

# GLOBAL DYNAMICS OF THE $2d$ NLS WITH WHITE NOISE POTENTIAL AND GENERIC POLYNOMIAL NONLINEARITY

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ABSTRACT. Using an approach introduced by Hairer-Labbé we construct a unique global dynamics for the NLS on  $\mathbb{T}^2$  with a white noise potential and an arbitrary polynomial nonlinearity. We build the solutions as a limit of classical solutions (up to a phase shift) of the same equation with smoothed potentials. This is an improvement on previous contributions of us and Debussche-Weber dealing with quartic nonlinearities and cubic nonlinearities respectively. The main new ingredient are space-time estimates for the approximate nonlinear solutions exploiting the time averaging effect for dispersive equations (the previous works were based only on fixed time spatial estimates).

## 1. INTRODUCTION

The aim of this work is to extend the result of [?] to an arbitrary polynomial nonlinearity. As announced in [?] this will require, in addition to the modified energies introduced in [?], a suitable use of the dispersive effect.

We therefore aim to solve, in a sense to be defined, the following Cauchy problem

$$(1.1) \quad i\partial_t u = \Delta u + \xi u - u|u|^p, \quad u(0, x) = u_0(x), \quad (t, x) \in \mathbb{R} \times \mathbb{T}^2,$$

where  $p \geq 2$  measures the strength of the nonlinear interaction and  $\xi(x, \omega)$  is the (zero mean value) space white noise which can be seen as the distribution of the random Fourier series

$$\xi(x, \omega) = \sum_{n \in \mathbb{Z}^2, n \neq 0} g_n(\omega) e^{in \cdot x},$$

where  $g_n(\omega)$  are standard complex gaussians such that  $\overline{g_n(\omega)} = g_{-n}(\omega)$  and otherwise independent. The assumption  $\overline{g_n(\omega)} = g_{-n}(\omega)$  implies that  $\xi$  is real valued, a fact which plays an important role in our analysis.

Along this paper we work for simplicity with defocusing NLS. However following [?] one can treat, with minor changes, the focusing case with a smallness assumption on the initial datum. Concerning the large data in the focusing case, it seems natural to expect blow-up similarly to the case of the unperturbed NLS with nonlinearity

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$p \geq 2$ .

Thanks to the work by Bourgain [?], we know how to construct the global dynamics of (??) if  $\xi$  is replaced by a smooth potential. Therefore a natural way to solve (??) is to regularize  $\xi$  and to try to pass to a limit in the regularized problems. As shown in [?, ?] such a passage to limit is possible for  $p \leq 3$  but only for well-prepared initial data. Therefore, we are interested in the solutions to the following regularization of (??)

$$(1.2) \quad i\partial_t u_\varepsilon = \Delta u_\varepsilon + \xi_\varepsilon(x, \omega) u_\varepsilon - u_\varepsilon |u_\varepsilon|^p, \quad u_\varepsilon(0, x) = u_0(x) e^{Y(x, \omega) - Y_\varepsilon(x, \omega)},$$

where  $\xi_\varepsilon = \chi_\varepsilon * \xi$ ,  $\varepsilon \in (0, 1)$  is a regularization of  $\xi$  with  $\chi_\varepsilon(x) = \varepsilon^{-2} \chi(x/\varepsilon)$ , where  $\chi(x)$  is smooth with a support in  $\{|x| < 1/2\}$  and  $\int_{\mathbb{T}^2} \chi = 1$ . As in [?, ?], in (??),  $Y = \Delta^{-1} \xi$  and  $Y_\varepsilon = \Delta^{-1} \xi_\varepsilon$  is its regularization.

The main result of the paper is the following one, which is an extension of the one proved in [?] where we were restricted to the powers  $p \in [2, 3]$ .

**Theorem 1.1.** *Assume  $p \geq 2$  and  $u_0(x)$  be such that  $e^{Y(x, \omega)} u_0(x) \in H^2(\mathbb{T}^2)$  a.s. Then there exists an event  $\Sigma \subset \Omega$  such that  $p(\Sigma) = 1$  and for every  $\omega \in \Sigma$  there exists*

$$v(t, x, \omega) \in \bigcap_{\gamma \in [0, 2)} \mathcal{C}(\mathbb{R}; H^\gamma(\mathbb{T}^2))$$

such that for every  $T > 0$  and  $\gamma \in [0, 2)$  we have:

$$(1.3) \quad \sup_{t \in [-T, T]} \|e^{iC_\varepsilon t} e^{Y_\varepsilon(x, \omega)} u_\varepsilon(t, x, \omega) - v(t, x, \omega)\|_{H^\gamma(\mathbb{T}^2)} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where  $C_\varepsilon = \mathbb{E}(|\nabla Y_\varepsilon(x, \omega)|^2)$  (this quantity is independent of  $x$ ) and  $u_\varepsilon(t, x, \omega)$  are solutions to (??). Moreover for  $\gamma \in [0, 1)$  and  $\omega \in \Sigma$  we have

$$(1.4) \quad \sup_{t \in [-T, T]} \| |u_\varepsilon(t, x, \omega)| - e^{-Y(x, \omega)} |v(t, x, \omega)| \|_{H^\gamma(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

The proof of Theorem ?? crucially relies on the modified energies and some results from our previous paper [?]. This makes that the present paper is not self-contained. As announced in [?], the new ingredient allowing to deal with general nonlinearities is the use of the dispersive effect which leads to Proposition ?? below. Proposition ?? displays a gain of regularity with respect to the Sobolev inequality used in [?], once a time averaging is performed. Note that we allow logarithmic losses in  $\varepsilon$  in our dispersive bounds. As already used in [?] these losses can be compensated by the polynomial in  $\varepsilon$  convergence of  $Y_\varepsilon$  to  $Y$  in the natural norms. At the best of our knowledge this is the first paper where the dispersive effect is used in order to achieve the global well-posedness of problems with white noise multiplicative potential. In particular Strichartz estimates are crucial along the proof Proposition ?? where, despite the paper [?], it is allowed on the r.h.s. a power  $\gamma$  smaller than 2 after time averaging.

