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# Analysis of the contact between cubic profiles

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## Abstract

The contact between cubic profiles with continuous relative curvatures has been analyzed and solutions in closed form were obtained for both symmetric and non symmetric conditions. Analytical solutions have allowed a complete parametric study to be performed, showing the effects of the gradient of relative curvature. The accuracy of approximate ‘equivalent’ Hertzian solutions has been verified.

**Keywords:** Cubic profile; Contact stresses; Non-Hertzian contacts.

## Notation

|          |  |
|----------|--|
| $E_i$    | modulus of elasticity for body $i$                                     |
| $\nu_i$  | Poisson’s ratio for body $i$   |
| $E^*$    | coupled modulus of elasticity in Johnson [1], $E^* = 2/A$ in Hills [2] |
| $p(x)$   | pressure function along the profile                                    |
| $P$      | total contact load per unit thickness                                  |
| $h(x)$   | <i>gap</i> function  |
| $k(x)$   | relative curvature function  |
| $c$      | relative curvature at first contact point $O$                          |
| $g$      | gradient of relative curvature (constant with cubic profile)           |
| $a$      | half length of the contact region                                      |
| $a_{Hz}$ | half length of the contact region for the Hertzian reference model     |
| $a_1$    | left extent of the contact region                                      |
| $a_2$    | right extent of the contact region                                     |
| $d$      | abscissa of the mid point of the contact region                        |
| $d_p$    | abscissa of the point where the resultant force is applied             |
| $\gamma$ | dimensionless gradient of relative curvature                           |

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## 1 Introduction

The Hertz's solution for the two-dimensional contact between two elastic bodies is based on the assumptions that the relative curvature of the profiles is unique in the contact region and the extension of contact region is small as compared to the radius of relative curvature. Under these hypotheses the gap function, representing the relative distance between the undeformed bodies in geometrical contact, can be represented by a symmetric quadratic function.

However, in several practical cases the condition of constant relative curvature in the whole contact region is questionable. For instance, in a heavy duty gear transmission, the contact region can become as large as the  $(20 - 30)\%$  the extension of the tooth profile. In these conditions, a non-negligible variation of relative curvature can be expected within the contact region.

As a first approximation, the relative curvature can be assumed to vary linearly with the position along the contact region. This can be modeled by assuming a third order polynomial (cubic) expression for the gap function. In the present paper, analytical solutions for the plane contact between two elastic bodies with several cubic gap functions have been obtained and then discussed. The contact between two smooth profiles, continuous up to the second derivative ( $C^2$  continuous), was considered.

The obtained results are interesting also for assessing numerical solutions of contact problems which can be found in the technical literature. Indeed, the use of cubic splines for approximating complex profiles is widely diffused, particularly when Finite Element analysis is employed. In fact, the cubic spline is one of the simplest smooth curve with either slope or curvature uniquely defined at any point. Finite Element solutions between profiles approximated by cubic splines can be found in Refs. [3, 4], and application to contact between meshing gears in Refs. [5, 6].

In the following sections, after the formulation of the contact problem in plane strain, symmetric and a not-symmetric cubic contact problems are analyzed and relevant solutions are obtained and discussed.

Reference Hertzian contacts are introduced for both cases, in order to discuss the effects of the relative curvature variation on the solution.

## 2 Definition of the contact problem

According to the Hertz's theory [1, 2], two linear elastic bodies are considered in contact in a small surface where friction is neglected. A two dimensional scheme in plane strain conditions is assumed.

In this paper only incomplete contact is considered, while the theoretical approach to partially and complete contact can be found in Ref. [7].

The distribution of the contact pressure (hereafter assumed a positive quan-

tity) is completely determined by the relative distance between the undeformed profiles when they reach the first geometrical contact. As shown in Fig. 1 (a), the position of the first point of contact  $O$  is assumed as the origin of a local Cartesian reference frame having the  $x$  axis on the common tangent and the  $y$  axis on the common normal. The gap function  $h(x)$  measures the relative distance of the two undeformed profiles in the  $y$  direction. The relative curvature of the undeformed profiles  $k(x)$  is defined as the curvature of the gap function and plays a fundamental role in the contact problem. The relative curvature can be obtained by the following expression:

$$k(x) \approx \frac{d^2}{dx^2} h(x) \quad (1)$$

which is an acceptable approximation if the slope of the gap function  $h'(x) = \frac{d}{dx} h(x)$  is much less than the unit within the contact region.

