

SOME RESULTS ON THE TWO-DIMENSIONAL DISSIPATIVE EULER EQUATIONS

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ABSTRACT. We make a review of some recent results concerning special solutions and behavior at infinity for 2D dissipative Euler equations. In particular, we give a simplified proof –in the space-periodic setting– of the uniform space/time boundedness of the first derivatives of the velocity, under suitable assumptions on the external force and on the dissipation (damping) coefficient. This is used to sketch the proof of existence of almost-periodic solutions.

1. Introduction. In this paper we summarize some results related with the long-time behavior of the Euler equations for incompressible fluids in two space dimensions. It is well-known that in the 2D case it is possible to prove, for smooth enough data, existence and uniqueness of smooth solution, for all positive times (see also the discussion in the next section for certain less-standard results). It is also clear that without any smoothing or dissipation, one cannot expect to have uniform boundedness of the energy and of other interesting quantities as the enstrophy or higher norms of the velocity. To this end we consider the so-called *dissipative Euler equations*

$$\begin{aligned} \partial_t u + \chi u + (u \cdot \nabla) u + \nabla p &= f & \text{in }]0, +\infty[\times \mathbf{T}, \\ \nabla \cdot u &= 0 & \text{in }]0, +\infty[\times \mathbf{T}, \end{aligned} \tag{1}$$

where $u = (u_1, u_2)$ is the velocity of the fluid with the initial condition $u(0) = u_0$, p is the kinematic pressure, $f = f(t, x)$ is the external force field, $\mathbf{T} := (\mathbf{R}/2\pi\mathbf{Z})^2$ is a two dimensional torus and all quantities are 2π -space periodic and with vanishing mean value. The damping term χu (with $\chi > 0$) models the bottom friction in some 2D oceanic models (when the system is considered in a bounded domain; in that case, the system is called the viscous Charney-Stommel barotropic ocean circulation model of the gulf stream) or the Rayleigh friction in the planetary boundary layer (with space-periodic boundary conditions). The positive constant χ is the Rayleigh friction coefficient (or the Ekman pumping/dissipation constant) or also the sticky viscosity, when the model is used to study motion in presence of rough boundaries, see for instance Gallavotti [10]. Early existence results can be found in Barilon, Constantin, and Titi [2], while links between the driven and damped 2D Navier-Stokes, attractors, and statistical solutions are proved in Ilyin, Miranville, and Titi [12] and Constantin and Ramos [8]. The model (1) represents (probably)

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the “weakest” dissipative modification of the Euler equations and results on the long-time behavior of the damped/driven Navier-Stokes do not directly pass to the limit as the “viscosity goes to zero,” hence a completely different treatment is required to study the problem without viscosity. This paper is aimed at sketching the fundamental steps needed to show existence of almost-periodic solutions and one key-result is that of showing a sort of asymptotic stability, cf. [15]. In order to use standard tools based on dissipation to construct almost-periodic solutions we need a control on the difference of two solutions. The presence of the nonlinear convection term seems to require an estimate on $\|\nabla u\|_{L^\infty}$. To this end we analyze the equation for the vorticity. Taking the curl of (1) (define $\xi := \text{curl } u := \partial_2 u_1 - \partial_1 u_2$ and $\phi := \text{curl } f$) one obtains

$$\partial_t \xi + \chi \xi + (u \cdot \nabla) \xi = \phi \quad \text{in }]0, \infty[\times \mathbf{T}, \quad (2)$$

which is a non-local scalar transport equation (with damping), which plays a fundamental role in the sequel.