In [?, ?] a different approach to the study of (??) is introduced. This approach is based on the construction of a suitable self adjoint realization of  $\Delta + \xi$ . Then the initial data in (??) is chosen in the domain of this self adjoint operator. Being in the domain of this self adjoint operator is the substitute of our assumption of well prepared initial data  $e^{Y(x, \omega)} u_0(x) \in H^2(\mathbb{T}^2)$ . At the best of our knowledge the

present paper is the first one where global well-posedness is proved for (??) with an arbitrary polynomial nonlinearity  $p$ , extending the papers [?] and [?]. Our proof is based on the approach introduced by Hairer-Labbé in [?].

For the sake of simplicity, we prove Theorem ?? in the context of the flat torus  $\mathbb{T}^2$ . However, it is quite likely that a similar result holds in the context of a general compact riemannian boundaryless manifolds. Indeed, the dispersive estimates can be extended to this setting in a relatively straightforward way. The stochastic analysis results from [?] can also be extended to this setting by some slightly more involved elaborations. We will address this question and some related issues in a forthcoming work.

Following Hairer-Labbé [?], we set

$$(1.5) \quad v_\varepsilon(t, x, \omega) = e^{iC_\varepsilon t} e^{Y_\varepsilon(x, \omega)} u_\varepsilon(t, x, \omega),$$

where  $C_\varepsilon$  is the constant appearing in Theorem ?. Then  $v_\varepsilon$  solves

$$(1.6) \quad i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon(x, \omega) + v_\varepsilon : |\nabla Y_\varepsilon|^2 : (x, \omega) - e^{-pY_\varepsilon} v_\varepsilon |v_\varepsilon|^p,$$

where

$$: |\nabla Y_\varepsilon|^2 : (x, \omega) = |\nabla Y_\varepsilon|^2(x, \omega) - C_\varepsilon$$

is the renormalized potential defined in [?]. As pointed out by one of the anonymous referees the change of unknown (??) resembles the so called Doss-Sussmann transformation, originally appearing in the context of stochastic ordinary differential equations (see [?, ?]).

Following Section 6 in [?] the proof of Theorem ?? follows from the next theorem concerning the behavior of  $v_\varepsilon(t, x, \omega)$ .

**Theorem 1.2.** *Assume  $p \geq 2$  and  $u_0(x)$  be such that  $e^{Y(x, \omega)} u_0(x) \in H^2(\mathbb{T}^2)$  a.s. Then there exists an event  $\Sigma \subset \Omega$  such that  $p(\Sigma) = 1$  and for every  $\omega \in \Sigma$  there exists*

$$v(t, x, \omega) \in \bigcap_{\gamma \in [0, 2)} \mathcal{C}(\mathbb{R}; H^\gamma(\mathbb{T}^2))$$

such that for every fixed  $T > 0$  and  $\gamma \in [0, 2)$  we have:

$$\sup_{t \in [-T, T]} \|v_\varepsilon(t, x, \omega) - v(t, x, \omega)\|_{H^\gamma(\mathbb{T}^2)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Here we have denoted by  $v_\varepsilon(t, x, \omega)$  for  $\omega \in \Sigma$  the unique global solution in the space  $\mathcal{C}(\mathbb{R}; H^2(\mathbb{T}^2))$  of the following problem:

$$(1.7) \quad i\partial_t v_\varepsilon = \Delta v_\varepsilon - 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon(x, \omega) + v_\varepsilon : |\nabla Y_\varepsilon|^2 : (x, \omega) - e^{-pY_\varepsilon} v_\varepsilon |v_\varepsilon|^p, \\ v_\varepsilon(0, x) = v_0(x) \in H^2(\mathbb{T}^2)$$

and  $v(t, x, \omega)$  denotes for  $\omega \in \Sigma$  the unique global solution in the space  $\mathcal{C}(\mathbb{R}; H^\gamma(\mathbb{T}^2))$ , for  $\gamma \in (1, 2)$ , of the following limit problem:

$$(1.8) \quad i\partial_t v = \Delta v - 2\nabla v \cdot \nabla Y(x, \omega) + v : |\nabla Y|^2 : (x, \omega) - e^{-pY} v |v|^p, \\ v(0, x) = v_0(x) \in H^2(\mathbb{T}^2)$$

where in both Cauchy problems (??) and (??)  $v_0(x) = e^{Y(x, \omega)} u_0(x)$ ,  $\omega \in \Sigma$ .

We quote the paper [?] where the same problem has been considered on  $\mathbb{R}^2$ . In particular in the euclidean setting the authors establish global well-posedness for subquadratic nonlinearity. It is worth mentioning that the problem on  $\mathbb{R}^2$  is considerably more difficult than the period setting  $\mathbb{T}^2$ . A first difficulty already appears in considering the existence and uniqueness of global solutions for the corresponding smoothed equation. In the periodic setting  $\mathbb{T}^2$  this issue can be settled by using the Bourgain's strategy (see [?]). On the contrary in the euclidean setting  $\mathbb{R}^2$  the smoothed equation involves unbounded perturbations, and as far as we know no dispersive estimates are available for the corresponding linear flows in order to get global well-posedness for the associated nonlinear problem via a contraction argument. We believe that the approach developed in this paper can be useful on  $\mathbb{R}^2$  as well in order to deal with superquadratic nonlinearities. We plan to pursue this problem in a future work. Let us also mention that  $3d$  case is a challenging one which goes beyond our present understanding of the problem.

**Notations.** For every  $s \in \mathbb{R}$  we denote  $s^+$  any number belonging to  $(s, s + \delta)$  for a suitable  $\delta > 0$ , similarly  $s^-$  denotes any number in  $(s - \delta, s)$  for a suitable  $\delta > 0$ . We shall denote by  $L^p, H^s, W^{s,p}$  the functional spaces  $L^p(\mathbb{T}^2), H^s(\mathbb{T}^2), W^{s,p}(\mathbb{T}^2)$ . In the sequel we shall denote by  $C$  a deterministic finite positive constant that can change from line to line and by  $C(\omega)$  a random variable defined on  $\Omega$  and finite a.s. that can change from line to line. We shall denote by  $C(\omega, T)$  a constant which is increasing w.r.t.  $T$  and finite for every  $(\omega, T) \in \Sigma \times \mathbb{R}^+$  for a suitable event  $\Sigma \subset \Omega$  of full measure. In the rest of the paper for shortness we will drop writing the  $\omega$  dependence of  $v_\varepsilon$  and  $Y_\varepsilon$ . For every  $a, b$  we denote by  $\int_a^b$  the integral w.r.t. time variable and  $\int_{\mathbb{T}^2}$  the integral on  $\mathbb{T}^2$ .

