mathcal{P} \xrightarrow{p} T, \mathcal{L})$ where $\mathcal{P} \to T$ is a Severi-Brauer variety and $\mathcal{L}$ is a line bundle on $\mathcal{P}$ whose restriction to the geometric fibers of $\mathcal{P} \to T$ has degree N. The isomorphism $\mathcal{B}(\operatorname{GL}_2/\boldsymbol{\mu}_N) \simeq \mathcal{B}(\operatorname{PGL}_2 \times \mathbb{G}_m)$ sends a pair $(\mathcal{P} \xrightarrow{p} T, \mathcal{L})$ to the pairs $(\mathcal{P} \xrightarrow{p} T, p_*(\mathcal{L} \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N}{2}}))$.

The map $\mathcal{W}_N^{\min} \to \mathcal{B}\,\mathrm{PGL}_2 \times \mathcal{B}\mathbb{G}_m$ can be factored as

$$\mathcal{W}_N^{\min} \longrightarrow \mathcal{F}_N^{\min} \longrightarrow \mathcal{B}(\mathrm{GL}_2/\boldsymbol{\mu}_N) \longrightarrow \mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m),$$

where the first map sends an object $((\mathcal{X},\mathcal{S}) \xrightarrow{f} \mathcal{P} \xrightarrow{p} T)$ to the pair $(\mathcal{P} \xrightarrow{p} T, (R^1 f_* \mathcal{O}_{\mathcal{X}})^\vee)$, from which we deduce that the composition sends a family of minimal Weierstrass fibrations over T to the pair

$$(\mathcal{P} \xrightarrow{p} T, p_*((R^1 f_* \mathcal{O}_{\mathcal{X}})^\vee \otimes \omega_{\mathcal{P}/T}^{\otimes \frac{N}{2}})).$$

□

3. Equivariant intersection theory on projective spaces

Set $V_k := \mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(k))$. The projective space $\mathbb{P}V_k$ can be naturally identified with the Hilbert scheme of k points on $\mathbb{P}^1$. In this section we consider two actions on $\mathbb{P}V_k$, namely:

- the GL_2-action inherited from V_k, on which GL_2 acts via $A \cdot f(x,y) := f(A^{-1}(x,y))$;
- the PGL_2-action inherited from the natural action of PGL_2 on $\mathbb{P}^1$.

The aim of this Section is to collect some basic facts on the integral Chow ring of $[\mathbb{P}V_k/G]$, where G is either GL_2 or PGL_2. We will divide our analysis in two parts, depending on whether V_k is a G-representation or not.

3.1. **First case.** As V_k is a GL_2-representation, the stack $[\mathbb{P}V_k/\mathrm{GL}_2]$ is a projective bundle over $\mathcal{B}\,\mathrm{GL}_2$. Similarly, for k even, the vector space V_k is a PGL_2-representation, where the action is defined as

$$A \cdot F(x,y) := \det(A)^{\frac{k}{2}} F(A^{-1}(x,y)).$$

Therefore, for $G = \mathrm{GL}_2$ or $G = \mathrm{PGL}_2$ and k even, we have that $\pi : [\mathbb{P}V_k/G] \to \mathcal{B}G$ is a projective bundle.

Let h be the hyperplane class. From an equivariant point of view, we can regard h as the class of the G-equivariant line bundle $\mathcal{O}_{\mathbb{P}V_k}(1)$. The following Proposition is just the usual projective bundle formula.

Proposition 3.1. *Assume that either $G = \mathrm{GL}_2$ or $G = \mathrm{PGL}_2$ and k is even. Then:*

1. *The integral Chow ring of $[\mathbb{P}V_k/G]$ is generated as $\mathrm{CH}^*(\mathcal{B}G)$-module by h^m for $m \leq k$.*
2. *We have $\pi_*(h^m) = s_{m-k}^G(V_k)$, where the latter denotes the G-equivariant Segre class of degree $m-k$ of V_k.*

3.2. **Second case.** For k odd, the vector space V_k is not a PGL_2-representation, hence the quotient stack $[\mathbb{P}V_k/\mathrm{PGL}_2]$ is not a projective bundle over $\mathcal{B}\,\mathrm{PGL}_2$. We have to treat this second case differently.

Let $\Sigma_k \subset \mathbb{P}V_k \times \mathbb{P}^1$ be the PGL_2-invariant subscheme defined as

$$\Sigma_k = \{(f,x) \text{ such that } f(x) = 0\}.$$

The line bundle $\mathcal{O}(\Sigma_k)$ is isomorphic to $\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}V_k}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(k)$, hence the latter admits a PGL_2-linearization. The canonical line bundle $\omega_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ admits a PGL_2-linearization as well. Then the isomorphism

$$\mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1 \simeq \mathrm{pr}_{1*}(\mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}V_k}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(k) \otimes \mathrm{pr}_2^*\omega_{\mathbb{P}^1}^{\frac{k-1}{2}})),$$

gives a PGL_2-linearization to the rank two vector bundle $\mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1$.

Proposition 3.2. *For $k \geq 0$ odd, let γ_1, γ_2 be the PGL_2-equivariant Chern classes of $\mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1$. Then we have*

$$\mathrm{CH}^*([\mathbb{P}V_k/\mathrm{PGL}_2]) \simeq \mathbb{Z}[\gamma_1,\gamma_2]/(\gamma_1^2 - 4\gamma_2 + c_2, \prod_{i=0}^{\frac{k-1}{2}} \left(\gamma_2 + \frac{(k-2i)^2-1}{4} c_2\right)),$$

and the generators of this ring as a module over $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2)$ are $\gamma_1^i\gamma_2^j$, where $i = 0,1$ and $j = 0,1,\ldots,\frac{k-1}{2}$.

Proof. Recall ([Pan98]) that $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2) \simeq \mathbb{Z}[c_2, c_3]/(2c_3)$, where c_i is the i^{th} Chern class of the vector bundle $[V_2/\,\mathrm{PGL}_2] \to \mathcal{B}\,\mathrm{PGL}_2$.

For k odd, in [ST21, Proposition 3.7] it is proved that the Chow ring of $\mathrm{CH}^*([\mathbb{P}V_k/\,\mathrm{PGL}_2])$ is isomorphic to

$$\mathbb{Z}[u,v]^{S_2}/(\prod_{i=0}^{k}((\frac{k+1}{2}-i)u+(\frac{-k+1}{2}+i)v))),$$

where u, v are the Chern roots of $\mathcal{O}_{\mathbb{P}V_k}(1) \otimes V_1$, so that $u+v=\gamma_1$ and $uv=\gamma_2$.

Moreover, we know from [ST21] that the pullback homomorphism $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2) \to \mathrm{CH}^*([\mathbb{P}V_k/\,\mathrm{PGL}_2])$ sends $c_2 \mapsto -(u-v)^2$ and $c_3 \mapsto 0$. This implies that

$$\gamma_1^2 = (u+v)^2 = (u-v)^2 + 4uv = 4uv - c_2 = 4\gamma_2 - c_2.$$

In particular $\mathrm{CH}^*([\mathbb{P}V_k/\,\mathrm{PGL}_2])$ is generated as a module by monomials of the form $(u+v)^i uv^j$, where i is either 0 or 1.

We can then rewrite the relation as follows:

$$\begin{aligned}
&\prod_{i=0}^{\frac{k-1}{2}}\left(\left(\left(\frac{k+1}{2}-i\right)u+\left(\frac{-k+1}{2}+i\right)v\right)\left(\left(\frac{-k+1}{2}+i\right)u+\left(\frac{k+1}{2}-i)v\right)\right)\\
&=\prod_{i=0}^{\frac{k-1}{2}}\left(uv+\left(\frac{k+1}{2}-i\right)\left(\frac{-k+1}{2}+i\right)(u-v)^2\right)\\
&=\prod_{i=0}^{\frac{k-1}{2}}\left(uv+\frac{(k-2i)^2-1}{4}c_2\right).
\end{aligned}$$

This shows that the monomials $(u+v)^i(uv)^j$ for $i \leq 1$ and $j \leq \frac{k-1}{2}$ actually generate $\mathrm{CH}^*([\mathbb{P}V_k/\,\mathrm{PGL}_2])$ as a module over $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2)$, as claimed. ☐

Remark 3.3. For $k=1$, we can see that the Chow ring of $[\mathbb{P}V_1/\,\mathrm{PGL}_2]$ is generated by the first Chern class of the normal bundle of the universal section which, coherently with the usual definition of psi classes, we denote ψ_1. The fact that ψ_1 is the unique generator follows from the formula of Proposition 3.2 but can also be seen as follows: on the universal conic $p : \mathcal{P} \to [\mathbb{P}V_1/\,\mathrm{PGL}_2]$ we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{P}} \longrightarrow \mathcal{O}(\sigma) \longrightarrow \sigma_*\sigma^*\mathcal{O}(\sigma) \longrightarrow 0 \tag{1}$$

where σ is the universal section. By pushing forward along p, we get an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{O} \longrightarrow p_*\mathcal{O}(\sigma) \longrightarrow \sigma^*\mathcal{O}(\sigma) \longrightarrow 0$$

which shows that the rank two bundle in the middle is an extension of the normal bundle of the universal section by the trivial line bundle. This implies that $\gamma_1 = \psi_1$ and $\gamma_2 = 0$.

