

# A Condensed Representation of Almost Normal Matrices<sup>☆</sup>

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## Abstract

In this paper we study the structure of almost normal matrices, that is the class of matrices for which there exists a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$ . Necessary and sufficient conditions for a matrix to belong to the class are given, and a canonical representation as a block tridiagonal matrix is shown. The approach is constructive and we show how, starting from a  $1 \times 1$  or  $2 \times 2$  matrix, we can generate almost normal matrices. Moreover, given an  $n \times n$  almost normal matrix we compute the block tridiagonal representation with a finite procedure described in the theorems.

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## 1. Introduction

Normal matrices play an important theoretical role in numerical linear algebra. A square complex matrix is called normal if

$$A^H A - A A^H = 0,$$

where  $A^H$  is the conjugate transpose of  $A$ . Over the years, many equivalent conditions have been found [13, 8], and it has been shown that the class of  $n \times n$  normal matrices can be partitioned according to the minimal degree  $s$  of a polynomial such that  $A^H = p_s(A)$ ,  $s \leq n - 1$ . A milestone in the study of normal( $s$ ) matrices are the results related to the Faber-Manteuffel theorem [11, 10, 9]. The theorem identifies the class of matrices for which the Arnoldi iteration can be carried out using an  $s$ -term recursion with the class of normal( $s$ ) matrices. Equivalently, the theorem proves that, applying the Arnoldi method to a normal( $s$ ) matrix, we end up with an  $(s + 2)$ -band Hessenberg matrix, independently of the starting vector.

The observation that band structure is preserved under  $QR$  steps, suggests to use implicit methods that, for small values of  $s$ , yield a considerable reduction of each  $QR$  iteration. However, for large  $s$ , for example for  $s = n - 1$ , which is the case of unitary matrices, Faber-Manteuffel theorem is not able to discover the possible hidden structure in the Hessenberg form. It is well known, for example, that unitary Hessenberg matrices have a rank-one structure in the upper triangular part [5]. This structure is revealed by considering that any normal matrix is such that  $A^H = p_\ell(A)/q_m(A)$  with  $p_\ell$  and  $q_m$  polynomials with degree  $\ell$  and  $m$  respectively. For unitary matrices  $q_m(A) = A$ , with  $m = 1$  and  $p_\ell(A) = I$ , with  $\ell = 0$ .

Despite it is simple [6] to prove that, if  $A^H = p_\ell(A)/q_m(A)$ , then all submatrices taken out above the  $\max\{|\ell - m|, 1\}$  diagonal of its Hessenberg form<sup>1</sup> have rank at most  $m$ , it is uncertain if the reverse implication is true. In [2] the authors try to extend the Faber-Manteuffel theorem to a wider class of matrices, namely the matrices  $A$  such that  $A^H = \frac{p_\ell(A)}{q_m(A)} + C_k$ , being  $C_k$  a rank- $k$  matrix. They determine sufficient conditions ensuring that the conjugate gradient method can be implemented using a short multiple recursion. Necessary conditions, although claimed, are not proven in the general case [1]. Given the hardness of studying this class of matrices, in this paper we consider the simpler case where, instead of

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<sup>1</sup>When  $\ell = 0$  and  $m = 1$  as in the case  $A$  is unitary, the rank-one structure extends up to the main diagonal.

rational functions, we have polynomials and a rank-one correction. In the literature, there are many results concerning the spectrum of a perturbed Hermitian matrix [16] or general normal matrices [18, 15], or examples of implicit  $QR$  algorithms for the computation of the eigenvalues of low rank perturbations of symmetric or Hermitian matrices [19, 22]. The notion of quasi-normal, or almost normal matrices, has been introduced earlier in the literature [21, 17]. In this paper we give a different definition of the class, and we are mainly interested in the description of a condensed representation that can eventually lead to algorithmic advantages for the computation of eigenvalues or for the solution of linear systems. More precisely, we say that an  $n \times n$  complex matrix  $A$  is *almost normal* if there exists a rank-one matrix  $C$  such that  $A^H - C$  commutes with  $A$ . In the nonderogatory case, if  $A$  is almost normal, then there exists a polynomial  $p_s(\cdot)$ ,  $s \leq n - 1$ , such that  $A^H = p_s(A) + C$ . However, in the derogatory case, this is not true because two matrices can commute without being polynomials in each other. In this paper we do not consider this polynomial formulation and the degree of the polynomial will not play a role, because we take the union over all the possible  $s$ . We investigate the structure of almost normal matrices with a constructive approach. Our approach has the same flavor of that carried out by Ikramov and Elsner in [14] and [7] for normal matrices and by Ghasemi Kamalvand and Ikramov [12] for low rank perturbations of normal matrices. The condensed form introduced in our paper can be obtained by a finite algorithmic procedure which is described in the theorems proven in the paper.

