# TWO-SIDED ASYMPTOTIC BOUNDS FOR THE COMPLEXITY OF SOME CLOSED HYPERBOLIC THREE-MANIFOLDS

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Abstract. We establish two-sided bounds for the complexity of two infinite series of closed orientable 3-dimensional hyperbolic manifolds, the Löbell manifolds and the Fibonacci manifolds.

## 1. Introduction

If $M$ is a compact 3-dimensional manifold, its *complexity* [11, 12] is a non-negative integer $c(M)$ which formally translates the intuitive notion of "how complicated" $M$ is. In particular, if $M$ is closed and irreducible and different from the 3-sphere $\mathbb{S}^3$, the projective 3-space $\mathbb{RP}^3$, and the lens space $L(3,1)$, its complexity $c(M)$ is equal to the minimum of the number of tetrahedra over all "singular" triangulations of $M$. (A singular triangulation of $M$ is a realization of $M$ as a union of tetrahedra with pairwise glued 2-faces). The complexity function has many natural properties, among which additivity under connected sum.

The task of computing the complexity $c(M)$ of a given manifold $M$ is extremely difficult. For closed $M$, the exact value is presently known only if $M$ belongs to the computer-generated tables of manifolds up to complexity 12, see [14]. Therefore the problem of finding "reasonably good" two-sided bounds for $c(M)$ is of primary importance. The first results of this kind were obtained in [15, 17], where an estimate on $c(M)$ was given in terms of the properties of the homology groups of $M$.

In the present paper we establish two-sided bounds on the complexity of two infinite series of closed orientable 3-dimensional hyperbolic manifolds, the Löbell manifolds and the Fibonacci manifolds. The upper

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bounds are obtained by constructing fundamental polytopes of these manifolds in hyperbolic space $\mathbb{H}^3$, while the lower bounds (which are only proved in an "asymptotic" fashion, see below) are based on the calculation of their volumes. We mention here that the Löbell manifolds are constructed from polytopes that generalize the right-angled dodecahedron, and the Fibonacci manifolds are constructed from polytopes that generalize the regular icosahedron.

Before turning to the statements and proofs of our estimates, we remark that, in the class of compact 3-manifolds with non-empty boundary, exact values of complexity are currently known for two infinite families. The first one consists of the manifolds which finitely cover the complement of the figure-eight knot or its "twin" (a different manifold with the same volume) [1, 13]. The second family consists of the manifolds $M$ such that $\partial M$ consists of $k \geqslant 0$ tori and a surface of genus $g \geqslant 2$, and $M$ admits an ideal triangulation with $g+k$ tetrahedra [6, 7].

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## 2. Löbell manifolds

In this section we obtain upper and lower bounds on manifold complexity for a certain infinite family of closed hyperbolic 3-manifolds which generalize the classical Löbell manifold. Recall that in order to give a positive answer to the question of the existence of "Clifford-Klein space forms" (that is, closed manifolds) of constant negative curvature, F. Löbell [10] constructed in 1931 the first example of a closed orientable hyperbolic 3-manifold. The manifold was obtained by gluing together eight copies of the right-angled 14-faced polytope (denoted by $R(6)$ and shown in Fig. 1 below) with an upper and a lower basis both being regular hexagons, and a lateral surface given by 12 pentagons, arranged similarly as in the dodecahedron.

An algebraic approach to constructing hyperbolic 3-manifolds from eight copies of a right-angled polytope was suggested in [19], as we will now describe. Let us fix in hyperbolic 3-space $\mathbb{H}^3$ a bounded polytope $R$, namely a compact and convex set homeomorphic to the 3-disc, with boundary given by a finite union of geodesic polygons, called the faces of $R$. The notions of vertex and edge of $R$ are defined in the obvious fashion. We further assume that $R$ is right-angled, namely that all the dihedral angles along the edges of $R$ are $\pi/2$, which easily implies that all the faces of $R$ have at least 5 edges and all vertices of $R$ are trivalent. Note that, according to Andreev's theorem [2], these combinatorial conditions on an abstract polytope are actually also sufficient for its realizability as a bounded right-angled polytope in $\mathbb{H}^3$.