Next we give an explicit description of the pushforward morphism along $\pi : [\mathbb{P}V_k/\,\mathrm{PGL}_2] \to \mathcal{B}\,\mathrm{PGL}_2$. For this, set

$$E_{n,m}(q) := (-1)^q \sum_{\substack{a=0\\a+b=2q+1}}^{m}\sum_{b=0}^{n} 2^{m-a}\binom{m}{a}\binom{n}{b}.$$

Lemma 3.4. *We have*

$$\pi_*(\gamma_1^m\gamma_2^n) = k^{-1}\sum_{0\leq q\leq n+\frac{m-k}{2}} E_{n,m}(q)\cdot s_{2(n-q)+m-k}^{\mathrm{PGL}_2}(V_{k-1})\cdot 2c_2^q.$$

Observe that the sum above is actually a scalar multiple of c_2: this is because every monomial containing c_3 that appears in a Segre class is killed by the multiplication by 2, hence the polynomial above lives in the ring $\mathbb{Z}[c_2]$. In this way the multiplication by the inverse of k can be understood literally, i.e. as the division of the scalar coefficient by k.

In particular, we are implying that such scalar coefficient is a multiple of k, because the whole expression belongs to the integral Chow ring.

Proof. Consider the commutative diagram of quotient stacks

$$\begin{array}{ccc} [\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\operatorname{PGL}_2] & \xrightarrow{\rho} & [\mathbb{P}V_k/\operatorname{PGL}_2] \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \pi} \\ [\mathbb{P}V_1/\operatorname{PGL}_2] & \xrightarrow{\pi'} & \mathcal{B}\operatorname{PGL}_2 \end{array} \tag{2}$$

where the top horizontal arrow is induced by the multiplication map. This map is finite of degree k, hence $\rho_*\rho^*\xi = k\xi$ and

$$k \cdot \pi_*\xi = \pi_*\rho_*\rho^*\xi = \pi'_*\mathrm{pr}_{1*}(\rho^*\xi).$$

As k is odd, multiplication by k is an injective group endomorphism of $\mathrm{CH}^*(\mathcal{B}\operatorname{PGL}_2)$. This argument shows that once we understand how the pullback homomorphism ρ^* and the composition $\pi'_*\mathrm{pr}_{1*}$ works, we also have an explicit formula for π_*.

We first have to compute the pullback of γ_1 and γ_2 to the Chow ring of $[\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\operatorname{PGL}_2]$. For this, observe that $\rho^*(V_1 \otimes \mathcal{O}_{\mathbb{P}V_k}(1)) = V_1 \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}V_{k-1}}(1)$. Recall from Proposition 3.1 and Remark 3.3 that $h = c_1^{\operatorname{PGL}_2}(\mathcal{O}_{\mathbb{P}V_{k-1}}(1))$ and $\psi_1 = c_1^{\operatorname{PGL}_2}(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1))$. Applying the usual formulas for the Chern classes, we deduce

$$\rho^*\gamma_1 = c_1^{\operatorname{PGL}_2}(V_1 \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}V_{k-1}}(1)) = \psi_1 + 2h$$
$$\rho^*\gamma_2 = c_2^{\operatorname{PGL}_2}(V_1 \otimes \mathrm{pr}_1^*\mathcal{O}_{\mathbb{P}V_1}(1) \otimes \mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}V_{k-1}}(1)) = h(h + \psi_1).$$

This implies that

$$\pi_*(\gamma_1^m\gamma_2^n) = k^{-1}\pi'_*\mathrm{pr}_{1*}((2h + \psi_1)^m h^n (h + \psi_1)^n).$$

The computation of pushforwards along $\mathrm{pr}_1 : [\mathbb{P}V_1 \times \mathbb{P}V_{k-1}/\operatorname{PGL}_2] \to [\mathbb{P}V_1/\operatorname{PGL}_2]$ is easy because $k-1$ is even, hence this map is the projection from a projective bundle. We deduce

$$\mathrm{pr}_{1*}(h^i\psi_1^j) = s_{i-k+1}^{\operatorname{PGL}_2}(V_{k-1})\psi_1^j.$$

Also the pushforward along $\pi' : [\mathbb{P}V_1/\operatorname{PGL}_2] \to \mathcal{B}\operatorname{PGL}_2$ is not hard to determine: consider the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}V_1 & \xrightarrow{\rho'} & \operatorname{Spec} k \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ [\mathbb{P}V_1/\operatorname{PGL}_2] & \xrightarrow{\pi'} & \mathcal{B}\operatorname{PGL}_2\,. \end{array}$$

Then the compatibility formula implies that $f^*\pi'_*(h) = \rho'_*g^*(h)$. To compute $g^*(\gamma_1)$, observe that $g^*(V_1 \otimes \mathcal{O}_{\mathbb{P}V_1}(1)) = \mathcal{O}_{\mathbb{P}V_1}(1)^{\oplus 2}$, hence $g^*(\gamma_1) = 2h$; this implies $f^*\pi'_*(\gamma_1) = \rho'_*g^*(\gamma_1) = 2$. In degree zero the pullback $f^* : \mathrm{CH}^0(\mathcal{B}\operatorname{PGL}_2) \to \mathrm{CH}^0(\operatorname{Spec} k)$ is an isomorphism, so we can conclude $\pi'_*\psi_1 = 2$.

The relation $\psi_1^2 = -c_2$ implies that $\pi'_{1*}(\psi_1^{2j}) = 0$ and $\pi'_{1*}(\psi_1^{2j+1}) = (-1)^j 2c_2^j$. We deduce

$$\pi'_*\mathrm{pr}_{1*}(h^i\psi_1^{2j}) = 0, \quad \pi'_*\mathrm{pr}_{1*}(h^i\psi_1^{2j+1}) = (-1)^j s_{i-k+1}^{\operatorname{PGL}_2}(V_{k-1})2c_2^j.$$

Putting all together, we obtain the claimed formulas for the pushforwards along $\pi : [\mathbb{P}V_k/\operatorname{PGL}_2] \to \mathcal{B}\operatorname{PGL}_2$. □

3.3. **Chern classes of representations.** Here we outline how to explicitly compute the Chern classes of the representations that appeared before. This also gives formulas for the Segre classes by formally inverting the total Chern class.

First, let us consider the case $G = \operatorname{GL}_2$. The integral Chow ring of $\mathcal{B}\operatorname{GL}_2$ is isomorphic to $\mathbb{Z}[c_1, c_2]$, where c_1 and c_2 are the Chern classes of the standard GL_2-representation E. Therefore, if ℓ_1 and ℓ_2 are the Chern roots of $E^\vee$, we have that $c_1 = -(\ell_1 + \ell_2)$ and $c_2 = \ell_1\ell_2$.

We have $V_m = \operatorname{Sym}^m E^\vee$, hence the Chern roots of this symmetric power are given by $j\ell_1 + (m-j)\ell_2$, where $0 \leq j \leq m$. From this we deduce that the total Chern class of V_m is equal to

$$\begin{aligned} c^{\operatorname{GL}_2}(V_m) &= \prod_{j=0}^{m}(1 + (j\ell_1 + (m-j)\ell_2)t) \\ &= \prod_{j \leq \frac{m}{2}}(1 + (j\ell_1 + (m-j)\ell_2)t)(1 + ((m-j)\ell_1 + j\ell_2)t) \\ &= \prod_{j \leq \frac{m}{2}}(1 - mc_1t + (j(m-j)c_1^2 + (2j-m)^2c_2)t^2). \end{aligned}$$

Let $\langle p(t)\rangle_d$ denote the coefficient in front of t^d in $p(t)$. Then we have proved the following:

Proposition 3.5.

$$c_d^{\operatorname{GL}_2}(V_m) = \left\langle \prod_{j \leq \frac{m}{2}}(1 - mc_1t + (j(m-j)c_1^2 + (2j-m)^2c_2)t^2) \right\rangle_d$$

Next, we consider the case $G = \operatorname{PGL}_2$. The vector space $V_{2m} = \operatorname{H}^0(\mathbb{P}^1, \mathcal{O}(2m))$ is a PGL_2-representation of rank $2m+1$, where the action is defined as $A \cdot f(x,y) := \det(A)^m f(A^{-1}(x,y))$. In what follows we will need explicit formulas for the PGL_2-equivariant Chern classes of V_{2m}. These has been computed by Fulghesu and Viviani in [FV11, Section 6].

Recall ([Pan98]) that $\operatorname{CH}^*(\mathcal{B}\operatorname{PGL}_2)$ is isomorphic to $\mathbb{Z}[c_2, c_3]/(2c_3)$, where $c_i^{\operatorname{PGL}_2}(V_2) = c_i$, and

Proposition 3.6 ([FV11, Corollary 6.3])**.**

$$c_d^{\operatorname{PGL}_2}(V_{2m}) = \begin{cases} \left\langle t\prod_{j=1}^m(t^2 + j^2c_2) + t^{\frac{m}{2}+1}\sum_{j=1}^{\frac{m}{2}}\binom{\frac{m}{2}}{j}(t^3 + c_2t)^{(\frac{m}{2}-j)}c_3^j \right\rangle_{2m+1-d} & \textit{if } m \textit{ is even,} \\ \left\langle t\prod_{j=1}^m(t^2 + j^2c_2) + t^{\frac{m-1}{2}}\sum_{j=1}^{\frac{m+1}{2}}\binom{\frac{m+1}{2}}{j}(t^3 + c_2t)^{(\frac{m+1}{2}-j)}c_3^j \right\rangle_{2m+1-d} & \textit{if } m \textit{ is odd.} \end{cases}$$

The PGL_2-equivariant Segre classes of V_{2m} can then be computed by formally inverting the total Chern classes of the PGL_2-representation.