Normal matrices are fully characterized by their Schur canonical form being diagonal. However, the Schur form does not characterize the class of almost normal matrices. In fact, if  $A$  is a nonderogatory almost normal matrix, that is  $A^H = p_s(A) + C$ , and we consider the Schur canonical form of  $A$ , we have

$$T^H = p_s(T) + \tilde{C}, \quad \text{where } \tilde{C} \text{ is a suitable rank-one matrix.}$$

Since  $T$  is upper triangular, we obtain that all the minors taken out from the strictly upper triangular part of  $T$  have rank at most one. However, this is not a sufficient condition for  $A$  to be almost normal.

The paper is organized as follows. In Section 2 we introduce the class of almost normal matrices and we give necessary conditions for a matrix to be in that class. In particular, we show that any almost normal matrix can be unitarily reduced to a matrix with a tridiagonal block structure, where the diagonal blocks are related to each other and the entries of the off diagonal blocks depend on those of the successive more external block. In Section 3 we show that the necessary conditions turn out to be sufficient as well. A recursive argument allows to border any almost normal matrix with suitable entries to obtain another (larger) matrix in the class. In Section 4 we prove that any almost normal matrix is a rank-one perturbation of a normal matrix and in Section 5 we give some concluding remarks.

## 2. Structure of the class: Necessary conditions

Let  $A$  be such that  $A^H = p_s(A) + C$ , where  $p_s(\cdot)$  is a polynomial with degree  $s$  and  $C$  is a rank-one matrix. If  $A$  is nonderogatory, this condition is equivalent to  $A^H - C$  commuting with  $A$ , that is

$$(A^H - C)A = A(A^H - C).$$

In this paper we study the class of almost normal matrices that we define as follows.

**Definition 1.** An  $n \times n$  complex matrix  $A$  is *almost normal* if there exists a matrix  $C$  with rank at most one such that

$$A^H A - A A^H = C A - A C. \quad (1)$$

Note that, since  $C$  has rank at most one, the matrix  $C A - A C$  has rank two or zero (in the latter case  $A$  is normal). Moreover, because matrix  $A^H A - A A^H$  appearing on the left-hand side of (1) is Hermitian, it can be diagonalized with a unitary transformation  $Q$ , that is  $Q^H (A^H A - A A^H) Q = D$ , where  $D$  is a real diagonal matrix. To keep the notation simple, we will still call  $A$  the matrix obtained applying unitary transformations which preserve the structural properties.

The matrix  $A^H A - A A^H$  has zero trace, and rank equal to zero or two, hence there exists a nonnegative real number  $\alpha \geq 0$  such that

$$A^H A - A A^H = \left[ \begin{array}{c|cc} O & & O \\ \hline O & \alpha & 0 \\ & 0 & -\alpha \end{array} \right]. \quad (2)$$

From (2), setting

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{22} \in \mathbb{C}^{2 \times 2} \quad (3)$$

we get the following blocks relations

$$A_{11}^H A_{11} - A_{11} A_{11}^H = A_{12} A_{12}^H - A_{21}^H A_{21}, \quad (4)$$

$$A_{11}^H A_{12} - A_{11} A_{21}^H = A_{12} A_{22}^H - A_{21}^H A_{22}, \quad (5)$$

$$A_{22}^H A_{22} - A_{22} A_{22}^H = A_{21} A_{21}^H - A_{12}^H A_{12} + \begin{bmatrix} \alpha & \\ & -\alpha \end{bmatrix}. \quad (6)$$

**Remark 1.** Note that equations (2), (4), (5) and (6) still hold whenever we apply to  $A$  unitary block diagonal transformations of this kind

$$\begin{bmatrix} P & O \\ O & S \end{bmatrix},$$

$P$  being an  $(n-2) \times (n-2)$  unitary matrix and  $S$  a  $2 \times 2$  phase matrix.

