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Infinite-Volume States with Irreducible Localization Sets for Gradient Models on Trees

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Abstract

We consider general classes of gradient models on regular trees with spin values in a countable Abelian group S such as $\mathbb{Z}$ or $\mathbb{Z}_q$. This includes unbounded spin models like the p-SOS model and finite-alphabet clock models. Under a strong coupling (low temperature) condition on the interaction, we prove the existence of families of distinct homogeneous tree-indexed Markov chain Gibbs states μ_A whose single-site marginals concentrate on a given finite subset $A \subset S$ of spin values. The existence of such states is a new and robust phenomenon which is of particular relevance for infinite spin models. These states are extremal in the set of homogeneous Gibbs states, and in particular cannot be decomposed into homogeneous Markov-chain Gibbs states with a single-valued concentration center. Whether they are also extremal in the set of all Gibbs states remains an open, challenging question. As a further application of the method we obtain the existence of new types of gradient Gibbs states with $\mathbb{Z}$-valued spins, whose single-site marginals do not localize, but whose correlation structure depends on the finite set A, where we provide explicit expressions for the correlation between the height-increments along disjoint edges.

Keywords Gibbs measure · Tree-indexed Markov chain · Localization · Delocalization · Brouwer fixed point theorem

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1 Introduction

1.1 Gradient Models on Lattices and Trees

Statistical mechanics models for $\mathbb{Z}$- or $\mathbb{R}$-valued height variables with gradient interactions have been studied in a number of variations. For homogeneous models with different types of base spaces (such as lattices and trees) and different interaction potentials, see [5, 8, 10, 11, 20, 24, 26, 29]. For disordered models with quenched randomness in the interaction, see [4, 6, 7, 9, 27].

In this paper we study S-valued gradient models on d-regular trees, whose interactions are defined by transfer operators given by an symmetric function $Q : S \to (0, \infty)$. Here, S is assumed to be a countable Abelian group which we think of as the local state space of the system. In particular, if $S \subset \mathbb{R}$ is infinite, then the local state space can be viewed as the height-dimension of the system. In concrete applications in statistical mechanics, we often encounter the case $S = \mathbb{Z}$ and the transfer operator is given by $Q(i) = \exp(-\beta U(|i|))$, where $U : S \to \mathbb{R}$ is a potential function prescribing the energetic cost of a spin configuration to make an increment of size $|i|$ along an edge of the tree, or more generally an edge of the supporting graph.

Important special cases for the choice of the potential are the p-SOS model with $U(|i|) = |i|^p$, for which exponents $p \in (0, \infty)$ are allowed. The most popular choices are $p = 1$ which corresponds to the classical SOS model, and $p = 2$ which gives the discrete Gaussian (see [2, 3] for an analysis on the lattice). In our present approach besides positivity and evenness of Q, however, we will make no assumption on monotonicity or convexity in the interaction, and treat the function Q as an infinite-dimensional parameter of the model.

The main interest of the study is in the construction and description of infinite-volume Gibbs measures (GM) given by the DLR-consistency equation, as well as in gradient Gibbs measures (GGM) in the case of non-compact local spin space S (for general background see [11, 14, 26]). GGMs are relevant as generalizations of the concept of GMs since they are suited to describe infinite-volume states which *do not* localize in any bounded region $A \subset S$. By contrast to our work, Lammers and Toninelli [25] provide a full description of localization and delocalization properties of gradient Gibbs measures for two specific instances of gradient models on the tree whose underlying potential function is strictly convex. These two instances are the *uniform graph homomorphism model*, allowing for height increments of size $\{\pm 1\}$, and *uniform monotone models on directed rooted graph*, which allow only configurations whose spin-values are non-increasing when going over from the parent to a child. For the uniform graph homomorphism model Lammers and Toninelli are able to employ martingale-based methods from [26] established for studying gradient models on the lattice $\mathbb{Z}^d$, where for the uniform monotone model they provide a classification of extremal Gibbs states by means of flows, which is possible by a suitable comparison to the geometric distribution. To the best of our knowledge, our more general set-up does not allow for this reasoning and hence our proof is based on the tree-specific boundary law formalism of Zachary [30]. In our present work we will only consider homogeneous (tree-automorphism invariant) measures. For some results on non-homogeneous measures on trees with homogeneous interactions we refer to [1, 6, 12, 13, 16].

1.2 Main New Result: Localization on Arbitrary Finite Sets

In a previous paper [15] two of us considered the case $S = \mathbb{Z}$ and proved the existence of localized Gibbs measures under a strong coupling condition formulated for Q, namely boundedness of $(d+1)/2$-norm and small deviation of Q from $1_{\{0\}}$ in terms of the $d+1$-norm. We showed that there are homogeneous states μ_i whose single-site marginals are concentrated around *single fixed heights* $i \in \mathbb{Z}$.

In the present paper we will extend this type of result to the case of arbitrary finite localization sets $A \subset S$, for the height variables, under appropriate strong coupling conditions. The strong-coupling states with non-singleton concentration sets are of a new type in the setup of unbounded spin models, and to our best knowledge have not been discussed before. To appreciate the result it is important to note that these new tree-indexed Markov chain Gibbs measures μ_A constructed in our present work are not convex combinations of each other, and in particular not of the spatially homogeneous measures μ_i with single-height concentration which were constructed in [15].

The existence result of Theorem 3.1 holds under an N-dependent strong-coupling condition on Q which provides existence of measures μ_A which concentrate on localization sets A of size $|A| \leq N$. It is particularly remarkable that under this condition the localization sets in these families can be arbitrarily spread out.

This existence result may look surprising but can be made plausible by seeing it as an infinite-dimensional generalization of a simpler phenomenon which is known to appear in the q-state Potts model on the tree. The homogeneous Markov chain states of the Potts model can described via explicit computations [23], due to the full invariance of the interaction under permutation in local spin-space. For results on the extremality and non-extremality of the Potts states, see [21]. In this way, the low temperature Potts model is the simplest illustrating example of our existence theorem for finite spins. Conversely it is also motivation to search for natural classes of interactions for which such A-localized states exist, which lead us to our theory. In the more difficult case of integer-valued spins, the results are new and the simplest interesting example of our theory could be seen to be the classical SOS-model for which we obtain: Any choice of a subset A of heights (the concentration set) comes with a transition temperature below which an infinite-volume Gibbs state μ_A exists which concentrates on A. Much different from the Potts model, there is no simplification from the concrete form of the interaction. Therefore we choose to formulate our theorem in the natural general framework, from which one can then easily derive properties for special models, see Appendix A. Note that our framework includes for instance also the interesting case of $\mathbb{Z}^d$-valued spins, and it suggests to formulate low-temperature conditions in terms of smallness of p-norms of transition operators. This use of p-norms is really for convenience; as the p-norms can be easily computed in given models, this easily transfers into explicit applications down to concrete numbers, see again Appendix A.

Our proof in the case of spins with values in an arbitrary countable Abelian group S is based on the boundary law description of Gibbs measures going back to Zachary [30]. This leads to a non-linear fixed point equation in the space $\ell^{d+1}(S)$ of $d+1$-summable functions $u : S \to \mathbb{R}$. In general explicit solutions are out of question, and for our proof we will develop a fixed point method, adapted to the type of A-dependent states we are hoping to find, see Sect. 5.2.

Our approach to study the infinite-dimensional fixed point problem is to break the problem into two parts: on the given finite concentration set A where we expect to find the large components, and a conditional problem away from it where we expect to find the small components. For the latter we devise a suitable (conditional) map on sequence space which we show to be a contraction, for the former we employ the Brouwer fixed point theorem, see (42). This leads us to quite explicit quantitative thresholds for the system parameters of given models for which we can prove existence of A-concentrated states, see Proposition 5.6. On the level of system parameters, the strong coupling condition on $Q = \exp(-\beta U)$ translates into the fact that the parameter β—which as usual should be interpreted as the inverse temperature—should be large enough, see Appendix A for a discussion along two concrete examples. For a discussion of uniqueness, see the Remarks 5.8 and 5.9.

1.3 Harvesting New Families of Delocalized Gradient Gibbs Measures

In the case $S = \mathbb{Z}$, next to proper Gibbs measures, another class of consistent measures, namely the gradient Gibbs measures (GGM) have received much attention, see [11, 17, 20, 26]. Gradient Gibbs measures (as opposed to Gibbs measures) are measures which are only defined on $\mathbb{Z}^V/\mathbb{Z}$, which is the space of infinite-volume height configurations modulo a joint height shift (as opposed to the state space of absolute heights $\mathbb{Z}^V$ itself). Their defining property is the validity of the DLR consistency equation, but read only modulo joint height-shift, for details see Subsect. 4.1. As a consequence of the first part of our work we also obtain new families of gradient Gibbs measures ν_A^q, where $q \geq 2$ is an integer and $A \subset \mathbb{Z}^q$. They have the delocalization property (see Theorem 4.5), and hence do not stem from homogeneous Markov chain GMs, which are localized as irreducible, aperiodic and positive recurrent Markov processes (see e.g. [15, Theorem 4]). Therefore they are completely different in character from the localized GMs.

We construct these states ν_A^q as follows. The idea is to relate a GGM ν to the Gibbs measures μ_A^q in an associated q-state clock model on $\mathbb{Z}_q$ with an interaction Q^q built from the original interaction Q on $\mathbb{Z}^V$. This is done via an edge-wise resampling procedure, see Subsect. 4.2. The concentration properties of the clock-measures μ_A^q we constructed in our first main Theorem 3.1, then carries over to an interesting A-dependent correlation structure for the gradient measures ν_A^q, see Corollary 4.7 and the discussion below.

The remainder of the paper is organized as follows: In Sect. 2 we define our models. Section 3 then contains our main results regarding Gibbs measures for arbitrary finite concentration sets $A \subset S$. Section 4 discusses existence of delocalized gradient states with A-dependent correlation structure. Finally, Sect. 5 contains the proofs.

2 Definitions

In this section, we review some definitions and known facts which are necessary in order to formulate our main result.

2.1 Spin Configurations on the Cayley Tree

Let $\Gamma^d = (V, E)$ denote the *d-regular tree* or *Cayley tree* of order $d \geq 2$, where V is the countably infinite set of vertices and $E \subset V \times V$ is the set of (unoriented) edges. The term *d-regular tree* means that the graph Γ^d is connected without cycles and each vertex $x \in V$ has exactly $d + 1$ nearest neighbors, i.e., vertices which are connected to x by an edge.

A *path* connecting two vertices $x, y \in V$ is an ordered list of n edges

$$\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, y\}$$

where any two consecutive edges share a common vertex. The length of the unique shortest path from x to y defines the *graph distance* $\mathrm{d}(x, y)$.

Besides the set of unoriented edges E, we also consider the set $\vec{E}$ of oriented edges, which consists of the ordered pairs (x, y) of vertices such that $\{x, y\} \in E$.

For any subset $\Lambda \subset V$, we denote by Λ^c the complement of Λ in V and by

$$\partial\Lambda := \{x \in \Lambda^c \mid \mathrm{d}(x, y) = 1 \text{ for some } y \in \Lambda\}$$

the *outer boundary* of Λ. By $\Lambda \Subset V$ we indicate that Λ is a finite subset of V.

We set

$$E_\Lambda := \{\{x, y\} \in E \mid x, y \in \Lambda\}, \qquad \vec{E}_\Lambda := \{(x, y) \in \vec{E} \mid x, y \in \Lambda\}.$$

and note that the pair (Λ, E_Λ) is a subgraph of Γ^d, which is a subtree if and only if it is connected.

Let $(S, +)$ be a countable Abelian group, which we think of as the *local state space* of our system. Important particular cases are given by the lattices $S = \mathbb{Z}^k$, $k \in \mathbb{N}$, and by the finite cyclic groups $S = \mathbb{Z}_q$, $q \in \mathbb{N}$. We see S as a discrete group and endow it with the measurable structure given by the whole power set $\mathcal{P}(S)$.

By $\ell^p(S)$, $1 \le p \le \infty$, we denote the space of p-summable real valued functions on S, which is a Banach space with the norm

$$\|u\|_p := \left(\sum_{i \in S} |u(i)|^p\right)^{\frac{1}{p}} \quad \text{for } 1 \le p < \infty, \qquad \|u\|_\infty := \sup_{i \in S} |u(i)|.$$

We recall that

$$\ell^p(S) \subset \ell^q(S) \quad \text{and} \quad \|u\|_q \le \|u\|_p \quad \text{if} \quad 1 \le p \le q \le \infty.$$

When the group S is finite, the spaces $\ell^p(S)$ are of course independent of p, as they all coincide with $\mathbb{R}^S$, but the p-norms on them are different. Convolution on S is denoted by

$$(u * v)(i) := \sum_{j \in S} u(i - j)v(j).$$

A *spin configuration* $\omega = (\omega_x)_{x \in V}$ is a map from the set of vertices V to the local state space S, and the set of all spin configurations is denoted by $\Omega := S^V$. For any subset $\Lambda \subset V$ and any $\omega \in \Omega$, we set $\Omega_\Lambda := S^\Lambda$ and denote by $\omega_\Lambda \in \Omega_\Lambda$ the restriction of ω to Λ.

We endow each Ω_Λ with the product σ-algebra $\mathcal{F}_\Lambda$ generated by the *spin projections* $\sigma_y : \Omega_\Lambda \to S$, $\sigma_y(\omega) = \omega_y$, where $y \in \Lambda$, and denote by $\mathcal{F} := \mathcal{F}_V$ the product σ-algebra on Ω.

The set of all probability measures on the space $(\Omega, \mathcal{F})$ is denoted by $\mathcal{M}_1(\Omega, \mathcal{F})$. We call a probability measure $\mu \in \mathcal{M}_1(\Omega, \mathcal{F})$ *(spatially) homogeneous* if it is invariant under all automorphisms $\varphi : V \to V$ of the tree, i.e., $\mu = \mu \circ \varphi_*^{-1}$, where $\varphi_* : \mathcal{F} \to \mathcal{F}$ is the map $\varphi_*(A) := \{\omega \circ \varphi^{-1} \mid \omega \in A\}$.

Given a spatially homogeneous probability measure μ, we denote by

$$\pi_\mu(i) := \mu(\sigma_x = i), \qquad P_\mu(i, j) := \mu(\sigma_x = j \mid \sigma_y = i), \qquad \forall i, j \in S,$$

the *single state marginal* and the *transition matrix* induced by μ. Here, $x \in V$ is any vertex and $\{x, y\} \in E$ is any edge, but the above objects do not depend on these choices because the measure μ is assumed to be spatially homogeneous.