4. Relations coming from $[\Delta_N^1/G]$

In this Section, we compute relations in the Chow ring of $\mathcal{W}_N^{\min}$ obtained excising $[\Delta_N^1/G]$. More precisely, we show that the ideal of relations obtained by excising this locus has a single generator (Lemma 4.1) and we give a recipe for computing it (see Proposition 4.2 and Proposition 4.3).

4.1. **Excision of $[\Delta^1_N/G]$.** Consider the localization exact sequence

$$\mathrm{CH}^G_*(\Delta^1_N) \to \mathrm{CH}^G_*(V'_{4N,6N}) \to \mathrm{CH}^*_G(V'_{4N,6N} \smallsetminus \Delta^1_N) \to 0.$$

We want to find generators for the ideal given by the image of the first map on the left. To do so, we construct an equivariant Chow envelope of Δ^1_N. Let

$$\phi : V'_{2N} \longrightarrow V'_{4N,6N} \tag{3}$$

be the map defined by $\phi(P) = (-3P^2, 2P^3)$. Observe that the image of ϕ lies in Δ^1_N.

Lemma 4.1. *The following hold true:*

1. *the map ϕ defines a G-equivariant bijective birational morphism $V'_{2N} \to \Delta^1_N$ that is an isomorphism away from the origin;*
2. *the pushforward morphism $\mathrm{CH}^*_G(V'_{2N}) \to \mathrm{CH}^*_G(\Delta^1_N)$ is surjective;*
3. *the image of ϕ_* is the ideal generated by $[\Delta^1_N]_G$.*

Proof. The map $(A, B) \mapsto -3B/2A$ defines an equivariant inverse to $V'_{2N} \to \Delta^1_N$ that is unramified away from the origin, so ϕ is bijective and is an isomorphism away from the origin. This also implies the surjectivity of the induced pushforward.

To prove the last point, observe that there is a well defined pullback morphism ϕ^* because $V'_{4N,6N}$ is smooth, and ϕ^* is clearly surjective because both $[V'_{2N}/G]$ and $[V'_{4N,6N}/G]$ are vector bundles over $\mathcal{B}G$. Therefore, for every cycle ζ in $\mathrm{CH}^*_G(V'_{2N})$, we have $\phi_*(\zeta) = \phi_*\phi^*(\zeta') = \phi_*(1)\cdot\zeta'$. This proves the last point. □

4.2. **The case $G = \mathrm{GL}_2$.** When N is odd, the group G is equal to GL_2. To compute $[\Delta^1_N]_{\mathrm{GL}_2}$ we can apply the localization formula ([EG98b, Theorem 2]). In general, this formula only gives an expression which is true up to cycles that are zero divisor. In our case we are lucky, as the equivariant Chow ring of $V'_{4N,6N}$ is a polynomial ring in the two variables c_1 and c_2, so the expression we obtain in the end hold true unconditionally.

Proposition 4.2. *For N odd, the image of $\mathrm{CH}^*_G(\Delta^1_N) \to \mathrm{CH}^*_G(V'_{4N,6N})$ is generated as an ideal by*

$$[\Delta^1_N]_G = \frac{c^G_{10N+2}(V'_{4N,6N})}{c^G_{2N+1}(V'_{2N})}.$$

Proof. Let $T \subset \mathrm{GL}_2$ be the maximal subtorus of diagonal matrices. The point in V'_{2dm} fixed by the T-action is the origin, whose tangent space is isomorphic to V'_{2dm} itself. Applying localization formula ([EG98b, Theorem 2]), we deduce that

$$\phi_*(1) = \frac{\phi_*([(0)]_T)}{c^T_{2N+1}(TV'_{2N,(0)})} = \frac{[(0)]_T}{c^T_{2N+1}(TV'_{2N,(0)})} = \frac{c^T_{10N+2}(V'_{4N,6N})}{c^T_{2N+1}(V'_{2N})}.$$

As the T-equivariant Chern classes of a GL_2-equivariant vector bundle are equal to the GL_2-equivariant ones, we obtain an expression for $[\Delta^1_N]_G$. By Lemma 4.1, this class generates the ideal $\mathrm{im}(\phi_*)$, and from the same Lemma we know that this ideal coincides with the image of $\mathrm{CH}^{*-8N-1}_G(\Delta^1_N) \to \mathrm{CH}^*_G(V'_{4N,6N})$. □

4.3. **The case $G = \mathrm{PGL}_2 \times \mathbb{G}_m$.** For N even, the group that we have to consider is $\mathrm{PGL}_2 \times \mathbb{G}_m$. Because of the fact that in $\mathrm{CH}^*(\mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m))$ there are zero divisors, e.g. the integer 2, we cannot apply the localization formula directly. To overcome this obstacle, we will use a trick introduced in [DL21].

Let $f : X' \to X$ be a $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant morphism between $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant schemes. Then by [DL21, Theorem 2.11] there exist $\mathrm{GL}_3 \times \mathbb{G}_m$-schemes Y and Y' with an equivariant morphism $Y' \to Y$ and a commutative diagram

$$\begin{array}{ccc} [X'/\,\mathrm{PGL}_2 \times \mathbb{G}_m] & \xrightarrow{\simeq} & [Y'/\,\mathrm{GL}_3 \times \mathbb{G}_m] \\ \downarrow & & \downarrow \\ [X/\,\mathrm{PGL}_2 \times \mathbb{G}_m] & \xrightarrow{\simeq} & [Y/\,\mathrm{GL}_3 \times \mathbb{G}_m]\,. \end{array} \tag{4}$$

We refer to the $\mathrm{GL}_3 \times \mathbb{G}_m$-scheme Y (resp. Y') as the $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of X (resp. X'), as in [DL21, Definition 1.10]. Let V'_{2dN} be the $\mathrm{PGL}_2 \times \mathbb{G}_m$-representation $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}(2dN)) \otimes L^{\otimes(-2d)}$, where L is the standard rank one representation of $\mathbb{G}_m$: our aim is to describe explicitly the $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of V'_{2dN}.

The affine space $\mathbb{A}^6$ is the parameter space of quadratic forms in three variables, and let D be the discriminant divisor, i.e. the divisor that parametrizes quadratic forms of rank ≤ 2. We regard $\mathbb{A}^6$ as a $\mathrm{GL}_3 \times \mathbb{G}_m$-scheme, where $\mathbb{G}_m$ acts trivially and GL_3 acts as $A \cdot q(x,y,z) = \det(A) q(A^{-1}(x,y,z))$. Observe that D is invariant with respect to this action.

Over $\mathbb{A}^6 \smallsetminus \{0\}$ we have an injective morphism of $\mathrm{GL}_3 \times \mathbb{G}_m$-equivariant free sheaves

$$\mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(dm-2)) \otimes L^{\otimes(-2d)} \otimes \mathcal{O}_{\mathbb{A}^5 \smallsetminus \{0\}} \longrightarrow \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(dm)) \otimes L^{\otimes(-2d)} \otimes \mathcal{O}_{\mathbb{A}^5 \smallsetminus \{0\}}, \tag{5}$$

where the GL_3-action on these sheaves is inherited from the natural action of GL_3 on $\mathbb{P}^2$, the latter regarded as the projectivization of the standard GL_3-representation.

The quotient of the map in (5) is denoted W'_{2dm} and by [DL21, Proposition 2.6] the restriction of W'_{2dm} to $\mathbb{A}^6 \smallsetminus D$ is the $\mathrm{GL}_3 \times \mathbb{G}_m$-counterpart of V'_{2dm}. Moreover, we adopt the notation $W'_{2dm,2em}$ for the direct sum of W'_{2dm} and W'_{2em}. The K-points in the total space of W'_{2dm} should be thought as pairs $(q, [f])$ where q is a non-zero ternary quadratic form on K, the polynomial f is a homogeneous form in three variables of degree dm and $[f] = [f']$ if and only if q divides the difference $f - f'$.

In this way we can also describe the counterpart of the equivariant map $\phi : V'_{2N} \to V'_{4N,6N}$ introduced in (3), which is the restriction to $\mathbb{A}^6 \smallsetminus D$ of the morphism

$$\psi : W'_{2N} \longrightarrow W'_{4N,6N}, \quad (q, [f]) \longmapsto (q, [-3f^2], [2f^3]).$$

In particular, this shows that the $\mathrm{GL}_3 \times \mathbb{G}_m$-equivariant fundamental class of $\psi(W'_{2N})$ is equal to the $\mathrm{PGL}_2 \times \mathbb{G}_m$-fundamental class of Δ^1_N.

As in the N odd case, we plan to use the localization formula to compute the image of $\mathrm{CH}^*_G(\Delta^1_N) \to \mathrm{CH}^*_G(V'_{4N,6N})$. To pass to integral coefficients however, it is convenient to work in an ambient space X such that $\mathrm{CH}^*([X/\,\mathrm{GL}_3 \times \mathbb{G}_m])$ is a free $\mathrm{CH}^*(\mathcal{B}\,\mathrm{GL}_3 \times \mathbb{G}_m)$-module. Therefore let $\mathbb{P}^5$ be the projectivization of $\mathbb{A}^6$. From [DL21, Definition 2.1] we know that there exists a locally free sheaf $\overline{W'}_{2dm}$ whose pullback along $\mathbb{A}^6 \smallsetminus \{0\} \to \mathbb{P}^5$ is equal to W'_{2dm}. Points in the total space of $\overline{W'}_{2dm}$ are pairs $([q], [f])$, where $[q] = [q']$ if and only if $q = \lambda q'$ for some invertible scalar λ. We also have an equivariant map $\overline{\psi} : \overline{W'}_{2N} \to \overline{W'}_{4N,6N}$, whose pullback along $\mathbb{A}^6 \smallsetminus \{0\} \to \mathbb{P}^5$ is equal to ψ.