We will denote henceforth by $G$ the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$, the isometry group of hyperbolic 3-space, generated by the reflections in the planes containing the faces of $R$. The following is an easy consequence of Poincaré's polyhedron theorem [5, 18]:

**Lemma 2.1.** *$R$ is a fundamental domain for $G$ and a presentation of $G$ is given by:*

- *A generator for each face of $R$;*
- *The relation $\rho^2$ for each generator $\rho$ and the relation $[\rho_1, \rho_2]$ for each pair of generators $\rho_1, \rho_2$ associated to faces sharing an edge.*

This result implies that $G$ is a discrete subgroup of isometries of $\mathbb{H}^3$. In particular, a subgroup $K$ of $G$ will act freely on $\mathbb{H}^3$ if and only if it is torsion-free. Moreover for each vertex $v$ of $R$ the stabilizer $\mathrm{Stab}_G(v)$ of $v$ in $G$ is isomorphic to the Abelian group $\mathbb{Z}/_2 \oplus \mathbb{Z}/_2 \oplus \mathbb{Z}/_2 = (\mathbb{Z}/_2)^3$ of order 8, which we will view as a vector space over the field $\mathbb{Z}/_2$. We next quote two lemmas proved in [19] and derive an easy consequence.

**Lemma 2.2.** *If $\varphi : G \to (\mathbb{Z}/_2)^3$ is an epimorphism, the following are equivalent:*

(1) $\mathrm{Ker}(\varphi)$ *is torsion-free;*
(2) *For each vertex of $R$, if $\rho_1, \rho_2, \rho_3$ are the reflections in the faces of $R$ incident to the vertex, $\varphi(\rho_1), \varphi(\rho_2), \varphi(\rho_3)$ are linearly independent over $\mathbb{Z}/_2$.*

We consider now in $(\mathbb{Z}/_2)^3$ the vectors $\alpha = (1, 0, 0)$, $\beta = (0, 1, 0)$, $\gamma = (0, 0, 1)$ and $\delta = \alpha + \beta + \gamma = (1, 1, 1)$, and we note that any three of them are linearly independent.

**Lemma 2.3.** *Let $\varphi : G \to (\mathbb{Z}/_2)^3$ be an epimorphism that maps each of the generating reflections of $G$ to one of the elements $\alpha$, $\beta$, $\gamma$, $\delta$. Then* $\mathrm{Ker}(\varphi)$ *is a subgroup of* $\mathrm{Isom}^+(\mathbb{H}^3)$, *the group of orientation-preserving isometries of hyperbolic 3-space.*

**Proposition 2.4.** *If an epimorphism $\varphi : G \to (\mathbb{Z}/_2)^3$ satisfies condition (2) of Lemma 2.2 and the hypothesis of 2.3 then the quotient $M = \mathbb{H}^3/\mathrm{Ker}(\varphi)$ is a closed orientable hyperbolic* 3*-manifold.*

*Proof.* Lemma 2.2 implies that $\mathrm{Ker}(\varphi)$ is torsion-free, so the quotient is a hyperbolic 3-manifold $M$ without boundary, and Lemma 2.3 implies that $M$ is orientable. Since $\mathrm{Ker}(\varphi)$ has index 8 in $G$ and $R$ is a fundamental domain for $G$, a fundamental domain for $\mathrm{Ker}(\varphi)$ is given by $\bigcup_{i=1}^{8} g_i(R)$ where $\{g_i : 1 \leqslant i \leqslant 8\}$ is a set of representatives of $G/\mathrm{Ker}(\varphi)$. Such a fundamental domain is compact, so $M$ is compact, whence closed. $\square$

If we now describe a homomorphism of $G$ by labelling each face of $R$ by the image of the reflection in the plane containing that face, a map $\varphi$ as in Proposition 2.4 gives an $\{\alpha,\beta,\gamma,\delta\}$-coloring of the faces of $R$ with the usual condition that adjacent faces should have different colors. On the other hand, Lemma 2.1 implies the converse, namely that any such coloring gives a map $\varphi$ as in Proposition 2.4, and hence a closed orientable hyperbolic manifold. As an example, the classical Löbell manifold [10] is obtained from the polytope $R(6)$ described above and shown in Fig. 1-left using the coloring shown on the right in the same figure.