2.2 Tree-Indexed Markov Chains

The notion of a tree-indexed Markov chain as given in Chap. 12 of [14] is based on the definition of the past of an oriented edge. Given any vertex $v \in V$ we write

$$^{v}\vec{E} := \{(x, y) \in \vec{E} \mid \mathrm{d}(y, v) = \mathrm{d}(x, v) + 1\}$$

for the oriented edges pointing away from v. The *past of an oriented edge* $(x, y) \in \vec{E}$ is then defined by

$$]-\infty, xy] := \{v \in V \mid (x, y) \in {}^{v}\vec{E}\}.$$

In other words, it consists of those vertices $v \in V$ for which the shortest path from v to y contains x. A probability measure μ on $(\Omega, \mathcal{F})$ is then called a *tree-indexed Markov chain* (or simply a *Markov chain*) if for all oriented edges $(x, y) \in \vec{E}$ and all $i \in S$ we have

$$\mu(\sigma_y = i \mid \mathcal{F}_{]-\infty,xy]}) = \mu(\sigma_y = i \mid \mathcal{F}_x) \quad \mu - \text{a.s.}$$

2.3 Transfer Operators and Gibbs Measures

In this paper, by a *transfer operator* on (Γ^d, S) we mean a function

$$Q : S \to (0, +\infty)$$

which is symmetric (i.e., $Q(-i) = Q(i)$ for every $i \in S$) and belongs to $\ell^{\frac{d+1}{2}}(S)$. A more precise name for such an object would be *spatially homogeneous positive symmetric transfer operator*. Often, transfer operators are given in terms of a suitable symmetric interaction function $U : S \to [0, +\infty)$ as

$$Q(i) = e^{-\beta U(i)}, \tag{1}$$

where $\beta > 0$ should be interpreted as the inverse of a temperature.

A transfer operator Q induces the *Markovian gradient specification*

$$\gamma = \{\gamma_\Lambda : \mathcal{F} \times \Omega \to [0, 1]\}_{\Lambda \Subset V}$$

by the assignment

$$\gamma_\Lambda(\sigma_\Lambda = \tilde{\omega} \mid \omega) = \frac{1}{Z_\Lambda(\omega_{\partial\Lambda})} \left(\prod_{\{x,y\} \in E_\Lambda} Q(\tilde{\omega}_x - \tilde{\omega}_y) \right) \prod_{\substack{\{x,y\} \in E \\ x \in \Lambda, y \in \partial\Lambda}} Q(\tilde{\omega}_x - \omega_y), \tag{2}$$

for every $\Lambda \Subset V$, $\tilde{\omega} \in \Omega_\Lambda$ and $\omega \in \Omega$. Here, the *partition function* Z_Λ gives for every $\omega \in \Omega$ the normalization constant $Z_\Lambda(\omega) = Z_\Lambda(\omega_{\partial\Lambda})$ turning $\gamma_\Lambda(\cdot \mid \omega)$ into a probability measure on $(\Omega, \mathcal{F})$. The assumptions $Q > 0$ and $Q \in \ell^{\frac{d+1}{2}}(S)$ guarantee that such a partition function does exist. See Lemma 1 in [15] (here the case $S = \mathbb{Z}^k$ is considered, but the proof immediately generalizes to the case of an arbitrary countable group S). The quantities $Q(\tilde{\omega}_x - \tilde{\omega}_y)$ and $Q(\tilde{\omega}_x - \omega_y)$ are well defined for $\{x, y\} \in E$ because Q is assumed to be symmetric.

Remark 2.1 (a) Note that if $\tilde{Q} = cQ$ for some $c > 0$, then the Markovian gradient specifications which are induced by Q and $\tilde{Q}$ coincide. We shall often find it useful to normalize Q by requiring $Q(0) = 1$.

(b) The notion **gradient** refers to the fact that the specification γ is measurable with respect to the field of height increments (gradients). We will elaborate on the details in Sect. 4.1 below.

A *Gibbs measure* for a specification γ (a transfer operator Q, respectively) is by definition a probability measure μ on $(\Omega, \mathcal{F})$, such that for all $\Lambda \Subset V$ and all $A \in \mathcal{F}$ the *Dobrushin–Lanford–Ruelle (DLR) equation*

$$\mu(A) = \int \gamma_\Lambda(A \mid \omega)\mu(\mathrm{d}\omega)$$

holds true.

We denote the (possibly empty) convex set of Gibbs measures on $(\Omega, \mathcal{F})$ for a specification γ by $\mathcal{G}(\gamma)$. If $\mathcal{G}(\gamma)$ is not empty, then each of its elements can be written in a unique way as a convex combination of extremal elements (see e.g. Thm. 7.26 in [14]). On the tree, each extremal Gibbs measure for a Markovian specification is a tree-indexed Markov chain (e.g. Thm. 12.6 in [14] for this statement in the case in which S is finite; the proof generalizes to countable local state spaces). Writing ex C for the set of extremal points of a convex set C and $\mathcal{MG}(\gamma)$ for the set of Gibbs measures for γ which are Markov chains, the above statement reads

$$\mathrm{ex}\,\mathcal{G}(\gamma) \subset \mathcal{MG}(\gamma) \subset \mathcal{G}(\gamma). \tag{3}$$

A homogeneous Markov chain Gibbs measure μ for a Markovian gradient specification γ induced by a transfer operator Q has the transition matrix

$$P_\mu(i, j) = \frac{u(j)Q(i-j)}{\sum_{k\in S} u(k)Q(i-k)}, \quad i, j \in S.$$

Here, the function $u : S \to (0, \infty)$ is a *boundary law*, i.e., satisfying the fixed-point equation (5.1) that uniquely characterizes μ. For the understanding of our main result, Theorem 3.1 below, it is not necessary to know about boundary laws, hence we will give a brief introduction to boundary laws not earlier than in Sect. 5.1 below to prepare the proof of Theorem 3.1.

Remark 2.2 Assume that the transfer operator Q on the discrete Abelian group S is normalized by $Q(0) = 1$ and satisfies $Q(i) < 1$ for every $i \in S \setminus \{0\}$. Then Q induces the translation invariant "distance function"

$$\mathrm{dist}_Q(i, j) := -\log Q(i-j), \qquad \forall i, j \in S,$$

where the quotes refer to the fact that the function $\mathrm{dist}_Q(i, j)$ is symmetric, non-negative, zero if and only if $i = j$, but in general does not satisfy the triangle inequality. It is a genuine distance function if Q satisfies the log-superadditivity condition $Q(i+j) \geq Q(i)Q(j)$. If we write $Q = e^{-U}$, where U is a non-negative symmetric function on S vanishing only at zero, then this is equivalent to the subadditivity of U. This condition is indeed fulfilled by many models, including the SOS-model and the Log-model which are analysed in Appendix A below, while it is not satisfied, e.g., for the p-SOS model with potential $U(i) = |i|^p$ when $p > 1$.

3 Main Result

In this section, we present the main result of this paper regarding the existence of Markov-chain Gibbs measures on the regular Cayley d-tree with a countable Abelian group S as local state space. The Gibbs measures we find localize on an arbitrary finite subset of S. We also discuss some of its immediate implications and applications.

3.1 Existence of Gibbs Measures Localizing on Finite Sets

In order to formulate the main existence result, we need to introduce some functions of the order d of the Cayley tree and of the cardinality n of the subsets of S on which our Gibbs measure will localize. We denote by $\rho = \rho(d, n)$ the unique positive number such that

$$(d-1)\rho^{d+1} + dn\rho^{d-1} - n = 0,$$

and we set

$$\eta(d, n) := \frac{\rho - \rho^d}{(\rho^{d+1} + n)^{\frac{d}{d+1}}}.$$

Note that ρ belongs to the interval $(0, 1)$, so $\eta(d, n)$ is a positive number. We actually have the bounds $\rho(d, n) < d^{-\frac{1}{d-1}}$ and

$$d^{-\frac{1}{d-1}}\left(1 - \frac{1}{d}\right)(n+1)^{-\frac{d}{d+1}} \le \eta(d, n) \le d^{-\frac{1}{d-1}}\left(1 - \frac{1}{d}\right)n^{-\frac{d}{d+1}} < 1, \tag{4}$$

as shown in Lemma B.1 in Appendix B below. We can now state the main existence result of the paper:

Theorem 3.1 *Let $d \ge 2$ and $N \ge 1$ be integers. Assume that the transfer operator $Q \in \ell^{\frac{d+1}{2}}(S)$ is normalized by $Q(0) = 1$ and satisfies the condition*

$$\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \le \eta(d, N). \tag{5}$$

Then for every $A \subset S$ with $1 \le |A| \le N$ the Markovian gradient specification which is induced by Q on the regular d-tree with local state space S admits a spatially homogeneous Markov-chain Gibbs measure μ such that, denoting by

$$\pi_\mu : S \to [0, 1], \qquad \pi_\mu(i) := \mu(\sigma_x = i),$$
$$\Delta : S \to [0, 1], \qquad \Delta(i) := P_\mu(i, i),$$

the functions giving the single-site marginals of μ and the diagonal elements of the transition matrix P_μ, we have:

$$\sum_{i \in A^c} \pi_\mu(i) < \theta \min_A \pi_\mu, \qquad \text{with } \theta := \left(d^{\frac{1}{d-1}}\rho(d, |A|)\right)^{d+1} \in (0, 1), \tag{6}$$

$$\|\Delta|_{A^c}\|_{\frac{d+1}{d-1}} < \rho(d, |A|)^{d-1} < \frac{1}{d} < \min_A \Delta. \tag{7}$$

Moreover, setting $\epsilon := \|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}}$ and $n := |A|$, the following estimates hold:

(i) $\|\Delta|_{A^c}\|_{\frac{d+1}{d-1}} \le c_1 \epsilon^{d-1}$, *where* $c_1 = c_1(d, n) := \frac{\rho(d,n)^{d-1}}{\eta(d,n)^{d-1}}$.

(ii) $\min_A \Delta > 1 - c_2 \epsilon$, *where* $c_2 = c_2(d, n) := (d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}$.

(iii) $\sum_{i \in A^c} \pi_\mu(i) \le c_3 \epsilon^{d+1}$, *where* $c_3 = c_3(d, n) := \frac{d^{\frac{d+1}{d-1}}\rho(d,n)^{d+1}}{n\,\eta(d,n)^{d+1}}$.

(iv) $(1 - c_5\epsilon)\frac{1}{|A|} \le \pi_\mu|_A \le (1 - c_4\epsilon)^{-1}\frac{1}{|A|}$, *where* $c_4 = c_4(d, n) := \frac{d+1}{d-1}c_2(d, n)$ *and* $c_5 = c_5(d, n) := c_4(d, n) + \frac{\rho(d,n)^{d+1}}{n\,\eta(d,n)^{d+1}}$.

(v) $\pi_\mu(i) \geq \frac{1}{|A|}(1 - c_6\,\epsilon)\left(\sum_{j\in A} Q(i-j)\right)^{d+1}$ *for every* $i \in A^c$*, where* $c_6 = c_6(d,n) := d\,c_4(d,n) + \frac{\rho(d,n)^{d+1}}{n\,\eta(d,n)^{d+1}}$.

Some comments are in order. First of all notice that the Assumption (5) constrains the cardinality of A but involves neither the specific choice of the Abelian group S nor the way in which A sits in S. Thanks to (4), (5) implies that $Q(i) < 1$ for every $i \in S \setminus \{0\}$.

Property (6) tells us that the spin values in A are preferred by the Gibbs measure μ: the probability that a given vertex is not in A is smaller than the probability that it is in the least likely of the spin values of A. The bounds (iii) and (iv) control how the probability distribution π_μ giving us the single state marginals converges to the equidistribution on A as the $\frac{d+1}{2}$-norm of $Q - \mathbb{1}_{\{0\}}$ tends to zero.

Property (7) and its asymptotic Q-dependent refinements (i) and (ii) tell us that A is the “lazy” set of the Gibbs measure μ. Indeed, in the case $d = 2$ (7) says that if a vertex is in a state i belonging to the set A, then its neighbouring vertices will prefer to remain in i with probability larger than $\frac{1}{2}$; otherwise, they will prefer to change their state with probability larger than $\frac{1}{2}$. When the order d increases, the probability threshold has the smaller value $\frac{1}{d}$: if the number $d+1$ of vertices that influence the state at a given vertex gets larger, then a change of state becomes more probable for states in A^c, but possibly also for those in A.

The asymptotic bounds (i) and (ii) moreover quantify how the “laziness” of A gets stronger and stronger as the $\frac{d+1}{2}$-norm of $Q - \mathbb{1}_{\{0\}}$ tends to zero. In the typical case of a function Q of the form (1), this corresponds to the low temperature asymptotic $\beta \to +\infty$. Both the probability of changing state if i is not in A and the probability of keeping the same state for i in A tend to one.

Given any state i, the probability to go from i to some state in A^c along some edge also tends to zero as $\epsilon := \|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}}$ tends to zero. By (ii), this is clear for states i in A. For states i in A^c it can be shown as follows. Thanks to the formula

$$P_\mu(i,j) = \frac{\pi_\mu(j)^{\frac{d}{d+1}}\,Q(i-j)}{\left(Q * \pi_\mu^{\frac{d}{d+1}}\right)(i)}, \tag{8}$$

which is discussed in Remark 5.5 below, the Hölder inequality and the bound (iii) imply for each $i \in S$

$$\begin{aligned} P_\mu(i, A^c) &:= \sum_{j\in A^c} P_\mu(i,j) = \frac{1}{\left(Q * \pi_\mu^{\frac{d}{d+1}}\right)(i)} \sum_{j\in A^c} \pi_\mu^{\frac{d}{d+1}}(j)Q(i-j) \\ &\leq \frac{1}{\left(Q * \pi_\mu^{\frac{d}{d+1}}\right)(i)} \|\pi_\mu|_{A^c}\|_1^{\frac{d}{d+1}} \|Q\|_{d+1} \leq \frac{c_7}{\left(Q * \pi_\mu^{\frac{d}{d+1}}\right)(i)} \|Q\|_{d+1}\,\epsilon^d, \end{aligned}$$

where $c_7 := c_3^{\frac{d}{d+1}}$. This already implies our claim for any fixed $i \in S$. Together with the lower bound (v), for $i \in A^c$ we obtain the further estimate

$$P_\mu(i, A^c) \leq c_7(1 - c_6\,\epsilon)^{-\frac{d}{d+1}} |A|^{\frac{d}{d+1}} \|Q\|_{d+1} \Big(\sum_{j\in A} Q(i-j)\Big)^{-d} \epsilon^d. \tag{9}$$

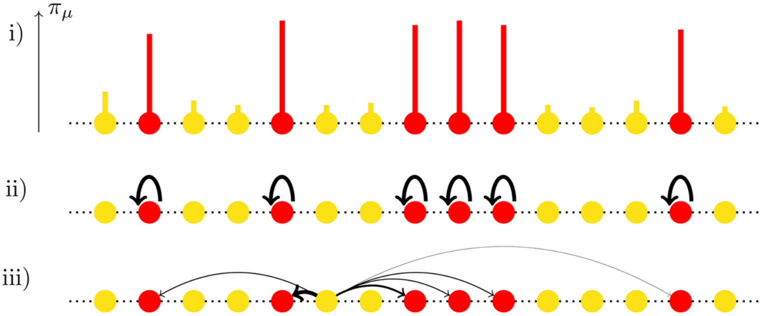


Fig. 1 The pictures show a part of the set $S = \mathbb{Z}$ with the bars in picture (i) marking the distribution of single-site marginals of the Gibbs measure μ from Theorem 3.1. Here, the red coloured (dark, if you are looking at a black and white version of this picture) circles with long bars belong to $A \Subset S$. If the chain is in a state in A, then it prefers to stay in this state, see (ii). On the other hand, being in a state which does not belong to A, the chain prefers jumps into states in A, with weights as indicated by the arrows in (iii). Under suitable decaying conditions on Q, (8) implies that shortest jumps are more likely (Color figure online)

Recalling that dist_Q denotes the "distance function" discussed in Remark 2.2, we find

$$\Big(\sum_{j\in A} Q(i-j)\Big)^{-d} \le \big(\max_{j\in A} Q(i-j)\big)^{-d} = \min_{j\in A} Q(i-j)^{-d} = \min_{j\in A} e^{-d\,\log Q(i-j)}$$
$$= \min_{j\in A} e^{d\,\mathrm{dist}_Q(i,j)} = e^{d\,\min_{j\in A}\mathrm{dist}_Q(i,j)} = e^{d\,\mathrm{dist}_Q(i,A)}.$$

Therefore, (9) implies that for every $i \in A^c$ we have the estimate

$$P_\mu(i, A^c) \le c_7(1-c_6\,\epsilon)^{-\frac{d}{d+1}}|A|^{\frac{d}{d+1}}\|Q\|_{d+1}e^{d\,\mathrm{dist}_Q(i,A)}\,\epsilon^d,$$

which tells us that as ϵ tends to zero $P_\mu(i, A^c)$ converges to zero uniformly for i in any subset of A^c whose elements have uniformly bounded distance from A.