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## 2. PRELIMINARY FACTS

We collect in this section some facts proved in [?] and some useful consequences that will be needed in the sequel. From now on we drop for shortness the  $\omega$  dependence of  $v_\varepsilon, Y_\varepsilon, |\nabla Y_\varepsilon|^2$  :

**Proposition 2.1.** *We have the following bound:*

$$(2.1) \quad \sup_{\varepsilon \in (0,1)} \|Y_\varepsilon(x)\|_{L^\infty} \leq C(\omega),$$

$$(2.2) \quad \|\nabla Y_\varepsilon(x)\|_{L^m} \leq C(\omega) |\log \varepsilon|, \quad \forall m \in [1, \infty) \quad \forall \varepsilon \in (0, \frac{1}{2})$$

$$(2.3) \quad \| |\nabla Y_\varepsilon|^2 : (x) \|_{L^m} \leq C(\omega) |\log \varepsilon|^2, \quad \forall m \in [1, \infty) \quad \forall \varepsilon \in (0, \frac{1}{2}).$$

For every  $T > 0$  we have the following estimates for the solutions  $v_\varepsilon(t, x)$  of (??):

$$(2.4) \quad \sup_{\varepsilon \in (0,1)} \|v_\varepsilon(t, x)\|_{L^\infty((0,T);H^1)} \leq C(\omega),$$

$$(2.5) \quad \|v_\varepsilon(t, x)\|_{L^\infty((0,T);H^{1+})} \leq C(\omega) \|v_\varepsilon\|_{L^\infty((0,T);H^2)}^{0+}, \quad \forall \varepsilon \in (0, \frac{1}{2})$$

$$(2.6) \quad \|v_\varepsilon(t, x)\|_{L^\infty((0, T); H^2)} \leq C(\omega) + C(\omega) \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0, T); L^2)}, \quad \forall \varepsilon \in (0, \frac{1}{2}).$$

*Proof.* The bounds (??), (??), (??) have been established in [?, Proposition 2.1]. The bound (??) has been established for  $p \in [2, 3]$  in [?, Proposition 3.1], however the proof extends after obvious modifications to the generic case  $p \geq 2$ . The estimate (??) follows by combining the inequality  $\|u\|_{H^{1+\theta}} \leq \|u\|_{H^1}^{1-\theta} \|u\|_{H^2}^\theta$ , where we can choose  $\theta$  arbitrarily small, with (??). The estimate (??) follows by the elliptic regularity estimate  $\|v_\varepsilon(t, x)\|_{H^2} \sim \|v_\varepsilon(t, x)\|_{L^2} + \|\Delta v_\varepsilon(t, x)\|_{L^2}$  in conjunction with (??), which implies that the weight  $e^{-Y_\varepsilon}$  are bounded from above and lower bounded away from zero uniformly w.r.t.  $\varepsilon$ . Moreover  $\|v_\varepsilon(t, x)\|_{L^2} \leq C(\omega)$  follows by (??).  $\square$

Next we introduce the family of operators:

$$(2.7) \quad H_\varepsilon u = \Delta u - 2\nabla u \cdot \nabla Y_\varepsilon(x) + u : |\nabla Y_\varepsilon|^2 : (x),$$

where as usual we drop the  $\omega$  dependence of the operators  $H_\varepsilon$ . In the sequel we shall need the following result.

**Proposition 2.2.** *We have the bound:*

$$(2.8) \quad \|(H_\varepsilon - \Delta)u\|_{L^2} \leq C(\omega) |\log \varepsilon|^C \|u\|_{H^{1+}}, \quad \forall \varepsilon \in (0, \frac{1}{2}).$$

*Proof.* It is sufficient to show the bounds

$$(2.9) \quad \|\nabla u \cdot \nabla Y_\varepsilon\|_{L^2} \leq C(\omega) |\log \varepsilon| \|u\|_{H^{1+}} \quad \text{and} \\ \|u : |\nabla Y_\varepsilon|^2 : \|_{L^2} \leq C(\omega) |\log \varepsilon|^2 \|u\|_{H^{1+}}.$$

We have for every  $\delta \in (0, 1)$

$$\|\nabla u \cdot \nabla Y_\varepsilon\|_{L^2} \leq C \|\nabla Y_\varepsilon\|_{L^{\frac{2}{\delta}}} \|\nabla u\|_{L^{\frac{2}{1-\delta}}} \leq C(\omega) |\log \varepsilon| \|u\|_{H^{1+\delta}}$$

where we have used (??) and the embedding  $H^\delta \subset L^{\frac{2}{1-\delta}}$ . The second bound in (??) follows by a similar argument

$$\|u : |\nabla Y_\varepsilon|^2 : \|_{L^2} \leq C \| : |\nabla Y_\varepsilon|^2 : \|_{L^{\frac{2}{\delta}}} \|u\|_{L^{\frac{2}{1-\delta}}} \leq C(\omega) |\log \varepsilon|^2 \|u\|_{H^{1+\delta}}$$

where we have used (??) and  $H^{1+\delta} \subset L^{\frac{2}{1-\delta}}$ .  $\square$

### 3. A PRIORI BOUNDS OF $v_\varepsilon$

We introduce the propagator  $S_\varepsilon(t)$  associated with the linear problem  $i\partial_t u = H_\varepsilon u$ , where  $H_\varepsilon$  is defined in (??). The main point of this section is Proposition ???. In order to prove it, we shall need Strichartz estimates with loss for the propagator  $S_\varepsilon(t)$ .