Moreover, the following results are accurate enough if the extension of the contact region,  $2a$  in Fig. 1 (b), is small as compared to the radius of relative curvature  $1/k(x)$  with  $x$  in the contact region.

External constraints on the two bodies are assumed to produce a relative movement which can be approximated, after the first contact in  $O$ , by a far field translation in the  $y$  direction, as schematically shown in Fig. 1 (a).

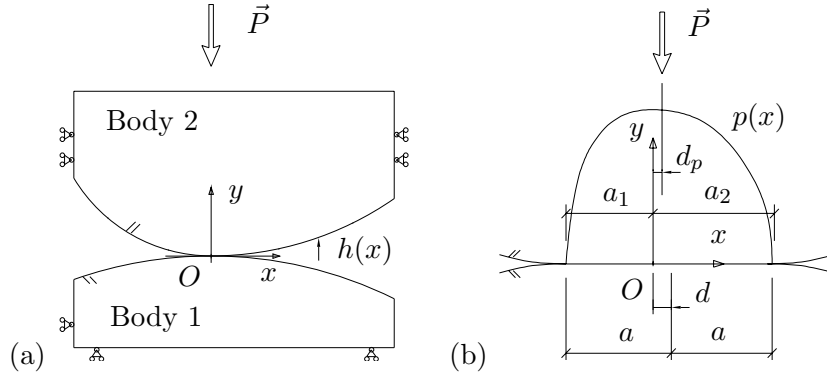


Figure 1: Scheme of two plane bodies in contact and related symbols. (a) Undeformed profiles and (b) deformed profiles.

Under these assumptions, the contact pressure is related to the gap function by the following integral equation [1]:

$$\frac{2}{\pi E^*} \int_{-a_1}^{a_2} \frac{p(s)}{x-s} ds = h'(x) \quad (2)$$

where the range  $[-a_1, a_2]$  indicates the region in which the pressure  $p(x)$  is applied,  $E_i$  and  $\nu_i$  are the elastic properties for the body  $i$  and, according to Johnson's notation [1],  $E^*$  is the coupled plane strain elastic modulus:

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \quad (3)$$

Fig. 1 (b) shows the main geometrical quantities for a general non-symmetric contact.

The solution for the integral equation 2 can be found in Refs. [8, 9]:

$$p(x) = \frac{1}{\pi \sqrt{(a_1 + x)(a_2 - x)}} \left( \frac{E^*}{2} \int_{-a_1}^{a_2} \frac{\sqrt{(a_1 + s)(a_2 - s)}}{x - s} h'(s) ds + P \right) \quad (4)$$

being  $P$  the total contact load per unit thickness

$$P = \int_{-a_1}^{a_2} p(s) ds \quad (5)$$

As shown in Refs. [1, 2], assuming for the gap function a bilinear expression ( $h(x) \propto |x|$ ), Eq. 4 gives the pressure of the wedge contact with the correct singular stress field.

For a symmetric quadratic gap function  $h(x) = \frac{c}{2}x^2$ , equation 4 leads to the classical plane strain Hertz's solution. In this case, the relative curvature of the profiles is constant in the contact region  $k(x) = c$  and  $c$  becomes the only geometrical quantity governing the solution.

In the contact between two smooth profiles with the gap function having a  $C^2$  continuity but not a unique relative curvature, a Hertz's approximation is commonly performed by assuming for the relative curvature the value in  $O(k(0))$ . However, as observed, several situations exist in which a non-negligible variation of relative curvature  $k(x)$  is found in the contact region  $x \in [a_1, a_2]$  and this approximation needs to be verified.

An abrupt change of relative curvature within the contact region was analyzed in Ref. [7] assuming for the gap function a  $C^1$  piecewise parabolic expression. However, also a continuous variation of relative curvature can affect the accuracy of the approximate Hertzian solution too.