A graphical illustration of the above discussion is given in Fig. 1.

Remark 3.2 For $|A| = 1$ and $S = \mathbb{Z}$, the existence of a Gibbs measure as in the above theorem has been proven in [15] under similar but not exactly equivalent assumptions on Q.

Remark 3.3 (*Uniqueness*) If $|A| = 1$ and $\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \le \eta(d,1)$, then we can further show that the spatially homogeneous Markov-chain Gibbs measure μ satisfying (7) is unique. See Remark 5.10 below. For larger sets A, we do not know whether the Gibbs measure μ satisfying (7) is necessarily unique under the assumption $\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \le \eta(d,|A|)$. By strengthening this assumption, we could get the following uniqueness statement: There exist positive numbers $\eta'(d,n)$ and $\delta(d,n)$ such that if A is a finite subset of S and $\|Q-\mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \le \eta'(d,|A|)$ then the Markovian gradient specification which is induced by Q on the regular d-tree with local state space S has a *unique* spatially homogeneous Markov-chain Gibbs measure μ whose single-site marginal probability distribution π_μ satisfies

$$\Big\|\pi_\mu - \tfrac{1}{|A|}\mathbb{1}_A\Big\|_1 < \delta(d,|A|).$$

See Remark 5.9 below.

Remark 3.4 (*Affine independence*) Assuming that $\|Q - \mathbb{1}_{\{0\}}\| \le \eta(d,N)$ holds, the above theorem gives us a family of Gibbs measures $\{\mu_A\}_{A\in\mathcal{A}_N}$, where $\mathcal{A}_N$ denotes the set of all subsets A of S with $1 \leq |A| \leq N$. Condition (7) implies that these measures are pairwise distinct. More is actually true: none of the measures in the above family is a convex combination of the other ones, so each μ_A should be though as irreducible. Indeed, this is a direct consequence of the fact that a non-trivial convex combination of spatially homogeneous Markov-chain Gibbs measures is never a Markov-chain Gibbs measure. In the case of a finite local state space S, this follows from Corollary 12.18 in [14], but the proof directly generalizes to the case of a countably infinite state space, as all occurring sums are finite by the normalizability assumption on our boundary laws and all terms are strictly positive by the assumption of positivity of Q. In particular, we obtain that when S is an infinite Abelian group and $\| Q - \mathbb{1}_{\{0\}} \| \leq \eta(d, 2)$ holds, then the convex set of all Gibbs measures $\mathcal{G}(\gamma)$ of the Gibbs specification induced by Q is infinite dimensional also after modding out the action on it which is given by translations on S.

The proof of Theorem 3.1 is based on an existence result for positive solutions $u \in \ell^{\frac{d+1}{d}}(S)$ of the normalized boundary law equation

$$u = (Q * u)^d,$$

which are suitably concentrated near the finite subset A. Boundary laws are discussed in Sect. 5.1 and the proof of the existence result, which is based on a combined use of the contraction mapping theorem and Brouwer's fixed point theorem, is discussed in Sect. 5.2 and Appendix B. How to derive Theorem 3.1 from this result is explained in Sect. 5.3.

The next two examples show how the Assumption (5) translates for some concrete models.

Example 3.5 (*SOS model*) Consider the case $S = \mathbb{Z}$ and $Q(i) = e^{-\beta|i|}$, where β is a positive parameter modelling the inverse temperature. Assumption (5) is fulfilled whenever

$$\beta \geq \frac{\log d}{d} + \frac{b}{d} + \log N,$$

where b is a suitable positive number. See Example A.1 in Appendix A for the computations leading to this and for more precise results. Note that for any fixed $N \geq 1$ the lower threshold for β has size of the order $\log N$ for $d \to \infty$. When $N = 1$, this threshold converges to zero and is asymptotic to $\frac{\log d}{d}$ for $d \to \infty$.

Example 3.6 (*Log potential*) Consider the case $S = \mathbb{Z}$ and $Q(i) = \frac{1}{(1+|i|)^\beta}$. One can check that (5) is fulfilled whenever

$$\beta \geq \frac{1}{\log 2} \left(\frac{\log d}{d} + \frac{b}{d} + \log N \right),$$

where b is a suitable positive number. See Example A.2 in Appendix A for details. Up to the multiplication by the factor $\frac{1}{\log 2}$, the asymptotic of this threshold is analogous to the one we found in Example 3.5 for the SOS model.

4 An Application to the Existence of Delocalized Gradient Gibbs Measures

In this section, we show how Theorem 3.1 implies the existence of suitable *gradient Gibbs measures* with *height-dimension* $\mathbb{Z}$. The sets A which appear as a discrete parameter of the measures and which played the role of localization sets for the *Gibbs measures* of the previous section will now acquire a different role. Indeed, for the delocalized gradient Gibbs measures we discuss in this section, there is no invariant single-site probability distribution in which the height variables would localize. Instead, the sets A will govern the structure of most probable increments along the edges, in a way that we will describe now. We first review the necessary definitions.

4.1 Gradient Gibbs Measures

The notion of a gradient Gibbs measure for lattice models has been established in [11] and further exploited in [26]. In this subsection we present an adaption to the situation on the tree, which is based on [15, 22]. Consider the case $S = \mathbb{Z}$, in which we interpret a spin configuration $\omega \in \Omega$ as a height configuration and denote the local state space $\mathbb{Z}$ as the *height-dimension* of the model.

Define the gradient projection $\nabla : \Omega \to \mathbb{Z}^{\vec{E}}; (\omega_x)_{x\in V} \mapsto (\omega_y - \omega_x)_{(x,y)\in\vec{E}}$. Then

$$\Omega^\nabla := \nabla(\Omega) = \{\zeta \in \mathbb{Z}^{\vec{E}} \mid \zeta_{(x,y)} = -\zeta_{(y,x)} \text{ for all } (x,y) \in \vec{E}\}$$

is the set of all *gradient configurations*. For any oriented edge $b = (x, y) \in \vec{E}$ let $\eta_b : \Omega^\nabla \to S$, $\eta_b(\zeta) := \zeta_b$ denote the *gradient spin projection* along b. By construction, $\eta_{(x,y)} \equiv -\eta_{(y,x)}$ whenever $(x, y) \in \vec{E}$. We endow Ω^∇ with the product σ-algebra $\mathcal{F}^\nabla$ generated by all gradient spin projections, i.e., $\mathcal{F}^\nabla = \sigma(\eta_b \mid b \in \vec{E})$.

Let $x_0 \in V$ be any fixed vertex. By connectedness of the tree and absence of cycles, prescription of any fixed height $s \in \mathbb{Z}$ at x_0 gives rise to a well-defined injective map

$$\Omega^\nabla \to \Omega, \qquad \zeta \mapsto \omega \quad \text{where } \nabla\omega = \zeta,\ \omega(x_0) = s.$$

A gradient configuration on the tree can be thus considered as a *relative height configuration* where two height configurations are equivalent iff one is obtained from the other one by a joint height shift $\theta_i(j) := j + i$. Hence we have the identification

$$\Omega^\nabla = \Omega/\mathbb{Z}.$$

Similar to the situation on the lattice [26], we may think of $\mathcal{F}^\nabla$ (or more precisely the σ-algebra on Ω generated by ∇) as the set of all events in $\mathcal{F}$ which are invariant under all joint height shifts θ_i.

To let the Markovian gradient specification γ for height configurations act on gradient configurations one has to consider that due to the absence of cycles on the tree the complement of any finite subtree (Λ, E_Λ) decomposes into disjoint subtrees. This means that

$$\zeta_{\Lambda^c} \in \Omega^\nabla_{\Lambda^c} := \{\zeta \in \mathbb{Z}^{\vec{E}^c_\Lambda} \mid \zeta_{(x,y)} = -\zeta_{(y,x)} \text{ for all } (x,y) \in \vec{E}^c_\Lambda\}$$

does not determine the relative heights at the boundary as an element of $\mathbb{Z}^{\partial\Lambda}/\mathbb{Z}$, i.e., up to a joint height shift at the boundary.

Thus, the appropriate outer gradient σ-algebra $\mathcal{T}^\nabla_\Lambda$ has to implement both the information on the gradient spin variables outside Λ and the information on the relative heights at the boundary. As the relative heights of the boundary are uniquely determined by the gradients inside $\Lambda \cup \partial\Lambda$ (each two vertices at the boundary are connected by a unique path in $\Lambda \cup \partial\Lambda$), these relative heights at the boundary can be expressed in terms of an $\mathcal{F}^\nabla$ measurable function $[\cdot]_{\partial\Lambda} : \Omega^\nabla \to \mathbb{Z}^{\partial\Lambda}/\mathbb{Z}$. Hence

Definition 4.1 (*Definition 2.3 in* [22]) The *gradient-σ-algebra outside* Λ is defined as

$$\mathcal{T}_\Lambda^\nabla := \sigma((\eta_b)_{b\cap\Lambda=\emptyset},\ [\eta]_{\partial\Lambda}) \subset \mathcal{F}^\nabla. \tag{10}$$

With that framework given, let us now make precise in which way a Markovian gradient specification γ acts on the gradients.

Remark 4.2 (*cp. Definition 2.4 in* [22]) A Markovian gradient specification γ can be regarded as a stochastic kernel $(\gamma_\Lambda)_{\Lambda\Subset V}$ from $(\Omega^\nabla, \mathcal{T}_\Lambda^\nabla)$ to $(\Omega^\nabla, \mathcal{F}^\nabla)$ by means of

$$\int F(\rho)\gamma_\Lambda(\mathrm{d}\rho \mid \zeta) = \int F(\nabla\varphi)\gamma_\Lambda(\mathrm{d}\varphi \mid \omega) \tag{11}$$

for all bounded $\mathcal{F}^\nabla$-measurable functions F, where $\omega \in \Omega$ is any height configuration with $\nabla\omega = \zeta$.

To ease readability, in the following we will write γ' instead of γ when we let the Markovian gradient specification γ act on the gradients. Also, we will use the notion **gradient Gibbs specification** to make precise that we let γ act on the gradients.

Finally, the DLR-equation for gradient measures on the tree reads:

Definition 4.3 (*Definition 2.5 in* [22]) A measure $\eta \in \mathcal{M}_1(\Omega^\nabla)$ is called a *gradient Gibbs measure (GGM)* if it satisfies the DLR equation

$$\int \eta(d\zeta)F(\zeta) = \int \eta(\mathrm{d}\zeta) \int \gamma'_\Lambda(\mathrm{d}\tilde{\zeta} \mid \zeta)F(\tilde{\zeta}) \tag{12}$$

for every finite subtree (Λ, L_Λ) and for all bounded continuous functions F on Ω^∇.

4.2 From Gibbs-Measures for Clock-Models to Integer-Valued Gradient Gibbs Measures

Consider the case $S = \mathbb{Z}$ and let $q \geq 2$ be an integer. Assume that $Q \in \ell^1(\mathbb{Z})$. Then

$$Q_q(\bar{i}) := \sum_{j\in\mathbb{Z}} Q(i + qj)$$

is a well defined function on the Abelian group $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$, which we think of as a “fuzzy” transfer operator on $\mathbb{Z}_q$. Then the Gibbsian specification γ^q on $(\mathbb{Z}_q)^V$ associated with Q_q via (2) describes a clock model. As shown in [15, 16, 22], any Gibbs measure on $(\mathbb{Z}_q)^V$ for Q_q can be assigned an (integer-valued) gradient Gibbs measure on Ω^∇. In this subsection we briefly summarize the construction as described in [16].

For any $\bar{i} \in \mathbb{Z}_q$ define a conditional distribution $\rho_Q^q(\cdot \mid \bar{i})$ on $\mathbb{Z}$ equipped with the power set $\mathcal{P}(\mathbb{Z})$ by

$$\rho_Q^q(j \mid \bar{i}) := \mathbb{1}_{\{\bar{i}\}}(j)\frac{Q(j)}{Q^q(\bar{i})}. \tag{13}$$

Then we can define a map $T_Q^q : \mathcal{M}_1(\mathbb{Z}_q^V, \mathcal{P}(\mathbb{Z}_q)^{\otimes V}) \to \mathcal{M}_1(\Omega^\nabla, \mathcal{F}^\nabla)$ from q-spin measures on vertices to integer-valued gradient measures in terms of the following two-step procedure:

$$T_Q^q(\mu)(\eta_\Lambda = \zeta_\Lambda) = \sum_{\bar{\omega}_\Lambda\in\mathbb{Z}_q^\Lambda} \mu(\bar{\sigma}_\Lambda = \bar{\omega}_\Lambda) \prod_{\substack{(x,y)\in{}^w\vec{L} \\ x,y\in\Lambda}} \rho_Q^q(\zeta_{(x,y)} \mid \bar{\omega}_y - \bar{\omega}_x), \tag{14}$$

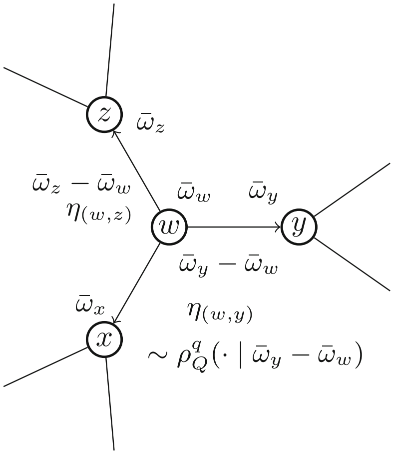


Fig. 2 Construction of the measure $T^q_Q(\mu^q)$: In the first step, a $\mathbb{Z}_q$-valued configuration $\bar\omega$ is drawn from μ^q. Conditional on the $\mathbb{Z}_q$-valued increment along the respective edge, the integer-valued gradient η is then distributed with respect to ρ^q_Q (13)

where $\Lambda \subset V$ is any finite connected set and $w \in V$ is an arbitrary fixed vertex.