Proposition 4.3. *For N even, the image of $\mathrm{CH}^*_G(\Delta^1_N) \to \mathrm{CH}^*_G(V'_{4N,6N})$ is generated as an ideal by*

$$\left. \frac{c^{\mathrm{GL}_3 \times \mathbb{G}_m}_{10N+2}(\overline{W'}_{4N,6N})}{c^{\mathrm{GL}_3 \times \mathbb{G}_m}_{2N+1}(\overline{W'}_{2N})} \right|_{h=0}.$$

This term is a polynomial in h and after evaluation at $h = 0$, it should be viewed as an element in $\mathbb{Z}[\tau_1, c_1, c_2, c_3]/(c_1, 2c_3)$, the G-equivariant Chow ring of $V'_{4N,6N}$.

Proof. The fact that $[\Delta^1_N]_G$ generates as an ideal the image of $\mathrm{CH}^*_G(\Delta^1_N) \to \mathrm{CH}^*_G(V'_{4N,6N})$ has already been proved in Lemma 4.1. Moreover, the previous discussion shows that $[\Delta^1_N]_G = [\psi(W'_{2N})]_{\mathrm{GL}_3 \times \mathbb{G}_m}$.

Let $T \subset \mathrm{GL}_3 \times \mathbb{G}_m$ be the maximal subtorus of pairs formed by diagonal matrices and an invertible scalar. The fixed points for the action of T on $\overline{W'}_{2dm}$ are of the form $([q],[0])$ where q is a monomial. Observe that the tangent space of $\overline{W'}_{2dm}$ at p is isomorphic to the direct sum $T\mathbb{P}^5_{[q]} \oplus \overline{W'}_{2dm,[q]}$. Moreover, the fundamental class of $([q],[0])$ in the equivariant Chow ring of $\overline{W'}_{2dm}$ is equal to the product $[[q]]_T \cdot c^T_{2dm+1}(\overline{W'}_{2dm,[q]})$. The localization formula ([EG98b, Theorem 2]) then gives us the equality

$$\begin{aligned}
\overline{\psi}_*(1) &= \overline{\psi}_*\left(\sum_{q=x_ix_j, i\leq j} \frac{[([q],[0])]_T}{c^T_{2N+6}(T\overline{W'}_{2N+1,[q]})}\right) \\
&= \sum_{q=x_ix_j, i\leq j} \frac{\overline{\psi}_*([([q],[0])]_T)}{c^T_5(T\mathbb{P}^5_{[q]})c_{2N+1}(\overline{W'}_{2dm,[q]})} \\
&= \sum_{q=x_ix_j, i\leq j} \frac{[[q]]_T \cdot c^T_{10N+2}(\overline{W'}_{4N,6N,[q]})}{c^T_5(T\mathbb{P}^5_{[q]})c_{2N+1}(\overline{W'}_{2dm,[q]})} = \frac{c^T_{10N+2}(\overline{W'}_{4N,6N})}{c^T_{2N+1}(\overline{W'}_{2N})}, \qquad (6)
\end{aligned}$$

where in the last equality we applied again the localization formula to obtain an expression in the equivariant Chow ring of $\overline{W'}_{4N,6N}$. This proves that the last term is not just a rational function but a polynomial, and that it coincides with $\overline{\psi}_*(1)$.

The T-equivariant top Chern classes of $\overline{W'}_{2dm}$ are equal to the $\mathrm{GL}_3 \times \mathbb{G}_m$-ones. Observe also that the last term in (6) can be regarded as a polynomial in h, the hyperplane class of $\mathbb{P}^5$, so that the element

$$\left.\frac{c^{\mathrm{GL}_3 \times \mathbb{G}_m}_{10N+2}(\overline{W'}_{4N,6N})}{c^{\mathrm{GL}_3 \times \mathbb{G}_m}_{2N+1}(\overline{W'}_{2N})}\right|_{h=c_1} \qquad (7)$$

is well defined, and it coincides with the pullback of $\overline{\psi}_*(1)$ along the $\mathbb{G}_m$-torsor $\mathbb{A}^6 \smallsetminus \{0\} \to \mathbb{P}^5$, which in turn is equal to $\psi_*(1)$. If we further restrict this cycle to $\mathbb{A}^6 \smallsetminus D$ (observe that this operation sends c_1 to zero), we get an explicit expression for the $\mathrm{GL}_3 \times \mathbb{G}_m$-fundamental class of $\psi(W'_{2N})$, which we already observed to be equal to the $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant fundamental class of Δ^1_N. □

We conclude this section explaining how to compute the Chern classes of $\overline{W'}_{2dm}$. For this, the basic ingredient is the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-1) \otimes \mathrm{Sym}^{m-2} E^\vee \otimes L^{\otimes(-m)} \to \mathrm{Sym}^m E^\vee \otimes L^{\otimes(-m)} \otimes \mathcal{O}_{\mathbb{P}^5} \to \overline{W'}_{2m} \to 0$$

of locally free sheaves on $\mathbb{P}^5$ (see [DL21, 2.3]), where E is the standard GL_3-representation and L is the standard $\mathbb{G}_m$-representation of weight one. This implies

$$\begin{aligned}
c^{\mathrm{GL}_3 \times \mathbb{G}_m}_{2m+1}(\overline{W'}_{2m}) &= \left\{\frac{c^{\mathrm{GL}_3 \times \mathbb{G}_m}(\mathrm{Sym}^m E^\vee \otimes L^{\otimes(-m)})}{c^{\mathrm{GL}_3 \times \mathbb{G}_m}(\mathrm{Sym}^{m-2} E^\vee \otimes L^{\otimes(-m)} \otimes \mathcal{O}(-1))}\right\}_{2m+1} \qquad (8) \\
&= \left\{\frac{\prod_{i+j\leq m}(1 + x(i\ell_1 + j\ell_2 + (m-i-j)\ell_3 - m\tau_1))}{\prod_{i'+j'\leq m-2}(1 + x(i'\ell_1 + j'\ell_2 + (m-2-i'-j')\ell_3 - m\tau_1 - h))}\right\}_{2m+1}
\end{aligned}$$

This expression in brackets should be interpreted as a formal series in x, from which we are extracting the coefficient in front of x^{2m+1}. Moreover, the symbols ℓ_1, ℓ_2 and ℓ_3 stands for the Chern roots of $E^\vee$, so that the elementary symmetric polynomial in ℓ_1, ℓ_2 and ℓ_3 of degree d is equal to $(-1)^d c_d$.

5. Relations coming from $[\Delta^2_N/G]$

In this Section we compute the relations in the Chow ring of $\mathcal{W}_N^{\min}$ coming from the excision of $[\Delta^2_N/G]$. We first define an equivariant stratification for Δ^2_N, which we leverage to compute the generators of the ideal of the relations. The final result is summarized in Proposition 5.4.

In the last part of the Section, we prove the first main result of the paper (Theorem 5.5).

5.1. An equivariant stratification of Δ^2_N.

First, we recall the definition of equivariant stratification in general.

Definition 5.1. Let X be a G-space. An equivariant stratification of X is a finite family $\{Z_\tau\}_{\tau\in J}$ of locally closed, pairwise disjoint, and equivariant subspaces of X such that $\bigcup_{\tau\in J} Z_\tau = X$ and

$$\overline{Z_\tau}\smallsetminus Z_\tau = \bigcup Z_{\tau'}.$$

Now, let G be either GL_2 or $\mathrm{PGL}_2\times\mathbb{G}_m$, so that $\mathbb{P}V_k$ is a G-scheme with the action defined at the beginning of Section 3. Let $\Sigma_k^{(m+1)}$ denote the $(m+1)$-thickening of the subscheme $\Sigma_k\subset \mathbb{P}V_k\times\mathbb{P}^1$ defined in Section 3.2, i.e. the subscheme defined by the ideal sheaf $\mathcal{I}_{\Sigma_k}^{m+1}\simeq \mathcal{O}_{\mathbb{P}V_k}(-m-1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(-(m+1)k)$. We then have a short exact sequence of G-equivariant sheaves

$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-m-1)\boxtimes\mathcal{O}_{\mathbb{P}^1}(-(m+1)k)\longrightarrow \mathcal{O}_{\mathbb{P}V_k\times\mathbb{P}^1}\longrightarrow i_*\mathcal{O}_{\Sigma_k^{(m+1)}}\longrightarrow 0.$$

We can twist the sequence above by $\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2d)$ and push everything down on $\mathbb{P}V_k$; if we further assume that $\mathrm{H}^1(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}(2d-(m+1)k)=0$, by cohomology and base chage we obtain the following short exact sequence of G-equivariant locally free sheaves on $\mathbb{P}V_k$:

$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-m-1)\otimes V_{2d-(m+1)k}\longrightarrow V_{2d}\otimes\mathcal{O}_{\mathbb{P}V_k}\longrightarrow \mathcal{P}_k^m(\mathcal{O}_{\mathbb{P}^1}(2d))\longrightarrow 0 \tag{9}$$

where we define $\mathcal{P}_k^m(\mathcal{O}_{\mathbb{P}^1}(2d))$ as the locally free sheaf $\mathrm{pr}_{1*}(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(2d)|_{\Sigma_k^{(m+1)}})$. This bundle coincides with the bundle of principal parts considered in [CK23].