The following gap functions with linearly variable relative curvature are considered:

- cubic gap function symmetric about the point  $O$ ,
- not symmetric cubic gap function.

The methods used to get the solutions and the results of the parametric analysis are reported in the following sections.

### 3 Symmetric cubic gap function

#### 3.1 Analytical solution

For a  $C^2$  symmetric cubic gap function the following expression was assumed:

$$h(x) = \frac{g}{6} |x|^3 \quad (6)$$

where the positive quantity  $g$  is the unique geometrical parameter. Using Eq. 1, the relative curvature can be obtained as follows:

$$k(x) = g |x| \quad (7)$$

indicating that  $g$  is the gradient of relative curvature in the  $x$  direction. In this symmetric condition  $a = a_1 = a_2$ , the relative curvature is zero at the center  $O$  ( $k(0) = 0$ ) and it attains the maximum values at the extremes  $k_{max} = k(a) = ga$ .

Eq. 4 can be rewritten as follows:

$$p(x) = \frac{1}{\pi \sqrt{1 - (\frac{x}{a})^2}} \left( \frac{E^*}{2} a^2 \int_{-1}^1 \frac{\sqrt{1 - \tau^2}}{\frac{x}{a} - \tau} \frac{g}{2} \tau |\tau| d\tau + \frac{P}{a} \right) \quad (8)$$

in which  $\tau$  is a dimensionless integration variable.

Before solving Eq. 8 (see the appendix A.1 for analytical details), the gradient of relative curvature  $g$  and the load  $P$  can be related to the extension of the contact region  $a$ :

$$P = \frac{E^*}{4} ga^3 \int_{-1}^1 \frac{|\tau|^3}{\sqrt{1 - \tau^2}} d\tau = \frac{E^*}{2} ga^3 \int_0^1 \frac{\tau^3}{\sqrt{1 - \tau^2}} d\tau = \frac{E^*}{3} ga^3 \quad (9)$$

By means of integration tables or symbolic software like Mathematica<sup>TM</sup> [10], also the contact stress distribution can be written in analytical form:

$$p(x) = \frac{3}{2\pi} \frac{P}{a} \left[ \sqrt{1 - \left(\frac{x}{a}\right)^2} + \left(\frac{x}{a}\right)^2 \log \frac{\sqrt{1 - (\frac{x}{a})^2} + 1}{|\frac{x}{a}|} \right] \quad (10)$$

### 3.2 Comparison with the equivalent Hertzian solution for the symmetric cubic gap function

As  $k(0) = 0$ , in this case the reference Hertzian solution cannot be defined. In order to analyze the properties of the solution, an equivalent Hertzian contact was introduced. The equivalence was defined by selecting a relative curvature  $c_{Hz}$  which produces the same extension of the contact region under the same load.

For the equivalent Hertzian solution, the following relationship holds:

$$P = \frac{\pi}{4} E^* a^2 c_{Hz} \quad (11)$$

Comparing Eqs. 9 and 11, the equivalent Hertzian curvature  $c_{Hz}$  and the pressure distribution  $p_{Hz}(x)$  can be obtained

$$c_{Hz} = \frac{4}{3\pi} ga \quad (12)$$

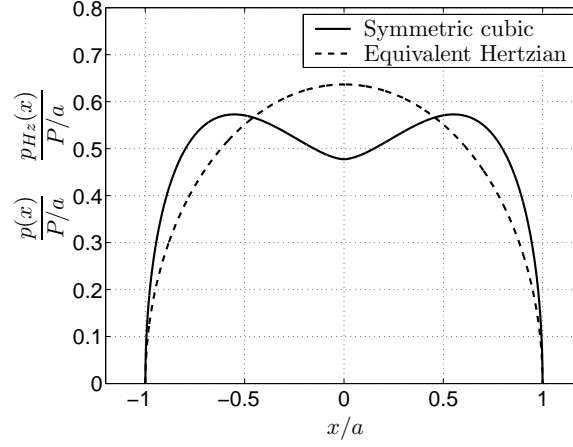


Figure 2: Pressure for a symmetric cubic profile compared to the equivalent Hertzian distribution.

$$p_{Hz}(x) = \frac{2}{\pi} \frac{P}{a} \sqrt{1 - \left(\frac{x}{a}\right)^2} \quad (13)$$

Fig. 2 shows the pressure distributions for the two solutions.

