Decay estimates for the supercritical 3-D Schrödinger equation with rapidly decreasing potential

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Abstract. We establish almost optimal decay estimate for the 3-D Schrödinger equation with non - negative potential decaying exponentially and nonlinearity of power $p > 1 + 2/3 = 5/3$. The key point is the introduction of appropriate analogue of the generators of the pseudoconformal group for the free Schrödinger equation.

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1. Introduction.

We consider the following Schrödinger type equation

$$(\mathrm{i}\partial_t + \Delta_x - W(x))u + |u|^{p-1}u = 0. \tag{1.1}$$

where $t \geq 1$ and $x \in \mathbb{R}^3$ and $W(x)$ is a non - negative potential.

Potential type perturbation for the classical Schrödinger equation

$$(\mathrm{i}\partial_t + \Delta_x)U + |U|^{p-1}U = 0$$

appears in natural way, after linearization around solitary type solutions, i.e. solution of the form

$$U = e^{i\omega t}\chi(x),$$

where $\chi \in H^1(\mathbb{R}^3)$ is a critical point of the functional

$$E(\chi) = \frac{1}{2}\|\nabla\chi\|_{L^2}^2 - \frac{1}{p+1}\int_{\mathbb{R}^3} |\chi(x)|^{p+1}dx$$

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subject to the constraint

$$\|\chi\|_{L^2}^2 = 1.$$

Restricting the attention to minimum of $E(\chi)$ subject to the same constraint one can work with so called ground states (see [6] and the references therein) and see the existence and the fact that ground states are positive, radial and exponentially decaying functions. The linearization of type

$$U = \left(u + e^{i\omega t}\chi\right)$$

leads to equation of type (1.1), where W is a bounded (possibly non self adjoint) operator in L^2 expressed in terms of $\chi(x)$.

For simplicity in this work we consider the case, when $W(x)$ is a real valued non-negative function having the same decay properties as the ground state $\chi(x)$. More precisely, we assume the following hypotheses on W:

(H1) W is a non - negative Schwartz function decaying exponentially at infinity and such that there exist positive constants $c_0 > c_1 > 0$ so that for any $x \in \mathbb{R}^3$

$$0 \le c_0 W(x) \le -\partial_r W(x),$$

and

$$W(x) + |\partial_r W(x)| \le C e^{-c_1|x|}.$$

Let Σ^s be the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R}^3)$ functions with respect to the norm

$$\|u\|_{\Sigma^s}^2 = \|u\|_{H^s(\mathbb{R}^3)}^2 + \||x|^s u\|_{L^2(\mathbb{R}^3)}^2. \tag{1.2}$$

It is well - known that $p = 5/3$ is a critical value for the existence of asymptotic profiles and dispersive estimates for small data solutions in case of potential $W = 0$ (see [11] for example). In this work we study the supercritical case $p > 5/3$ and obtain the following decay estimate.

Theorem 1.1. *Assume (H1) and the parameters s, p satisfy $p > 5/3, s > 3/2$. Then there is a constant $\epsilon_0 > 0$, so that for any $\delta > 0$ one can find a constant $C_0 = C_0(\varepsilon_0, \delta) > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the solution to* (1.1) *satisfies the inequality*

$$\|u(t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{C_0}{t^{3/2-\delta}}\epsilon, \tag{1.3}$$

provided

$$\|u(1)\|_{\Sigma^s} \le \epsilon.$$

There is a long list of results concerning the Strichartz type estimates, $L^p - L^q$ estimates and similar dispersive estimates for potential (or magnetic) type perturbations of the linear Schrödinger or wave equation. (see [16], [18], [7], [2], [14] for some of these results). However these type of estimates seem to have non – obvious application if one tries to get the almost optimal decay rate for the nonlinear supercritical Scrödinger equation. The classical approach developed in [11] is based on the use of the generators of the pseudoconformal transform that enable to get the optimal decay for the case of potential $W = 0$. Since no reasonable definition of these generators is available for the case of potential we are forced to use another approach.

Our approach is based on a direct application of the pseudoconformal transform. After this transform the global Cauchy problem with initial data at $t = 1$ becomes local Cauchy problem for the Schrödinger equation with time dependent potential. More precisely, we shall need an estimate of the solution to the problem

$$i\partial_T v + \Delta v - T^{-2} W\left(\frac{X}{T}\right) v = G, \quad T \in (0,1)$$

Taking $s \in [3/2, 2]$, for any $\delta > 0$ we shall be able to find positive constant $C = C(s,\delta)$ so that

$$\|v(T,\cdot)\|_{H^s_X} \leq C|T|^{3/2-s-\delta}\|v(1,\cdot)\|_{H^s_X} + + C|T|^{3/2-s-\delta}\|G\|_{L^1((T,1);H^s_X)}.$$

This is our key estimate to derive local existence result for the nonlinear Schrödinger equation after the pseudoconformal transform.

2. Pseudo - conformal transform.

The pseudo – conformal transform is defined as follows

$$(t, x, u) \Longrightarrow (T, X, v),$$

where $t = 1/T, x = X/T$ and

$$v(T,X) = \frac{1}{T^{n/2}}\overline{u}(\frac{1}{T}, \frac{X}{T})e^{\mathrm{i}\frac{X^2}{4T}}. \tag{2.1}$$

Then we have the relation

$$\mathrm{i}\partial_T v(T,X) + \Delta_X v(T,X) = T^{-2-\frac{n}{2}} e^{\mathrm{i}\frac{X^2}{4T}} \left(-\mathrm{i}\partial_t \overline{u}(t,x) + \Delta_x \overline{u}(t,x)\right)|_{t=1/T, x=X/t}.$$