Moreover, it is well-known that (by the Biot-Savart formula) the velocity can be reconstructed from the vorticity by recalling that $-\Delta u = \nabla^\perp \xi$. Basic Calderon-Zygmund or Schauder estimates for the Poisson equations allow us to state that ∇u and ξ are at the same level of regularity in L^p spaces ($1 < p < \infty$) or in Hölder spaces $C^{0,\alpha}$. Roughly speaking (full details are given in [5, 6]) the L^p -setting, with $p < +\infty$ is too weak, while the $C^{0,\alpha}$ setting seems too strong in order to obtain uniform estimates. This suggest to use a more precise functional framework and in particular to employ the following well-known potential theoretic result:

$$\exists C_0 = C_0(\mathbf{T}) > 0 : \quad \|\nabla u\|_{L^\infty(\overline{\mathbf{T}})} \leq C_0 \|\xi\|_{C_D(\overline{\mathbf{T}})}, \quad (3)$$

to show boundedness of the gradient of u . We recall that the set of Dini-continuous functions $C_D(\overline{\mathbf{T}}) \subset C(\overline{\mathbf{T}})$ is the subset of continuous functions $f : \overline{\mathbf{T}} \rightarrow \mathbf{R}$ such that

$$\|f\|_{C_D(\overline{\mathbf{T}})} := \|f\|_{L^\infty(\overline{\mathbf{T}})} + [f]_{C_D} := \|f\|_{L^\infty(\overline{\mathbf{T}})} + \int_0^{\sqrt{22}\pi} \omega(f, \sigma) \frac{d\sigma}{\sigma} < +\infty,$$

where

$$\omega(f, \sigma) := \sup \left\{ |f(x) - f(y)| : x, y \in \overline{\mathbf{T}}, 0 < |x - y| < \sigma, \right\}.$$

The main reason for the use of this functional space to study the vorticity stems in the uniform estimate proved in Proposition 1, which –together with (3)– gives the requested bound. We emphasize that the first use of these spaces for the vorticity of Euler equations dates back to Beirão da Veiga [4] in the context of global well-posedness of the 2D problem. In questions of stability the role of Dini-continuous vorticity has been first recognised by Koch [14], while recent results on global attractors are those proved in [5]. Close relationship between Dini and critical Besov spaces is analyzed in [11]. We consider Stepanov almost-periodic solutions (see [1] for further details), which seems the most natural setting for problems related with the Euler equations. If X is a Banach space we define $L^2_{uloc}(X)$ as the space of *uniformly locally square integrable X -valued functions*

$$L^2_{uloc}(X) := \left\{ v \in L^2_{loc}(\mathbf{R}; X) : \sup_{t \in \mathbf{R}} \int_t^{t+1} \|v(s)\|_X^2 ds < \infty \right\}.$$

Next, we say that $v \in L^2_{uloc}(X)$ belongs to $\mathcal{S}^2(X)$ or is Stepanov almost-periodic (with values in X) if and only if the set of the time-translates of v is relatively compact with respect to the $L^2_{uloc}(X)$ -topology.

The main result of this paper is then the following

Theorem 1.1. *Let be given a divergence-free external force $f \in \mathcal{S}^2(L^2(\mathbf{T}))$ with $\operatorname{curl} f \in L^\infty(\mathbf{R}; C_D(\overline{\mathbf{T}}))$. There exists $\chi_0 = \chi_0(f) > 0$ such that if $\chi > \chi_0$, then there exists an almost-periodic solution $u \in \mathcal{S}^2(L^2(\mathbf{T}))$ to the dissipative Euler equations (1).*

Remark 1. The condition on χ can be also read as a smallness condition on f . Moreover, by standard results due to Dafermos [9], obtained by compact embedding and interpolation, the solution u will belong also to $\mathcal{S}^2(H^1(\mathbf{T}))$.

Remark 2. Appropriate modifications of the calculations from the next sections can be used to handle also the more general case of a bounded smooth domain $\Omega \subset \mathbf{R}^2$ for the problem endowed with the boundary condition $u \cdot n = 0$ on $\partial\Omega$, see [6] for full details.

The same approach can be also used (with some additional technical steps) to prove, in the case of a time-independent force, the following result concerning the existence of a global attractor, see [5] for full details.