**Proposition 3.1.** *For every  $T > 0$  we have the following bound:*

$$(3.1) \quad \|S_\varepsilon(t)\varphi\|_{L^\infty((0, T); H^s)} \leq C(\omega) |\log \varepsilon|^C \|\varphi\|_{H^s}, \quad s \in [0, 2].$$

Moreover for every  $r, q \in (2, \infty)$  such that  $\frac{2}{r} + \frac{2}{q} = 1$  we have

$$(3.2) \quad \|S_\varepsilon(t)\varphi\|_{L^r((0, T); L^q)} \leq C(\omega, T) |\log \varepsilon|^C \|\varphi\|_{H^{\frac{1}{r}}+}.$$

*Proof.* Estimate (??) is established in [?]. For the proof of (??), we follow the argument of [?] which is closely related to the analysis in [?, ?, ?, ?]. The basic strategy is to perform a perturbative argument with respect to the evolution  $\exp(it\Delta)$  by a partition on small time intervals which makes the perturbation  $H_\varepsilon - \Delta$  better but which losses some regularity on the data because of the summation on the small time intervals. An additional difficulty resolved in [?] is coming from the fact that a frequency localisation of  $(H_\varepsilon - \Delta)(u)$  does not imply a frequency localisation of  $u$ .

Recall the following semi-classical Strichartz estimates from [?] on a generic compact riemannian surface  $(M, g)$

$$(3.3) \quad \|e^{it\Delta_g}\varphi\|_{L^r((t_0, t_0 + \frac{1}{N}); L^q(M))} \leq C\|\Delta_N\varphi\|_{L^2(M)}, \quad \frac{1}{r} + \frac{1}{q} = \frac{1}{2}, \quad r > 2, \quad t_0 \in \mathbb{R}.$$

where  $\Delta_g$  denotes the Laplace-Beltrami operator on  $(M, g)$  and  $\Delta_N$  is the associated Littlewood-Paley projection at dyadic frequency  $N$ . This bound will play a crucial role along the rest of the proof in the specific case  $M = \mathbb{T}^2$ .

Let

$$\text{Id} = \sum_{N\text{-dyadic}} \Delta_N$$

be a Littlewood-Paley partition of the unity. Therefore the issue is to bound

$$(3.4) \quad \|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r((0, T); L^q)}.$$

In order to evaluate (??), we distinguish two cases according to the sizes of  $N_1$  and  $N_2$  and to sum up on  $N_1, N_2$ .

*First case:*  $N_1 \geq N_2$

In this case we split the interval  $[0, T]$  in an essentially disjoint union of intervals of size  $N_1^{-1}$  as

$$(3.5) \quad [0, T] = \bigcup_j I_j$$

and we aim to estimate  $\|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r(I_j; L^q)}$ . Suppose that  $I_j = [a, b]$ . Then following [?] (see also [?]), for  $t \in [a, b]$  we can write

$$(3.6) \quad \begin{aligned} \Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi &= \Delta_{N_1} e^{i(t-a)\Delta} S_\varepsilon(a) \Delta_{N_2} \varphi \\ &\quad + i \int_a^t \Delta_{N_1} e^{i(t-\tau)\Delta} (H_\varepsilon - \Delta) S_\varepsilon(\tau) \Delta_{N_2} \varphi d\tau. \end{aligned}$$

We now estimate each term in the right hand-side of (??). Using (??), we estimate the first term as follows for  $\delta > 0$ :

$$\begin{aligned} \|\Delta_{N_1} e^{i(t-a)\Delta} S_\varepsilon(a) \Delta_{N_2} \varphi\|_{L^r(I_j; L^q)} &\leq C N_1^{-\frac{1}{r} - \delta} \|S_\varepsilon(a) \Delta_{N_2} \varphi\|_{H^{\frac{1}{r} + \delta}} \\ &\leq C(\omega) |\log \varepsilon|^C N_1^{-\frac{1}{r} - \delta} \|\varphi\|_{H^{\frac{1}{r} + \delta}} \end{aligned}$$

where we have used (??). Now we estimate the second term in the right hand-side of (??). Using the Minkowski inequality and again (??), we can write for every  $\delta > 0$ :

$$\begin{aligned}
& \left\| \int_a^t \Delta_{N_1} e^{i(t-\tau)\Delta} (H_\varepsilon - \Delta) S_\varepsilon(\tau) \Delta_{N_2} \varphi d\tau \right\|_{L^r(I_j; L^q)} \\
& \leq C \int_{I_j} \|(H_\varepsilon - \Delta) S_\varepsilon(\tau) \Delta_{N_2} \varphi\|_{L^2} d\tau \\
& \leq C(\omega) |\log \varepsilon|^C N_1^{-1} N_2^{1+\frac{\delta}{2}} N_2^{-\frac{1}{r}-\delta} \|\varphi\|_{H^{\frac{1}{r}+\delta}}
\end{aligned}$$

where we have used (??) and (??). Summarizing we get

$$\|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r(I_j; L^q)} \leq C(\omega) |\log \varepsilon|^C (N_1^{-\frac{1}{r}-\delta} + N_1^{-1} N_2^{1-\frac{1}{r}-\frac{\delta}{2}}) \|\varphi\|_{H^{\frac{1}{r}+\delta}}$$

and hence using that the number of  $I_j$  is smaller than  $TN_1$ , taking the  $r$ 'th power of the previous bound and summing on  $j$ , we get the estimate

$$\|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r((0,T); L^q)} \leq C(\omega) T^{\frac{1}{r}} |\log \varepsilon|^C (N_1^{-\delta} + N_1^{-1+\frac{1}{r}} N_2^{1-\frac{1}{r}-\frac{\delta}{2}}) \|\varphi\|_{H^{\frac{1}{r}+\delta}}$$

and hence

$$(3.7) \quad \sum_{N_2 \leq N_1} \|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r((0,T); L^q)} \leq C(\omega) T^{\frac{1}{r}} |\log \varepsilon|^C \|\varphi\|_{H^{\frac{1}{r}+\delta}}$$

where we have used

$$\sum_{N_2 \leq N_1} (N_1^{-\delta} + N_1^{-1+\frac{1}{r}} N_2^{1-\frac{1}{r}-\frac{\delta}{2}}) < \infty.$$