Since $\overline{u}$ satisfies the equation

$$(-\mathrm{i}\partial_t + \Delta_x - W(x))\overline{u} + |\overline{u}|^{p-1}\overline{u} = 0,$$

we get

$$\mathrm{i}\partial_T v(T,X) + \Delta_X v(T,X) = T^{-2-\frac{n}{2}}\left(W(x)\overline{u}(t,x) - |u(t,x)|^{p-1}\overline{u}(t,x)\right) e^{\mathrm{i}\frac{X^2}{4T}}$$
$$= T^{-2} W\left(\frac{X}{T}\right) v(T,X) - T^{\frac{np-n-4}{2}}|v(T,X)|^{p-1}v(T,X).$$

Hence

$$\mathrm{i}\partial_T v + \Delta_X v - T^{-2} W\left(\frac{X}{T}\right) v + T^{\frac{np-n-4}{2}}|v|^{p-1}v = 0 \tag{2.2}$$

for $0 < T \leq 1$ and $X \in \mathbb{R}^n$. Note that the L^2 norm of u is constant, since u satisfied the nonlinear Schrödinger equation (1.1). On the other hand, the transform (2.1) preserves the L^2 norm, so we have

$$\frac{d}{dT}\int_{\mathbb{R}^n} \left(|v|^2\right) dX = 0.$$

In this way we obtain the following lemma.

Lemma 2.1. *If u satisfies the linear Schrödinger equation*

$$(\mathrm{i}\partial_t + \Delta_x - W(x))u = F \text{ for } t \geq 1 \text{ and } x \in \mathbb{R}^n, n \geq 3, \tag{2.3}$$

then $v(T,X)$ defined according to (2.1) and

$$G(T,X) = \frac{1}{T^{2+n/2}} \overline{F}(\frac{1}{T}, \frac{X}{T}) e^{\mathrm{i}\frac{X^2}{4T}}. \tag{2.4}$$

satisfy

$$\mathrm{i}\partial_T v + \Delta_X v - W_T(X) v = G(T,X), \tag{2.5}$$

where

$$W_T(X) = T^{-2} W\left(\frac{X}{T}\right). \tag{2.6}$$

3. H^2 estimate.

Setting

$$v_2(T,X) = \left(-\Delta + T^{-2} W\left(\frac{X}{T}\right)\right) v(T,X),$$

we have the following equation satisfied by v_2

$$i\partial_T v_2 + \Delta v_2 - T^{-2} W\left(\frac{X}{T}\right) v_2 = iW_1 v + G_2,$$

where

$$W_1(T,X) = \left[\partial_T \left(T^{-2} W\left(\frac{X}{T}\right)\right)\right]$$

and

$$G_2 = \left(-\Delta + T^{-2} W\left(\frac{X}{T}\right)\right) G.$$

Let us take $n = 3$. Applying the Strichartz estimates of Theorem 6.8, we get

$$\|v_2(T,\cdot)\|_{L^2_X} \leq C\left(\|v_2(1,\cdot)\|_{L^2_X} + \|W_1 v\|_{L^2((T,1);L^{6/5}_X)} + \|G_2\|_{L^1((T,1);L^2_X)}\right)$$

The assumption (H_1) guarantees that

$$W_1(T,X) \leq C \frac{W_0(T,X)}{T}, \tag{3.1}$$

where

$$W_0(T,X) = \frac{e^{-c_2|X|/T}}{T^2}.$$

We have a generalization of this estimate, given in the Lemma below, where typically we shall assume that $b(s) = 1 + |\log s|$ or $b(s) = s^{-\delta}$ with $\delta > 0$ small.

Lemma 3.1. *Suppose $b(s)$ is a positive continuous function in $(0,\infty)$, such that b is decreasing in $(0,1)$, satisfies the estimate $b(s^2) \le Cb(s), s \in (0,1)$ and*

$$\lim_{s\searrow 0} s^\varepsilon b(s) = 0$$

for any $\varepsilon > 0$ such that $\varepsilon < \min(c_2, a/2))$

and satisfies one of the following assumptions

a) b is decreasing in $(1,\infty)$, and

$$\lim_{s\nearrow \infty} s^\varepsilon b(s) = \infty$$

for some $\varepsilon > 0$ such that $\varepsilon < \min(c_2, a/2))$

or

b) b is increasing in $(1,\infty)$, satisfies the estimate $b(s^{-1}) \le Cb(s), s \in (0,1)$ and

$$\lim_{s\nearrow \infty} s^\varepsilon b(s) = \infty$$

for any $\varepsilon > 0$ such that $\varepsilon < \min(c_2, a/2))$.

Then for $T \in (0,1)$ we have the estimate

$$b(|X|)W_1(T,X) \le C\frac{b(T)\widetilde{W_0}(T,X)}{T^{1-a}|X|^a}, \quad a \in (0,1), \tag{3.2}$$

where

$$\widetilde{W_0}(T,X) = \frac{e^{-(c_2-\varepsilon)|X|/T}}{T^2}.$$

Proof. It is sufficient to verify the inequality

$$\frac{|X|^a b(|X|)}{b(T)T^a} \le Ce^{\varepsilon|X|/T}.$$

If $T^2 < |X| < T$, then

$$\frac{|X|^a b(|X|)}{T^a b(T)} \le \frac{|X|^a b(T^2)}{T^a b(T)} \le C.$$

If $|X| \le T^2$, then the condition

$$\lim_{s\searrow 0} s^\varepsilon b(s) = 0$$

implies $s^\varepsilon b(s) \le C$ so

$$\frac{|X|^a b(|X|)}{T^a b(T)} \le \frac{|X|^{a-\varepsilon}}{T^a b(T)} \le \frac{T^{a-2\varepsilon}}{b(T)} \le C.$$

For $T < |X| < 1$ we have

$$\frac{|X|^a b(|X|)}{T^a b(T)} \le \frac{|X|^a}{T^a} \le Ce^{\varepsilon|X|/T}$$

and the same argument works if $|X| > 1 > T$ and b is decreasing everywhere, i.e. a) holds. For b) we can separate the cases $1 < |X| < 1/T$ and $|X| > 1/T$. If $1 < |X| < 1/T$ then $b(|X|) \leq b(1/T) \leq Cb(T)$ and we have

$$\frac{|X|^a b(|X|)}{T^a b(T)} \leq \frac{C|X|^a}{T^a} \leq C_1 e^{\varepsilon|X|/T}.$$