Theorem 1.2. *Let be given $f \in H^1(\mathbf{T})$ such that $\phi = \operatorname{curl} f \in C_D(\overline{\mathbf{T}})$. There exists $\chi_0(f) > 0$ such that if $\chi > \chi_0$, then, there exists a global attractor $\mathcal{A} \subset C(\overline{\mathbf{T}})$, for the dissipative 2D Euler equations (1).*

Remark 3. Also Thm. 1.2 holds true in a bounded smooth domain $\Omega \subset \mathbf{R}^2$ and moreover the Hausdorff dimension of \mathcal{A} turns out to be finite, cf. [5]

2. Existence of weak solutions. In this section we recall some basic results on existence and uniqueness of weak solutions, proved in Bessaih and Flandoli [7], by adapting classical results by Yudovich [16] and Bardos [3]. Let \mathcal{V} be the space of infinitely differentiable, periodic, divergence-free, and with vanishing mean value vector-fields on \mathbf{T} . We introduce the usual Hilbert space H defined as the closure of \mathcal{V} with respect to the norm $|\cdot|$ of $L^2(\mathbf{T})^2$, with the inner product of $L^2(\mathbf{T})^2$, denoted in the sequel by $\langle \cdot, \cdot \rangle$. As usual, V is the closure of \mathcal{V} with respect to the norm $\|\cdot\|$ of $H^1(\mathbf{T})^2$. Identifying H with its dual H' , and H' with the corresponding natural subspace of V' , we have the standard Gelfand triple $V \subset H \subset V'$ with continuous and dense injections. (For simplicity we denote the dual pairing between V and V' by the same symbol as for the inner product of H .)

Definition 2.1. We say that the vector field $u \in C(0, \infty; H) \cap L_{loc}^\infty(0, \infty; V)$, with $\partial_t u \in L_{loc}^2(0, \infty; V')$, is a weak solution to (1) on $[0, \infty[$ if the following properties hold $\forall v \in \mathcal{V}$ and all $t \geq t_0 \geq 0$:

$$\begin{aligned} \|u(t)\|^2 &\leq \|u(t_0)\|^2 e^{-\chi(t-t_0)} + \chi^{-1} \int_{t_0}^t \|f(s)\|^2 e^{-\chi(t-s)} ds, \\ |u(t)|^2 + 2\chi \int_{t_0}^t |u(s)|^2 ds &\leq |u(t_0)|^2 + \int_{t_0}^t \langle f(s), u(s) \rangle ds, \\ \langle u(t) - u(t_0), v \rangle + \chi \int_{t_0}^t \langle u(s), v \rangle ds &+ \int_{t_0}^t \langle (u(s) \cdot \nabla) u(s), v \rangle ds = \int_{t_0}^t \langle f(s), v \rangle ds. \end{aligned}$$

We have the following result:

Theorem 2.2. *Let be given $u_0 \in V$ and $f \in L_{loc}^1(0, +\infty; V)$. Then, there exists at least a weak solution to (1). Moreover, if $\operatorname{curl} u_0 \in L^\infty(\mathbf{T})$ and $\operatorname{curl} f \in L_{loc}^1(0, +\infty; L^\infty(\mathbf{T}))$, such a solution is unique.*

Proof. The proof of this result is classically based on a vanishing-viscosity approximation. The Navier-Stokes equations are considered for $\nu > 0$

$$\begin{aligned} \partial_t u^\nu + \chi u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu &= f \quad \text{in }]0, T[\times \mathbf{T}, \\ \nabla \cdot u^\nu &= 0 \quad \text{in }]0, T[\times \mathbf{T}, \end{aligned}$$

for which existence of Leray-Hopf weak solutions in $[0, T]$ for any positive T is well-known. Next, by using the vorticity equation for $\xi^\nu = \text{curl } u^\nu$ it is easy to prove (along Galerkin approximation) that

$$\frac{d}{dt} |\xi^\nu(t)|^2 + \chi |\xi^\nu(t)|^2 + \nu |\nabla \xi^\nu(t)|^2 \leq \frac{1}{\chi} |\phi|^2,$$

which can be used to show an uniform bound for the vorticity in $L^2(\mathbf{T})$. Then, with this it is possible to show that the limit $u := \lim_{\nu \rightarrow 0^+} u^\nu$ is a weak solution to the dissipative Euler equations.