*Second case:*  $N_1 \leq N_2$

We consider again the splitting (??) but this time the intervals  $I_j$  are of size  $N_2^{-1}$ . Again we consider (??) and we estimate each term of the right hand-side. Since  $N_2^{-1} \leq N_1^{-1}$ , we can combine (??) with (??) and we estimate the first term at the right hand-side of (??) as

$$\|\Delta_{N_1} e^{i(t-a)\Delta} S_\varepsilon(a) \Delta_{N_2} \varphi\|_{L^r(I_j; L^q)} \leq C(\omega) |\log \varepsilon|^C N_2^{-\frac{1}{r}-\delta} \|\varphi\|_{H^{\frac{1}{r}+\delta}},$$

where  $\delta > 0$ . Next, as above, we can estimate the second term at the right hand-side of (??) as

$$\begin{aligned}
& \left\| \int_a^t \Delta_{N_1} e^{i(t-\tau)\Delta} (H_\varepsilon - \Delta) S_\varepsilon(\tau) \Delta_{N_2} \varphi d\tau \right\|_{L^r(I_j; L^q)} \\
& \leq C(\omega) |\log \varepsilon|^C N_2^{-1} N_2^{1+\frac{\delta}{2}} N_2^{-\frac{1}{r}-\delta} \|\varphi\|_{H^{\frac{1}{r}+\delta}}.
\end{aligned}$$

Summarizing we get

$$\|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r(I_j; L^q)} \leq C(\omega) |\log \varepsilon|^C (N_2^{-\frac{1}{r}-\delta} + N_2^{-\frac{1}{r}-\frac{\delta}{2}}) \|\varphi\|_{H^{\frac{1}{r}+\delta}}$$

and as above, using that the number of  $I_j$  is smaller than  $TN_2$ , taking the  $r$ 'th power of the previous bound and summing on  $j$ , we get the estimate

$$\|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r((0,T); L^q)} \leq C(\omega) T^{\frac{1}{r}} |\log \varepsilon|^C (N_2^{-\delta} + N_2^{-\frac{\delta}{2}}) \|\varphi\|_{H^{\frac{1}{r}+\delta}}.$$

Hence we get

$$(3.8) \quad \sum_{N_1 \leq N_2} \|\Delta_{N_1} S_\varepsilon(t) \Delta_{N_2} \varphi\|_{L^r((0,T); L^q)} \leq C(\omega) T^{\frac{1}{r}} |\log \varepsilon|^C \|\varphi\|_{H^{\frac{1}{r}+\delta}}$$

since

$$\sum_{N_1 \leq N_2} (N_2^{-\delta} + N_2^{-\frac{\delta}{2}}) < \infty.$$

We conclude by combining (??) and (??) with the Minkowski inequality.  $\square$

As a consequence we get the following result.

**Proposition 3.2.** *For every  $T > 0$  we have the following estimates:*

$$(3.9) \quad \|S_\varepsilon(t)\varphi\|_{L^4((0,T);W^{\frac{3}{4}-,4})} \leq C(\omega, T) |\log \varepsilon|^C \|\varphi\|_{H^1}$$

and

$$(3.10) \quad \left\| \int_0^t S_\varepsilon(t-s)f(s)ds \right\|_{L^4((0,T);W^{\frac{3}{4}-,4})} \leq C(\omega, T) |\log \varepsilon|^C \|f\|_{L^1((0,T);H^1)}.$$

*Proof.* Notice that (??) follows by combining (??) with the Minkowski inequality. Next we focus on the proof of (??). Notice that for every  $\varepsilon_0 \in (0, 1)$ , there exists  $q \in (1, \infty)$  such that the following Gagliardo-Nirenberg inequality occurs:

$$\|u\|_{W^{\frac{3}{4}-\varepsilon_0,4}} \leq C \|u\|_{L^q}^{\frac{1}{2}} \|u\|_{H^{\frac{3}{2}}}^{\frac{1}{2}}$$

and hence by integration in time and Hölder inequality in time we get

$$\begin{aligned} \|S_\varepsilon(t)\varphi\|_{L^4((0,T);W^{\frac{3}{4}-\varepsilon_0,4})}^4 &\leq C \|S_\varepsilon(t)\varphi\|_{L^2((0,T);L^q)}^2 \|S_\varepsilon(t)\varphi\|_{L^\infty((0,T);H^{\frac{3}{2}})}^2 \\ &\leq C(\omega, T) |\log \varepsilon|^C \|S_\varepsilon(t)\varphi\|_{L^r((0,T);L^q)}^2 \|\varphi\|_{H^{\frac{3}{2}}}^2 \leq C(\omega, T) |\log \varepsilon|^C \|\varphi\|_{H^{\frac{1}{2}}}^2 \|\varphi\|_{H^{\frac{3}{2}}}^2 \end{aligned}$$

where  $q, r$  are Strichartz admissible and we have used (??), (??).

Notice that for initial datum  $\varphi = \Delta_N \varphi$  which is spectrally localized at dyadic frequency  $N$  we get from the previous bound

$$\|S_\varepsilon(t)\Delta_N \varphi\|_{L^4((0,T);W^{\frac{3}{4}-\varepsilon_0,4})} \leq C(\omega, T) |\log \varepsilon|^C \|\Delta_N \varphi\|_{H^{1-}}.$$

We conclude (??) by summing on  $N$ .  $\square$

Next we get the following bound on the nonlinear solutions  $v_\varepsilon$  to (??).

**Proposition 3.3.** *For every  $T > 0$  we have the following bound:*

$$(3.11) \quad \|v_\varepsilon(t, x)\|_{L^4((0,T);W^{\frac{3}{4}-,4})} \leq C(\omega, T) |\log \varepsilon|^C (1 + \|v_\varepsilon(t, x)\|_{L^\infty((0,T);H^2)}^+).$$

*Proof.* By combining Proposition ?? with the integral formulation associated with (??) we get:

$$\begin{aligned} \|v_\varepsilon\|_{L^4((0,T);W^{\frac{3}{4}-,4})} &\leq C(\omega, T) |\log \varepsilon|^C \|v_\varepsilon(0)\|_{H^1} + C(\omega, T) |\log \varepsilon|^C \int_0^T \|e^{-pY_\varepsilon} v_\varepsilon |v_\varepsilon|^p\|_{H^1} \\ &\leq C(\omega, T) |\log \varepsilon|^C \|v_\varepsilon(0)\|_{H^1} + C(\omega, T) |\log \varepsilon|^C \int_0^T \|v_\varepsilon\|_{H^1} \|v_\varepsilon\|_{L^\infty}^p \|e^{-pY_\varepsilon}\|_{L^\infty} \\ &\quad + C(\omega, T) |\log \varepsilon|^C \int_0^T \|\nabla Y_\varepsilon\|_{L^2} \|e^{-pY_\varepsilon}\|_{L^\infty} \|v_\varepsilon\|_{L^\infty}^{p+1} \end{aligned}$$

and we conclude by using the Sobolev embedding  $H^{1+} \subset L^\infty$ , (??), (??), (??), (??).  $\square$

We conclude this section with the following key estimate.