If $|X| \geq 1/T$, then $b(T) \geq C^{-1}b(1/T) \geq C^{-1}b(1) = C_1$ and hence

$$\frac{|X|^a b(|X|)}{T^a b(T)} \leq C|X|^{2a} b(|X|) \leq Ce^{\varepsilon|X|} \leq Ce^{\varepsilon|X|/T}.$$

□

Note that

$$\|\widetilde{W_0}(T,\cdot)\|_{L^{n/2}} = O(1). \tag{3.3}$$

Applying the estimate of the previous Lemma (with $n = 3, b(T) = |\log T|$) we get

$$\|W_1 v\|_{L^2((T,1);L^{6/5}_X)} \leq C \left\| \frac{b(\tau)\widetilde{W_0}(\tau, X)}{\tau^{1/2} b(|X|)|X|^{1/2}} v \right\|_{L^2((T,1);L^{6/5}_X)}.$$

Using the fact that

$$\|b(\tau)\tau^{-1/2}\|^2_{L^2((T,1))} = \int_T^1 \log^2(\tau) \frac{d\tau}{\tau} \sim |\log T|^3 = b(T)^2 |\log T|,$$

combined with the Hölder inequality

$$\left\| \widetilde{W_0}(\tau, X) g(x) \right\|_{L^{6/5}_X} \leq \|\widetilde{W_0}(T,\cdot)\|_{L^{3/2}_X} \|g(x)\|_{L^{6/5}_X} = C\, \|g(x)\|_{L^{6/5}_X},$$

we get

$$\|W_1 v\|_{L^2((T,1);L^{6/5}_X)} \leq Cb(T)|\log T|^{1/2} \left\| \frac{v}{b(|X|)|X|^{1/2}} \right\|_{L^\infty((T,1);L^6_X)} \leq$$

$$\leq Cb(T)|\log T|^{1/2} \|v\|_{L^\infty((T,1);L^\infty_X)}$$

so

$$\|W_1 v\|_{L^2((T,1);L^{6/5}_X)} \leq C|\log T|^2 \|v\|_{L^\infty((T,1);L^\infty_X)}.$$

In this way we arrive at

$$\begin{aligned} \|(-\Delta + W_T)v(T,\cdot)\|_{L^2_X} \leq C\|v(1,\cdot)\|_{H^2_X} + C|\log T|^2 \|v(1,\cdot)\|_{L^2_X} + \\ + C\|(-\Delta + W_T)G\|_{L^1((T,1);L^2_X)}, \end{aligned} \tag{3.4}$$

where W_t is defined according to (2.6).

4. Interpolation between $s = 2$ and $s = 0$.

First we take any $\delta \in (0,1)$ and $v(1,x) = 0$ and consider the operator

$$M : G \in L^1((\delta,1); L^2(\mathbb{R}^3)) \longrightarrow v \in L^\infty((\delta,1); L^2(\mathbb{R}^3))$$

such that v solves the equation

$$i\partial_T v + \Delta v - T^{-2} W\left(\frac{X}{T}\right) v = G,$$

with zero data at $T = 1$. We take another small parameter $\delta_1 \in (0,1)$ and define for any complex z with $\mathrm{Re} z \in [0,1]$ the following operator

$$\mathbb{U}(z) = \mathbb{U}_{\delta_1}(z) = (-\Delta + W_T)^z T(\delta_1 - \Delta + W_T)^{-z}$$

$$\mathbb{U}(z) : L^1((\delta,1); L^2(\mathbb{R}^3)) \longrightarrow v \in L^\infty((\delta,1); L^2(\mathbb{R}^3)).$$

This is analytic operator – valued operator. Using the fact that

$$(-\Delta + W_T)^{is} : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3)$$

is bounded operator for real s due to the spectral theorem and using the charge conservation law for the Schrödinger equation with real valued potential, we see that

$$\|\mathbb{U}_{\delta_1}(z)F\|_{L^\infty((\delta,1);L^2(\mathbb{R}^3))} \leq C\|F\|_{L^1((\delta,1);L^2(\mathbb{R}^3))} \tag{4.1}$$

provided $\mathrm{Re} z = 0$. Note that the constant $C > 0$ is independent of G, δ, δ_1.

Using the estimate (3.4), we see that

$$\|\mathbb{U}_{\delta_1}(z)F\|_{L^\infty((\delta,1);L^2(\mathbb{R}^3))} \leq C\|F\|_{L^1((\delta,1);L^2(\mathbb{R}^3))} \tag{4.2}$$

provided $\mathrm{Re} z = 1$ and again the constant $C > 0$ is independent of G, δ, δ_1.

Applying the Stein interpolation theorem (in this simple case the three lines lemma) we see that for any $s \in (0,2)$ we have

$$\|\mathbb{U}_{\delta_1}(s/2)F\|_{L^\infty((\delta,1);L^2(\mathbb{R}^3))} \leq C\|F\|_{L^1((\delta,1);L^2(\mathbb{R}^3))} \tag{4.3}$$

and using the definition of $\mathbb{U}_{\delta_1}(z)$ we set

$$F = (\delta_1 - \Delta + W_T)^{s/2} G, \quad G \in L^1((\delta,1); H^s(\mathbb{R}^3))$$

and get

$$\|(-\Delta + W_T)^{s/2} v(T,\cdot)\|_{L^\infty(\delta,1);L^2_X)} \leq C\|(\delta_1 - \Delta + W_T)^{s/2} G\|_{L^1((\delta,1);L^2_X)}. \tag{4.4}$$

with constant $C > 0$ is independent of G, δ, δ_1, letting δ and δ_1 to tend to zero we find

$$\|(-\Delta + W_T)^{s/2} v(T,\cdot)\|_{L^\infty(0,1);L^2_X)} \leq C\|(-\Delta + W_T)^{s/2} G\|_{L^1((0,1);L^2_X)}. \tag{4.5}$$