The uniqueness in the case of a bounded vorticity for the Euler equations is more delicate, and it is based on the inequality proved in [16].

$$\exists C > 0, \text{ independent of } p : \quad \|u\|_{L^p(\mathbf{T})} \leq C \sqrt{p} \|u\|_{W^{1,2}(\mathbf{T})} \quad \forall p \geq 2.$$

□

Since we have a unique solution of (1) we can prove better regularity on it simply by using representation formulas. It is well-known that if $\xi \in L^\infty(\mathbf{T})$, then this is not enough to have $\nabla u \in L^\infty(\mathbf{T})$ (being the endpoint estimate) hence Lipschitz characteristics. The boundedness of the vorticity implies that the velocity is Lip-Log (called also quasi-Lipschitz) and then that the characteristics are unique and Hölder continuous. In particular, the following result is well-known, see for instance [13].

Lemma 2.3. *Let $\|\xi\| := \sup_{(s,y) \in [0,T] \times \mathbf{T}} |\xi(s,y)|$, then there exists a constant $c > 0$ such that, for all $x, x_1 \in \mathbf{T}$ such that $|x - x_1| < 1$*

$$|u(t, x) - u(t, x_1)| \leq c \|\xi\| |x - x_1| [1 - \log(|x - x_1|)].$$

If $U(s, t, x)$ denotes the solution of the Cauchy problem

$$\begin{cases} \frac{dU(t, s, x)}{dt} = u(t, U(t, s, x)), \\ U(s, s, x) = x, \end{cases} \quad (4)$$

then, defining δ as follows $\delta := e^{-c\|\xi\|T}$, it holds

$$|U(s, t, x) - U(s_1, t_1, x_1)| \leq c \|\xi\| |t - t_1| + e(1 + e c \|\xi\|) (|x - x_1|^\delta + |s - s_1|^\delta).$$

In order to have Lipschitz characteristics, it would be enough, to have bounded gradient of the velocity, which will follow from Dini-continuous vorticity.

Moreover, remaining in the setting of Hölder functions it follows (by direct computation) that the composition of Dini and of an Hölder continuous functions is again a Dini-continuous function.

Lemma 2.4. *Let be given $f \in C_D(\overline{\mathbf{T}})$ and $U \in C^{0,\delta}(\overline{\mathbf{T}})$, then the following estimate for the Dini's semi-norm holds true:*

$$[f \circ U]_{C_D} \leq \frac{1}{\delta} [f]_{C_D} + \frac{2}{\delta} \log[U]_\delta (\sqrt{22}\pi)^{\delta-1}.$$

Since the Hölder exponent of the characteristics decreases with time, we first fix an interval $[0, T]$ and the previous lemma allows to control the Dini-norm of the vorticity, by using the representation formula obtained by following the characteristics in the equation for the vorticity

$$\xi(t, x) = \xi_0(U(0, t, x)) e^{-\chi t} + \int_0^t \phi(s, U(s, t, x)) e^{-\chi(t-s)} ds, \quad t \in [0, T]. \quad (5)$$

By using Lemma 2.4, formula (5), and by reasoning as in [4, 14] one can easily show that if $\xi_0 \in C_D(\overline{\mathbf{T}})$ and $\phi \in L^1_{\text{loc}}(0, +\infty; C_D(\overline{\mathbf{T}}))$, then $\xi \in L^\infty(0, T; C_D(\overline{\mathbf{T}}))$, for all positive T .