**Remark 2.** Note that some matrices satisfying equation (2) might not be almost normal, in fact it might not exist a matrix  $C$  such that (1) holds.

Next theorem, gives necessary conditions on the structure of the blocks of an almost normal (not normal) matrix.

**Theorem 1.** Let  $A$  be an  $n \times n$ ,  $n \geq 3$  almost normal matrix, not normal, such that equations (1) and (2) hold for a given matrix  $C$  with rank at most one. Partition  $A$  as in (3) and  $C$  accordingly, that is  $C = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} [\mathbf{y}_1^H, \mathbf{y}_2^H]$ , for some vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2$ , with  $\|\mathbf{x}_1\|_2^2 + \|\mathbf{x}_2\|_2^2 = 1$ . Then we have

$$(a) \mathbf{x}_1 = 0, \mathbf{y}_1 = 0, \mathbf{x}_2 = \begin{bmatrix} \zeta \\ -\omega \end{bmatrix}, \text{ and } \mathbf{y}_2^H = \kappa [\omega, -\zeta], \text{ for a nonzero constant } \kappa, \text{ and } \zeta, \omega \text{ such that } |\zeta|^2 + |\omega|^2 = 1.$$

(b)  $A_{12}$  and  $A_{21}$  have rank at most one and

$$A_{12} = [\omega \mathbf{d}, \zeta \mathbf{d}], \quad A_{21} = \begin{bmatrix} \zeta \mathbf{f}^H \\ \omega \mathbf{f}^H \end{bmatrix}, \quad \mathbf{d}, \mathbf{f} \in \mathbb{C}^{n-2}. \quad (7)$$

$$(c) \text{ Denoting } A_{22} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ we have that } b_{11} = b_{22}, \text{ and moreover } \omega^2 b_{12} - \zeta^2 b_{21} = \alpha / \kappa.$$

**PROOF.** Since  $A$  is not normal,  $C$  has rank exactly one, so  $\mathbf{x}$  and  $\mathbf{y}$  are different from zero. For the same reason  $\alpha$  is strictly positive. Rewriting equation (1) and (2) in terms of the blocks of  $A$  and of the generators of  $C$  we obtain four matrix equations:

$$\mathbf{x}_1 (\mathbf{y}_1^H A_{11} + \mathbf{y}_2^H A_{21}) - (A_{11} \mathbf{x}_1 + A_{12} \mathbf{x}_2) \mathbf{y}_1^H = 0 \quad (8)$$

$$\mathbf{x}_1 (\mathbf{y}_1^H A_{12} + \mathbf{y}_2^H A_{22}) - (A_{11} \mathbf{x}_1 + A_{12} \mathbf{x}_2) \mathbf{y}_2^H = 0 \quad (9)$$

$$\mathbf{x}_2 (\mathbf{y}_1^H A_{11} + \mathbf{y}_2^H A_{21}) - (A_{21} \mathbf{x}_1 + A_{22} \mathbf{x}_2) \mathbf{y}_1^H = 0 \quad (10)$$

$$\mathbf{x}_2 (\mathbf{y}_1^H A_{12} + \mathbf{y}_2^H A_{22}) - (A_{21} \mathbf{x}_1 + A_{22} \mathbf{x}_2) \mathbf{y}_2^H = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}. \quad (11)$$

To prove (a) assume by contradiction that  $\mathbf{x}_1 \neq 0$ . From equation (8) and (9), applying Lemma 9 in the Appendix, we have

$$\mu \mathbf{y}_1^H = (\mathbf{y}_1^H A_{11} + \mathbf{y}_2^H A_{21}) \quad (12)$$

$$\mu \mathbf{y}_2^H = (\mathbf{y}_1^H A_{12} + \mathbf{y}_2^H A_{22}), \quad (13)$$

where  $\mu = (A_{11}\mathbf{x}_1 + A_{12}\mathbf{x}_2)^H \mathbf{x}_1$ .