In a similar way one can consider the map

$$M_0 : f \in L^2(\mathbb{R}^3) \longrightarrow v \in L^\infty((\delta,1); L^2(\mathbb{R}^3))$$

such that v solves the equation

$$i\partial_T v + \Delta v - T^{-2} W\left(\frac{X}{T}\right) v = 0,$$

with data $v(1,X) = f(X)$ at $T = 1$. The estimate (3.4) shows that we have

$$\|(-\Delta + W_T)v(T,\cdot)\|_{L^2_X} \leq C\|f\|_{H^2_X} + C|\log T|^2\|f\|_{L^2_X}. \tag{4.6}$$

Applying interpolation argument as above, we find

$$\|(-\Delta + W_T)^{s/2}v(T,\cdot)\|_{L^2_X} \leq C(1 + |\log T|)^s\|f\|_{H^s_X} \tag{4.7}$$

for any $s \in [0,2]$.

In this way we obtain the estimate.

Theorem 4.1. *Assume $s \in [0,2]$. Then the solution to the equation*

$$i\partial_T v + \Delta v - T^{-2}W\left(\frac{X}{T}\right)v = G$$

satisfies the inequality

$$\begin{aligned}\|(-\Delta + W_T)^{s/2}v(T,\cdot)\|_{L^2_X} \leq C\,(1 + |\log T|)^s\,\|v(1,\cdot)\|_{H^s_X} + \\ + C\|(-\Delta + W_T)^{s/2}G\|_{L^1((T,1);L^2_X)}.\end{aligned} \tag{4.8}$$

One can show that

$$\|(-\Delta + W_T)^{s/2}f\|_{L^2_X} \sim \|(-\Delta)^{s/2}f\|_{L^2_X},$$

for $0 \leq s < 3/2$. This fact is established in [9], [8] (see also the section 7 below where this is verified for completeness).

Then we arrive at

Theorem 4.2. *Assume $s \in [0,3/2)$. Then the solution to the equation*

$$i\partial_T v + \Delta v - T^{-2}W\left(\frac{X}{T}\right)v = G$$

satisfies the inequality

$$\begin{aligned}\|(-\Delta)^{s/2}v(T,\cdot)\|_{L^2_X} \leq C\,(1 + |\log T|)^s\,\|v(1,\cdot)\|_{H^s_X} + \\ + C\|(-\Delta)^{s/2}G\|_{L^1((T,1);L^2_X)}.\end{aligned} \tag{4.9}$$

For $s = 2$ we can use the maximum principle for $-\Delta + W$ and see that

$$\|(-\Delta + W)^{-1}f\|_{L^2} \leq \|(-\Delta)^{-1}f\|_{L^2}$$

so

$$\|(-\Delta + W)^{-1}\Delta f\|_{L^2} \leq C\|f\|_{L^2} \tag{4.10}$$

and by duality

$$\|\Delta(-\Delta + W)^{-1}f\|_{L^2} \leq C\|f\|_{L^2}.$$

From this estimate we find

$$\|\Delta f\|_{L^2} = \|\Delta(-\Delta + W)^{-1}(-\Delta + W)f\|_{L^2} \leq C\|(-\Delta + W)f\|_{L^2} \tag{4.11}$$

Thus we can obtain the following

Theorem 4.3. *Assume $s \in [3/2, 2]$. Then the solution to the equation*

$$i\partial_T v + \Delta v - T^{-2} W \left(\frac{X}{T}\right) v = G$$

satisfies the inequality

$$\|(-\Delta)^{s/2} v(T,\cdot)\|_{L^2_X} \leq C|T|^{3/2-s-\delta} \|v(1,\cdot)\|_{H^s_X} + \\ +C|T|^{3/2-s-\delta} \|(1-\Delta)^{s/2} G\|_{L^1((T,1);L^2_X)}. \tag{4.12}$$

Proof. It is sufficient to verify the estimate for $s = 2$ and then to apply interpolation argument between $s = 2$ and $s < 3/2$ (established in the previous theorem). For $s = 2$ we use (3.4) as well as (4.11) and see that

$$\|(-\Delta) v(T,\cdot)\|_{L^2_X} \leq C\|v(1,\cdot)\|_{H^2_X} + C|\log T|^2 \|v(1,\cdot)\|_{L^2_X} + \\ +C\|(-\Delta)G\|_{L^1((T,1);L^2_X)} + C\|WG\|_{L^1((T,1);L^2_X)}. \tag{4.13}$$

Now the estimate

$$\|Wf\|_{L^2} \leq \frac{C}{T^{1/2+\delta}} \|(-\Delta)^{3/2+\delta} f\|_{L^2}.$$

Hence the desired estimate with $s = 2$ is fulfilled. This completes the proof. □

Corollary 4.4. *Assume $s \in [3/2, 2]$. Then for any $\delta > 0$ one can find positive constant $C = C(s, \delta)$ so that the solution to the equation*

$$i\partial_T v + \Delta v - T^{-2} W \left(\frac{X}{T}\right) v = G, \quad T \in (0,1)$$

satisfies the inequality

$$\|v(T,\cdot)\|_{H^s_X} \leq C|T|^{3/2-s-\delta} \|v(1,\cdot)\|_{H^s_X} + \\ +C|T|^{3/2-s-\delta} \|G\|_{L^1((T,1);H^s_X)}. \tag{4.14}$$

5. Proof of Theorem 1.1

Our goal is to solve the nonlinear problem (2.2)

$$\mathrm{i}\partial_T v + \Delta_X v - T^{-2} W \left(\frac{X}{T}\right) v + T^{\frac{3p-7}{2}} |v|^{p-1} v = 0$$

with initial data

$$v(1, X) = \varphi(X) \in H^s, s > 3/2.$$

We shall assume that we deal with small initial data, i.e.