The comparison is general as, with the dimensionless quantities represented in the Fig. 2, the curves are independent from geometry, load and elastic properties.

For the symmetric cubic contact, the pressure distribution attains the maximum values  $p_{max}$  at two points ( $|x| = x_{max}$ ) symmetric about the origin. Indeed the zero relative curvature at  $O$  produces a kind of local conforming contact and a local minimum of  $p(x)$ . Moreover, the maximum pressure is lower than the equivalent Hertzian maximum.

The main properties of the solution are reported hereafter:

$$x_{max} \approx 0.55 a$$

$$p_{max} = p(x_{max}) \approx 0.57 \frac{P}{a}$$

$$p(0) = \frac{3}{2\pi} \frac{P}{a} \approx 0.48 \frac{P}{a}$$

while

$$p_{Hzmax} = p_{Hz}(0) = \frac{2}{\pi} \frac{P}{a} \approx 0.64 \frac{P}{a} \quad (14)$$

In common with every contacts between two smooth profiles, the derivative of the pressure distribution is singular at the boundaries of the contact region.

The following relationships hold:

$$\lim_{\frac{x}{a} \rightarrow +1^-} \frac{\frac{d}{dx}p(x)}{\frac{d}{dx}p_{Hz}(x)} = \lim_{\frac{x}{a} \rightarrow -1^+} \frac{\frac{d}{dx}p(x)}{\frac{d}{dx}p_{Hz}(x)} = \frac{3}{2} \quad (15)$$

thus indicating that the singularity of the pressure derivative at the edges of the contact is of the same kind than the equivalent Hertzian solution but stronger.

Eq. 9 shows that  $P \propto a^3$  if the gap function is cubic symmetric. It is interesting to observe that for the bilinear symmetric gap function (the wedge problem)  $P \propto a$ , and for the symmetric parabolic gap function (Hertz's problem)  $P \propto a^2$ . This result can be explained by considering that the higher is the power of the function defining the symmetric gap function the more conforming is the contact between the elastic bodies.

## 4 Not symmetric cubic gap function

### 4.1 Analytical solution

The following expression for a the general not symmetric gap function was assumed

$$h(x) = \frac{g}{6}x^3 + \frac{c}{2}x^2 \quad (16)$$

indicating that the problem depends on two geometrical parameters:

- $g$  the gradient of the relative curvature,
- $c = k(0)$  the relative curvature at the point  $O$  of the first contact.

The two parameters are independent from each other, however, as discussed below, they may vary in limited ranges (depending also on the load and material properties) in order to guarantee convex gap function, inside the contact region.

Due to the asymmetry, a shift of the contact region has to be expected which is indicated by the abscissa  $d$ , of the mid point of the contact region:

$$d = \frac{a_2 - a_1}{2} \quad (17)$$

as shown in Fig. 1 (b). For uniformity of notation, the half-width of the contact region is indicated with the usual letter  $a$  :

$$a = \frac{a_2 + a_1}{2} \quad (18)$$

Assuming the pressure distribution given by Eq. 4 with the boundary conditions:  $p(-a_1) = p(a_2) = 0$ , the following equation can be derived (see the appendix A.1 for analytical details):

$$\frac{gd^2}{2a} + \frac{cd}{a} + \frac{ga}{4} = 0 \quad (19)$$



giving the unique meaningful solution (with  $|d| < a$ ) :

$$d = \sqrt{\left(\frac{c}{g}\right)^2 - \frac{a^2}{2}} - \frac{c}{g} \quad (20)$$

Positive  $d$  values are predicted for negative  $g$ , thus indicating that the center of the contact region moves toward the side where the relative curvature is lower. Relation 20 is not purely geometrical as the load  $P$  and material properties influence the extension of the contact region  $a$ .