In particular, if we specialize this short exact sequence to the cases $(d,m)=(2N,3)$, $(3N,5)$, we get short exact sequences

$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-4)\otimes V_{4(N-k)}\longrightarrow V_{4N}\otimes\mathcal{O}_{\mathbb{P}V_k}\longrightarrow \mathcal{P}_k^3(\mathcal{O}_{\mathbb{P}^1}(4N))\longrightarrow 0$$
$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-6)\otimes V_{6(N-k)}\longrightarrow V_{6N}\otimes\mathcal{O}_{\mathbb{P}V_k}\longrightarrow \mathcal{P}_k^5(\mathcal{O}_{\mathbb{P}^1}(6N))\longrightarrow 0$$

Define $\mathbb{L}^{\otimes 2}$ as follows:

- for N odd, it is defined as $\det(E)^{\otimes(N-1)}$, where E is the standard representation of GL_2;
- for N even, it is defined as $L^{\otimes(-2)}$, the rank one representation of $\mathbb{G}_m$ of weight -2.

There is an action of $\mathrm{GL}_2/\boldsymbol{\mu}_N$ on $\mathbb{P}V_k$: for N odd, we have $\mathrm{GL}_2/\boldsymbol{\mu}_N\simeq\mathrm{GL}_2$, and for N even we have $\mathrm{GL}_2/\boldsymbol{\mu}_N\simeq\mathrm{PGL}_2\times\mathbb{G}_m$; the action of these two groups on $\mathbb{P}V_k$ are the ones mentioned at the beginning of Section 3.

We have then short exact sequences of $\mathrm{GL}_2/\boldsymbol{\mu}_N$-equivariant locally free sheaves

$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-4)\otimes V_{4(N-k)}\otimes\mathbb{L}^{\otimes 4}\longrightarrow V_{4N}\otimes\mathbb{L}^{\otimes 4}\otimes\mathcal{O}_{\mathbb{P}V_k}\longrightarrow \mathcal{P}_k^3(\mathcal{O}_{\mathbb{P}^1}(4N))\otimes\mathbb{L}^{\otimes 4}\longrightarrow 0$$
$$0\longrightarrow \mathcal{O}_{\mathbb{P}V_k}(-6)\otimes V_{6(N-k)}\otimes\mathbb{L}^{\otimes 6}\longrightarrow V_{6N}\otimes\mathbb{L}^{\otimes 6}\otimes\mathcal{O}_{\mathbb{P}V_k}\longrightarrow \mathcal{P}_k^5(\mathcal{O}_{\mathbb{P}^1}(6N))\otimes\mathbb{L}^{\otimes 6}\longrightarrow 0$$

Let Z_k denote the total space of the locally free sheaf

$$\left(\mathcal{O}_{\mathbb{P}V_k}(-4)\otimes V_{4(N-k)}\otimes\mathbb{L}^{\otimes 4}\right)\oplus\left(\mathcal{O}_{\mathbb{P}V_k}(-6)\otimes V_{6(N-k)}\otimes\mathbb{L}^{\otimes 6}\right).$$

Then Z_k is a G-equivariant vector subbundle of $V'_{4N,6N} \times \mathbb{P}V_k$ and we have an equivariant morphism

$$p_k : Z_k \longrightarrow V'_{4N,6N}$$

whose image correspond to the invariant subscheme of pair of forms (A, B) such that there exists a form H of degree k with A vanishing with order ≥ 4 along $H = 0$ and B vanishing with order ≥ 6 along $H = 0$.

Moreover, this morphism is one-to-one on the locally closed subscheme of pairs (A, B) which satisfy the previous condition together with the further restraint that there exists no form H' of degree $k+1$ such that A (resp. B) vanish with order 4 (resp. 6) along $H' = 0$.

Lemma 5.2. *The image of the pushforward* $\mathrm{CH}^{*-9}([\Delta^2_N/G]) \longrightarrow \mathrm{CH}^*([V'_{4N,6N}/G])$ *is equal to the sum of the images of the equivariant pushforwards* p_{k*}*, for* $k = 1, \ldots, N$*.*

Proof. Set $\Delta^2_{N,k} := \mathrm{im}(p_k)$, so that we have an equivariant stratification of Δ^2_N given by

$$\Delta^2_N = \Delta^2_{N,1} \supset \Delta^2_{N,2} \supset \cdots \supset \Delta^2_{N,N-1} \supset \Delta^2_{N,N}.$$

Observe that the induced maps $Z_k \smallsetminus p_k^{-1}(\Delta^2_{N,k+1}) \longrightarrow (\Delta^2_{N,k} \smallsetminus \Delta^2_{N,k+1})$ are equivariant Chow envelopes of the strata. We can then apply [DLFV21, Lemma 3.3] and conclude the proof. □

We have reduced the problem of computing the relations coming from Δ^2_N to determining the images of several pushforwards. The generators of the Chow groups of $[Z_k/G]$ are easier to compute, and so are their pushforwards. Indeed, consider the diagram

$$\begin{array}{ccc} [V'_{4N,6N} \times \mathbb{P}V_k/G] & \xrightarrow{\mathrm{pr}_1} & [V'_{4N,6N}/G] \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \\ [\mathbb{P}V_k/G]\,. & & \end{array}$$

Then we have the following.

Lemma 5.3. *The image of* p_{k*} *is generated as an ideal by all the cycles of the form* $\mathrm{pr}_{1*}([Z_k]_G \cdot \mathrm{pr}_2^*\eta)$*, where* η *ranges among all the generator of* $\mathrm{CH}^*([\mathbb{P}V_k/G])$ *as* $\mathrm{CH}^*(\mathcal{B}G)$*-module.*

Proof. Write p_k as the composition of the closed embedding $i : Z_k \hookrightarrow V'_{4N,6N} \times \mathbb{P}V_k$ followed by the projection $\mathrm{pr}_1 : V'_{4N,6N} \times \mathbb{P}V_k \to V'_{4N,6N}$. Observe that the Chow ring of $[Z_k/G]$ is generated as a module over $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2)$ by the pullback of generators of $\mathrm{CH}^*(\mathbb{P}V_k)$, i.e. by elements of the form $i^*\mathrm{pr}_2^*\eta$. We deduce that the image of p_{k*} is generated as an ideal by

$$p_{k*}(i^*\mathrm{pr}_2^*\eta) = \mathrm{pr}_{1*}i_*(i^*\mathrm{pr}_2^*\eta) = \mathrm{pr}_{1*}([Z_k]_G \cdot \mathrm{pr}_2^*\eta),$$

as claimed. □

5.2. Computation of the fundamental class of Z_k. .

The subvariety $Z_k \subset V'_{4N,6N} \times \mathbb{P}V_k$ has codimension $10k$ and its equivariant fundamental class is equal to the equivariant top Chern class of the vector bundle

$$\left(\mathcal{P}^3_k(\mathcal{O}_{\mathbb{P}^1}(4N)) \otimes \mathbb{L}^{\otimes 4}\right) \oplus \left(\mathcal{P}^5_k(\mathcal{O}_{\mathbb{P}^1}(6N)) \otimes \mathbb{L}^{\otimes 6}\right),$$

which is equal to the product of the top Chern classes of the two factors. We write

$$c^G_{2dk}(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN)) \otimes \mathbb{L}^{\otimes 2d}) = \sum_{i=0}^{2dk} c^G_i(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN)))(2d\zeta_1)^{2dk-i},$$

where we set $\zeta_1 = -\tau_1$ for $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and $\zeta_1 = (N-1)c_1/2$ for $G = \mathrm{GL}_2$. In this way, we have reduced our computation of the fundamental class of Z_k to determining the G-equivariant Chern classes of $\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN))$. For this we use (9), which tells us that

$$c_i^G(\mathcal{P}^{2d-1}_k(\mathcal{O}_{\mathbb{P}^1}(2dN))) = \sum_{j=0}^{i} c_j^G(V_{2dN}) s^G_{i-j}(V_{2d(N-k)} \otimes \mathcal{O}_{\mathbb{P}^k}(-2d)).$$

Define ξ_1 as $h = c_1^G(\mathcal{O}_{\mathbb{P}V_k}(1))$ for $G = \mathrm{GL}_2$ or $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and k even, and as $\gamma_1/2$ for $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and k odd, where $\gamma_1 = c_1^{\mathrm{PGL}_2}(\mathcal{O}_{\mathbb{P}V_k}(2))$. Applying the formula for the Segre classes of tensor products, we obtain

$$\begin{aligned} s^G_{i-j}(V_{2d(N-k)} \otimes \mathcal{O}_{\mathbb{P}^k}(-2d)) &= \sum_{\ell=0}^{i-j} (-1)^{i-j-l} \binom{2d(N-k)+i-j}{2d(N-k)+\ell} s^G_\ell(V_{2d(N-k)})(-2d\xi_1)^{i-j-\ell} \\ &= \sum_{\ell=0}^{i-j} \binom{2d(N-k)+i-j}{2d(N-k)+\ell} s^G_\ell(V_{2d(N-k)})(2d\xi_1)^{i-j-\ell} \end{aligned}$$

Putting everything together, we get the following expression for the G-equivariant fundamental class of Z_k:

$$\prod_{d=2}^{3} \sum \binom{2d(N-k)+i_d-j_d}{2d(N-k)+\ell_d} (2d)^{2dk-j_d-\ell_d} c_j^G(V_{2dN}) s^G_{\ell_d}(V_{2d(N-k)}) \xi_1^{i_d-j_d-\ell_d} \zeta_1^{2dk-i} \tag{10}$$

where the sum index runs over all the triples (i_d, j_d, ℓ_d) such that $j_d + \ell_d \le i_d \le 2dk$, for $d = 2, 3$.