Remark 4. By using the Schauder's fixed point theorem (employed in two slightly different manners in Ref. [4, 14]) it is possible also to show that $\xi \in C([0, T]; C_D(\overline{\mathbf{T}}))$, for all $T > 0$, but this is not needed here.

For our purposes the continuity is not so important, but what will be relevant is the following result.

Proposition 1. *Let $u_0 \in V$ such that $\xi_0 \in C_D(\overline{\mathbf{T}})$ and $\phi \in L^\infty(0, +\infty; C_D(\overline{\mathbf{T}}))$. Then, for large enough $\chi > 0$, the Dini-norm of ξ is uniformly bounded over $[0, +\infty[$.*

Proof. We are assuming that we have a unique solution $\xi \in L^\infty(0, T; C_D(\overline{\mathbf{T}}))$ of the transport equation (2), for any given $T > 0$. Then for a.e. $t \in [0, T]$ it follows $\nabla u(t, \cdot) \in L^\infty$ and U is Lipschitz continuous (especially in the space variable) and the Lip-norm depends on the Dini-norm of ξ . More precisely, we have the estimate

$$|\nabla U(s, t, x)| \leq e^{\int_s^t \sup_{y \in \mathbf{T}} |\nabla u(\tau, y)| d\tau} \quad \text{for } (s, t, x) \in [0, T]^2 \times \mathbf{T}, \quad (6)$$

but, since the bound on $\|\nabla u\|_{L^\infty}$ depends on $\|\xi(t)\|_{C_D}$, it may depend on $T > 0$. To show an uniform bound we first observe that the L^∞ bound for the vorticity (shown also in [7]) follows directly from (5) and it is independent of T :

$$\|\xi(t)\|_{L^\infty} \leq \|\xi_0\|_{L^\infty} e^{-\chi t} + \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \frac{1 - e^{-\chi t}}{\chi}.$$

We estimate the Dini-continuity of $\eta = \xi e^{\chi t}$ on $[0, T]$. Observe that, for η we have the representation formula $\eta(t, x) = \xi_0(U(0, t, x)) + \int_0^t \phi(s, U(s, t, x)) e^{\chi s} ds$, and

$$\|\eta(t)\|_{L^\infty} \leq \|\xi_0\|_{L^\infty} + \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \frac{e^{\chi t} - 1}{\chi}.$$

Moreover, we observe that $[\eta(t)]_{C_D} = [\xi(t)]_{C_D} e^{\chi t}$, and we split it as follows:

$$\begin{aligned} [\eta(t)]_{C_D} &:= \int_0^1 \sup_{|x-y| \leq \rho} |\eta(t, x) - \eta(t, y)| \frac{d\rho}{\rho} \\ &\leq \int_0^1 \sup_{|x-y| \leq \rho} |\xi_0(U(0, t, x)) - \xi_0(U(0, t, y))| \frac{d\rho}{\rho} \\ &\quad + \int_0^t \int_0^1 \sup_{|x-y| \leq \rho} |\phi(s, U(s, t, x)) - \phi(s, U(s, t, y))| e^{\chi s} \frac{d\rho}{\rho} ds \\ &=: B_1 + B_2. \end{aligned} \quad (7)$$

By making a change of variable by means of the unitary diffeomorphism $U(0, t, x)$ we have that

$$\begin{aligned} B_1 &\leq \int_0^1 \sup_{|x-y| \leq \rho \|\nabla U(0, t, \cdot)\|_{L^\infty}} |\xi_0(x) - \xi_0(y)| \frac{d\rho}{\rho} \\ &\leq \int_0^1 \sup_{|x-y| \leq \rho} |\xi_0(x) - \xi_0(y)| \frac{d\rho}{\rho} + 2\|\xi_0\|_{L^\infty} \int_1^{\|\nabla U(0, t, \cdot)\|_{L^\infty}} \frac{d\rho}{\rho} \\ &\leq [\xi_0]_{C_D} + 2\|\xi_0\|_{L^\infty} \log \|\nabla U(0, t, \cdot)\|_{L^\infty}, \end{aligned}$$

and, by appealing to (6), we get

$$B_1 \leq [\xi_0]_{C_D} + 2\|\xi_0\|_{L^\infty} \int_0^t \|\nabla u(s)\|_{L^\infty} ds \leq [\xi_0]_{C_D} + 2C_0\|\xi_0\|_{L^\infty} \int_0^t \|\eta(s)\|_{C_D} ds.$$