By integrating the pressure, the total contact load  $P$  can be written as follows:

$$P = \frac{\pi}{4} E^* a^2 (c + gd) \quad (21)$$

where the quantity in brackets is the relative curvature at the mid-point ( $x = d$ ) of the contact region.

By substituting Eq. 20 into Eq. 21, the total contact load can be expressed as follows:

$$P = \frac{\pi}{4} E^* a^2 \sqrt{c^2 - \frac{(ga)^2}{2}} \quad (22)$$

If the load is imposed, the extension of the contact region  $a$  can be found by solving a third order equation deduced by Eq. 22 (see appendix A.3 for details), and, consequently, the off-set  $d$  can be calculated from Eq. 20.

By Eq. 4, the following expression for the pressure distribution can be obtained:

$$p(x) = \frac{2}{\pi} \frac{P}{a} \frac{c}{c + gd} \sqrt{1 - \left(\frac{x-d}{a}\right)^2} \left[1 + \frac{g}{2c} (x+d)\right] \quad (23)$$

The pressure distribution has an unique maximum which is located at the position given by:

$$x_{max} = \frac{d}{2} - \frac{c + \sqrt{(c+dg)^2 + 2(ga)^2}}{2g} \quad (24)$$

Due to the asymmetry, if the relative movement of the two bodies is constrained as shown in Fig. 1 (a), the resultant of the pressure,  $P$ , has to be applied in a position  $d_P$  differing from either the origin  $O$  or the mid point of the contact region. The abscissa  $d_P$  can be obtained by equating the moment of  $P$  therein applied with the moment produced by the pressure distribution (see appendix A.2)

$$d_P = d - \frac{8dc + ga^2 + 4gd^2}{8(c + gd)} \quad (25)$$

It is worth noting that this quantity (along with the positions of the maximum  $x_{max}$  and of the mid-point of the contact region  $d$ ) increases when the

load is increased. Alternatively, if the position of the load is set, a tilt between the two bodies must be imposed as discussed in Ref. [7]. However, in many practical situations (for instance in the contact between two matching gear teeth), the relative motion of the bodies is imposed by external constraints thus the model developed in this section appears more realistic.

## 4.2 Discussion and parametric analysis

In order to highlight the effect of the relative curvature gradient, the reference Hertzian contact was defined by assuming for the relative curvature the value at  $O$ ,  $k(0) = c$ . The extension of the contact region for the reference Hertzian contact is given by:

$$a_{Hz} = \sqrt{\frac{4P}{\pi E^* c}} \quad (26)$$

As demonstrated in the following,  $a_{Hz}$  is always smaller than the corresponding value  $a$  for the cubic contact. However, the solution for the non symmetric contact approaches the reference Hertzian solution when reducing the load. Moreover, even at relatively high load,  $a_{Hz}$  can be regarded as a reasonable underestimation of  $a$ .

The following dimensionless gradient of relative curvature is introduced:

$$\gamma = \frac{a_{Hz}}{c} g \quad (27)$$

The quantity  $\gamma$  collects the main properties of the problem: the gradient of relative curvature  $g$ , the relative curvature at the beginning of the contact  $c$ , and an approximate value of the contact region extension  $a_{Hz}$  (a quantity depending on material properties and load level). For this reason  $\gamma$  is a meaningful parameter by which the properties of the solution can be completely analyzed. Indeed, coupling Eqs. 27 and 22, the following relationship between  $a$ ,  $\gamma$  and  $a_{Hz}$  can be obtained:

$$\frac{\gamma^2}{2} \left( \frac{a}{a_{Hz}} \right)^6 - \left( \frac{a}{a_{Hz}} \right)^4 + 1 = 0 \quad (28)$$

which can be reduced to a third-order algebraic equation giving a real solution under the condition: (see appendix A.3)

$$|\gamma| \leq \hat{\gamma} = \sqrt[4]{\frac{16}{27}} \approx 0.8774 \quad (29)$$