5.3. **Relations from Δ^2_N.** We are going to compute generators as an ideal for the image of

$$\mathrm{CH}^{*-9}([\Delta^2_N/G]) \longrightarrow \mathrm{CH}^*([V'_{4N,6N}/G]). \tag{11}$$

Consider again the diagram

$$\begin{array}{ccc} [V'_{4N,6N} \times \mathbb{P}V_k/G] & \xrightarrow{\mathrm{pr}_1} & [V'_{4N,6N}/G] \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow \\ [\mathbb{P}V_k/G] & \xrightarrow{\quad\pi\quad} & \mathcal{B}G. \end{array} \tag{12}$$

From Lemma 5.3 we know that the image of (11) is generated by the cycles $\mathrm{pr}_{1*}([Z_k]_G \cdot \mathrm{pr}_2^*\eta)$, where η ranges among all the generators of $\mathrm{CH}^*([\mathbb{P}V_k/G])$ as $\mathrm{CH}^*(\mathcal{B}G)$-module.

Let us rewrite the formula for $[Z_k]_G$ contained in (10) as

$$\sum C_k(i,j,\ell)\xi_1^{i-j-\ell}$$

where $j + \ell \le i \le 10k$, and the coefficients are

$$C_k(i,j,\ell) = \zeta_1^{10k-i} \cdot \left(\sum \prod_{d=2}^{3} \binom{2d(N-k)+i_d-j_d}{2d(N-k)+\ell_d} (2d)^{2dk-j_d-\ell_d} c^G_{j_d}(V_{2dN}) s^G_{\ell_d}(V_{2d(N-k)}) \right).$$

The sum above is taken over all the triples (i_d, j_d, ℓ_d), $d = 2, 3$, such that

$$j_d + \ell_d \le i_d \le 2dk, \quad \sum_{d=2}^{3} (i_d, j_d, \ell_d) = (i, j, \ell).$$

Pullbacks along the vertical arrows of the diagram (12) induce isomorphism of Chow rings. Thus, after identifying the Chow rings on the top of the diagram with the respective ones on the bottom, we have

$$\mathrm{pr}_{1*}([Z_k]_G \cdot \mathrm{pr}_2^* \eta) = \sum C_k(i,j,\ell)\pi_*(\xi_1^{i-j-\ell} \cdot \eta) \tag{13}$$

Note that, in the equality above, we are allowed to apply the projection formula because the coefficients $C_k(i,j,\ell)$ are cycles pulled back from $\mathcal{B}G$.

For $G = \mathrm{GL}_2$ we know from Proposition 3.1 that the Chow ring of $[\mathbb{P}V_k/\mathrm{GL}_2]$ is generated by powers of the hyperplane class h and we have $\zeta_1 = (N-1)c_1/2$ and $\xi_1 = h$. Therefore, applying Proposition 3.1 we get the following expression explicit expression for (13) when $\eta = h^m$:

$$\begin{aligned} f_{k,m} := \sum &((N-1)c_1/2)^{10k-i} s^{\mathrm{GL}_2}_{i-j-\ell-(k-m)}(V_k) \\ &\cdot \left(\sum \prod_{d=2}^{3} \binom{2d(N-k)+i_d-j_d}{2d(N-k)+\ell_d} (2d)^{2dk-j_d-\ell_d} c^{\mathrm{GL}_2}_{j_d}(V_{2dN}) s^{\mathrm{GL}_2}_{\ell_d}(V_{2d(N-k)}) \right), \end{aligned} \tag{14}$$

where $(i,j,\ell) := (i_2,j_2,\ell_2) + (i_3,j_3,\ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d,j_d,\ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2dk$.

For $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and k even, we have a similar picture: the only difference is that $\zeta_1 = -\tau_1$, hence an explicit expression for (13) when $\eta = h^{m'}$ is given by

$$\begin{aligned} g_{k,m'} := \sum &(-\tau_1)^{10k-i} s^{\mathrm{PGL}_2}_{i-j-\ell-(k-m')}(V_k) \\ &\cdot \left(\sum \prod_{d=2}^{3} \binom{2d(N-k)+i_d-j_d}{2d(N-k)+\ell_d} (2d)^{2dk-j_d-\ell_d} c^{\mathrm{PGL}_2}_{j_d}(V_{2dN}) s^{\mathrm{PGL}_2}_{\ell_d}(V_{2d(N-k)}) \right). \end{aligned} \tag{15}$$

where again $(i,j,\ell) := (i_2,j_2,\ell_2) + (i_3,j_3,\ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d,j_d,\ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2dk$.

Finally, for $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and k odd, we know from Proposition 3.2 that the Chow ring of the stack $[\mathbb{P}V_k/\mathrm{PGL}_2 \times \mathbb{G}_m]$ is generated as a module over $\mathrm{CH}^*(\mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m))$ by monomials of the form $\gamma_1^m \gamma_2^n$, where $m \in \{0,1\}$ and $n \leq \frac{k-1}{2}$. Moreover, for k odd we have $\xi_1 = \frac{\gamma_1}{2}$, hence

$$\mathrm{pr}_{1*}([Z_k]_G \cdot \gamma_1^m \gamma_2^n) = \sum 2^{-(i-j-\ell)} C_k(i,j,\ell)\pi_*(\gamma_1^{m+i-j-\ell}\gamma_2^n)$$

Write $m' = 2n + m$, where m is either 0 or 1. Applying Lemma 3.4, we get the following explicit expression for the pushforwards:

$$\begin{aligned} g_{k,m'} := \sum &k^{-1} 2^{-(i-j-\ell)} (-\tau_1^{10k-i}) \\ &\cdot \left(\sum \prod_{d=2}^{3} \binom{2d(N-k)+i_d-j_d}{2d(N-k)+\ell_d} (2d)^{2dk-j_d-\ell_d} c^{\mathrm{PGL}_2}_{j_d}(V_{2dN}) s^{\mathrm{PGL}_2}_{\ell_d}(V_{2d(N-k)}) \right) \\ &\cdot \left(\sum_{q \leq n + \frac{m+i-j-\ell-k}{2}} E_{n,m+i-j-\ell}(q) \cdot s^{\mathrm{PGL}_2}_{2(n-q)+m+i-j-\ell-k}(V_{k-1}) 2c_2^q \right) \end{aligned} \tag{16}$$

where, as before, we set $(i,j,\ell) := (i_2,j_2,\ell_2) + (i_3,j_3,\ell_3)$ and the sum is taken over all the pairs of triples of positive numbers $\{(i_d,j_d,\ell_d)\}_{d=2,3}$ such that $j_d + \ell_d \leq i_d \leq 2dk$. The quantity $E_{n,m}(q)$ is the one defined just before Lemma 3.4.

Putting all together, we deduce the following.

Proposition 5.4. *The image of the pushforward* $\mathrm{CH}^{*-9}([\Delta_N^2/G]) \to \mathrm{CH}^*([V'_{4N,6N}/G])$ *is generated by:*

(1) when N is odd, by the cycles $f_{k,m}$ described in (14) for $1 \leq k \leq N$ and $0 \leq m \leq k$;
(2) when N is even, by the cycles $g_{k,m'}$ described in (15) and (16) for $1 \leq k \leq N$ and $0 \leq m' \leq k$.

5.4. **Proof of the main result.** We have all the ingredients necessary to prove our main result. Indeed, we know from Proposition 2.3 that the stack $\mathcal{W}_N^{\min}$ is isomorphic to $[V'_{4N,6N} \smallsetminus (\Delta_N^1 \cup \Delta_N^2)/G]$, hence we have a localization exact sequence

$$\mathrm{CH}([(\Delta_N^1 \cup \Delta_N^2)/G]) \longrightarrow \mathrm{CH}([V'_{4N,6N}/G]) \longrightarrow \mathrm{CH}(\mathcal{W}_N^{\min}) \longrightarrow 0.$$

The image of the map on the left is equal to the sum of the images of the maps $\mathrm{CH}_*([\Delta_N^i/G]) \to \mathrm{CH}_*([V'_{4N,6N}/G])$ for $i = 1, 2$, which have been computed in Lemma 4.1 and Proposition 5.4.

The integral Chow ring of $[V'_{4N,6N}/G]$ is isomorphic to the one of $\mathcal{B}G$, where the isomorphism is induced by the pullback morphism along the map $[V'_{4N,6N}/G] \to \mathcal{B}G$. When N is odd, we have $G = \mathrm{GL}_2$ and $\mathrm{CH}^*(\mathcal{B}\,\mathrm{GL}_2) \simeq \mathbb{Z}[c_1, c_2]$, with c_1 and c_2 the Chern classes of the universal rank two vector bundle.