The assignment (14) describes a two-step procedure, where in the first step $\mathbb{Z}_q$-valued configurations are drawn from μ and in the second step integer-valued gradients are edgewise independently sampled conditioned on the $\mathbb{Z}_q$-valued increment along the respective edge. See also Fig. 2.

Then the following holds true without any assumption on spatial homogeneity:

Theorem 4.4 (Theorem 4.1 in [22], Theorem 2 in [16]) T^q_Q *maps Gibbs measures on* $\mathbb{Z}_q^V$ *for the fuzzy specification* γ^q *to gradient Gibbs measures on* Ω^∇ *for the gradient Gibbs specification* γ' (11).

The fact that Gibbs measures are mapped to gradient Gibbs measures as described in Theorem [18] is a rare example for the preservation of the quasilocal Gibbs property, as it occurs throughout the whole phase diagram. In general, local maps tend to destroy the Gibbs property in strong coupling regions, see e.g., [18, 28].

The map T^q_Q as defined in (14) has two important properties: First, as we will see below, any integer-valued gradient Gibbs measure $\eta \in T^q_Q(\mathcal{G}(\gamma^q))$ is delocalized.

Second, for any gradient Gibbs measure ν^q which is given as the image of a homogeneous Markov-chain Gibbs measure μ^q on $\mathbb{Z}_q$ we can identify both the period q and the distribution of the underlying Markov chain μ from ν^q up to certain symmetries. This motivates to call such a gradient Gibbs measure ν^q a *delocalized gradient Gibbs measure of height-period* q.

The general delocalization statement of Theorem 4.5 below directly follows from Proposition 1 in [16] for the specific case of ν being constructed from a Markov-chain Gibbs measure in combination with extremal decomposition in $\mathcal{G}(\gamma^q)$. It crucially employs the nature of ν restricted to a branch of the tree being a random walk in a random environment. The less general identifiability result has already been proved in [15].

Theorem 4.5 *Let* $q \geq 2$ *be an integer. Then any* ν *in* $T^q_Q(\mathcal{G}(\gamma^q)) \subset \mathcal{G}(\gamma')$ *delocalizes in the sense that* $\nu(W_n = k) \overset{n\to\infty}{\to} 0$ *for any total increment* W_n *along a path of length* n *and any* $k \in \mathbb{Z}$.

Note that Theorem 4.5 holds without the assumption of homogeneity, while for the identifiability result below we have to restrict to homogeneous measures.

Theorem 4.6 (Theorem 5 and Corollary 1 in [15]) *Let $q \geq 2$ be an integer. Let $\nu^q \in T_Q^q(\mathcal{G}(\gamma^q))$ be such that $\nu^q = T_Q^q(\mu^q)$ for some homogeneous Markov-chain Gibbs measure μ^q on $\mathbb{Z}_q^V$. Then the period q is uniquely determined by ν up to integer-valued multiples. Moreover, the distribution of μ^q is uniquely determined by ν^q up to a joint height shift θ_i on $\mathbb{Z}_q$.*

Proof of Theorem 4.5 By Proposition 1 in [16], we already know that for any (not necessarily homogeneous) q-state Markov-chain Gibbs measure $\mu^q \in \mathcal{G}(\gamma^q)$ the associated integer-valued gradient Gibbs measure $T_Q^q(\mu^q)$ delocalizes in the sense of $T_Q^q(\mu^q)(W_n = k) \overset{n\to\infty}{\to} 0$ for any fixed $k \in \mathbb{Z}$.

Now let $\mu \in \mathcal{G}(\gamma^q)$ be any Gibbs measure on $(\mathbb{Z}_q)^V$. By extremal decomposition, we have a unique probability measure w_μ on $(\mathrm{ex}\,\mathcal{G}(\gamma^q), \mathrm{ev}\,\mathrm{ex}\,\mathcal{G}(\gamma^q))$, such that

$$\mu(\cdot) = \int_{\mathrm{ex}\,\mathcal{G}(\gamma)} w_\mu(\mathrm{d}\tilde{\mu})\tilde{\mu}(\cdot). \tag{15}$$

Here, $\mathrm{ev}\,\mathrm{ex}\,\mathcal{G}(\gamma)$ denotes the *evaluation σ-algebra* on $\mathrm{ex}\,\mathcal{G}(\gamma)$ generated by the *evaluations* of the form $\pi_A : \tilde{\mu} \mapsto \tilde{\mu}(A)$, where $A \in \mathcal{P}(\mathbb{Z}_q)^V$ is a fixed event.

Recalling the definition of T_Q^q in (14), linearity of the integral gives

$$\begin{aligned} T_Q^q(\mu)(W_n = k) &= T_Q^q\left(\int_{\mathrm{ex}\,\mathcal{G}(\gamma)} w_\mu(\mathrm{d}\tilde{\mu})\tilde{\mu}\right)(W_n = k) \\ &= \int_{\mathrm{ex}\,\mathcal{G}(\gamma)} w_\mu(\mathrm{d}\tilde{\mu}) \left(T_Q^q(\tilde{\mu})(W_n = k)\right) \overset{n\to\infty}{\to} 0, \end{aligned} \tag{16}$$

by Proposition 1 in [15] and dominated convergence (e.g., Corollary 6.26 in [19]) with integrable majorant $g(\tilde{\mu}) = 1$. □

4.3 Existence of Height-Periodic Gradient Gibbs Measures

The existence result for localized Gibbs measures of Theorem 3.1 above implies an existence criterion for an associated family of height-periodic gradient Gibbs measures:

Corollary 4.7 *Consider the d-regular tree with $d \geq 2$. Let the integer $q \geq 2$ be a fixed height-period and let $Q \in \ell^1(\mathbb{Z})$ be a spatially homogeneous positive transfer operator normalized by $Q(0) = 1$. Let $N \in \{1, \dots, q-1\}$ and assume that the normalized fuzzy transfer operator Q_q on $\mathbb{Z}_q$ satisfies*

$$\|Q_q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \leq \eta(d, N).$$

Then for every $A \subset \mathbb{Z}_q$ with $1 \leq |A| \leq N$ there exists a spatially homogeneous q-periodic delocalized gradient Gibbs measure ν_A^q of the form

$$\nu_A^q = T_{Q_q}(\mu_A),$$

where μ_A is the homogeneous Markov-chain Gibbs measure on $\mathbb{Z}_q$ with lazy set A given by Theorem 3.1.

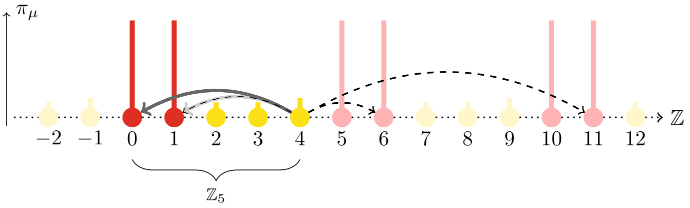


Fig. 3 The gradient Gibbs measure ν_A^5 associated to the subset $A := \{0, 1\} \subset \mathbb{Z}_5$: two main transitions of the fuzzy chain μ with a lighter and a darker colour saturation and the distribution of single-site marginals π_μ of μ concentrated on $\{0, 1\}$. The lighter coloured jump from 4 to 1 of the chain μ allows jumps of height $-3 + 5\mathbb{Z}$ for ν, whose conditional distribution is according to $\rho_Q^5(\cdot \mid \bar{2})$ (see (13)). Three of these possible jumps are marked by the dashed black arrows

Remark 4.8 The similar statement can be proved for the q-Potts model in zero external field by explicit computations, as they can be reduced to one-dimensional equations in that case. This relies on the very special property that in the Potts model we necessarily have

$$\pi = x\mathbb{1}_A + y\mathbb{1}_{A^c},$$

which follows from the boundary law equation using permutation symmetry, see [23]. This does not hold true anymore for more general clock models, and the equations become truly q-dimensional.

An illustration of the construction of such a gradient Gibbs measure ν in the case $q = 5$ is given in Fig. 3.

How do the period q and the concrete choice of the lazy set $A \subset \mathbb{Z}_q$ affect the associated gradient Gibbs measure ν_A^q? The answer lies in (the proof of) Theorem 4.6 above: Considering the sequence of empirical distributions of increments along a branch of the tree gives in particular the sequence of empirical distributions of increments of the homogeneous fuzzy chain μ_A^q. By the ergodic Theorem for Markov chains this sequence converges. The knowledge of the limit is equivalent to the knowledge of the stationary distribution on $\mathbb{Z}_q$ modulo cyclic shift, from which the set A can be read off. In particular, also the period q can be recovered. For more details, see also the proof of Corollary 1 in [16].

More can be said in the present case. Consider the joint empirical distribution along a branch of the tree $x_1, x_2, \dots$ for fuzzy spin values and integer-valued increments of the form

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{\bar{\sigma}_{x_i},\sigma_{x_{i+1}}-\sigma_{x_i}} \tag{17}$$

which is a random measure on $\mathbb{Z}_q \times \mathbb{Z}$. It is important in the case of delocalized gradient Gibbs measures to consider fuzzy spins $\bar{\sigma}_{x_i}$ in the first entry, as the empirical measures for spins σ_{x_i} would not converge.

We claim that there is the ν_A^q-a.s. convergence

$$\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{\bar{\sigma}_{x_i},\sigma_{x_{i+1}}-\sigma_{x_i}}\right)(\bar{a}, c) \overset{n\to\infty}{\longrightarrow} \frac{\pi_{\mu_A^q}(\bar{a})\pi_{\mu_A^q}(\bar{a}+\bar{c})^{\frac{d}{d+1}}}{Q^q * \pi_{\mu_A^q}^{\frac{d}{d+1}}(\bar{a})} Q(c). \tag{18}$$

Before we prove this statement, let us discuss what it tells us about the correlation structure of the gradient state. First note that jump probabilities of increment size c, for fixed mod q fuzzy classes $\bar{a}, \bar{c}$ depend only on the multiplicative factor $Q(c)$, which strongly suppresses large jumps. On the other hand, recall that by the concentration bounds of Theorem 3.1 the mod q fuzzy measure $\pi_{\mu_A^q}$ concentrates strongly on the set $A \subset \mathbb{Z}_q$ where it equals up to small errors the equidistribution. So, (18) means that the delocalized measure ν_A^q inherits a structure from the underlying measure μ_A^q, in which fuzzy jumps occur mostly from A to A, while arbitrarily large jumps in $\mathbb{Z}$ occur, but are suppressed and modulated via the summable Q. An example is discussed in Fig. 3.

Finally, to prove the a.s. convergence (18) for a fixed pair $(\bar{a}, c)$, denote by $\bar{c}$ for the mod-q class of c and use the hidden Markov model structure (14) of the gradient measure to write the l.h.s. of (18) in the product form

$$\frac{|\Lambda_n(\bar{a},\bar{c})|}{n} \times \frac{1}{|\Lambda_n(\bar{a},\bar{c})|} \sum_{i \in \Lambda_n(\bar{a},\bar{c})} Y_i(c) \tag{19}$$

where $\Lambda_n(\bar{a},\bar{c}) = \{1 \le i \le n, (\bar{\sigma}_i, \bar{\sigma}_{i+1} - \bar{\sigma}_i) = (\bar{a},\bar{c})\}$. Here the variables $Y_i(c)$ are independent Bernoulli with success probability $\rho_Q^q(c \mid \bar{c}) = Q(c)/Q^q(\bar{c})$.

By the Birkhoff a.s. ergodic theorem applied to the first factor in (19) which we recognize as the pair empirical distribution of the irreducible hidden Markov chain μ_A^q, there is a set of full measure for μ_A^q (and hence for ν_A^q) such that the first term in the product converges to its expectation. On this full measure set, in particular $|\Lambda_n(\bar{a},\bar{c})| \uparrow \infty$ by positivity of Q^q, and conditionally on that we can apply the SLLN for the independent variables $Y_j(c)$ to see that also the second term converges to its expectation $\frac{Q(c)}{Q^q(\bar{c})}$. Plugging in these expectations the claimed a.s. limit of (18) follows.

As suggested by a referee, besides the considerations on the empirical distributions, we also provide an expression similar to (18) to describe the covariance between two gradient spin variables $\eta_{(x,y)}$ and $\eta_{(u,v)}$ at disjoint edges (x, y) and (u, v). We will restrict to the case where (x, y) and (u, v) point in the opposite direction in that $n+1 = d(x,v) = d(x,u)+1$, where the other case can be treated similarly. Note that by construction the measure ν_A^q has a zero tilt, i.e., $\nu(\eta_{(x,y)}) = 0 = \nu(\eta_{(u,v)})$. Let $(\bar{\sigma}_w)_{w \in V}$ denote the $\mathbb{Z}_q$-valued spin variables of the internal fuzzy Markov chain μ_A with transition matrix P_μ. Then conditioning on the spin value at site x gives:

$$\begin{aligned}
&Cov(\eta_{(x,y)}, \eta_{(u,v)}) \\
&\quad = \sum_{s,t \in \mathbb{Z}} st \left(\sum_{\bar{a} \in \mathbb{Z}_q} \pi(\bar{a}) \nu_A^q(\eta_{(x,y)} = s \mid \bar{\sigma}_x = \bar{a}) \sum_{\bar{b} \in \mathbb{Z}_q} (P_\mu)^n(\bar{a},\bar{b}) \nu_A^q(\eta_{(u,v)} = t \mid \bar{\sigma}_u = \bar{b}) \right) \\
&\quad = \sum_{\bar{a} \in \mathbb{Z}_q} \pi(\bar{a}) \sum_{s \in \mathbb{Z}} \frac{s Q(s) \pi^{\frac{d}{d+1}}(\bar{a}+\bar{s})}{\sum_{k \in \mathbb{Z}} Q(k) \pi^{\frac{d}{d+1}}(\bar{a}+\bar{k})} \sum_{\bar{b} \in \mathbb{Z}_q} (P_\mu)^n(\bar{a},\bar{b}) \sum_{t \in \mathbb{Z}} \frac{t Q(t) \pi^{\frac{d}{d+1}}(\bar{b}+\bar{t})}{\sum_{k \in \mathbb{Z}} Q(k) \pi^{\frac{d}{d+1}}(\bar{b}+\bar{k})}.
\end{aligned}$$

This shows explicitly how the localization property of the single-site marginal, and the exponential decorrelation of the fuzzy chain along a branch, enter into the correlation structure of the full gradients.