In order to produce a contact on a single interval under compressive pressure only, the following conditions have to be fulfilled :

$$c > 0$$

$$|g| \leq \sqrt[4]{\frac{16}{27}} \sqrt{\frac{4Pc}{\pi E^*}} \quad (30)$$

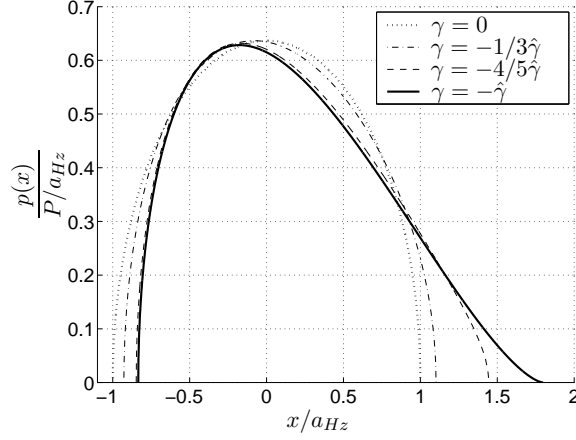


Figure 3: Pressure distributions for asymmetric cubic contact for different gradients of relative curvature.

Pressure distributions obtained by Eq. 23 are shown in Fig. 3 (values normalized with the quantity  $P/a_{Hz}$ ) for several values of the dimensionless gradient of relative curvature. Solutions for negative  $\gamma$  values have been represented as the curves for positive values can be easily obtained by symmetry.

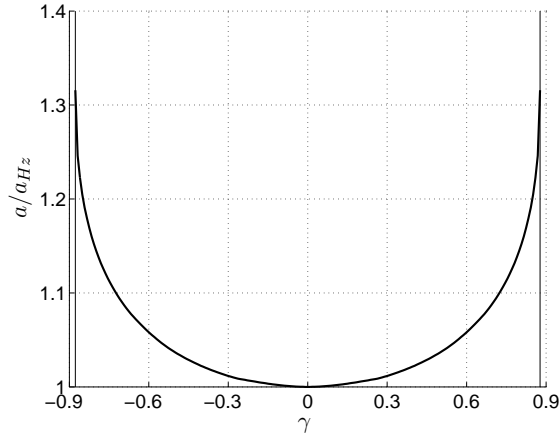


Figure 4: Effect of  $\gamma$  on the extension of the contact region.

For  $\gamma = 0$  the reference Hertzian solution is obtained. By increasing  $|\gamma|$  a rightward shift of the pressure distribution is observed thus demonstrating that the mid-point of contact region tends to migrate toward the zones with lower relative curvature. As usual, the slope of the pressure distribution is unbounded at the extremes of the contact region for  $|\gamma| < \hat{\gamma}$ . It is worth

noting that for  $|\gamma| = \hat{\gamma}$  the slope of the pressure tends to zero at the extreme with the minimum curvature.

In Fig. 4 the extension of the contact region is plotted versus the gradient of relative curvature. As anticipated  $a \geq a_{Hz}$  but, even for relatively high values of the contact load,  $a \approx a_{Hz}$ .

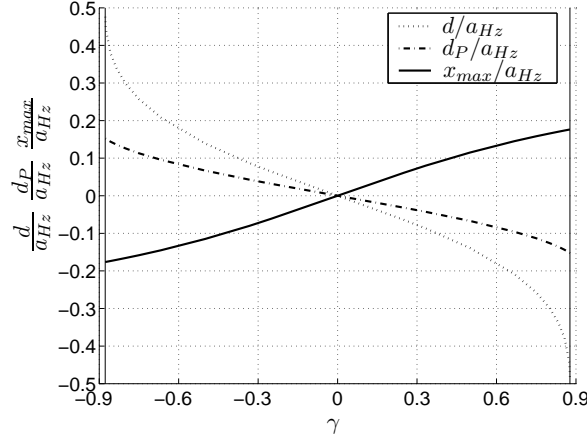


Figure 5: Non symmetric properties of the solution versus  $\gamma$ .