$$\|\varphi\|_{H^s_X} \leq \varepsilon.$$

We shall apply the contraction mapping principle for the Banach space suggested by the estimate (4.14). Indeed, taking $s = 3/2 + \delta$, with $\delta > 0$, consider the norm

$$|||\, v \,|||_\delta = \sup_{0 \leq T \leq 1} T^{2\delta} \|v(T,\cdot)\|_{H^s_X} \tag{5.1}$$

and the corresponding Banach space B_δ. The estimates

$$\| |v(T,\cdot)|^p \|_{H^s_X} \leq C \|v(T,\cdot)\|^p_{H^s_X} \tag{5.2}$$

$$\begin{aligned} \| |v(T,\cdot) - w(T,\cdot)|^p \|_{H^s_X} \leq C \|v(T,\cdot)\|^{p-1}_{H^s_X} \|v(T,\cdot) - w(T,\cdot)\|_{H^s_X} + \\ + C \|w(T,\cdot)\|^{p-1}_{H^s_X} \|w(T,\cdot) - w(T,\cdot)\|_{H^s_X} \end{aligned} \tag{5.3}$$

are fulfilled for any $s > 3/2$. Possible reference for these estimates is Theorem 1, Section 5.4.3 in [17]. Using the estimate of Corollary 4.4, one can define the sequence $v_k \in B_\delta$ so that v_0 is a solution to the linear Cauchy problem

$$\mathrm{i}\partial_T v_0 + \Delta_X v_0 - T^{-2} W \left(\frac{X}{T} \right) v_0 = 0$$

with initial data

$$v_0 = \varphi(X).$$

Then given any $v_k \in B_\delta$ we define v_{k+1} as the unique solution to

$$\mathrm{i}\partial_T v_{k+1} + \Delta_X v_{k+1} - T^{-2} W \left(\frac{X}{T} \right) v_{k+1} + T^{\frac{3p-7}{2}} |v_k|^{p-1} v_k = 0$$

with initial data

$$v(1,X) = \varphi(X) \in H^s.$$

Applying the estimate of Corollary 4.4 as well as (5.2), we find

$$||| \, v_{k+1} \, |||_{2\delta} \leq C\varepsilon + C \, ||| \, v_k \, |||_{2\delta} \int_T^1 \tau^{3p-7-2p\delta} d\tau.$$

The assumption $p > 5/3$ guarantees that (taking $\delta > 0$ small enough)

$$\int_T^1 \tau^{3p-7-2p\delta} d\tau \leq C < \infty,$$

so

$$||| \, v_{k+1} \, |||_{2\delta} \leq C\varepsilon + C \, ||| \, v_k \, |||^p_{2\delta} \, .$$

From this estimate we easily get

$$||| \, v_k \, |||_{2\delta} \leq C_1 \varepsilon. \tag{5.4}$$

In a similar way, we can use (5.3) and derive

$$\begin{aligned} ||| \, v_{k+1} - v_k \, |||_{2\delta} \leq C \, ||| \, v_k - v_{k-1} \, |||_{2\delta} \left(||| \, v_k \, |||^{p-1}_{2\delta} + ||| \, v_{k-1} \, |||^{p-1}_{2\delta} \right) \leq \\ \leq C\varepsilon^{p-1} \, ||| \, v_k - v_{k-1} \, |||_{2\delta}, \end{aligned}$$

so taking $\varepsilon > 0$ small enough, we can apply contraction mapping principle and find a solution

$$v \in B_\delta \subset L^\infty([0,1]; H^s).$$

Turning back to the pseudoconformal transform (2.1), we see that for $T = 1$ we have

$$v(1,x) = \overline{u}(1,x) e^{\mathrm{i}\frac{x^2}{4}}. \tag{5.5}$$

and it is easy to see that the map

$$\psi(x) \Longrightarrow \varphi(x) = \overline{\psi(x)} e^{\mathrm{i}\frac{x^2}{4}}$$

maps Σ^s in H^s and

$$\|\varphi\|_{H^s} \leq C\|\psi\|_{\Sigma^s}.$$

This completes the proof of Theorem 1.1.

6. Resolvent and Strichartz type estimates

In this section we discuss briefly the dispersive and Strichartz type estimates using resolvent estimates. This link is possible in view of the following result due to Kato.

Theorem 6.1. *(Kato* [12]*) Let H be a self-adjoint operator on the Hilbert space $\mathcal{X}$, and for $\mu \in \mathbb{R}, \Im\,\mu \neq 0,$ let*

$$(H-\mu)^{-1},$$

denote the resolvent. Suppose that A is a closed, densely defined operator, possibly unbounded, from $\mathcal{X}$ into a Hilbert space $\mathcal{Y}$. Suppose that

$$\Gamma := \sup\{\|A((H-\mu)^{-1})A^* f\|_{\mathcal{Y}}; \Im\mu \neq 0, f \in D(A^*), \|f\|_{\mathcal{X}} = 1\} < \infty.$$

Then A is H-smooth and

$$\|A\|_H^2 := \sup\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \|Ae^{-itH}f\|_{\mathcal{Y}}^2 dt; f \in \mathcal{X}, \|f\|_{\mathcal{X}} = 1\right\} \leq \frac{\Gamma^2}{\pi^2}.$$

Typical application for Schrödinger equation is the choice $\mathcal{X} = L^2$ and A is the multiplication operator

$$< x >^{-s}: f(x) \in L^2 \longrightarrow < x >^{-s} f(x) \in L^2_s,$$

where here and below for any real s

$$L^2_s = \{f \in L^2_{loc}, < x >^s f \in L^2\}.$$

The study of the resolvent estimates is closely connected with the resonances of the operator

$$-\Delta + W(|x|), x \in \mathbb{R}^3.$$

Definition 6.2. A real number λ is called a strong resonance of $-\Delta + W(|x|)$ if there exists $u \in L^2_{-a}(\mathbb{R}^3)$ with $a > 1/2$, so that $u(x)$ is not identically zero and $-\Delta u + W(|x|)u = \lambda u$ in distribution sense in $\mathbb{R}^3$.