Concerning B_2 , by making the change of variables by means of $U(s, t, x)$, we have

$$\begin{aligned} B_2 &\leq \int_0^t \int_0^1 \sup_{|x-y| \leq \rho \|\nabla U(s, t, \cdot)\|_{L^\infty}} |\phi(s, x) - \phi(s, y)| \frac{d\rho}{\rho} e^{\chi s} ds \\ &\leq \int_0^t [\phi(s)]_{C_D} e^{\chi s} ds + 2\|\phi(s)\|_{L^\infty} \int_0^t \int_1^{\|\nabla U(s, t, \cdot)\|_{L^\infty}} \frac{d\rho}{\rho} e^{\chi s} ds \\ &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \log \|\nabla U(s, t, \cdot)\|_{L^\infty} e^{\chi s} ds \\ &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \log \|\nabla U(s, t, \cdot)\|_{L^\infty} e^{\chi s} ds \\ &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \int_s^t \|\nabla u(\tau)\|_{L^\infty} e^{\chi s} d\tau ds. \end{aligned}$$

Changing the order of integration in the last integral we have

$$\begin{aligned} B_2 &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \int_0^t e^{\chi s} ds + 2 \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \int_0^\tau \|\nabla u(\tau)\|_{L^\infty} e^{\chi s} ds d\tau \\ &\leq \sup_{t \geq 0} [\phi(t)]_{C_D} \frac{e^{\chi t}}{\chi} + \frac{2C_0}{\chi} \sup_{t \geq 0} \|\phi(t)\|_{L^\infty} \int_0^t \|\eta(\tau)\|_{C_D} d\tau. \end{aligned}$$

Collecting all the estimates and by defining $\Phi := \sup_{t \geq 0} \|\phi(t)\|_{C_D}$ we arrive at

$$\|\eta(t)\|_{C_D} \leq \|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} e^{\chi t} + 2C_0 \left[\|\xi_0\|_{C_D} + \frac{\Phi}{\chi} \right] \int_0^t \|\eta(s)\|_{C_D} ds.$$

By using Gronwall lemma and by coming back to the variable ξ we get

$$\begin{aligned} \|\xi(t)\|_{C_D} &\leq \left[\|\xi_0\|_{C_D} + \frac{2\Phi}{\chi} - \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)} \right] e^{t \left[\frac{2C_0(\Phi + \|\xi_0\|_{C_D}\chi)}{\chi} - \chi \right]} \\ &\quad + \frac{2\Phi\chi}{\chi^2 - 2C_0(\Phi + \|\xi_0\|_{C_D}\chi)} \end{aligned}$$

which is uniformly bounded on $[0 + \infty[$ if $2C_0\Phi + 2C_0\|\xi_0\|_{C_D}\chi - \chi^2 < 0$, that is if $\chi > \chi_0 := C_0\|\xi_0\|_{C_D} + \sqrt{C_0^2\|\xi_0\|_{C_D}^2 + 2C_0\Phi}$, ending the proof. \square

We are now ready to proceed to the proof of the main result. The first step consists in proving existence of weak solution defined for all $t \in \mathbf{R}$. This is classically obtained by constructing solutions of the following problems

$$\begin{cases} \partial_t u_n + \chi u_n + (u_n \cdot \nabla) u_n + \nabla p_n = f & \text{in }]-n, +\infty[\times \mathbf{T}, \\ \nabla \cdot u_n = 0 & \text{in }]-n, +\infty[\times \mathbf{T}, \\ u_n(-n) = 0 & \text{in } \mathbf{T}. \end{cases} \quad (8)$$

The results of the previous section show the following result.