**Proposition 3.4.** *We have the following bound for a suitable  $\eta \in (0, 1)$  and for every  $T > 0$ :*

$$\|v_\varepsilon(t, x)\|_{L^2((0,T);W^{1,4})}^2 \leq C(\omega, T) |\log \varepsilon|^C (1 + \|v_\varepsilon(t, x)\|_{L^\infty((0,T);H^2)}^\eta).$$

*Proof.* We have the bound for time independent functions:

$$\|u\|_{W^{1,4}} \leq C \|u\|_{W^{\frac{3}{4}-,4}}^{\frac{2}{3}-} \|u\|_{H^2}^{\frac{1}{3}+}.$$

Hence by integration in time and by choosing  $u = v_\varepsilon$  we get

$$\begin{aligned} \|v_\varepsilon\|_{L^2((0,T);W^{1,4})}^2 &\leq C \|v_\varepsilon\|_{L^{\frac{4}{3}-}((0,T);W^{\frac{3}{4}-,4})}^{\frac{4}{3}-} \|v_\varepsilon\|_{L^\infty((0,T);H^2)}^{\frac{2}{3}+} \\ &\leq CT^{\frac{2}{3}-} \|v_\varepsilon\|_{L^4((0,T);W^{\frac{3}{4}-,4})}^{\frac{4}{3}-} \|v_\varepsilon\|_{L^\infty((0,T);H^2)}^{\frac{2}{3}+}. \end{aligned}$$

We conclude by Proposition ??.

□

#### 4. PROOF OF THEOREM ??

We aim at proving the following bound for every given  $T > 0$ :

$$(4.1) \quad \|v_\varepsilon(t, x)\|_{L^\infty((0,T);H^2)} \leq |\log \varepsilon|^{C(\omega,T)}, \quad \forall \varepsilon \in (0, \frac{1}{2}).$$

Recall that the bound (??) has been achieved in [?] in the case  $2 \leq p \leq 3$  (see Proposition 4.5 in [?]). The main point is that we get the bound (??) for every  $p \geq 2$ . Once (??) is achieved then Theorem ?? can be proved exactly as in [?, Section 5].

We can now establish (??). In order to do that we recall some notations from [?]. Denote by  $\mathcal{H}_\varepsilon, \mathcal{F}_\varepsilon$  and  $\mathcal{G}_\varepsilon$  the energies introduced along [?, Proposition 4.1] which satisfy

$$(4.2) \quad \frac{d}{dt}(\mathcal{F}_\varepsilon(v_\varepsilon) - \mathcal{G}_\varepsilon(v_\varepsilon)) = -\mathcal{H}_\varepsilon(v_\varepsilon).$$

For the sake of completeness we recall the definition of those energies:

- $\mathcal{F}_\varepsilon(v_\varepsilon) = \int_{\mathbb{T}^2} |\Delta v_\varepsilon|^2 e^{-2Y_\varepsilon} - 4\text{Re} \int_{\mathbb{T}^2} \Delta v_\varepsilon \nabla Y_\varepsilon \cdot \nabla \bar{v}_\varepsilon e^{-2Y_\varepsilon}$   
 $+ 4 \sum_{i=1}^2 \int_{\mathbb{T}^2} (|\partial_i v_\varepsilon|^2) (\partial_i Y_\varepsilon)^2 e^{-2Y_\varepsilon} + 8\text{Re} \int_{\mathbb{T}^2} \partial_1 Y_\varepsilon \partial_2 Y_\varepsilon \partial_1 v_\varepsilon \partial_2 \bar{v}_\varepsilon e^{-2Y_\varepsilon}$   
 $+ 2\text{Re} \int_{\mathbb{T}^2} v_\varepsilon : |\nabla Y_\varepsilon|^2 : \nabla \bar{v}_\varepsilon \cdot \nabla (e^{-2Y_\varepsilon}) + 2\text{Re} \int_{\mathbb{T}^2} \Delta v_\varepsilon \bar{v}_\varepsilon : |\nabla Y_\varepsilon|^2 : e^{-2Y_\varepsilon}$   
 $+ \int_{\mathbb{T}^2} |v_\varepsilon|^2 (|\nabla Y_\varepsilon|^2)^2 e^{-2Y_\varepsilon};$
- $\mathcal{G}_\varepsilon(v_\varepsilon) = - \int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 |v_\varepsilon|^p e^{-(p+2)Y_\varepsilon} - 2\text{Re} \int_{\mathbb{T}^2} v_\varepsilon \nabla (|v_\varepsilon|^p) \cdot \nabla \bar{v}_\varepsilon e^{-(p+2)Y_\varepsilon}$   
 $+ \frac{p}{4} \int_{\mathbb{T}^2} |\nabla (|v_\varepsilon|^2)|^2 |v_\varepsilon|^{p-2} e^{-(p+2)Y_\varepsilon}$   
 $+ \frac{2}{p+2} \int_{\mathbb{T}^2} |v_\varepsilon|^{p+2} : |\nabla Y_\varepsilon|^2 : e^{-(p+2)Y_\varepsilon} + 2p\text{Re} \int_{\mathbb{T}^2} v_\varepsilon |v_\varepsilon|^{p-1} \nabla Y_\varepsilon \cdot \nabla \bar{v}_\varepsilon e^{-(p+2)Y_\varepsilon};$

$$\begin{aligned}
\bullet \quad \mathcal{H}_\varepsilon(v_\varepsilon) &= - \int_{\mathbb{T}^2} |\nabla v_\varepsilon|^2 \partial_t (|v_\varepsilon|^p) e^{-(p+2)Y_\varepsilon} \\
&- 2\operatorname{Re} \int_{\mathbb{T}^2} \partial_t v_\varepsilon \nabla (|v_\varepsilon|^p) \cdot \nabla \bar{w}(t) e^{-(p+2)Y_\varepsilon} - \frac{p}{4} \int_{\mathbb{T}^2} |\nabla (|v_\varepsilon|^2)|^2 \partial_t (|v_\varepsilon|^{p-2}) e^{-(p+2)Y_\varepsilon} \\
&\quad + 2p\operatorname{Re} \int_{\mathbb{T}^2} \partial_t (v_\varepsilon |v_\varepsilon|^p) \nabla Y_\varepsilon \cdot \nabla \bar{v}_\varepsilon e^{-(p+2)Y_\varepsilon}.
\end{aligned}$$

An important point is to obtain the following modification of [?, Proposition 4.3] which, concerning the estimate of  $\mathcal{H}_\varepsilon(v_\varepsilon)$ , gains on the power of  $\|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}$  by exploiting the averaging in the time variable.