Therefore, the generators c_1 and c_2 of $\mathrm{CH}^*(\mathcal{W}_N^{\min})$ are by construction the Chern classes of the pullback of the universal rank two vector bundle on $\mathcal{B}\,\mathrm{GL}_2$. The map $\mathcal{W}_N^{\min} \to \mathcal{B}\,\mathrm{GL}_2$ is induced by the rank two vector bundle $\mathcal{E}_N$ of Definition 2.12 (see Proposition 2.11), hence the pullback of the universal vector bundle is equal to $\mathcal{E}_N$.

Similarly, for N even we have $G = \mathrm{PGL}_2 \times \mathbb{G}_m$ and the integral Chow ring of the associated classifying stack is isomorphic to $\mathbb{Z}[\tau_1, c_2, c_3]/(2c_3)$.

The generator τ_1 is the first Chern class of the pullback of the universal line bundle on $\mathcal{B}\mathbb{G}_m$, which by Proposition 2.13 is equal to $\mathcal{L}_N$. The other two generators c_2 and c_3 are by definition the pullback of the generators of $\mathrm{CH}^*(\mathcal{B}\,\mathrm{PGL}_2)$, which are the Chern classes of the rank three vector bundle $(\mathcal{P} \xrightarrow{p} B) \longmapsto p_*(\omega_{\mathcal{P}/B}^\vee)$. The pullback of the latter is by definition the rank three vector bundle $\mathcal{E}_N$ of Definition 2.12.

Putting all together, we obtain our first main result.

Theorem 5.5. *Suppose that the ground field has characteristic $\neq 2, 3$. Then*

(1) for N odd we have

$$\mathrm{CH}^*(\mathcal{W}_N^{\min}) \simeq \mathbb{Z}[c_1, c_2]/I_N$$

where the ideal of relations I_N is generated by the polynomials $f_{k,m}$ described in (14) for $1 \leq k \leq N$ and $0 \leq m \leq k$, together with the fundamental class $[\Delta_N^1]_{\mathrm{GL}_2}$. The degree of $f_{k,m}$ is $9k + m$ and the degree of $[\Delta_N^1]_{\mathrm{GL}_2}$ is $8N + 1$. The generators c_1 and c_2 are the Chern classes of the rank two vector bundle $\mathcal{E}_N$ introduced in Definition 2.10.

(2) for N even, we have

$$\mathrm{CH}^*(\mathcal{W}_N^{\min}) \simeq \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, I_N)$$

where the ideal of relations I_N is generated by the polynomials $g_{k,m'}$ described in (15) and (16) for $1 \leq k \leq N$ and $0 \leq m' \leq k$, together with the fundamental class $[\Delta_N^1]_{\mathrm{PGL}_2 \times \mathbb{G}_m}$. The degree of $g_{k,m'}$ is $9k + m'$ and the degree of $[\Delta_N^1]_{\mathrm{PGL}_2 \times \mathbb{G}_m}$ is $8N + 1$. The generator τ_1 is the first Chern class of the line bundle $\mathcal{L}_N$ introduced in Definition 2.12, and the generators c_2 and c_3 are the Chern classes of the rank three vector bundle $\mathcal{E}_N$ introduced in Definition 2.12.

Note the relations appearing in the Theorem above can be made fully explicit: one can apply Proposition 4.2 and Proposition 4.3 for computing the fundamental class of Δ_N^1, and Proposition 3.5 and Proposition 3.6 to obtain explicit expressions for the Chern and Segre classes of the representations appearing in $f_{k,m}$ and $g_{k,m'}$. Plugging these formulas into the relations, one get the desired description. This is exactly what we will do in the next Section for $N = 1, 2$.

6. Integral Chow rings of stacks of rational elliptic surfaces and elliptic K3 surfaces

In this Section we compute the integral Chow ring of $\mathcal{W}_1^{\min}$, the moduli stack of rational elliptic surfaces, and of $\mathcal{W}_2^{\min}$, the moduli stack of elliptic K3 surfaces. The two main results are Theorem 6.1 and Theorem 6.2.

6.1. **The case $N = 1$.** A Weierstrass fibration $X \to \mathbb{P}^1$ with fundamental invariant $N = 1$ is a rational surface, obtained by blowing up $\mathbb{P}^2$ along the base locus of a pencil of cubics. Equivalently, we can think of X as the blow-up of a Del Pezzo surface of degree 1 along the anticanonical divisor.

The stack $\mathcal{W}_1^{\min}$ is not Deligne-Mumford because of the presence of objects with infinite dimensional automorphism group [PS21, Remark 4.5].

Theorem 6.1. *Suppose that the ground field k has characteristic $\neq 2, 3$ and set $r_6 = 576(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3)$. Then we have*

$$\mathrm{CH}^*(\mathcal{W}_1^{\min}) \simeq \mathbb{Z}[c_1, c_2]/(6c_1c_2r_6, c_1^3r_6, c_1^2c_2r_6),$$

where c_1 and c_2 are the Chern classes of the rank two vector bundle $\mathcal{E}_1$ introduced in Definition 2.10.

Proof. This is a straightforward application of Theorem 5.5. To compute explicitly the Chern classes of the representations involved, one can use Proposition 3.5. The Segre classes are then obtained by formally inverting the total Chern classes. Then one can plug in these expressions into the formulas given in (14) and into the formula given in Proposition 4.2. After performing these computations with Mathematica, we obtain:

$$[\Delta_1^1]_{\mathrm{GL}_2} = -3456c_1c_2(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3);$$
$$f_{1,0} = -576c_1^3(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3);$$
$$f_{1,1} = -576c_1^2c_2(30c_1^6 + 151c_1^4c_2 + 196c_1^2c_2^2 + 64c_2^3).$$

This concludes the proof. □

6.2. **The case $N = 2$.** The stack $\mathcal{W}_2^{\min}$ can be regarded as the stack of lattice-polarized elliptic K3 surfaces, as explained in the Introduction of [CK23]. The coarse space of this moduli stack is particularly interesting and it has been the subject of much work (see for instance [MOP17, PY20]). Here we determine its integral Chow ring.

Theorem 6.2. *Suppose that the ground field has characteristic $\neq 2, 3$. Then we have*

$$\mathrm{CH}^*(\mathcal{W}_2^{\min}) \simeq \mathbb{Z}[\tau_1, c_2, c_3]/(2c_3, r_9, r_{10}, r_{18}, r_{19})$$

where

$$r_9 = 1152(691c_2^4\tau_1 - 38005c_2^3\tau_1^3 + 309568c_2^2\tau_1^5 - 497520c_2\tau_1^7 + 124416\tau_1^9),$$
$$r_{10} = 1152(30c_2^5 - 6811c_2^4\tau_1^2 + 133495c_2^3\tau_1^4 - 481528c_2^2\tau_1^6 + 327600c_2\tau_1^8 - 20736\tau_1^{10}),$$
$$r_{18} = 1152c_2^5(108314154642930c_2^4 + 1045672c_2^3\tau_1^2 - 89483c_2^2\tau_1^4 + 35c_2\tau_1^6 - 4\tau_1^8),$$
$$r_{19} = 2304c_2^6\tau_1(118203201c_2^3 + 180502c_2^2\tau_1^2 - 7c_2\tau_1^4 + 4\tau_1^6).$$

The generator τ_1 is the first Chern class of the line bundle $\mathcal{L}_2$ (see Definition 2.12), the other generators c_2 and c_3 are Chern classes of the rank three vector bundle $\mathcal{E}_2$ (see Definition 2.12), whose first Chern class vanishes.

We will prove this Theorem by applying Theorem 5.5 and by explicitly computing the relations in terms of the generators τ_1, c_2 and c_3.

Lemma 6.3.

$$[\Delta_2^1]_{\mathrm{PGL}_2\times\mathbb{G}_m} = -995328\tau_1(9c_2^2+160c_2\tau_1^2+256\tau_1^4)(100c_2^6+5369c_2^5\tau_1^2 + 74074c_2^4\tau_1^4+400257c_2^3\tau_1^6+972972c_2^2\tau_1^8+1061424c_2\tau_1^{10}+419904\tau_1^{12})$$

Proof. Instead of applying directly the formula of Proposition 4.3, we first compute $[\Delta_2^1]_G$ modulo c_3, and then we conclude the computation modulo 2. This trick is inspired by [FV11].

The homomorphism of algebraic groups $\mathrm{SL}_2\times\mathbb{G}_m \to \mathrm{PGL}_2\times\mathbb{G}_m$ induces a morphism of stacks

$$\mathcal{B}(\mathrm{SL}_2\times\mathbb{G}_m)\to\mathcal{B}(\mathrm{PGL}_2\times\mathbb{G}_m).$$

By taking the pullback along this map we get a homomorphism of rings

$$\mathrm{CH}^*(\mathcal{B}(\mathrm{PGL}_2\times\mathbb{G}_m))\simeq\mathbb{Z}[c_2,c_3,\tau_1]/(2c_3)\to\mathrm{CH}^*(\mathcal{B}(\mathrm{SL}_2\times\mathbb{G}_m))\simeq\mathbb{Z}[c_2,\tau_1],$$

that sends τ_1 to τ_1, the class c_2 to $4c_2$ and c_3 is sent to zero. The pullback of $[\Delta_2^1]_G$ along this map is equal to $[\Delta_2^1]_{\mathrm{SL}_2\times\mathbb{G}_m}$, hence if we compute this last class and we substitute c_2 with $c_2/4$ we get an expression of $[\Delta_2^1]_G$ that holds up to multiples of c_3.