5 Proof of Theorem 3.1

5.1 Boundary Laws and Gibbs Measures

As established in [30], tree-indexed Markov-chain Gibbs measures for nearest-neighbour interactions and a countable local state space can be described in terms of the solutions to a recursive system of boundary law equations on the tree. In this subsection, we briefly outline this formalism for the specific case of spatially homogeneous Gibbs measures for gradient interactions on the d-regular tree.

Definition 5.1 A spatially homogeneous boundary law for a transfer operator Q is a positive function $u \in \ell^{\frac{d+1}{d}}(S)$ such that

$$u = c\,(Q * u)^d \tag{20}$$

for some $c > 0$.

Remark 5.2 If (u, c) is a solution of (20) and a is any positive number, then $v := au$ satisfies

$$v = c'(Q * v)^d$$

with $c' = a^{1-d}c$, and hence v is also a boundary law. Boundary laws differing by a multiplicative constant are considered to be equivalent. By multiplying u by a suitable constant, we can always assume that $c = 1$ in (20).

Now, the relation between boundary laws and tree-indexed Markov chains reads:

Theorem 5.3 (See Theorem 3.2 in [30]) *Let Q be a transfer operator. Then for the Markov specification γ associated to Q we have:*

(i) Each spatially homogeneous boundary law u for Q defines a unique spatially homogeneous tree-indexed Markov-chain Gibbs measure $\mu \in \mathcal{MG}(\gamma)$ with marginals

$$\mu(\sigma_{\Lambda\cup\partial\Lambda} = \omega) = \frac{1}{Z_\Lambda} \prod_{y\in\partial\Lambda} u(\omega_y) \prod_{\substack{\{x,y\}\in L \\ \{x,y\}\cap\Lambda\neq\emptyset}} Q(\omega_y - \omega_x), \tag{21}$$

for any connected set $\Lambda \Subset V$ and $\omega \in S^{\Lambda\cup\partial\Lambda}$, where Z_Λ is the normalization constant which turns μ into a probability measure.

(ii) Conversely, every spatially homogeneous tree-indexed Markov-chain Gibbs measure $\mu \in \mathcal{MG}(\gamma)$ admits a representation of the form (21) in terms of a spatially homogeneous boundary law u which is unique up to a constant positive factor.

We note that the boundary law equation guarantees that (21) describes a projective family of finite-volume marginals, whereas the summability condition $u \in \ell^{\frac{d+1}{d}}(S)$ gives us the finiteness of these finite-volume marginals.

From (21) and (20), we can easily determine the single-site marginals and the transition matrices of the spatially homogeneous Gibbs measure that is determined by a boundary law:

Proposition 5.4 *Let u be a spatially homogeneous boundary law for the transfer operator Q and let μ be the corresponding spatially homogeneous tree-indexed Markov-chain Gibbs measure. Then*

$$\pi_\mu(i) = \frac{u(i)^{\frac{d+1}{d}}}{\|u^{\frac{d+1}{d}}\|_1}, \qquad P_\mu(i, j) = \frac{u(j)Q(i-j)}{(Q * u)(i)},$$

for every $i, j \in S$.

Remark 5.5 From the identities of Proposition 5.4 we deduce the formula

$$P_\mu(i,j) = \frac{\pi_\mu(j)^{\frac{d}{d+1}} Q(i-j)}{\big(Q * \pi_\mu^{\frac{d}{d+1}}\big)(i)}$$

which we used in (8).

5.2 Existence of Solutions of the Boundary Law Equation

Let $d \geq 2$ be a positive integer and $Q \in \ell^{\frac{d+1}{2}}(S)$ be a positive function, which we normalize by assuming that $Q(0) = 1$. In this section, we wish to discuss the existence of positive solutions $u \in \ell^{\frac{d+1}{d}}(S)$ of the normalized boundary law equation

$$u = (Q * u)^d. \tag{22}$$

More precisely, given a finite subset $A \subset S$ we wish to find a solution $u \in \ell^{\frac{d+1}{d}}(S)$ close to $\mathbb{1}_A$ provided that $\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}}$ is small enough. The existence and uniqueness of such a solution could be deduced from the implicit mapping theorem starting from the fact that for $Q = \mathbb{1}_{\{0\}}$ the function $u = \mathbb{1}_A$ is indeed a solution. See Remark 5.9 below for more details. However, a naive application of this argument would require a very strong smallness assumption on $Q - \mathbb{1}_{\{0\}}$ and would not give the precise information on $u - \mathbb{1}_A$ that we need for proving Theorem 3.1. In order to obtain a better quantitative result we argue as follows.

First, it is convenient to set $u = x^d$ and rewrite the Eq. (22) as

$$x = Q * x^d, \tag{23}$$

where x is a positive element of $\ell^{d+1}(S)$. We split Q as

$$Q = \mathbb{1}_{\{0\}} + q,$$

and rewrite (23) as

$$x - x^d = q * x^d. \tag{24}$$

The reformulation (24) shows that every positive solution x takes values in the interval $(0, 1)$.

We fix a finite subset $A \subset S$ and look for solutions $x \in \ell^{d+1}(S)$ of (24) which are close to 1 on A and close to 0 on its complement A^c. More precisely, we denote by

$$\lambda_d := d^{-\frac{1}{d-1}} \in (0,1) \tag{25}$$

the point at which the function

$$\varphi_d : [0, +\infty) \to \mathbb{R}, \qquad \varphi_d(r) = r - r^d,$$

achieves its maximum and look for solutions $x \in \ell^{d+1}(S)$ of (24) such that

$$A = \{i \in S \mid \lambda_d < x(i) < 1\}, \qquad A^c = \{i \in S \mid 0 < x(i) < \lambda_d\}. \tag{26}$$

Our strategy for finding a solution of (24) satisfying (26) will be to split the fixed point equation (23) into two coupled fixed point equations for functions on A^c and on A. The first fixed point equation will have a unique solution by the contraction mapping theorem, whereas the fixed point of the second equation will be found by the Brouwer fixed point theorem, using the fact that A is a finite set. A drawback of this strategy is that in the latter step we do not get uniqueness, but the advantage will be that the smallness assumption for q will be rather weak.

In order to describe this assumption, recall that in Sect. 3.1 we defined $\rho = \rho(d, n)$ to be the unique positive number such that

$$(d-1)\rho^{d+1} + dn\rho^{d-1} - n = 0, \tag{27}$$

and $\eta = \eta(d, n) \in (0, 1)$ to be the number

$$\eta(d, n) = \frac{\rho - \rho^d}{(\rho^{d+1} + n)^{\frac{d}{d+1}}}. \tag{28}$$

Here is our existence result for solutions of (23) satisfying the conditions (26).

Proposition 5.6 *Let $Q \in \ell^{\frac{d+1}{2}}(S)$ be a positive function with $Q(0) = 1$ and set $q := Q - \mathbb{1}_{\{0\}}$. Assume that*

$$\|q\|_{\frac{d+1}{2}} \le \eta(d, n) \tag{29}$$

for some integers $d \ge 2$ and $n \ge 1$. Then for every subset $A \subset S$ with $|A| = n$ there exists a positive function $\overline{x} \in \ell^{d+1}(S)$ such that

$$\overline{x} = Q * \overline{x}^d, \qquad \|\overline{x}|_{A^c}\|_\infty \le \|\overline{x}|_{A^c}\|_{d+1} < \rho(d, n) < \lambda_d < \overline{x}|_A < 1. \tag{30}$$

Moreover:

- *(i)* $\|\overline{x}|_{A^c}\|_{d+1} \le \frac{\rho(d,n)}{\eta(d,n)} \|q\|_{\frac{d+1}{2}}$;
- *(ii)* $0 < 1 - \overline{x}|_A \le \frac{d^{\frac{d}{d-1}} - d}{d-1}(1+n)^{\frac{d}{d+1}} \|q\|_{\frac{d+1}{2}}$.
- *(iii)* $\overline{x}(i) \ge (1 - \frac{d}{d-1}(d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}\epsilon) \sum_{j\in A} Q(i-j)$ *for every* $i \in A^c$.

Proof Given functions $x_0 : A^c \to \mathbb{R}$ and $x_1 : A \to \mathbb{R}$, we denote by

$$x_0 \sqcup x_1 : S \to \mathbb{R}$$

the function mapping $i \in A^c$ to $x_0(i)$ and $i \in A$ to $x_1(i)$. We start by fixing an arbitrary $x_1 \in [\lambda_d, 1]^A$ and look for functions x of the form $x = x_0 \sqcup x_1$ which solve (23) on A^c, i.e.

$$x_0 = \big(Q * (x_0^d \sqcup x_1^d)\big)\big|_{A^c}. \tag{31}$$

Equivalently, we are looking for the fixed points of the map

$$F_{x_1} : \ell^{d+1}(A^c) \to \ell^{d+1}(A^c), \qquad F_{x_1}(x_0) = \big(Q * (x_0^d \sqcup x_1^d)\big)\big|_{A^c},$$

which is well defined because of the Young inequality

$$\|Q * (x_0^d \sqcup x_1^d)\|_{d+1} \le \|Q\|_{\frac{d+1}{2}} \|x_0^d \sqcup x_1^d\|_{\frac{d+1}{d}} = \|Q\|_{\frac{d+1}{2}} \|x_0 \sqcup x_1\|_{d+1}^d.$$

Given $r > 0$, set

$$X_r := \{x_0 \in \ell^{d+1}(A^c) \mid x_0 \ge 0,\ \|x_0\|_{d+1} \le r\}.$$

We now check which condition on r guarantees that F_{x_1} maps X_r to itself. If x_0 is in X_r then $F_{x_1}(x_0) \geq 0$ and using again the Young inequality we find

$$\begin{aligned}\|F_{x_1}(x_0)\|_{d+1} &= \|x_0^d + q * (x_0^d \sqcup x_1^d)\|_{d+1} \leq \|x_0^d\|_{d+1} + \|q * (x_0^d \sqcup x_1^d)\|_{d+1} \\ &\leq \|x_0\|_{d(d+1)}^d + \|q\|_{\frac{d+1}{2}} \|x_0^d \sqcup x_1^d\|_{\frac{d+1}{d}} \leq \|x_0\|_{d+1}^d + \|q\|_{\frac{d+1}{2}} \|x_0 \sqcup x_1\|_{d+1}^d \\ &= \|x_0\|_{d+1}^d + \|q\|_{\frac{d+1}{2}} \left(\|x_0\|_{d+1}^{d+1} + \|x_1\|_{d+1}^{d+1}\right)^{\frac{d}{d+1}} \\ &\leq r^d + \|q\|_{\frac{d+1}{2}} (r^{d+1} + |A|)^{\frac{d}{d+1}} = r^d + \|q\|_{\frac{d+1}{2}} (r^{d+1} + n)^{\frac{d}{d+1}},\end{aligned}$$

where we have also used the inequality $|x_1| \leq 1$ and the following consequence of the monotonicity of the ℓ^p norms: $\|x_0\|_{d(d+1)} \leq \|x_0\|_{d+1}$. Therefore, F_{x_1} maps X_r to itself provided that

$$r^d + \|q\|_{\frac{d+1}{2}} (r^{d+1} + n)^{\frac{d}{d+1}} \leq r.$$

This condition can be equivalently rewritten as

$$\|q\|_{\frac{d+1}{2}} \leq f_{d,n}(r), \tag{32}$$

where $f_{d,n} : [0, +\infty) \to \mathbb{R}$ is the function

$$f_{d,n}(r) := \frac{r - r^d}{(r^{d+1} + n)^{\frac{d}{d+1}}}. \tag{33}$$

Next note that

$$F_{x_1}(x_0) = F_0(x_0) + (q * (0_{A^c} \sqcup x_1^d))|_{A^c}, \tag{34}$$

where 0_{A^c} denote the zero function on A^c. The map F_0 is the composition of the maps

$$\ell^{d+1}(A^c) \to \ell^{\frac{d+1}{d}}(A^c), \qquad x_0 \mapsto x_0^d,$$

and

$$\ell^{\frac{d+1}{d}}(A^c) \to \ell^{d+1}(A^c), \qquad y_0 \mapsto (Q * (y_0 \sqcup 0_A)|_{A^c}.$$

By the mean value theorem, the first map has Lipschitz constant dr^{d-1} on the r-ball of $\ell^{d+1}(A^c)$. The second map is linear with operator norm not exceeding

$$\|Q\|_{\frac{d+1}{2}} = \left(1 + \|q\|_{\frac{d+1}{2}}^{\frac{d+1}{2}}\right)^{\frac{2}{d+1}}.$$

Therefore, the restriction of the map F_{x_1} to X_r is a contraction if r satisfies (32) and

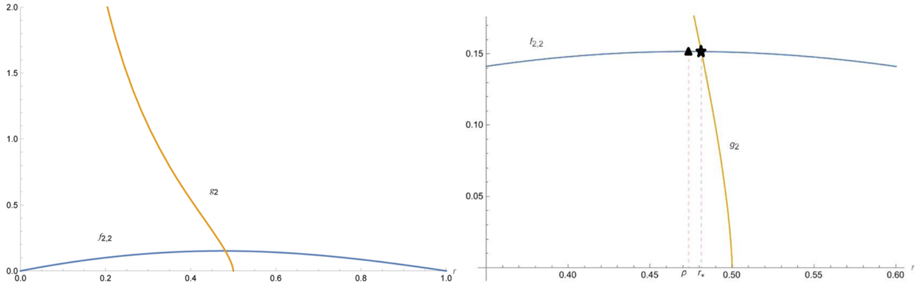
$$d\left(1 + \|q\|_{\frac{d+1}{2}}^{\frac{d+1}{2}}\right)^{\frac{2}{d+1}} r^{d-1} < 1.$$

This condition forces r to belong to the interval $(0, \lambda_d)$ and can be equivalently rewritten as

$$\|q\|_{\frac{d+1}{2}} < g_d(r), \tag{35}$$

where $g_d : (0, \lambda_d] \to \mathbb{R}$ is the function

$$g_d(r) := \left(\left(\frac{\lambda_d}{r}\right)^{\frac{d^2-1}{2}} - 1\right)^{\frac{2}{d+1}}. \tag{36}$$



(a) $r \in (0, 1)$ (b) Zoomed in

Fig. 4 An illustration of the set-up and statement of Lemma 5.7: Plot (**a**) shows the graphs of the functions $f_{d,n}$ and g_d in the case $d = n = 2$, i.e., $\lambda_2 = 0.5$. Plot (**b**) is a zoomed-in version of plot (**a**), in which the graph of $f_{2,2}$ is supplemented with two marks. The left (triangular shaped) mark indicates the unique maximum of $f_{2,2}$, obtained at $\rho \approx 0.473$. Hence, $\eta(2, 2) = f_{2,2}(\rho) \approx 0.152$. The right (star shaped) mark is at the unique point of intersection of the two graphs, which happens at $r_* \approx 0.481$. We see that the number ρ, where $f_{2,2}$ achieves its maximum, is smaller than the number r_* at which $f_{2,2}$ and g_2 coincide

The next lemma describes some useful properties of the functions $f_{d,n}$ and g_d. See also Fig. 4 for an illustration.