Theorem 6.3. *(see Theorem IX.2 in* [9]*) Suppose the potential $W(r)$ is a positive decreasing function, such that there exist positive constants C^*, ε so that (H_1) is fulfilled. Then zero is not a strong resonance for $-\Delta + W(|x|)$.*

Remark 6.4. Since W is an exponentially decaying and real valued, the above result implies that $-\Delta + W(|x|)$ has no resonances .

In order to verify resolvent estimate of the perturbed operator $-\Delta + W(|x|)$, denote

$$R_0(\mu) = (-\Delta - \mu^2)^{-1},$$

the resolvent of the operator $-\Delta$, and set $R_0^+(\mu) = R_0(\mu)$ if $\Im\mu > 0$ and respectively $R_0^-(\mu) = R_0(\mu)$ for $\Im\mu < 0$. Classical resolvent estimate (limiting absorbtion principle) is the following one

$$\lim_{\Im\mu \searrow 0} \| < x >^{-s} (-\Delta - \mu^2)^{-1} < x >^{-s} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \tag{6.1}$$

where $s > 1$, $\Re\mu \geq 0$, and the constant C is independent of μ. We have also the estimate

$$\| < x >^{-s} \nabla(-\Delta - \mu^2)^{-1} < x >^{-s} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)},$$

where

$$s > \frac{1}{2}, \ \Re\mu \geq 0,$$

so we can claim that the operators

$$< x >^{-s} (-\Delta - \mu^2)^{-1} < x >^{-s}$$

are compact ones in L^2 provided $s > 1$ and $\Im\mu \geq 0$.

Hence

$$\| < x >^{-s} R_0^+(\mu) < x >^{-s} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}, \quad s > 1, \ \Re\mu \geq 0, \Im\mu = 0. \tag{6.2}$$

Set

$$R(\mu) = (-\Delta + W - \mu^2)^{-1}, \quad A(\mu) =< x >^{-s} (-\Delta - \mu^2)^{-1} W < x >^{s} .$$

One have the following compactness result

Lemma 6.5. *The operators $A(\mu)$ are compact in the space $B(L^2, L^2)$, for*

$$\Im\mu \geq 0, \quad s > 1.$$

Moreover the following estimate is satisfied:

$$\|A(\mu)\|_{B(L^2,L^2)} \to 0,$$

as $\Im\mu \geq 0, \Re\mu \to \infty$.

This Lemma is a well-known standard result so we give only the idea of the proof. It suffices to notice that $< x >^{-s} (-\Delta - \mu^2)^{-1} < x >^{-s}$ is continuous and compact as an operator in $B(L^2, L^2)$, in the zone $\Im\mu \geq 0$. Since the potential W is such that

$$< x >^{s} W < x >^{s}$$

is bounded in L^2, we have the desired result.

Lemma 6.6. *Let us assume that the potential W satisfies (H_1). For any $s>1$ the weighted resolvent operator $<x>^{-s} R^+(\mu) <x>^{-s}$ has a continuous extension from $\Im\mu>0$ to $\Im\mu\geq 0$. Moreover there exists a real constant $C>0$ such that the following estimate is true:*

$$\| <x>^{-s} R^+(\mu) <x>^{-s} f\|_{L^2} \leq C\|f\|_{L^2}. \tag{6.3}$$

for any Schwartz function f.

Proof. The result is well - known, so we briefly sketch the idea. The perturbed resolvent $R(\mu^2)=(-\Delta+W-\mu^2)^{-1}$ satisfies in $\Im\mu>0$ the relation

$$(-\Delta+W-\mu^2)^{-1} = \left(I+(-\Delta-\mu^2)^{-1}W\right)^{-1}(-\Delta-\mu^2)^{-1} \tag{6.4}$$

provided the operator $(I-(P_0-\mu^2)^{-1}W)$ is invertible. This relation implies

$$\begin{aligned} &<x>^{-s} (-\Delta+W-\mu^2)^{-1} <x>^{-s} = \\ &\left(I+ <x>^{-s} (-\Delta-\mu^2)^{-1} W <x>^{s}\right)^{-1} <x>^{-s} (-\Delta-\mu^2)^{-1} <x>^{-s}. \end{aligned} \tag{6.5}$$

We can apply Fredholm Theory and the Theorem 6.3 that shows 0 is not resonance, so we are able to say that the operator

$$\left(I+ <x>^{-s} (-\Delta-\mu^2)^{-1} W <x>^{+s}\right)^{-1},$$

is continuous in $\Im\mu\geq 0$. □

Once resolvent estimate is established, one can use the approach from [5] and derive the Strichartz type estimate. for the corresponding inhomogeneous Cauchy problem

$$i\partial_t u - \Delta u = F \quad , \quad u(0)=f. \tag{6.6}$$

We shall call the pair $(\frac{1}{p},\frac{1}{q})$ sharp admissible (see [13] for this notion and the properties of sharp admissible pairs), if it satisfies the condition:

$$\frac{n}{4}=\frac{1}{p}+\frac{n}{2q} \quad , \quad 2\leq p\leq\infty, (p,q,n)\neq(2,\infty,2). \tag{6.7}$$

If $n=3$, then we can chose the end point

$$p^*=2, q^*=6$$

as admissible couple. Moreover, (6.7) becomes

$$\frac{3}{4}=\frac{1}{p}+\frac{3}{2q} \quad , \quad 2\leq p\leq\infty. \tag{6.8}$$