The same argument of Proposition 4.2 shows that

$$[\Delta_2^1]_{\mathrm{SL}_2\times\mathbb{G}_m} = \frac{c_{22}^{\mathrm{SL}_2\times\mathbb{G}_m}(V'_{8,12})}{c_5^{\mathrm{SL}_2\times\mathbb{G}_m}(V'_4)} = \frac{c_9^{\mathrm{SL}_2\times\mathbb{G}_m}(V'_8)c_{13}^{\mathrm{SL}_2\times\mathbb{G}_m}(V'_{12})}{c_5^{\mathrm{SL}_2\times\mathbb{G}_m}(V'_4)}.$$

The representation V'_{2m} is equal to $\mathrm{Sym}^{2m}E^\vee\otimes L^{\otimes(-m)}$, where E is the standard SL_2-representation and L is the standard $\mathbb{G}_m$-representation (of weight one). If ℓ_1 and ℓ_2 denote the Chern roots of $E^\vee$ and τ_1 is the first Chern class of L, we see that the Chern roots of V'_{2m} are of the form $i\ell_1+(2m-i)\ell_2-m\tau_1$, for $i=0,\dots,2m$. As the product of the Chern roots is equal to the top Chern class, after some computations and after plugging in the relations $\ell_1+\ell_2=0$ and $\ell_1\ell_2=c_2$, we get

$$[\Delta_2^1]_{\mathrm{SL}_2\times\mathbb{G}_m} = -1019215872(9c_2^2+40c_2\tau_1^2+16\tau_1^4)(6400c_2^6\tau_1+85904c_2^5\tau_1^3 + 296296c_2^4\tau_1^5+400257c_2^3\tau_1^7+243243c_2^2\tau_1^9+66339c_2\tau_1^{11}+6561\tau_1^{13}) \tag{17}$$

We replace c_2 with $4c_2$, thus obtaining

$$-995328\tau_1(9c_2^2+160c_2\tau_1^2+256\tau_1^4)(100c_2^6+5369c_2^5\tau_1^2 + 74074c_2^4\tau_1^4+400257c_2^3\tau_1^6+972972c_2^2\tau_1^8+1061424c_2\tau_1^{10}+419904\tau_1^{12}) \tag{18}$$

We have $[\Delta_2]_G^1 = (18)+c_3\eta$, where η belongs to $\mathbb{Z}[c_2,c_3]/(2)$. In particular, the class of Δ_2^1 modulo 2 is equal to $c_3\eta$. For computing the class of Δ_2^1 modulo 2, we first find an element ξ such that

$$\xi\cdot c_5^{\mathrm{GL}_3\times\mathbb{G}_m}(\overline{W}'_4) = c_{22}^{\mathrm{GL}_3\times\mathbb{G}_m}(\overline{W}'_{8,12}). \tag{19}$$

This task is accomplished by direct computations of the top Chern classes using (8), and then reduction modulo 2: we find a polynomial ξ' such that $\xi=h\xi'$ satisfies the condition (19). We are still not done, because the ring $\mathrm{CH}^*([\mathbb{P}^5/\mathrm{GL}_3\times\mathbb{G}_m])\otimes\mathbb{Z}/2$ is isomorphic to

$$\mathbb{Z}[c_2,c_3,h]/(2,h^3(c_1c_2+c_3+c_1^2h+c_2h+h^3))$$

hence the reduction modulo 2 of

$$\frac{c_{22}^{\mathrm{GL}_3\times\mathbb{G}_m}(\overline{W'}_{8,12})}{c_5^{\mathrm{GL}_3\times\mathbb{G}_m}(\overline{W'}_4)} \tag{20}$$

is equal to ξ only up to annihilators of $c_5^{\mathrm{GL}_3\times\mathbb{G}_m}(\overline{W'}_4)$. This top Chern class is equal modulo 2 to $h^2(c_1c_2+c_3+c_1^2h+c_2h+h^3)$, so if ξ'' is an annihilator of this element, it must be a multiple of h (this can also be checked directly using the tautological exact sequence on $\mathbb{P}^5$). This shows that the reduction modulo

2 of (20) is divisible by h. As the reduction modulo 2 of $[\Delta_2^1]_G$ is equal to (20) evaluated at $h = 0$ (see Proposition 4.3), we deduce that this reduction is zero, hence $[\Delta_2^1]_G$ is equal to the expression in (18). $\square$

According to Theorem 5.5, we need to compute five other relations. The first two are obtained as follows: let $Z_1 \subset V'_{8,12} \times \mathbb{P}V_1$ be the subscheme of triples (A, B, p) where p is a point of $\mathbb{P}^1$ and the form A (resp. the form B) vanishes in p with order ≥ 4 (resp. ≥ 6). Let γ_1 be the generator of the $\mathrm{PGL}_2 \times \mathbb{G}_m$-equivariant Chow ring of $\mathbb{P}V_1$ as a module over $\mathrm{CH}^*(\mathcal{B}(\mathrm{PGL}_2 \times \mathbb{G}_m))$. Then the first two relations are given by

$$g_{1,0} := \mathrm{pr}_{1*}[Z_1]_{\mathrm{PGL}_2 \times \mathbb{G}_m}, \quad g_{1,1} := \mathrm{pr}_{1*}([Z_1]_{\mathrm{PGL}_2 \times \mathbb{G}_m} \cdot \mathrm{pr}_2^*\gamma_1),$$

where pr_1 (resp. pr_2) is the projection on the first (resp. second) factor.

Formulas for these two relations are given by (16) with $N = 2$, $k = 1$, $m \in \{0, 1\}$ and $n = 0$. To make these expressions completely explicit we have to plug in the formulas for Chern classes and Segre classes of V_4, V_6, V_8 and V_{12}, which can be extracted from Proposition 3.6. After some computations with Mathematica, we get

$$g_{1,0} = -1152(691c_2^4\tau_1 - 38005c_2^3\tau_1^3 + 309568c_2^2\tau_1^5 - 497520c_2\tau_1^7 + 124416\tau_1^9) \tag{21}$$

$$g_{1,1} = -1152(30c_2^5 - 6811c_2^4\tau_1^2 + 133495c_2^3\tau_1^4 - 481528c_2^2\tau_1^6 + 327600c_2\tau_1^8 - 20736\tau_1^{10}) \tag{22}$$

Let us recall how the other three relations are obtained: let $Z_2 \subset V'_{8,12} \times \mathbb{P}V_2$ be the subscheme of triples $(p_1 + p_2, A, B)$ such that $p_1 + p_2$ is a dimension zero subscheme of $\mathbb{P}^1$ of length two and A (resp. B) vanishes along $p_1 + p_2$ with order ≥ 4 (resp. 6). If h denotes the hyperplane section of $\mathbb{P}V_2$ and pr_i the projection on the i^{th}-factor, then the cycles

$$g_{2,0} := \mathrm{pr}_{1*}[Z_2]_{\mathrm{PGL}_2 \times \mathbb{G}_m}, \quad g_{2,1} := \mathrm{pr}_{1*}([Z_2]_{\mathrm{PGL}_2 \times \mathbb{G}_m} \cdot \mathrm{pr}_2^*h), \quad g_{2,2} := \mathrm{pr}_{1*}([Z_2]_{\mathrm{PGL}_2 \times \mathbb{G}_m} \cdot \mathrm{pr}_2^*h^2)$$

are the three relations we are looking for.

Formulas for these relations are given in Proposition 5.4: they correspond to the cases $N = 2$, $k = 2$ and $0 \leq m \leq 2$. Observe that in this case the representation $V_{2d(N-k)}$ is trivial, hence the only non-zero Segre class is the one of degree zero, which is equal to one. This means that in the summation we can impose $\ell_d = 0$ for $d = 2, 3$.

To make the formulas completely explicit, we only need to plug in the values of the Chern classes of V_8 and V_{12} and of the Segre classes of V_2, which are computed as before using Proposition 3.6. After some computations with Mathematica, we get

$$\begin{aligned}
g_{2,0} = &- 11943936(38562300c_2^9 - 109363770c_2^8\tau_1^2 + 134699250c_2^7\tau_1^4 - 303690446c_2^6\tau_1^6 + 312766535c_2^5\tau_1^8 \\
&- 259047756c_2^4\tau_1^10 + 192326864c_2^3\tau_1^{12} - 128471616c_2^2\tau_1^{14} + 87091200c_2\tau_1^{16} - 11943936\tau_1^{18}), \\
g_{2,1} = &23887872c_2\tau_1(37514745c_2^8 - 64489645c_2^7\tau_1^2 + 97095345c_2^6\tau_1^4 - 170891502c_2^5\tau_1^6 + 142583080c_2^4\tau_1^8 \\
&- 114176800c_2^3\tau_1^{10} + 78779520c_2^2\tau_1^{12} - 54743040c_2\tau_1^{14} + 23887872\tau_1^{16}), \\
g_{2,2} = &- c_2 \cdot g_{2,0}.
\end{aligned}$$

These five relations, together with the fundamental class $[\Delta_2^1]_{\mathrm{PGL}_2 \times \mathbb{G}_m}$ computed in Lemma 6.3, are all we need to compute the integral Chow ring of $\mathcal{W}_2^{\min}$.

A quick computation with Mathematica shows that $[\Delta_2^1]_{\mathrm{PGL}_2 \times \mathbb{G}_m}$ belongs to the ideal generated by (21) and (22). After further simplifying it via Mathematica, we obtain the presentation given in Theorem 6.2.

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