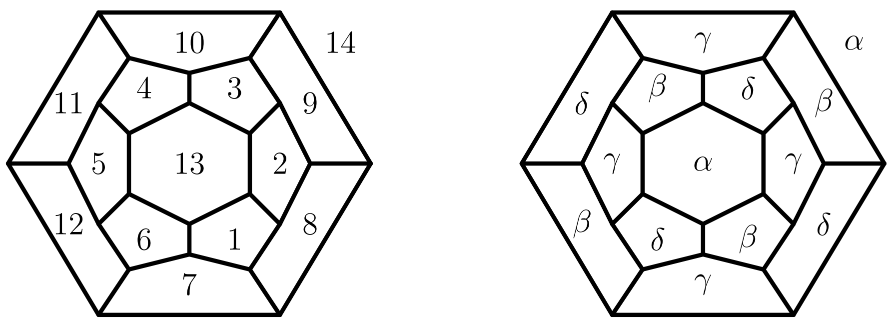


FIGURE 1. The polytope $R(6)$ and a coloring of its faces.

We generalize this example by considering for each $n \geqslant 5$ the right-angled polytope $R(n)$ in $\mathbb{H}^3$ with $(2n+2)$ faces, two of which (viewed as the upper and lower bases) are regular $n$-gons, while the lateral surface is given by 2n pentagons, arranged as one easily imagines. Note that $R(5)$ is the right-angled dodecahedron. As in Fig. 1 for $R(6)$, we number the faces of $R(n)$ so that:

- The upper and lower bases have numbers $2n+1$ and $2n$ respectively;
- The pentagons adjacent to the upper basis are cyclically numbered $1,\ldots,n$;
- The pentagons adjacent to the lower basis are cyclically numbered $n+1,\ldots,2n$ in the same verse as the previous ones, with pentagon $n+1$ adjacent to pentagons 1 and $n$.

Now define $g_i \in \mathrm{Isom}(\mathbb{H}^3)$ as the reflection in the plane containing the $i$-th face of $R(n)$, and $G(n)$ as the group generated $\{g_i\}_{i=1}^{2n+2}$. According to Lemma 2.1 a presentation of $G(n)$ is obtained by adding the relations:

$$\begin{array}{ll}
g_i^2 & i=1,\ldots,2n+2 \\
[g_{2n+1},g_i] & i=1,\ldots,n \\
[g_{2n+2},g_{n+i}] & i=1,\ldots,n \\
[g_i,g_{i+1}] & i=1,\ldots,2n-1 \\
[g_1,g_n] & \\
[g_i,g_{n+i}] & i=1,\ldots,n \\
[g_{n+1},g_{2n}] & \\
[g_i,g_{n+1+i}] & i=1,\ldots,n-1.
\end{array}$$

We now define the class *Löbell manifolds* of order $n$ as follows:

$$\mathcal{L}(n)=\Big\{\mathbb{H}^3/\mathrm{Ker}(\varphi): \begin{array}{l}\varphi: G(n)\to(\mathbb{Z}/_2)^3 \text{ epimorphism}, \\ \mathrm{Ker}(\varphi)<\mathrm{Isom}^+(\mathbb{H}^3), \\ \mathrm{Ker}(\varphi)\text{ is torsion-free}\Big\}.\end{array}$$

Each element of $\mathcal{L}(n)$ is a closed orientable hyperbolic 3-manifold with volume equal to 8 times the volume of $R(n)$. According to the above discussion (and the 4-color theorem!) the set $\mathcal{L}(n)$ is non-empty for all $n\geqslant 5$. The classical Löbell manifold constructed in [10] and described above belongs to $\mathcal{L}(6)$. We now provide an upper bound for the complexity of the elements of $\mathcal{L}(n)$.

**Lemma 2.5.** *For all $n\geqslant 5$ and $M\in\mathcal{L}(n)$ we have $c(M)\leqslant 32(2n-1)$.*

*Proof.* By construction $M$ is built by gluing together in pairs the faces of 8 copies of $R(n)$. Each face is a $k$-gon with $k\in\{5,n\}$ and we can subdivide it into $k-2$ triangles by inserting $k-3$ diagonals, in such a way that the gluing respects the subdivision. The boundary of each of the 8 copies of $R(n)$ is now subdivided into $2(n-2)+2n(5-2)=4(2n-1)$ triangles. Taking the cone from the center, we can subdivide the copy of $R(n)$ itself into the same number of tetrahedra, which yields the desired upper bound immediately. □

To give lower complexity estimates we will employ the hyperbolic volume. Let us denote by $\ell_n$ the common value of $\mathrm{vol}(M)$ as $M$ varies in $\mathcal{L}(n)$, and recall the definition of the *Lobachevskii function*

$$\Lambda(x)=-\int_0^x \log|2\sin(t)|\,\mathrm{d}t.$$

The following was established in [20]:

**Theorem 2.6.** *For all $n\geqslant 5$ we have*

$$\ell_n=4n\left(2\Lambda(\theta)+\Lambda\left(\theta+\frac{\pi}{n}\right)+\Lambda\left(\theta-\frac{\pi}{n}\right)-\Lambda\left(2\theta-\frac{\pi}{2}\right)\right),$$

*where*

$$\theta = \frac{\pi}{2} - \arccos\left(\frac{1}{2\cos(\pi/n)}\right).$$

This theorem implies that the volume of the classical Löbell manifold is equal to $48.184368\ldots$ . In addition, it allows us to determine the asymptotic behavior of $\ell_n$ as $n$ tends to infinity. Indeed, in the limit the angle $\theta$ of the statement tends to $\pi/6$, and using the fact that $\Lambda$ is continuous and odd we have:

**Corollary 2.7.** *As $n$ tends to $\infty$ we have*

$$\ell_n \sim 10n \cdot v_3$$

*where $v_3 = 2\Lambda(\pi/6) = 1.014\ldots$ is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.*

To apply this result we establish the following general fact:

**Proposition 2.8.** *If $M$ is a closed orientable hyperbolic manifold then*

$$\mathrm{vol}(M) < c(M) \cdot v_3.$$

*Proof.* Denote $c(M)$ by $k$. Since $M$ is irreducible and not one of the exceptional manifolds $\mathbb{S}^3$, $\mathbb{RP}^3$, and $L(3,1)$, there exists a realization of $M$ as a gluing of $k$ tetrahedra. Denoting by $\Delta$ the abstract tetrahedron, this realization induces continuous maps $\sigma_i : \Delta \to M$ for $i = 1, \ldots, k$ describing how the tetrahedra appear in $M$ after the gluing. Note that each $\sigma_i$ is injective on the interior of $\Delta$ but perhaps not on the boundary. Since the gluings used to pair the faces of the tetrahedra in the construction of $M$ are simplicial, we see that $\sum_{i=1}^{k} \sigma_i$ is a singular 3-cycle, which of course represents the fundamental class $[M] \in H_3(M;\mathbb{Z})$.

Let us now consider the universal covering $\mathbb{H}^3 \to M$. Since $\Delta$ is simply connected, we can lift $\sigma_i$ to a map $\widetilde{\sigma}_i : \Delta \to \mathbb{H}^3$. We denote now by $\widetilde{\tau}_i : \Delta \to \mathbb{H}^3$ the simplicial map which agrees with $\widetilde{\sigma}_i$ on the vertices, where geodesic convex combinations are used in $\mathbb{H}^3$ to define the notion of 'simplicial.' Let $\tau_i : \Delta \to M$ by the composition of $\widetilde{\tau}_i$ with the projection $\mathbb{H}^3 \to M$. One can easily see that $\sum_{i=1}^{k} \tau_i$ is again a singular 3-cycle in $M$. Using this fact and taking convex combinations in $\mathbb{H}^3$ one can actually see that the cycles $\sum_{i=1}^{k} \sigma_i$ and $\sum_{i=1}^{k} \tau_i$ are homotopic. In particular, since the first cycle represents $[M]$, the latter also does, which implies that $\bigcup_{i=1}^{k} \tau_i(\Delta)$ is equal to $M$, otherwise $\sum_{i=1}^{k} \tau_i$ would be homotopic to a map with 2-dimensional image.

Let us now note that $\widetilde{\tau}_i(\Delta)$ is a compact geodesic tetrahedron in $\mathbb{H}^3$, therefore its volume is less than $v_3$, see [3]. Moreover the volume of

$\tau_i(\Delta)$ is at most equal to the volume of $\widetilde{\tau}_i(\Delta)$, because the projection $\mathbb{H}^3 \to M$ is a local isometry, and the volume of $M$ is at most the sum of the volumes of the $\tau_i(\Delta)$'s, because we have shown above that $M$ is covered by the $\tau_i(\Delta)$'s. This concludes the proof. $\square$

**Corollary 2.9.** *For sufficiently large $n$ and $M \in \mathcal{L}(n)$ we have $c(M) > 10n$.*

The following theorem summarizes the complexity bounds we have obtained in the present section:

**Theorem 2.10.** *For sufficiently large $n$ and $M \in \mathcal{L}(n)$ the complexity of $M$ satisfies the following inequalities:*

$$10n < c(M) \leqslant 32(2n-1).$$

## 3. Fibonacci manifolds

In this section we consider the compact orientable hyperbolic 3-manifolds whose fundamental groups are the Fibonacci groups, introduced by J. Conway in [4]. There is one such group $F(2,n)$ for each $n \geqslant 3$, and a presentation of it is given by

$$F(2,n) = \langle x_1, x_2, \ldots, x_n : \ x_i x_{i+1} x_{i+2}^{-1}, \ i = 1, \ldots, n \rangle$$

where indices are understood modulo $n$. It was shown in [8] that for each $n \geqslant 4$ the group $F(2,2n)$ is isomorphic to a discrete cocompact subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, the group of orientation-preserving isometries of hyperbolic 3-space. We will need below to refer to several details of the construction of [8], so we recall it here.

We fix $n \geqslant 4$ and we first define the order-$n$ antiprism $A(n)$ as the polytope whose boundary is constructed as follows:

- Take $2n$ triangles and two polygons with $n$ faces;
- Attach a different triangle to each edge of each of the two $n$-polygons;
- Glue together the objects thus obtained by matching the free edges of the triangles (there are two circles consisting of $n$ edges to glue together, so there is essentially only one way to do so).

Now we define $Y(n)$ as the polytope obtained from $A(n)$ by attaching an $n$-pyramid to each of the bases. In particular, $Y(5)$ is the icosahedron. We remark that in general $Y(n)$ has $2n+2$ vertices, $6n$ edges, and $4n$ triangular faces, and we denote the vertices by $Q$, $R$, $P_1, \ldots, P_{2n}$ and the faces by $F_1, \ldots, F_n$, and $F_1^*, \ldots, F_n^*$, as shown in Fig. 2 for $n = 4$.

We define now a face-pairing on $Y(n)$ under which each face $F_i$ is glued to the face $F_i^*$ via a simplicial homeomorphism $s_i : F_i \longrightarrow F_i^*$.

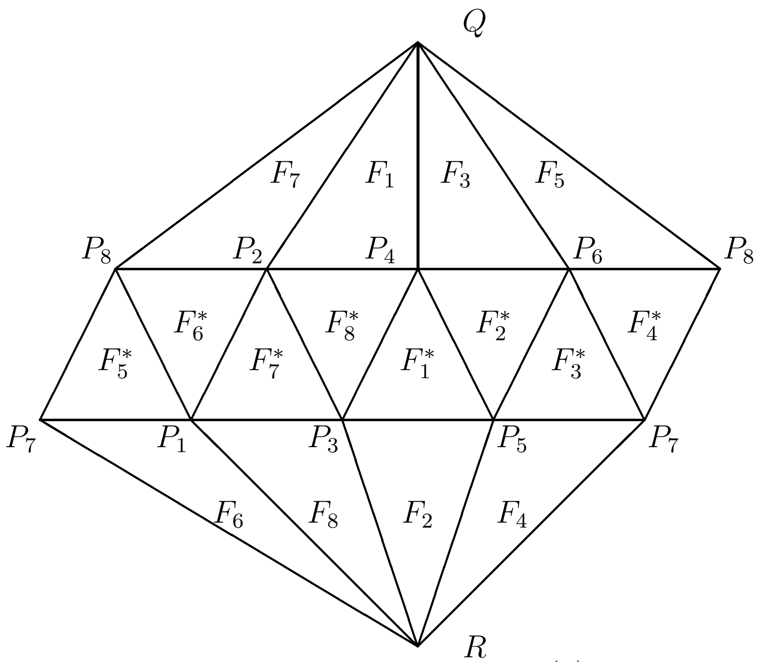


FIGURE 2. The polytope $Y(4)$.