**Proposition 4.1.** *For a suitable  $\gamma \in (1, 2)$  we have the bound:*

$$\int_0^T |\mathcal{H}_\varepsilon(v_\varepsilon(s))| ds \leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^\gamma.$$

*Proof.* By using the Hölder inequality, the Leibnitz rule and the diamagnetic inequality  $|\partial_t |u|| \leq |\partial_t u|$  we get that the first three terms in  $\mathcal{H}_\varepsilon(v_\varepsilon)$  can be estimated by:

$$\int_{\mathbb{T}^2} |\partial_t v_\varepsilon| |\nabla v_\varepsilon|^2 |v_\varepsilon|^{p-1} e^{-(p+2)Y_\varepsilon} \leq C(\omega) \|\partial_t v_\varepsilon\|_{L^2} \|\nabla v_\varepsilon\|_{L^4}^2 \|v_\varepsilon\|_{L^\infty}^{p-1}.$$

where we have used (??). By using the equation solved by  $v_\varepsilon(t, x)$  and the Sobolev embedding  $H^{1+} \subset L^\infty$  we get from the estimate above after integration in time:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^2} |\partial_t v_\varepsilon| |\nabla v_\varepsilon|^2 |v_\varepsilon|^{p-1} e^{-(p+2)Y_\varepsilon} \\
&\quad \leq C(\omega) \|\Delta v_\varepsilon\|_{L^\infty((0,T);L^2)} \|\nabla v_\varepsilon\|_{L^2((0,T);L^4)}^2 \|v_\varepsilon\|_{L^\infty((0,T);H^{1+})}^{p-1} \\
&\quad + C(\omega) \|\nabla v_\varepsilon \cdot \nabla Y_\varepsilon\|_{L^\infty((0,T);L^2)} \|\nabla v_\varepsilon\|_{L^2((0,T);L^4)}^2 \|v_\varepsilon\|_{L^\infty((0,T);H^{1+})}^{p-1} \\
&\quad + C(\omega) \|v_\varepsilon : |\nabla Y_\varepsilon|^2 : \|_{L^\infty((0,T);L^2)} \|\nabla v_\varepsilon\|_{L^2((0,T);L^4)}^2 \|v_\varepsilon\|_{L^\infty((0,T);H^{1+})}^{p-1} \\
&\quad + C(\omega) \|e^{-pY_\varepsilon} v_\varepsilon |v_\varepsilon|^p\|_{L^\infty((0,T);L^2)} \|\nabla v_\varepsilon\|_{L^2((0,T);L^4)}^2 \|v_\varepsilon\|_{L^\infty((0,T);H^{1+})}^{p-1} \\
&\quad = I + II + III + IV.
\end{aligned}$$

Combining (??), (??), (??) and Proposition ?? we get

$$\begin{aligned}
I &\leq C(\omega, T) |\log \varepsilon|^C \|\Delta v_\varepsilon\|_{L^\infty((0,T);L^2)} \|v_\varepsilon\|_{L^\infty((0,T);H^2)}^{0+} (1 + \|v_\varepsilon\|_{L^\infty((0,T);H^2)}^\eta) \\
&\leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^{1+\eta^+}.
\end{aligned}$$

By combining now Hölder inequality, (??) and Proposition ?? we get

$$II \leq C(\omega, T) |\log \varepsilon|^C \|\nabla Y_\varepsilon\|_{L^4} \|\nabla v_\varepsilon\|_{L^\infty((0,T);L^4)} (1 + \|v_\varepsilon\|_{L^\infty((0,T);H^2)})^{\eta^+}$$

and hence by (??) and Sobolev embedding  $H^1 \subset L^4$  we conclude

$$\begin{aligned}
II &\leq C(\omega, T) |\log \varepsilon|^C (1 + \|v_\varepsilon\|_{L^\infty((0,T);H^2)})^{1+\eta^+} \\
&\leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^{1+\eta^+}
\end{aligned}$$

where we used at the last step (??). We also get

$$III \leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^{1+\eta^+}$$

whose proof is identical to the estimate of the term  $II$  given above, except that we use (??) instead of (??). For the term  $IV$  we get by (??), (??), (??), (??) and Proposition ??

$$\begin{aligned} IV &\leq C(\omega, T) |\log \varepsilon|^C \|v_\varepsilon\|_{L^\infty((0, T); L^2(p+1))}^{p+1} (1 + \|v_\varepsilon\|_{L^\infty((0, T); H^2)})^{\eta^+} \\ &\leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0, T); L^2)}^{\eta^+}, \end{aligned}$$

where we used at the last step the Sobolev embedding  $H^1 \subset L^{2(p+1)}$ . Concerning the last term in the expression of  $\mathcal{H}_\varepsilon(v_\varepsilon)$  we can estimate it as follows:

$$\begin{aligned} &\int_{\mathbb{T}^2} |\partial_t v_\varepsilon| |v_\varepsilon|^p |\nabla Y_\varepsilon| |\nabla v_\varepsilon| e^{-(p+2)Y_\varepsilon} \\ &\leq C(\omega) \|v_\varepsilon\|_{L^{8p}}^p \|\partial_t v_\varepsilon\|_{L^2} \|\nabla Y_\varepsilon\|_{L^8} \|\nabla v_\varepsilon\|_{L^4} \\ &\leq C(\omega) |\log \varepsilon| \|\partial_t v_\varepsilon\|_{L^2} \|\nabla v_\varepsilon\|_{L^4} \end{aligned}$$