Lemma 5.7 *Let $\rho = \rho(d, n)$ be the unique positive solution of the Eq.* (27). *Then ρ belongs to the interval* $(0, \lambda_d)$ *(see* (25)*). The function $f_{d,n}$ defined in* (33) *is strictly increasing on the interval* $[0, \rho]$, *strictly decreasing on* $[\rho, +\infty)$ *and strictly concave on* $[0, 1]$. *Its maximum is the number*

$$\eta(d, n) = f_{d,n}(\rho(d, n))$$

which is introduced in (28)*. The function g_d, defined in* (36)*, is strictly decreasing on* $(0, \lambda_d]$ *and there exists a number $r_* \in (\rho, \lambda_d)$ such that*

$$f_{d,n} < g_d \quad \text{on } (0, r_*), \qquad g_d < f_{d,n} \quad \text{on } (r_*, \lambda_d].$$

The proof of this Lemma is elementary but rather delicate. The interested reader can find the details in Appendix B. We now proceed with the proof of Proposition 5.6. By the above lemma and our Assumption (29), we can find a number $r_q \in [0, \rho]$ such that

$$f_{d,n}(r_q) = \|q\|_{\frac{d+1}{2}}.$$

Then the equality holds in (32) with $r = r_q$ and hence F_{x_1} maps X_{r_q} to itself. Since r_q belongs to $[0, \rho]$, the above lemma implies that

$$\|q\|_{\frac{d+1}{2}} = f_{d,n}(r_q) < g_d(r_q),$$

so $r = r_q$ satisfies (35). We conclude that F_{x_1} is a contraction on X_{r_q} and hence has a unique fixed point $\xi_0(x_1)$ in X_{r_q}. In particular, we have

$$\|\xi_0(x_1)\|_\infty \leq \|\xi_0(x_1)\|_{d+1} \leq r_q < \rho(d, n),$$

so $x_0 = \xi_0(x_1)$ takes values in $[0, \rho(d, n))$ and is a solution of (31).

Note that by (34)

$$\xi_0(x_1) = (\mathrm{id} - F_0)^{-1}\big((q * (0_{A^c} \sqcup x_1^d))|_{A^c}\big),$$

where the map $\mathrm{id} - F_0$ is a homeomorphism from X_{r_q} to its image thanks to the fact that F_0 has Lipschitz constant less than 1 on X_{r_q}. From the above identity we deduce that the map

$$\xi_0 : [\lambda_d, 1]^A \to \ell^{d+1}(A^c)$$

is continuous.

Let $x_1 \in [\lambda_d, 1]^A$. The function $x = \xi_0(x_1) \sqcup x_1$ is a solution of (23) if and only if x_1 satisfies the equation

$$x_1 = \bigl(Q * (\xi_0(x_1)^d \sqcup x_1^d)\bigr)\big|_A, \tag{37}$$

which can be rewritten as

$$\varphi_d(x_1) = \bigl(q * (\xi_0(x_1)^d \sqcup x_1^d)\bigr)\big|_A. \tag{38}$$

We set

$$\mu_d := \max_{r \in [0,+\infty)} \varphi_d(r) = \varphi_d(\lambda_d) = \lambda_d\bigl(1 - \tfrac{1}{d}\bigr), \tag{39}$$

and claim that

$$0 \le q * (\xi_0(x_1)^d \sqcup x_1^d) < \mu_d \qquad \text{on } A, \tag{40}$$

for every $x_1 \in [\lambda_d, 1]^A$. The first inequality is clear. In order to prove the second one, we use the upper bound

$$\begin{aligned}
\|q * (\xi_0(x_1)^d \sqcup x_1^d)\|_\infty &\le \|q\|_{d+1} \|\xi_0(x_1)^d \sqcup x_1^d\|_{\frac{d+1}{d}} = \|q\|_{d+1} \|\xi_0(x_1) \sqcup x_1\|_{d+1}^d \\
&= \|q\|_{d+1} \bigl(\|\xi_0(x_1)\|_{d+1}^{d+1} + \|x_1\|_{d+1}^{d+1}\bigr)^{\frac{d}{d+1}} \le \|q\|_{d+1} (r_q^{d+1} + |A|)^{\frac{d}{d+1}} \\
&= \|q\|_{d+1} (r_q^{d+1} + n)^{\frac{d}{d+1}}.
\end{aligned}$$

By the bound $\|q\|_{d+1} \le \|q\|_{\frac{d+1}{2}}$ and our choice of r_q we have

$$\begin{aligned}
\|q * (\xi_0(x_1)^d \sqcup x_1^d)\|_\infty &\le \|q\|_{\frac{d+1}{2}} (r_q^{d+1} + n)^{\frac{d}{d+1}} = f_{d,n}(r_q)(r_q^{d+1} + n)^{\frac{d}{d+1}} \\
&= \varphi_d(r_q) < \varphi_d(\lambda_d) = \mu_d,
\end{aligned} \tag{41}$$

as claimed in (40). Thanks to (40), we can rewrite (38) as

$$x_1 = \psi\bigl(\bigl(q * (\xi_0(x_1)^d \sqcup x_1^d)\bigr)\big|_A\bigr),$$

where $\psi : [0, \mu_d] \to [\lambda_d, 1]$ is the inverse of the restriction of φ_d to the interval $[\lambda_d, 1]$, on which φ_d is strictly decreasing. Therefore, $x_1 \in [\lambda_d, 1]^A$ satisfies (37) if and only if it is a fixed point of the continuous map

$$G : [\lambda_d, 1]^A \to [\lambda_d, 1]^A, \qquad G(x_1) := \psi\bigl(\bigl(q * (\xi_0(x_1)^d \sqcup x_1^d)\bigr)\big|_A\bigr). \tag{42}$$

By Brouwer's fixed point theorem, G has a fixed point $\overline{x}_1$ on the n-dimensional cube $[\lambda_d, 1]^A$. Setting $\overline{x}_0 := \xi_0(\overline{x}_1)$, we obtain that $\overline{x} := \overline{x}_0 \sqcup \overline{x}_1$ is a positive solution of (23) and as such takes values in $(0, 1)$.

By the strict upper bound in (40) and by the properties of ψ, we have $\overline{x}_1 > \lambda_d$. Since $\overline{x}_0$ belongs to X_{r_q} with $r_q < \rho(d, n)$, we have

$$\|\overline{x}_0\|_\infty \le \|\overline{x}_0\|_{d+1} < \rho(d, n).$$

We conclude that $\overline{x} \in \ell^{d+1}(S)$ is a positive solution of (30).

There remains to prove the bounds (i), (ii) and (iii). By construction,

$$\|\overline{x}|_{A^c}\|_{d+1} = \|\overline{x}_0\|_{d+1} \le r_q. \tag{43}$$

Since $f_{d,n}$ is strictly increasing and concave on $[0, \rho]$, its restriction to $[0, \rho]$ has an inverse which is strictly increasing and convex on $[0, \eta(d, n)]$. The convexity of this inverse implies the inequality

$$r_q = \big(f_{d,n}|_{[0,\rho]}\big)^{-1}\big(\|q\|_{\frac{d+1}{2}}\big) \le \frac{\rho(d,n)}{\eta(d,n)}\|q\|_{\frac{d+1}{2}},$$

so (i) follows from (43).

Since the restriction of φ_d to $[\lambda_d, 1]$ is strictly decreasing and concave, its inverse ψ_d is strictly decreasing and concave on $[0, \mu_d]$. The concavity of ψ_d implies

$$\psi_d(s) \ge 1 - \frac{1-\lambda_d}{\mu_d}s = 1 - \frac{d^{\frac{d}{d-1}} - d}{d-1}s \qquad \forall s \in [0, \mu_d]. \tag{44}$$

By the first inequality in (41), we have

$$\|q * \overline{x}^d\|_\infty \le \|q\|_{\frac{d+1}{2}}(r_q^{d+1} + n)^{\frac{d}{d+1}} \le (1+n)^{\frac{d}{d+1}}\|q\|_{\frac{d+1}{2}}.$$

Together with the fact that ψ_d is decreasing and satisfies the concavity inequality (44), the above upper bound implies

$$\overline{x}|_A = \overline{x}_1 = \psi_d\big((q * \overline{x}^d)|_A\big) \ge 1 - \tfrac{d^{\frac{d}{d-1}}-d}{d-1}(1+n)^{\frac{d}{d+1}}\|q\|_{\frac{d+1}{2}},$$

proving (ii).

The map $F_{\overline{x}_1}$ is monotonically increasing on the subset of non-negative functions in $\ell^{d+1}(A^c)$, with respect to the standard partial order of functions. Since the fixed point $\overline{x}_0$ of $F_{\overline{x}_1}$ satisfies $\overline{x}_0 \ge 0$, we have

$$\overline{x}|_{A^c} = \overline{x}_0 = F_{\overline{x}_1}(\overline{x}_0) \ge F_{\overline{x}_1}(0) = \big(Q * (0_{A^c} \sqcup \overline{x}_1^d)\big)|_{A^c}.$$

By evaluating at $i \in A^c$ and using (ii), we obtain the lower bound

$$\begin{aligned}
\overline{x}(i) &\ge \big(Q * (0_{A^c} \sqcup \overline{x}_1^d)\big)(i) = \sum_{j\in A} Q(i-j)\overline{x}_1^d(j)\\
&\ge \left(1 - \tfrac{d^{\frac{d}{d-1}}-d}{d-1}(1+n)^{\frac{d}{d+1}}\|q\|_{\frac{d+1}{2}}\right)^d \sum_{j\in A} Q(i-j)\\
&\ge \left(1 - \tfrac{d}{d-1}(d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}\|q\|_{\frac{d+1}{2}}\right)\sum_{j\in A} Q(i-j),
\end{aligned}$$

where in the last step we have used the Bernoulli inequality $(1+x)^d \ge 1 + dx$ for every $x \ge -1$ and $d \ge 1$. This proves (iii) and concludes the proof of Proposition 5.6. □

Remark 5.8 (*Uniqueness for* $|A| = 1$) Consider the standard partial order on the space of real valued functions. It is easy to show that the map $G : [\lambda_d, 1]^A \to [\lambda_d, 1]^A$ is monotonically decreasing. Indeed, the fact that the map $(x_0, x_1) \mapsto F_{x_1}(x_0)$ is monotonically increasing implies that if $x_1 \le x_1'$ then $F_{x_1}^n(x_0) \le F_{x_1'}^n(x_0)$ for every $n \in \mathbb{N}$ and every x_0. Taking the limit in n, we deduce that the map ξ_0 which associates to every $x_1 \in [\lambda_d, 1]^A$ the unique fixed point of F_{x_1} is also monotonically increasing, and so is the map $x_1 \mapsto (q * (\xi_0(x_1) \sqcup x_1^d))|_A \in$

$[0, \mu_d]^A$. From the fact that the function ψ_d is montonically decreasing on $[0, \mu_d]$, we deduce that G is monotonically decreasing, as claimed. When $|A| = 1$, this implies that G has a unique fixed point . In this case, the solution $\overline{x}$ of (30) is unique.

Remark 5.9 (*Uniqueness for* $|A| > 1$) If $n = |A| > 1$, we do not know whether the solution of (30) is unique under the assumption $\|q\|_{\frac{d+1}{2}} \leq \eta(d, n)$. By assuming a stronger smallness assumption on $\|q\|_{\frac{d+1}{2}}$, we surely have existence and uniqueness of a solution $x \in \ell^{d+1}(S)$ of the equation $x = Q * x^d$ which is sufficiently close to $\mathbb{1}_A$ in the $(d+1)$-norm. This follows from the implicit mapping theorem applied to the continuously differentiable map

$$H : \ell^{\frac{d+1}{2}}(S) \times \ell^{d+1}(S) \to \ell^{d+1}(S), \qquad H(Q, x) = x - Q * x^d.$$

Indeed, $H(\mathbb{1}_{\{0\}}, \mathbb{1}_A) = 0$ and the differential of H with respect to the second variable at $(\mathbb{1}_{\{0\}}, \mathbb{1}_A)$ is the linear operator

$$d_2 H(\mathbb{1}_{\{0\}}, \mathbb{1}_A)[y] = \big(1 - d\mathbb{1}_A\big)y,$$

which is an isomorphism on $\ell^{d+1}(S)$ because $d \neq 1$. From Theorem 5.3 and the first identity in Proposition 5.4, we then deduce that there exists positive numbers $\eta'(d, n)$ and $\delta(d, n)$ such that if $\|q\|_{\frac{d+1}{2}} \leq \eta'(d, n)$ and $A \subset S$ has n elements, then the Markovian gradient specification which is induced by Q on the regular d-tree with local state space S has a unique spatially homogeneous Markov-chain Gibbs measure μ whose single-site marginal probability distribution π_μ satisfies

$$\left\|\pi_\mu - \tfrac{1}{|A|}\mathbb{1}_A\right\|_1 < \delta(d, |A|),$$

as claimed in Remark 3.2. The bounds η' and δ which one gets from standard quantitative versions of the implicit function theorem are much worse than the ones appearing in Theorem 3.1. In order to obtain better bounds, one can look for a solution $x \in \ell^{d+1}(S)$ of the equation $x = Q * x^d$ which is close to $\mathbb{1}_A$ by considering the fixed point of the map

$$(x_0, x_1) \mapsto \Big(\big(Q * (x_0^d \sqcup x_1^d)\big)|_{A^c}, \psi_d\big(\big(q * (x_0^d \sqcup x_1^d)\big)|_A\big)\Big),$$

which can be shown to be a contraction on a suitable closed subset of $\ell^{d+1}(A^c) \times \ell^\infty(A)$ if $\|q\|_{\frac{d+1}{2}}$ is small enough. In this way, one gets an existence and uniqueness statement as above but with bounds which are not too much worse than those in Theorem 3.1.