Proposition 2. *Under the hypotheses of Thm. 1.1, for $\chi > \sqrt{2C_0\Phi}$ the unique weak solution of (8) satisfies $u_n \in L^\infty(-n, +\infty; V)$ and $\text{curl } u_n \in L^\infty(-n, +\infty; C_D(\mathbf{T}))$.*

By extending u_n to zero for $t < -n$ and by standard compactness tools it follows that $u_n \xrightarrow{*} u$ in $L^\infty(\mathbf{R}; V)$ where u is a weak solution to the dissipative Euler equations on the whole line. The uniform bounds on $\|\nabla u_n\|_{L^\infty}$ imply also that, for χ large enough

$$\exists C_2 = C_2(f, \chi) : \quad \sup_{t \in \mathbf{R}} \|\nabla u(t)\|_{L^\infty(\mathbf{T})} \leq C_2 < +\infty, \quad (9)$$

With the above estimate at hand we can give an outline of an existence result for almost-periodic solutions.

Sketch of the Proof of Thm. 1.1. The condition that f is $S^2(H)$ -almost-periodic reads: for any sequence $\{r_m\}$ there exists a sub-sequence $\{r_{m_k}\}$ and a function $\tilde{f}(t, x)$ such that

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} |f(\tau + r_{m_k}) - \tilde{f}(\tau)|^2 d\tau \rightarrow 0.$$

As in [15, §4], we proceed by contradiction. Therefore, there is a weak solution u to (1) and a sequence $\{h_m\}$ such that

$$\sup_{t \in \mathbf{R}} \int_t^{t+1} |f(\tau + h_m) - \tilde{f}(\tau)|^2 d\tau \rightarrow 0,$$

and there exist three sequences $\{t_k\}$, $\{h_{m_k}\}$, $\{h_{n_k}\}$ and a positive constant $\delta_0 > 0$ such that

$$\int_{t_k}^{t_k+1} |u(s + h_{m_k}) - u(s + h_{n_k})|^2 ds \geq \delta_0, \quad \forall k \in \mathbf{N}. \quad (10)$$

Since f is $S^2(H)$ -almost-periodic, there exist $f^*(x, t)$ such that

$$\begin{aligned} \sup_{t \in \mathbf{R}} \int_t^{t+1} |f(\tau + t_k + h_{m_k}) - f^*(\tau)|^2 d\tau &\rightarrow 0, \\ \sup_{t \in \mathbf{R}} \int_t^{t+1} |f(\tau + t_k + h_{n_k}) - f^*(\tau)|^2 d\tau &\rightarrow 0. \end{aligned}$$

By defining the maps $u_1^k(s) := u(s + t_k + h_{m_k})$ and $u_2^k(s) := u(s + t_k + h_{n_k})$, inequality (10) can be rewritten as follows

$$\delta_0 \leq \int_{t_k}^{t_k+1} |u_1^k(s - t_k) - u_2^k(s - t_k)|^2 ds = \int_0^1 |u_1^k(s) - u_2^k(s)|^2 ds. \quad (11)$$

Using the a priori bounds on u we can extract a sub-sequence $\{u_i^{k_i}\}$ of $\{u_i^k\}$, $i = 1, 2$, strongly convergent to u_i in $L^2_{loc}(\mathbf{R}; H)$, for $i = 1, 2$, respectively. Hence, we can pass to the limit in (11) to get

$$\delta_0 \leq C \int_0^1 |u_1(s) - u_2(s)|^2 ds. \quad (12)$$

By studying the difference $u_1 - u_2$ on the interval $[t_0, 1]$, with $t_0 < 0$, one can show (by using (9)) that $\int_0^1 |u_1(s) - u_2(s)|^2 ds$ can be made smaller than any positive constant, by taking t_0 sufficiently small. This shows a contradiction and ends the proof. \square

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