The non-symmetric properties of the solution are depicted in Fig. 5. As already observed, negative  $\gamma$  produces positive values of both  $d$  and  $d_P$ . On the contrary, the abscissa of the maximum pressure  $x_{max}$  has the same sign of  $\gamma$ . It can be noted also that  $d_P \approx d/2$  even at high  $\gamma$  values.

As qualitatively shown in Fig. 3, the maximum values of the dimensionless pressure are not significantly affected by  $\gamma$ . The maximum pressure is plotted versus the gradient of relative curvature in Fig. 6 showing that the maximum dimensionless pressure varies less than 1% changing  $\gamma$  in the whole range. The peak value of the maximum dimensionless pressure is attained for the reference Hertzian solution when  $\gamma = 0$ .

## 5 Conclusions

Symmetric and non symmetric contacts between cubic profiles with a  $C^2$  continuity have been analyzed and analytical solutions obtained. The analytical solutions allowed for a complete parametric study of the problem.

In the symmetric contact, an equivalent Hertzian problem was defined in order to produce a contact region with the same extension under the same load. It was verified that the maximum contact pressure is always smaller than the equivalent Hertzian maximum.

For the non symmetric contact, the effect of the gradient of relative curvature was analyzed. In this case, the reference Hertzian solution was defined by assuming, for the relative curvature, the value in the first point of con-

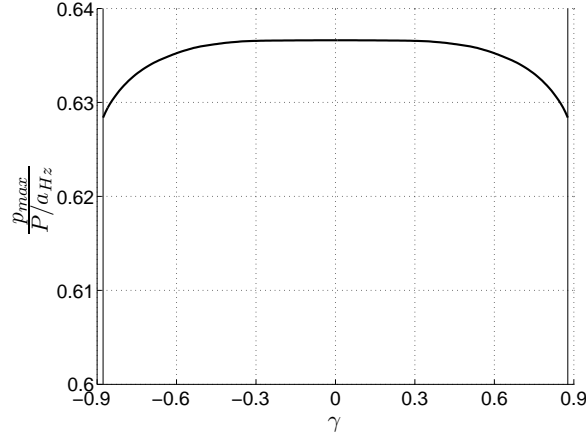


Figure 6: Effect of  $\gamma$  on the maximum dimensionless pressure.

tact. This simplified scheme is usually adopted when the gradient of relative curvature is neglected and its predictions are accurate when the contact extension is small. It was shown that the gradient of relative curvature may vary in a limited range, depending on the load and on the local relative curvature, if the contact has to be constrained in a single interval. Within these limits, excluding a small shift of the contact region due to the asymmetry, the gradient of relative curvature does not produce strong modification of the reference Hertzian solution. In particular, as compared to the reference Hertzian solution:

- the contact region is slightly larger,
- the maximum pressure is almost coincident.

In general it can be concluded that, provided the gradient of relative curvature does not induce tensile pressure in the solution, the reference Hertzian approximation gives always a reasonable conservative estimation of the contact parameters.

## A Mathematical details

In this appendix the main issues of the paper are briefly demonstrated.

### A.1 Derivation of the pressure distributions

From the quadrature solution 4 of the integral equation 2, using the positions

$$\hat{x} = x - d, \quad \xi = \frac{\hat{x}}{a}, \quad \delta = \frac{d}{a} \quad (31)$$

the following dimensionless form is achieved:

$$p(\xi) = \frac{1}{\pi\sqrt{(1-\xi^2)}} \left( \frac{E^*}{2} \int_{-1}^1 \frac{\sqrt{1-\tau^2}}{\xi-\tau} \hat{h}'(\tau) d\tau + \frac{P}{a} \right) \quad (32)$$

where  $\hat{h}'(\xi)$  is the dimensionless derivative of the gap function, for cubic symmetric

$$\hat{h}'(\xi) = \frac{g}{2} a^2 \xi |\xi| \quad (33)$$

while for cubic non symmetric

$$h'(\xi) = \frac{ga^2}{2} (\xi + \delta)^2 + ca(\xi + \delta) \quad (34)$$

which contains the unknown quantity  $\delta$ .