5.3 Proof of Theorem 3.1

Building on Theorem 5.3 and Proposition 5.4, we now show how Theorem 3.1 follows from Proposition 5.6. We assume that

$$\|q\|_{\frac{d+1}{2}} = \|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} \leq \eta(d, N)$$

and we fix a subset A with $1 \leq n := |A| \leq N$. Let $\overline{x} \in \ell^{d+1}(S)$ be a solution of (30), whose existence is guaranteed by Proposition 5.6. Let $u = \overline{x}^d \in \ell^{\frac{d+1}{d}}(S)$ be the corresponding solution of the boundary law equation (20) with $c = 1$. By Theorem 5.3 and Proposition 5.4, the boundary law u induces a spatially homogeneous Markov-chain Gibbs measure $\mu \in \mathcal{MG}(\gamma)$ whose single-site marginal distribution π_μ and transition matrix P_μ are given by

$$\pi_\mu(i) = \frac{u(i)^{\frac{d+1}{d}}}{\|u^{\frac{d+1}{d}}\|}, \qquad P_\mu(i, j) = \frac{u(j)Q(i-j)}{(Q * u)(i)}.$$

From the identities $u = \overline{x}^d$ and $\overline{x} = Q * \overline{x}^d$, we find

$$\pi_\mu(i) = \frac{\overline{x}(i)^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}}, \qquad P_\mu(i,j) = \frac{\overline{x}(j)^d Q(i-j)}{\overline{x}(i)}. \tag{45}$$

Then (30) implies, setting $\theta := \theta(d,n) := (\frac{\rho(d,n)}{\lambda_d})^{d+1}$,

$$\|\pi_\mu|_{A^c}\|_1 = \frac{\|\overline{x}|_{A^c}\|_{d+1}^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} < \frac{\rho(d,n)^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} < \theta \frac{(\min_A \overline{x})^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} = \theta \min_A \frac{\overline{x}^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} = \theta \min_A \pi_\mu,$$

proving (6) in Theorem 3.1. By (45), the diagonal elements $\Delta(i) = P_\mu(i,i)$ of the transition matrix are given by

$$\Delta(i) = \overline{x}(i)^{d-1}.$$

The bounds

$$\|\overline{x}|_{A^c}\|_\infty \leq \|\overline{x}|_{A^c}\|_{d+1} < \rho(d,n) < \lambda_d = d^{-\frac{1}{d-1}} < \overline{x}|_A < 1, \tag{46}$$

from (30) translate into

$$\|\Delta_{A^c}\|_\infty \leq \|\Delta|_{A^c}\|_{\frac{d+1}{d-1}} < \rho(d,n)^{d-1} < \frac{1}{d} < \Delta|_A < 1,$$

proving (7) in Theorem 3.1.

Remark 5.10 As shown in Remark 5.8, if $|A| = 1$ then the solution $\overline{x} \in \ell^{d+1}(S)$ of (30) is unique. Together with the uniqueness of the boundary law which is determined by a Gibbs measure (see again Theorem 5.3), this implies that in the case $|A| = 1$ the above μ is the unique spatially homogeneous Markov-chain Gibbs measure μ whose transition matrix P_μ satisfies (7).

In the following proof of statements (i)–(v) of Theorem 3.1, we use the abbreviations

$$\epsilon := \|q\|_{\frac{d+1}{2}} = \|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}}, \qquad \rho = \rho(d,n), \qquad \eta = \eta(d,n).$$

By statement (i) in Proposition 5.6 we have

$$\|\Delta|_{A^c}\|_{\frac{d+1}{d-1}} = \|\overline{x}|_{A^c}\|_{d+1}^{d-1} \leq \frac{\rho^{d-1}}{\eta^{d-1}}\epsilon^{d-1},$$

proving assertion (i) in Theorem 3.1. By statement (ii) in Proposition 5.6 we have, using the Bernoulli inequality,

$$\Delta|_A = \overline{x}_A^{d-1} \geq \left(1 - \frac{d^{\frac{d}{d-1}} - d}{d-1}(1+n)^{\frac{d}{d+1}}\epsilon\right)^{d-1} \geq 1 - \left(d^{\frac{d}{d-1}} - d\right)(1+n)^{\frac{d}{d+1}}\epsilon,$$

which proves statement (ii) in Theorem 3.1.

There remains to prove the bounds (iii), (iv) and (v) on the single-site marginal distribution π_μ. By (46), the $(d+1)$-norm of $\overline{x}$ has the lower bound

$$\|\overline{x}\|_{d+1}^{d+1} \geq \|\overline{x}|_A\|_{d+1}^{d+1} \geq \lambda_d^{d+1}|A|.$$

By statement (i) in Proposition 5.6 we have

$$\|\pi_\mu|_{A^c}\|_1 = \frac{\|\overline{x}|_{A^c}\|_{d+1}^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} \leq \frac{\left(\frac{\rho}{\eta}\right)^{d+1}\epsilon^{d+1}}{\lambda_d^{d+1}|A|} = \frac{d^{\frac{d+1}{d-1}}\rho^{d+1}}{\eta^{d+1}n}\epsilon^{d+1},$$

which proves assertion (iii) in Theorem 3.1. From statement (ii) in Proposition 5.6 we obtain, using the Bernoulli inequality,

$$\begin{aligned}\|\overline{x}\|_{d+1}^{d+1} \geq \|\overline{x}|_A\|_{d+1}^{d+1} &\geq \left(1 - \frac{d^{\frac{d}{d-1}} - d}{d-1}(1+n)^{\frac{d}{d+1}}\epsilon\right)^{d+1} |A| \\ &\geq \left(1 - \frac{d+1}{d-1}\big(d^{\frac{d}{d-1}} - d\big)(1+n)^{\frac{d}{d+1}}\epsilon\right)|A|,\end{aligned}$$

and hence

$$\pi_\mu|_A = \frac{\overline{x}|_A^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} \leq \frac{1}{\|\overline{x}\|_{d+1}^{d+1}} \leq \left(1 - \frac{d+1}{d-1}\big(d^{\frac{d}{d-1}} - d\big)(1+n)^{\frac{d}{d+1}}\epsilon\right)^{-1} \frac{1}{|A|},$$

which proves the right-hand side inequality in statement (iv) of Theorem 3.1. On the other hand, using statement (i) in Proposition 5.6 we have

$$\begin{aligned}\|\overline{x}\|_{d+1}^{d+1} = \|\overline{x}|_A\|_{d+1}^{d+1} + \|\overline{x}_{A^c}\|_{d+1}^{d+1} &\leq |A| + \frac{\rho^{d+1}}{\eta^{d+1}}\epsilon^{d+1} \\ &= \left(1 + \frac{\rho^{d+1}}{n\eta^{d+1}}\epsilon^{d+1}\right)|A|,\end{aligned}$$

and hence, using also the inequality $\epsilon \leq \eta(d, N) \leq 1$ (see (62)),

$$\|\overline{x}\|_{d+1}^{-(d+1)} \geq \left(1 - \frac{\rho^{d+1}}{n\eta^{d+1}}\epsilon^{d+1}\right)\frac{1}{|A|} \geq \left(1 - \frac{\rho^{d+1}}{n\eta^{d+1}}\epsilon\right)\frac{1}{|A|}. \tag{47}$$

Using again statement (ii) in Proposition 5.6 and the Bernoulli inequality we obtain

$$\begin{aligned}\pi_\mu|_A = \frac{\overline{x}|_A^{d+1}}{\|\overline{x}\|_{d+1}^{d+1}} &\geq \left(1 - \frac{d^{\frac{d}{d-1}} - d}{d-1}(1+n)^{\frac{d}{d+1}}\epsilon\right)^{d+1}\left(1 - \frac{\rho^{d+1}}{n\eta^{d+1}}\epsilon\right)\frac{1}{|A|} \\ &\geq \left(1 - \frac{d+1}{d-1}\big(d^{\frac{d}{d-1}} - d\big)(1+n)^{\frac{d}{d+1}}\epsilon\right)\left(1 - \frac{\rho^{d+1}}{n\eta^{d+1}}\epsilon\right)\frac{1}{|A|} \\ &\geq \left(1 - \Big(\frac{d+1}{d-1}\big(d^{\frac{d}{d-1}} - d\big)(1+n)^{\frac{d}{d+1}} + \frac{\rho^{d+1}}{n\eta^{d+1}}\Big)\epsilon\right)\frac{1}{|A|}.\end{aligned}$$

This proves the left-hand inequality in statement (iv) of Theorem 3.1. By (47), statement (iii) in Proposition 5.6 and a last application of the Bernoulli inequality, we obtain for every $i \in A^c$ the lower bound

$$\begin{aligned}\pi_\mu(i) &\geq \frac{1}{|A|}\left(1 - \tfrac{\rho^{d+1}}{n\eta^{d+1}}\epsilon\right)\left(1 - \tfrac{d}{d-1}(d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}\epsilon\right)^{d+1}\Big(\sum_{j\in A} Q(i-j)\Big)^{d+1} \\ &\geq \frac{1}{|A|}\left(1 - \tfrac{\rho^{d+1}}{n\eta^{d+1}}\epsilon\right)\left(1 - \tfrac{d(d+1)}{d-1}(d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}\epsilon\right)\Big(\sum_{j\in A} Q(i-j)\Big)^{d+1} \\ &\geq \frac{1}{|A|}\left(1 - \big(\tfrac{\rho^{d+1}}{n\eta^{d+1}} + \tfrac{d(d+1)}{d-1}(d^{\frac{d}{d-1}} - d)(1+n)^{\frac{d}{d+1}}\big)\epsilon\right)\Big(\sum_{j\in A} Q(i-j)\Big)^{d+1}.\end{aligned}$$

This proves statement (v) of Theorem 3.1 and concludes the proof of Theorem 3.1.

Table 1 The threshold $\underline{\beta}(d,n)$ for the SOS model

d	$n = 1$	$n = 2$	$n = 10$
2	1.953	2.367	3.396
3	1.400	1.870	3.038
6	0.810	1.366	2.719

A Appendix: Examples

In this Appendix, we study how the Assumption (5) of Theorem 3.1 translates for the concrete models of Examples 3.5 and 3.6. In the study of these models, we shall make use of the fact that the function η which is defined at the beginning of Sect. 3.1 satisfies the bounds

$$\underline{c}\, d\, n^d \le \eta(d,n)^{-(d+1)} \le \overline{c}\, d\, n^d \qquad \forall n \ge 1, \quad \forall d \ge 2, \tag{48}$$

for suitable positive numbers $\underline{c}$ and $\overline{c}$, as proven in Lemma B.1 in Appendix B.

Example A.1 (*SOS model*) Consider the case $S = \mathbb{Z}$ and $Q(i) = e^{-\beta|i|}$, where β is a positive parameter modelling the inverse temperature. Then

$$\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} = \left(\frac{2}{e^{\frac{d+1}{2}\beta} - 1}\right)^{\frac{2}{d+1}},$$

and the Assumption (5) reads

$$\beta \ge \underline{\beta}(d,n) := \frac{2}{d+1}\log\left(1 + 2\cdot\eta(d,n)^{-\frac{d+1}{2}}\right).$$

Table 1 lists some approximate values of the threshold $\underline{\beta}$.

The asymptotic behaviour of this threshold for d and/or n tending to infinity can be determined as follows. By (48), $\underline{\beta}$ has the bounds

$$\frac{2}{d+1}\log\left(1 + 2\sqrt{\underline{c}d}n^{\frac{d}{2}}\right) \le \underline{\beta}(d,n) \le \frac{2}{d+1}\log\left(1 + 2c\sqrt{\underline{c}d}n^{\frac{d}{2}}\right).$$

By the inequalities

$$\log x \le \log(1+x) \le \log x + \frac{1}{x}, \qquad \forall x > 0,$$

we obtain the bounds

$$\frac{1}{d+1}\big(2\log(2\sqrt{\underline{c}}) + \log d + d\log n\big) \le \frac{1}{d+1}\left(2\log(2\sqrt{\overline{c}}) + \log d + d\log n + \frac{1}{\sqrt{\overline{c}}}\right),$$

and hence

$$\frac{\log d}{d} - \frac{a}{d} + \frac{2}{3}\log n \le \underline{\beta}(d,n) \le \frac{\log d}{d} + \frac{b}{d} + \log n \qquad \forall d \ge 2,\ \forall n \ge 1,$$

for suitable positive numbers a and b.

Example A.2 (*Log potential*) Consider the case $S = \mathbb{Z}$ and $Q(i) = \frac{1}{(1+|i|)^\beta}$. Then

$$\|Q - \mathbb{1}_{\{0\}}\|_{\frac{d+1}{2}} = \left(2\zeta\left(\tfrac{d+1}{2}\beta\right) - 2\right)^{-\frac{2}{d+1}},$$

Table 2 The threshold $\underline{\beta}(d,n)$ for the model with log potential

d	$n=1$	$n=2$	$n=10$
2	2.974	3.527	4.946
3	2.145	2.773	4.402
6	1.240	1.994	3.924

where ζ denotes the Riemann zeta function. The Assumption (5) now becomes

$$\beta \geq \underline{\beta}(d,n) := \frac{2}{d+1}\zeta^{-1}\left(1+\tfrac{1}{2}\eta(d,n)^{\frac{d+1}{2}}\right), \tag{49}$$

where $\zeta^{-1} : (1,+\infty) \to (1,+\infty)$ denotes the inverse of the restriction of the Riemann zeta function to the interval $(1,+\infty)$, on which this function is strictly monotonically decreasing with image $(1,+\infty)$. Table 2 lists some approximate values of the threshold $\underline{\beta}$.

We now determine the asymptotics of $\underline{\beta}(d,n)$ for d and/or n tending to infinity. On the interval $(1,+\infty)$, the Riemann zeta functions satisfies the bounds

$$1+\frac{1}{2^s} \leq \zeta(s) = 1+\frac{1}{2^s}+\sum_{k=3}^{\infty}\frac{1}{k^s} \leq 1+\frac{1}{2^s}+\int_2^{\infty}\frac{dx}{x^s} = 1+\left(1+\tfrac{2}{s-1}\right)\frac{1}{2^s}. \tag{50}$$

If $\overline{s}$ is the unique number in $(1,+\infty)$ such that $\zeta(\overline{s}) = \frac{3}{2}$, we have

$$\zeta(s) \leq 1+\frac{c}{2^s} \qquad \forall s \in [\overline{s},+\infty), \tag{51}$$

where $c := 1+\frac{2}{\overline{s}-1}$. From the lower bound in (50) we deduce the bound

$$\zeta^{-1}(1+r) \geq \frac{\log\frac{1}{r}}{\log 2} \qquad \forall r \in (0,+\infty). \tag{52}$$

Similarly, the upper bound (51) implies

$$\zeta^{-1}(1+r) \leq \frac{\log c+\log\frac{1}{r}}{\log 2} \qquad \forall r \in \left(0,\tfrac{1}{2}\right]. \tag{53}$$

By (49), (52) and (48) we find

$$\begin{aligned}
\underline{\beta}(d,n) &\geq \frac{2}{(d+1)\log 2}\left(\log 2+\tfrac{1}{2}\log\eta^{-(d+1)}\right)\\
&\geq \frac{2}{(d+1)\log 2}\left(\log 2+\tfrac{1}{2}\log\underline{c}+\tfrac{1}{2}\log d+\tfrac{d}{2}\log n\right)\\
&\geq \frac{1}{\log 2}\left(\frac{\log d}{d}-\frac{a}{d}+\frac{2}{3}\log n\right),
\end{aligned}$$

for a suitable positive number a. Similarly, (49), (53), (48) and the bound $\frac{1}{2}\eta(d,n)^{\frac{d+1}{2}} < \frac{1}{2}$ imply

$$\begin{aligned}
\underline{\beta}(d,n) &\leq \frac{2}{(d+1)\log 2}\left(\log c+\log 2+\tfrac{1}{2}\log\eta^{-(d+1)}\right)\\
&\leq \frac{2}{(d+1)\log 2}\left(\log c+\log 2+\tfrac{1}{2}\log\overline{c}+\tfrac{1}{2}\log d+\tfrac{d}{2}\log n\right)\\
&\leq \frac{1}{\log 2}\left(\frac{\log d}{d}+\frac{b}{d}+\log n\right),
\end{aligned}$$

for a suitable number $b > 0$. We conclude that the threshold $\underline{\beta}$ satisfies the lower and upper bounds

$$\frac{1}{\log 2}\left(\frac{\log d}{d} - \frac{a}{d} + \frac{2}{3}\log n\right) \le \underline{\beta}(d,n) \le \frac{1}{\log 2}\left(\frac{\log d}{d} + \frac{b}{d} + \log n\right)$$

for every $d \ge 2$ and $n \ge 1$.