Then we have

Theorem 6.7. *If (p,q) and $(\tilde p,\tilde q)$ satisfy (6.8), then the solution to the Cauchy problem*

$$\begin{aligned} i\partial_t u + \triangle u - Wu = F \quad , \quad (t,x)\in(1,\infty)\times\mathbb{R}^3_x, \\ u(1,x)=f(x), \end{aligned} \tag{6.9}$$

satisfies the estimate:

$$\|u\|_{L^p((1,\infty);L^q_x)} + \|u\|_{C([1,\infty);L^2)} \leq C\left(\|F\|_{L^{\tilde{p}'}((1,\infty);L^{\tilde{q}'}_x)} + \|f\|_{L^2}\right). \quad (6.10)$$

Using the pseudoconformal transform, we make the substitution

$$\begin{aligned} v(T,X) &= \frac{1}{T^{3/2}}\overline{u}(\frac{1}{T}, \frac{X}{T})e^{\mathrm{i}\frac{X^2}{4T}}, \\ H(T,X) &= \frac{1}{T^{2+3/2}}\overline{F}(\frac{1}{T}, \frac{X}{T})e^{\mathrm{i}\frac{X^2}{4T}} \\ h(X) &= f(X)e^{\mathrm{i}\frac{X^2}{4}}. \end{aligned} \quad (6.11)$$

and see that v is a solution to the following Cauchy problem

$$\begin{aligned} i\partial_T v + \Delta_X v - T^{-2}W\left(\frac{X}{T}\right)v = H \quad , \quad (T,X) \in (0,1)) \times \mathbb{R}^3_x, \\ u(1,x) = h(x). \end{aligned} \quad (6.12)$$

A simple computation shows that

$$\|u\|_{L^p((1,\infty);L^q_x)} = \|v\|_{L^p((0,1);L^q_X)},$$

and

$$\|F\|_{L^{\tilde{p}'}((1,\infty);L^{\tilde{q}'}_x)} = \|H\|_{L^{\tilde{p}'}((0,1);L^{\tilde{q}'}_X)}$$

provided the couples (p,q) and $(\tilde{p},\tilde{q})$ are admissible ones. Since

$$\|u\|_{L^\infty((1,\infty);L^2)} = \|v\|_{L^\infty((0,1);L^2)},$$

we can take

$$p^* = 2, q^* = 6$$

as admissible couple and we arrive at the following.

Theorem 6.8. *If $n = 3$, then the solution to the Cauchy problem* (6.12) *satisfies the estimates:*

$$\|v\|_{L^2((0,1);L^6_X)} + \|v\|_{L^\infty((0,1);L^2_X)} \leq C\left(\|H\|_{L^2((0,1);L^{6/5}_X)} + \|h\|_{L^2_X}\right). \quad (6.13)$$

and

$$\|v\|_{L^2((0,1);L^6_X)} + \|v\|_{L^\infty((0,1);L^2_X)} \leq C\left(\|H\|_{L^1((0,1);L^2_X)} + \|h\|_{L^2_X}\right). \quad (6.14)$$

7. Equivalence of $\dot{H}^s_W$ and $\dot{H}^s$

Here we follow the argument of section 5 in [8] To show that $\dot{H}^s_{W_T} = \dot{H}^s$ for $s < \frac{3}{2}$ we will first prove the following

Lemma 7.1. $\dot{H}^1_{W_T} = \dot{H}^1$.

Proof. The positivity of W_T implies

$$\left\|(-\Delta+W_T)^{\frac12}f\right\|_{L^2}^2=\langle(-\Delta+W_T)f,f\rangle_{L^2}\geq\langle(-\Delta)f,f\rangle_{L^2}=\left\|(-\Delta)^{\frac12}f\right\|_{L^2}^2.$$

The assumption (H1) implies

$$W_T(X)=T^{-2}W\left(\frac{X}{T}\right)\leq\frac{C}{|X|^2}.$$

The Hardy inequality yields

$$\langle(W_Tf,f\rangle_{L^2}\leq C\left\|(-\Delta)^{\frac12}f\right\|_{L^2}^2$$

so

$$\left\|(-\Delta+W_T)^{\frac12}f\right\|_{L^2}^2=\langle(-\Delta+W_T)f,f\rangle_{L^2}\leq C\left\|(-\Delta)^{\frac12}f\right\|_{L^2}^2.$$

This completes the proof. □

Lemma 7.2. *For $0\leq s<3/2$ we have*

$$\left\|(-\Delta+W_T)^{\frac{s}{2}}f\right\|_{L^2}^2\sim\left\|(-\Delta)^{\frac{s}{2}}f\right\|_{L^2}^2.$$

Proof. Take $1<s<\frac32$. We shall use the identity

$$\left\|(-\Delta+W)^{\frac{s}{2}}f\right\|_{L^2}^2=\left((-\Delta+W)^{s-1}f,(-\Delta+W)f\right)_{L^2}$$

$$=\left((-\Delta+W)^{s-1}f,(-\Delta)f\right)_{L^2}+\left((-\Delta+W)^{s-1}f,Wf\right)_{L^2}=$$

$$=\left((-\Delta)^{1-\frac{s}{2}}(-\Delta+W)^{s-1}f,(-\Delta)^{\frac{s}{2}}f\right)_{L^2}+\left(|W|^{1-\frac{s}{2}}(-\Delta+W)^{s-1}f,W^{\frac{s}{2}}f\right)_{L^2}.\quad(7.1)$$

Let we set

$$I_1=\left((-\Delta)^{1-\frac{s}{2}}(-\Delta+W)^{s-1}f,(-\Delta)^{\frac{s}{2}}f\right)_{L^2},$$

$$I_2=\left(W^{1-\frac{s}{2}}(-\Delta+W)^{s-1}f,W^{\frac{s}{2}}f\right)_{L^2}.$$

Now we can apply the lemma 7.1 and using the fact that $\frac12<2-s<1$, we get $\left\|(-\Delta)^{\frac{2-s}{2}}g\right\|_{L^2}\leq C\left\|(-\Delta+V)^{\frac{2-s}{2}}g\right\|_{L^2}$. Taking now $g=(-\Delta+V)^{s-1}f$, we get

$$\left\|(-\Delta)^{\frac{2-s}{2}}(-\Delta+W)^{s-1}f\right\|_{L^2}\leq C\left\|(-\Delta+W)^{\frac{s}{2}}f\right\|_{L^2}.\quad(7.2)$$

Hence

$$|I_1|\leq\|f\|_{\dot H^s_W}\|f\|_{\dot H^s}.\quad(7.3)$$

Further we need the following

Lemma 7.3. *We have the estimate*

$$\left\|W^{\frac{s}{2}}f\right\|_{L^2}\leq C\|f\|_{\dot H^s},$$

where $W=W_T=T^{-2}W(X/T)$ and $0\leq s<\frac32$.