In this paper only incomplete contact is considered for which the pressure at the edges of the contact region is zero. Therefore putting  $p = 0$  for  $\xi = \pm 1$  the following equations can be found:

$$\int_{-1}^1 \frac{\hat{h}'(\tau)}{\sqrt{1-\tau^2}} d\tau = 0 \quad (35)$$

$$P = -\frac{E^*}{2} a \int_{-1}^1 \frac{\tau}{\sqrt{1-\tau^2}} \hat{h}'(\tau) d\tau \quad (36)$$

For symmetric gap, Eq. 35 is trivial, while it produces a relation between  $d$  and  $a$  as Eq. 19 for non symmetric gaps. Equation 36 relates load  $P$  and material properties  $E^*$  to  $a$  and  $\delta$  as in 21, while  $P$ ,  $E^*$  to  $a$ , for symmetric profile as in Eq. 9.

By substituting 36 in 32 with a little of algebra expression

$$p(\xi) = -\frac{E^*}{2\pi} \sqrt{1-\xi^2} \int_{-1}^1 \frac{\hat{h}'(\tau)}{\sqrt{1-\tau^2}(\tau-\xi)} d\tau \quad (37)$$

can be obtained. This result is very important in contact mechanics, a similar procedure is proposed in Ref. [7]. A more comprehensive treatise can be found in Ref. [8].

## A.2 Load off-set $d_P$

In order to find the position of the load off-set, as shown in Ref. [7], the moment of the pressure distribution  $p(x)$  about the origin has to be calculated

$$M = \int_{-a_1}^{a_2} p(s) s ds = \int_{-a}^a p(\hat{s}) (\hat{s} + d) d\hat{s} = Pd + \int_{-a}^a p(\hat{s}) \hat{s} d\hat{s} \quad (38)$$

After dividing by  $P$ , and putting the integrals in dimensionless form, the expression of  $d_P$  can be rewritten

$$d_P = d + a \frac{\int_{-1}^1 p(\tau) \tau d\tau}{\int_{-1}^1 p(\tau) d\tau} \quad (39)$$

From Eq. 36 it follows

$$d_P = d + \frac{a^2 E^*}{2P} \int_{-1}^1 \frac{\tau^2}{\sqrt{1-\tau^2}} \hat{h}'(\tau) d\tau \quad (40)$$

and, remembering equation 35

$$d_P = d - \frac{a^2 E^*}{2P} \int_{-1}^1 \hat{h}'(\tau) \sqrt{1-\tau^2} d\tau \quad (41)$$

### A.3 Discussion and solutions of Eq. 28 relating $a/a_{Hz}$ to $\gamma$

By indicating  $\alpha = (a/a_{Hz})^2$ , Eq. 28 can be rewritten as

$$\alpha^3 - \frac{2}{\gamma^2} \alpha^2 + \frac{2}{\gamma^2} = 0 \quad (42)$$

The roots of Eq. 42 are

$$\alpha_k = \frac{2}{3\gamma^2} \left[ 1 + 2 \cos \left( \frac{1}{3} \arccos \left( 1 - \frac{27}{8} \gamma^4 \right) + \frac{2}{3} (k-1) \pi \right) \right], \quad k = 1, 2, 3. \quad (43)$$

Real and positive values of  $\alpha$  are obtained only if

$$1 - \frac{27}{8} \gamma^4 \geq -1 \quad \Rightarrow \quad |\gamma| \leq \hat{\gamma} = \sqrt[4]{\frac{16}{27}} \quad (44)$$

The only solution of interest is that for  $k = 3$ , here expanded

$$\alpha_3 = \frac{2}{3\gamma^2} \left[ 1 - \cos \left( \frac{\arccos(1 - \frac{27}{8} \gamma^4)}{3} \right) + \sqrt{3} \sin \left( \frac{\arccos(1 - \frac{27}{8} \gamma^4)}{3} \right) \right] \quad (45)$$

This solution has the following properties:

$$\begin{aligned} \alpha_3(\gamma) &> 1, \quad \gamma \neq 0 \\ \alpha_3(0) &= 1 \end{aligned} \quad (46)$$

While the two other solutions do not fulfill these conditions.

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