where we have used (??), (??), the Sobolev embedding  $H^1 \subset L^{8p}$  and (??). Next we replace  $\partial_t v_\varepsilon$  by using the equation solved by  $v_\varepsilon$  and, thanks to the following time-independent Gagliardo-Nirenberg inequality

$$(4.3) \quad \|\nabla u\|_{L^4}^2 \leq C \|\nabla u\|_{L^2} \|\Delta u\|_{L^2},$$

we can continue the estimate above as follows:

$$\begin{aligned} \dots &\leq C(\omega) |\log \varepsilon| \|\Delta v_\varepsilon\|_{L^2} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\quad + C(\omega) |\log \varepsilon| \|\nabla v_\varepsilon \cdot \nabla Y_\varepsilon\|_{L^2} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\quad + C(\omega) |\log \varepsilon| \|v_\varepsilon : |\nabla Y_\varepsilon|^2 : \|_{L^2} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\quad + C(\omega) |\log \varepsilon| \|e^{-pY_\varepsilon} v_\varepsilon |v_\varepsilon|^p\|_{L^2} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla v_\varepsilon\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

and by the Sobolev embedding  $H^1 \subset L^4$  and (??), (??)

$$\begin{aligned} \dots &\leq C(\omega) |\log \varepsilon| \|\Delta v_\varepsilon\|_{L^2}^{\frac{3}{2}} + C(\omega) |\log \varepsilon| \|\nabla v_\varepsilon\|_{L^4} \|\nabla Y_\varepsilon\|_{L^4} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\quad + C(\omega) |\log \varepsilon| \|v_\varepsilon\|_{L^4} \| : |\nabla Y_\varepsilon|^2 : \|_{L^4} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} + C(\omega) |\log \varepsilon| \|v_\varepsilon |v_\varepsilon|^p\|_{L^2} \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \\ &\leq C(\omega) |\log \varepsilon| \|\Delta v_\varepsilon\|_{L^2}^{\frac{3}{2}} + C(\omega) |\log \varepsilon|^2 \|\Delta v_\varepsilon\|_{L^2}^{\frac{3}{2}} + C(\omega) |\log \varepsilon|^3 \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} + C(\omega) |\log \varepsilon| \|\Delta v_\varepsilon\|_{L^2}^{\frac{1}{2}} \end{aligned}$$

where we have used (??) and (??). Summarizing we get from the computation above and by (??)

$$\int_0^T \int_{\mathbb{T}^2} |\partial_t v_\varepsilon| |v_\varepsilon|^p |\nabla Y_\varepsilon| |\nabla v_\varepsilon| e^{-(p+2)Y_\varepsilon} \leq C(\omega, T) |\log \varepsilon|^C + \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^2}^{\frac{3}{2}+}.$$

□

Next we shall also need the following bound from [?, Proposition 4.4].

**Proposition 4.2.** *For every  $\mu > 0$  there exists a random variable  $C(\omega, \mu)$  such that:*

$$(4.4) \quad \left| \mathcal{F}_\varepsilon(v_\varepsilon) - \int_{\mathbb{T}^2} |\Delta v_\varepsilon|^2 e^{-2Y_\varepsilon} \right| < \mu \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^2}^2 + C(\omega, \mu) |\log \varepsilon|^4$$

and

$$(4.5) \quad |\mathcal{G}_\varepsilon(v_\varepsilon)| < \mu \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^2}^2 + C(\omega, \mu) |\log \varepsilon|^4.$$

We have now all tools to prove (??). By integration in time of (??) and by combining Proposition ?? (where we choose  $\mu$  small enough in order to absorb on the l.h.s. the term  $\|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^2$ ) with Proposition ?? we get

$$\|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^2 \leq C(\omega, T) |\log \varepsilon|^C + C \|e^{-Y_\varepsilon} \Delta v_\varepsilon\|_{L^\infty((0,T);L^2)}^\gamma, \quad \gamma < 2$$

and hence, by recalling (??), we conclude (??).

#### REFERENCES

- [1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations*, Geom. and Funct. Anal. 3 (1993) n.2, 107–156
- [2] N. Burq, P. Gérard, N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. 126 (2004) n.3, 569–605
- [3] A. Debussche, J. Martin, *Solution to the stochastic Schrödinger equation on the full space*, Nonlinearity, 32 (2019) n. 4, 1147–1174
- [4] A. Debussche, H. Weber, *The Schrödinger equation with spatial white noise potential*, Electron. J. Probab., 23 (2018) n. 28, 16 pp.
- [5] H. Doss, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B (N.S.), 13 (1977) n. 2, 99–125
- [6] M. Gubinelli, B. Ugurcan, I. Zachhuber, *Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions*, Stoch. Partial Differ. Equ. Anal. Comput., 8 (2020) n. 1, 82–149
- [7] J.L. Journé, A. Soffer, C. Sogge, *Decay estimates for Schrödinger operators* Comm. Pure Appl. Math. 44 (1991) n.5, 573–604
- [8] M. Hairer, C. Labbé, *A simple construction of the continuum parabolic Anderson model on  $\mathbf{R}^2$* , Electron. Commun. Probab., 20 (2015) n. 43, 11 pp.
- [9] H. Koch, N. Tzvetkov, *On the local well-posedness of the Benjamin-Ono equation in  $H^s$* , Int. Math. Res. Not. 26 (2003), 1449–1464
- [10] A. Mouzard, I. Zachhuber, *Strichartz inequalities with white noise potential on compact surfaces*, arXiv:2104.07940 [math.AP]
- [11] G. Staffilani, D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Comm. Partial Differential Equations 27 (2002) n.7-8,1337–1372
- [12] H. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probability, 6 (1978) n.1, 19–41
- [13] N. Tzvetkov, N. Visciglia, *Two dimensional nonlinear Schrödinger equation with spatial white noise potential and fourth order nonlinearity*, Stoch PDE: Anal Comp (2022). <https://doi.org/10.1007/s40072-022-00251-z>
- [14] D. Tataru, *Strichartz estimates for operators with nonsmooth coefficients and the nonlinear wave equation*, Amer. J. Math. 122 (2000) n.2, 349–376

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