B Appendix: Proof of Lemma 5.7 and of the Bounds on η

For the sake of simplicity, we omit subindices and use the abbreviations $f = f_{d,n}$, $g = g_d$, $\varphi = \varphi_d$, $\lambda = \lambda_d$ throughout this section.

Proof of Lemma 5.7 The identity

$$f'(r) = \frac{(1-d)r^{d+1} - dnr^{d-1} + n}{(r^{d+1}+n)^{\frac{d}{d+1}+1}}$$

shows that f is strictly increasing on the interval $[0,\rho]$ and strictly decreasing on the interval $[\rho,+\infty)$, where $\rho = \rho(d,n)$ is the unique positive solution of (27). Since

$$f'(\lambda) = \frac{(1-d)\lambda^2}{d(r^{d+1}+n)^{\frac{d}{d+1}+1}}$$

is negative, the number ρ at which f achieves its global maximum belongs to the interval $(0,\lambda)$.

From the identity

$$f''(r) = \frac{(d-1)r^{2d+1} + 3ndr^{2d-1} - n(d^2+d+1)r^d - n^2d(d-1)r^{d-1}}{(r^{d+1}+n)^{\frac{d}{d+1}+2}},$$

and the Decartes rule of signs, we deduce that f'' changes sign exactly once on $(0,+\infty)$, so the fact that the number

$$f''(1) = -\frac{(d-1)(n^2d + (d-1)n - 1)}{(r^{d+1}+1)^{\frac{d}{d+1}+2}}$$

is negative implies that f is strictly concave on $[0,1]$.

The function g is positive on $(0,\lambda)$, where it strictly decreases from $+\infty$ to $g(\lambda) = 0$. Since $f(0) = 0$ and $f(\lambda) > 0$, there exist solutions $r_* \in (0,\lambda)$ of the equation $f(r_*) = g(r_*)$. Since g is strictly convex on $(0,\lambda]$ and f is strictly concave on this interval, the solution r_* is unique and we have

$$g > f \quad \text{on } (0,r_*), \qquad g < f \quad \text{on } (r_*,\lambda]. \tag{54}$$

There remains to prove that r_* is larger than ρ. Thanks to (54), it sufficies to find a number $\sigma \in [\rho,\lambda)$ such that $f(\sigma) < g(\sigma)$.

We set $\rho = \lambda(1-\theta)$, where θ belongs to the interval $(0,1)$, and rewrite (27) as

$$n = \frac{(d-1)\lambda^2}{d}\,\frac{(1-\theta)^{d+1}}{1-(1-\theta)^{d-1}}.$$

From the Bernoulli inequality

$$(1+x)^p \ge 1 + px \qquad \forall x \ge -1,\ \forall p \ge 1,$$

we deduce the bound

$$n \geq \frac{(d-1)\lambda^2}{d}\frac{1-(d+1)\theta}{(d-1)\theta} = \frac{\lambda^2}{d}\frac{1-(d+1)\theta}{\theta},$$

which can be reformulated as

$$\theta \geq \frac{\lambda^2}{dn+\lambda^2(d+1)}.$$

We conclude that $\rho \leq \sigma$ where

$$\sigma := \left(1-\frac{\lambda^2}{dn+\lambda^2(d+1)}\right)\lambda = \left(1+\frac{\lambda^2}{d(n+\lambda^2)}\right)^{-1}\lambda < \lambda.$$

In the remaining part of the proof, we show that $f(\sigma) < g(\sigma)$. Using again the Bernoulli inequality we find

$$g(\sigma) = \left(\left(1+\frac{\lambda^2}{d(n+\lambda^2)}\right)^{\frac{d^2-1}{2}}-1\right)^{\frac{2}{d+1}} \geq \left(\frac{d^2-1}{2d}\frac{\lambda^2}{n+\lambda^2}\right)^{\frac{2}{d+1}},$$

so it is enough to prove the inequality

$$f(\sigma) < \left(\frac{d^2-1}{2d}\frac{\lambda^2}{n+\lambda^2}\right)^{\frac{2}{d+1}}. \tag{55}$$

We first deal with the case $d = 2$, in which λ and σ have the values

$$\lambda = \frac{1}{2}, \qquad \sigma = \frac{4n+1}{8n+3}.$$

Since

$$f(\sigma) = \frac{\sigma-\sigma^2}{(\sigma^3+n)^{\frac{2}{3}}} < \frac{\sigma-\sigma^2}{n^{\frac{2}{3}}} = \frac{16n^2+12n+2}{n^{\frac{2}{3}}(8n+3)^2},$$

(55) will be proven if we can show that

$$\frac{16n^2+12n+2}{n^{\frac{2}{3}}(8n+3)^2} < \left(\frac{3}{4}\frac{1}{4n+1}\right)^{\frac{2}{3}}.$$

By raising both sides to the power 3, the above inequality is easily seen to be equivalent to

$$2^7(4n+1)^2(8n^2+6n+1)^3 < 9n^2(8n+3)^6. \tag{56}$$

Since

$$9n^2(8n+3)^6 > 8n^2(64n^2+48n+9)^3 > 8n^2(64n^2+48n+8)^3 = 2^{12}n^2(8n^2+6n+1)^3,$$

(56) is implied by the inequality

$$(4n+1)^2 < 32n^2,$$

which is indeed true for every $n \geq 1$, being equivalent to

$$(4n-1)^2 > 2.$$

This proves (55) in the case $d = 2$. The case $d \geq 3$ can be dealt with by starting from the weaker bound

$$f(\sigma) = \frac{\sigma - \sigma^d}{(\sigma^{d+1} + n)^{\frac{d}{d+1}}} < \frac{\sigma}{n^{\frac{d}{d+1}}} = \frac{\lambda}{n^{\frac{d}{d+1}}} \left(1 + \frac{\lambda^2}{d(n+\lambda^2)}\right)^{-1}.$$

By the above upper bound on $f(\sigma)$, (55) holds true if we can prove the inequality

$$\left(\frac{d^2-1}{2d} \frac{\lambda^2}{n+\lambda^2}\right)^2 > \frac{\lambda^{d+1}}{n^d} \left(1 + \frac{\lambda^2}{d(n+\lambda^2)}\right)^{-(d+1)}. \tag{57}$$

Using the identity $\lambda = d^{-\frac{1}{d-1}}$ and the Bernoulli inequality, the right-hand side of (57) can be estimated in the following way:

$$\begin{aligned} \frac{\lambda^{d+1}}{n^d} \left(1 + \frac{\lambda^2}{d(n+\lambda^2)}\right)^{-(d+1)} &= \frac{\lambda^2}{dn^d} \left(1 + \frac{\lambda^2}{d(n+\lambda^2)}\right)^{-(d+1)} \\ \leq \frac{\lambda^2}{dn^d} \left(1 + \frac{(d+1)\lambda^2}{d(n+\lambda^2)}\right)^{-1} &= \frac{\lambda^2}{n^d} \cdot \frac{n+\lambda^2}{dn + (2d+1)\lambda^2}. \end{aligned}$$

Therefore, (57) is implied by

$$\left(\frac{d^2-1}{2d} \frac{\lambda^2}{n+\lambda^2}\right)^2 > \frac{\lambda^2}{n^d} \cdot \frac{n+\lambda^2}{dn + (2d+1)\lambda^2}. \tag{58}$$

A simple algebraic manipulation shows that (58) is equivalent to

$$d^2 p_{d,n}(d) + (n + 2\lambda^2)d + \lambda^2 > 0, \tag{59}$$

where

$$p_{d,n}(x) := (n + 2\lambda^2)x^3 + \lambda^2 x^2 - 2(n + 2\lambda^2)x - 2\left(\lambda^2 + \frac{2(n+\lambda^2)^3}{\lambda^2 n^d}\right).$$

We shall prove that $p_{d,n}(x) > 0$ for every $x \geq 3$, $d \geq 3$ and $n \geq 1$, which implies (59) and by the above discussion (55) for every $d \geq 3$. By the Decartes rule of signs, the polynomial $p_{d,n}$ has precisely one positive real root α, and

$$p_{d,n}(x) < 0 \quad \text{on } [0, \alpha), \qquad p_{d,n}(x) > 0 \quad \text{on } (\alpha, +\infty).$$

Therefore, it is enough to prove

$$p_{d,n}(3) > 0 \qquad \forall d \geq 3,\ \forall n \geq 1. \tag{60}$$

Using the inequalities $n \geq 1$ and $d \geq 3$, we find

$$\begin{aligned} p_{d,n}(3) &= 21n + 49\lambda^2 - 4\frac{(n+\lambda^2)^3}{\lambda^2 n^d} \geq 21 + 49\lambda^2 - 4\frac{(n+\lambda^2)^3}{\lambda^2 n^3} \\ &= 21 + 49\lambda^2 - \frac{4}{\lambda^2}\left(1 + \frac{\lambda^2}{n}\right)^3 \geq 21 + 49\lambda^2 - \frac{4}{\lambda^2}\left(1 + \lambda^2\right)^3 \\ &= 9 - \frac{4}{\lambda^2} + 37\lambda^2 - 4\lambda^4. \end{aligned} \tag{61}$$

From the fact that the sequence $\lambda_d = d^{-\frac{1}{d-1}}$ is monotonically increasing and converges to 1 we deduce

$$\frac{1}{3} \le \lambda_d^2 < 1 \qquad \forall d \ge 3.$$

Therefore, (61) implies

$$p_{d,n}(3) \ge 9 - 12 + \frac{37}{3} - 4 = \frac{16}{3} > 0,$$

as we wished to prove. □

We conclude this Appendix by proving the bounds (4) and (48) for the quantity $\eta(d,n)$.

Lemma B.1 *For every pair of integers $d \ge 2$ and $n \ge 1$ the quantity $\eta(d,n)$ satisfies the bounds*

$$d^{-\frac{1}{d-1}}\left(1-\tfrac{1}{d}\right)(n+1)^{-\frac{d}{d+1}} \le \eta(d,n) \le d^{-\frac{1}{d-1}}\left(1-\tfrac{1}{d}\right)n^{-\frac{d}{d+1}}, \tag{62}$$

$$\underline{c}\, d\, n^d \le \eta(d,n)^{-(d+1)} \le \overline{c}\, d\, n^d, \tag{63}$$

for suitable positive numbers $\underline{c}$ and $\overline{c}$.

Proof By (39), the function $\varphi(r) = r - r^d$ achieves its maximum at $\lambda = d^{-\frac{1}{d-1}}$, where it has the value $\lambda(1-\frac{1}{d})$. From this fact, we obtain the bound

$$\eta(d,n) = \frac{\rho - \rho^d}{(\rho^{d+1}+n)^{\frac{d}{d+1}}} \le \lambda\left(1-\tfrac{1}{d}\right)n^{-\frac{d}{d+1}},$$

which gives us the right-hand side estimate in (62). From the fact that $\eta(d,n)$ is the maximum of the function f and that λ is smaller than one, we obtain also the lower bound

$$\eta(d,n) \ge \frac{\lambda - \lambda^d}{(\lambda^{d+1}+n)^{\frac{d}{d+1}}} = \frac{\lambda\left(1-\frac{1}{d}\right)}{(\lambda^{d+1}+n)^{\frac{d}{d+1}}} \ge d^{-\frac{1}{d-1}}\left(1-\tfrac{1}{d}\right)(n+1)^{-\frac{d}{d+1}},$$

which is the left-hand side estimate in (62).

The sequence

$$\lambda_d = d^{-\frac{1}{d-1}} = e^{-\frac{1}{d-1}\log d}$$

is increasing from the value $\frac{1}{2}$ it takes for $d=2$ towards the value 1 of its limit for $d \to \infty$. From this fact and the identity

$$(\lambda - \lambda^d)^{d+1} = \lambda^{d+1}\left(1-\tfrac{1}{d}\right)^{d+1} = \frac{\lambda^2}{d}\left(1-\tfrac{1}{d}\right)^{d+1}$$

we deduce that

$$\frac{1}{32\,d} \le (\lambda - \lambda^d)^{d+1} \le \frac{1}{e\,d}, \tag{64}$$

where we have used also the fact that the sequence $\left(1-\frac{1}{d}\right)^{d+1}$ is increasing from the value $\frac{1}{8}$ it takes for $d=2$ to the value $\frac{1}{e}$ of its limit. Moreover, we have

$$(\lambda^{d+1}+n)^d = n^d\left(1+\tfrac{\lambda^2}{dn}\right)^d \le n^d\left(1+\tfrac{1}{dn}\right)^d \le n^d\left(1+\tfrac{1}{d}\right)^d \le e\,n^d, \tag{65}$$

where we have used the fact that the sequence $\left(1+\frac{1}{d}\right)^d$ is increasing and converges to e. Since $\eta(d,n)$ is the maximum of f, (64) and (65) imply

$$\eta(d,n)^{d+1} \geq f(\lambda)^{d+1} = \frac{(\lambda-\lambda^d)^{d+1}}{(\lambda_d^{d+1}+n)^d} \geq \frac{1}{32\,d\,e\,n^d}. \tag{66}$$

On the other hand, since λ maximizes the function φ, we have by (64)

$$\eta(d,n)^{d+1} \leq \frac{(\lambda-\lambda^d)^{d+1}}{n^d} \leq \frac{1}{e\,d\,n^d}. \tag{67}$$

The bounds (66) and (67) imply that (63) holds with $\underline{c}=e$ and $\overline{c}=32e$. □

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Declarations

Competing interests The authors have no competing interests to declare that are relevant to the content of this article.

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