We specify the action of $s_i$ by describing its action on the vertices. Namely, for odd $i$ we choose $s_i$ so that

$$s_i : QP_{i+1}P_{i+3} \longrightarrow P_{i+2}P_{i+3}P_{i+4}$$

whereas for even $i$ we choose it so that

$$s_i : RP_{i+1}P_{i+3} \longrightarrow P_{i+2}P_{i+3}P_{i+4}.$$

Note that if we choose an orientation of $Y(n)$ and orient the $F_i$ and $F_i^*$ accordingly, all the $s_i$'s are orientation-reversing homeomorphisms. This implies that the quotient of $Y(n)$ under the face-pairing is a manifold except perhaps at the vertices, and that the projection restricted to each open edge of $Y(n)$ is injective. In particular, we can describe how the various edges of $Y(n)$ are cyclically arranged around an edge in the quotient. These cycles are actually easy to describe: for odd $i$ we have

$$QP_{i+1} \xrightarrow{s_i} P_{i+2}P_{i+3} \xrightarrow{s_{i-1}^{-1}} P_iP_{i+2} \xrightarrow{s_{i-2}^{-1}} QP_{i+1}, \tag{1}$$

and for even $i$ we have

$$RP_{i+1} \xrightarrow{s_i} P_{i+2}P_{i+3} \xrightarrow{s_{i-1}^{-1}} P_iP_{i+2} \xrightarrow{s_{i-2}^{-1}} RP_{i+1}. \tag{2}$$

It was shown in [8] that $Y(n)$ can be realized in a unique way (up to isometry) as a compact polytope in hyperbolic space $\mathbb{H}^3$ in such a way that:

- Each of the faces of $Y(n)$ is an equilateral triangle;
- The sums of the dihedral angles corresponding to the cycles (1) and (2) is $2\pi$;
- $Y(n)$ has a cyclic symmetry of order $n$ with axis $QR$ and an orientation-reversing symmetry which permutes $Q$ and $R$.

We will henceforth identify $Y(n)$ with such a realization in $\mathbb{H}^3$. Since all the faces of $Y(n)$ are congruent, each face-pairing $s_i$ can be realized in a unique fashion as an orientation-preserving isometry of $\mathbb{H}^3$, and we will denote this isometry also by $s_i$. The condition that the total dihedral angle around the edge-cycles (1) and (2) is $2\pi$ easily implies that

$$s_i s_{i+1} = s_{i+2}, \qquad i = 1, \ldots, 2n \tag{3}$$

where indices are understood modulo $2n$. More precisely, Poincaré's polyhedron theorem [5, 18] implies that:

- The subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ generated by the $s_i$'s is isomorphic to $F(2, 2n)$;
- This group is discrete and torsion-free, and $Y(n)$ is a fundamental domain for its action on $\mathbb{H}^3$;
- The quotient of $\mathbb{H}^3$ under this action is a 3-manifold.

We will denote from now on by $M(n)$ the closed hyperbolic 3-manifold thus obtained, and call it the $n$-th *Fibonacci manifold*. It was remarked in [9] that $M(n)$ is the $n$-fold cyclic covering of $\mathbb{S}^3$ branched over the figure-eight knot $4_1$.

**Lemma 3.1.** For $n \geqslant 4$ we have $c(M(n)) \leqslant 3n$.

*Proof.* For each triangular face of $Y(n)$ not containing $Q$ we can construct the tetrahedron with vertex at $Q$ and basis at that face. This gives a decomposition of $Y(n)$ into $3n$ tetrahedra, whence a singular triangulation of $M(n)$ with $3n$ tetrahedra, whence the conclusion at once. □

To estimate the complexity of $M(n)$ from below we use the following formula for its volume established in [16]:

**Theorem 3.2.** *For $n \geqslant 4$ we have*

$$\mathrm{vol}(M(n)) = 2n(\Lambda(a_n + b_n) + \Lambda(a_n - b_n)),$$

*where $b_n = \pi/n$ and $a_n = (1/2) \cdot \arccos(\cos(2a_n) - 1/2)$.*

This result allows us to determine the asymptotic behavior of the volume of the Fibonacci manifold $M(n)$. Indeed, as $n$ tends to infinity, we see that $b_n$ converges to 0 and $a_n$ converges to $\pi/6$. Using the fact that $\Lambda$ is continuous we deduce the following:

**Corollary 3.3.** *As $n$ tends to $\infty$ we have*

$$\mathrm{vol}(M(n)) \sim 2n \cdot v_3$$

*where $v_3 = 2\Lambda(\pi/6) = 1.014\ldots$ is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.*

This result together with Proposition 2.8 implies:

**Corollary 3.4.** *For sufficiently large $n$ we have $c(M(n)) > 2n$.*

The following theorem summarizes the complexity bounds we have obtained in the present section:

**Theorem 3.5.** *For sufficiently large $n$ the complexity of the Fibonacci manifold $M(n)$ satisfies the following inequalities:*

$$2n < c(M(n)) \leqslant 3n.$$

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