Proof. Applying the Hölder inequality for Lorentz spaces and using the fact that $\left\| |W|^{\frac{s}{2}} \right\|_{L^{(\frac{3}{s},\infty)}} \leq C \left\| W \right\|^{\frac{s}{2}}_{L^{(\frac{3}{2},\infty)}} \leq C_0^{\frac{s}{2}}$, we get

$$\left\| W^{\frac{s}{2}} f \right\|_{L^2} \leq C \left\| W^{\frac{s}{2}} \right\|_{L^{(\frac{3}{s},\infty)}} \left\| f \right\|_{L^{(q,2)}} , \tag{7.4}$$

$$\frac{1}{2} = \frac{s}{3} + \frac{1}{q} \quad , \quad q = 6 \in (2, \infty) . \tag{7.5}$$

Now we can apply the Sobolev's embedding (see [3]) $\dot{H}^s \subset L^{(q,2)}$ for $\frac{1}{2} = \frac{s}{3} + \frac{1}{q}$ and we get $\left\| |W|^{\frac{s}{2}} f \right\|_{L^2} \leq C_1 \left\| f \right\|_{\dot{H}^s}$. □

Now we are ready to estimate the term I_2. We have

$$|I_2| \leq \left\| W^{\frac{2-s}{2}} (-\Delta + W)^{s-1} f \right\|_{L^2} \left\| (W)^{\frac{s}{2}} f \right\|_{L^2} . \tag{7.6}$$

Since $2 - s \in (0, \frac{3}{2})$, we can apply Lemma 5.2 and get

$$\left\| |W|^{\frac{2-s}{2}} (-\Delta + W)^{s-1} f \right\|_{L^2} \leq \left\| (-\Delta)^{\frac{2-s}{2}} (-\Delta + W)^{s-1} f \right\|_{L^2} \tag{7.7}$$

and $\left\| |W|^{\frac{s}{2}} f \right\|_{L^2} \leq C \left\| f \right\|_{\dot{H}^s}$. We estimate the right hand side of (7.7) using (7.2) and find

$$\left\| W^{\frac{2-s}{2}} (-\Delta + W)^{s-1} f \right\|_{L^2} \leq C \left\| (-\Delta + W)^{\frac{s}{2}} f \right\|_{L^2} . \tag{7.8}$$

From (7.2) (7.7) and (7.8) we obtain

$$|I_2| \leq C \left\| f \right\|_{\dot{H}^s_W} \left\| f \right\|_{\dot{H}^s} . \tag{7.9}$$

This estimate, (7.2) and (7.1) lead to

$$\left\| f \right\|^2_{\dot{H}^s_W} \leq C \left\| f \right\|_{\dot{H}^s_W} \left\| f \right\|_{\dot{H}^s} .$$

Hence

$$\left\| f \right\|_{\dot{H}^s_W} \leq C \left\| f \right\|_{\dot{H}^s} , \tag{7.10}$$

for $0 \leq s < \frac{3}{2}$.

To show the opposite inequality, we use the fact that W_T is a non - negative potential decaying faster than $|x|^{-2}$ at infinity, so one can apply Theorem 1.1 of the work [19] and get the following estimate of the heat kernel $K_W(t,x,y)$ of the heat operator

$$\partial_t - \Delta + W$$

$$|K_W(t,x,y)| \leq C K_0(t,x,y), \tag{7.11}$$

where

$$K_0(t,x,y) = ct^{-3/2} e^{-c|x-y|^2/(4t)}$$

is the heat kernel of the free heat operator $\partial_t - \frac{1}{c}\Delta$.

It is important to notice that the potential $W = W_T$ depends o the parameter $T \in (0,1)$, but the constant C in (7.11) is independent of this parameter, since we have the inequality

$$|W_T(x)| = \left|T^{-2}W(x/T)\right| \leq \frac{C}{|x|^3}$$

with some constant C independent of $T \in (0,1)$. Given any sectorial operator A with spectrum $\sigma(A)$ satisfying

$$z \in \sigma(A) \Longrightarrow \Re z \geq 0,$$

we can define the negative powers of A as follows (see for example section 1.4 in [10])

$$A^{-k} = \frac{1}{\Gamma(k)} \int_0^\infty t^{k-1} e^{-At} dt. \tag{7.12}$$

Choosing $k = 1$,

$$A = -\Delta + W, \quad A_0 = -\frac{1}{c}\Delta$$

and comparing the kernels of

$$e^{-At}, \quad e^{-A_0 t}$$

by the aid of the estimate (7.12), we see

$$\|(-\Delta + W)^{-1} f\|_{L^2} \leq C \|(-\Delta)^{-1} f\|_{L^2}.$$

This estimate shows that the operator

$$(-\Delta + W)^{-1}(-\Delta)$$

is L^2 bounded, so its dual

$$(-\Delta)(-\Delta + W)^{-1}$$

is also L^2 bounded and we see that

$$\|(-\Delta) f\|_{L^2} \leq C \|(-\Delta + W) f\|_{L^2}.$$

Hence, by interpolation

$$\|f\|_{\dot{H}^s_W} \leq C \, \|f\|_{\dot{H}^s}, \tag{7.13}$$

for $0 \leq s \leq 2$.

This completes the proof. □

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