Then from (12) and (13) it follows that  $\mathbf{y}^H$  is a left eigenvector of  $A$  corresponding to the eigenvalue  $\mu$ . From (1) and (2), we have

$$\left[ \begin{array}{c|cc} O & O & \\ \hline O & \alpha & 0 \\ & 0 & -\alpha \end{array} \right] = \mathbf{x}\mathbf{y}^H A - A\mathbf{x}\mathbf{y}^H = (\mu I - A) \mathbf{x}\mathbf{y}^H \quad (14)$$

which is a contradiction because  $\alpha \neq 0$ , and the matrix on the left hand side of equation (14) has rank exactly two while the right hand side is a rank-one matrix. Hence  $\mathbf{x}_1 = 0$ . We can repeat the same reasoning assuming  $\mathbf{y}_1 \neq 0$  and using equations (8) and (10), obtaining that  $\mathbf{y}_1$  must be zero.

Substituting  $\mathbf{x}_1 = 0$  and  $\mathbf{y}_1 = 0$  into equation (11) we get

$$\mathbf{x}_2 \mathbf{y}_2^H A_{22} - A_{22} \mathbf{x}_2 \mathbf{y}_2^H = \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix}. \quad (15)$$

Multiplying on the left by  $\mathbf{y}_2^H$  and on the right by  $\mathbf{x}_2$  we get that the left hand side of (15) vanishes, and we obtain the following relation between  $\mathbf{y}_2$  and  $\mathbf{x}_2$

$$\mathbf{y}_2^H \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} \mathbf{x}_2 = 0.$$

Taking  $\mathbf{y}_2^H = (\bar{y}_{n-1}, \bar{y}_n)$  and  $\mathbf{x}_2 = (\zeta, -\omega)^T$ , with  $|\zeta|^2 + |\omega|^2 = 1$ , we have

$$\bar{y}_{n-1}\zeta + \bar{y}_n\omega = 0,$$

which implies  $\mathbf{y}_2^H = \kappa[\omega, -\zeta]$ , for some nonzero constant  $\kappa$ . Thus (a) is proven.

Rewriting equations (9) and (10) by setting  $\mathbf{x}_1 = 0$  and  $\mathbf{y}_1 = 0$ , since  $\mathbf{x}_2 \neq 0$  and  $\mathbf{y}_2 \neq 0$ , we have

$$A_{12}\mathbf{x}_2 = 0, \quad \mathbf{y}_2^H A_{21} = 0, \quad (16)$$

meaning that the matrices  $A_{12}$  and  $A_{21}$  have rank at most 1. Because of relation (16) we have

$$A_{12} = [\omega \mathbf{d}, \zeta \mathbf{d}], \quad A_{21} = \begin{bmatrix} \zeta \mathbf{f}^H \\ \omega \mathbf{f}^H \end{bmatrix},$$

where  $\mathbf{d}$  and  $\mathbf{f}$  are  $(n-2)$ -vectors. Thus (b) is proven.

Part (c) of the theorem is proven imposing relation (15) on the entries of the matrix  $A_{22}$ .  $\square$

The fact that  $A_{12}$  and  $A_{21}$  are rank-one matrices with the same proportional parameters is useful to prove that the structure of the blocks  $A_{12}$  and  $A_{21}$  can be further simplified. This is the first step towards the block tridiagonalization of  $A$ .

**Theorem 2.** *Let  $A$  be an almost normal matrix of size  $n$ ,  $n \geq 3$ , satisfying equation (2), and let  $A_{12}$  and  $A_{21}$  have the form described by equation (7), with  $|\omega|^2 + |\zeta|^2 = 1$ . Then  $A$  can be unitarily reduced to a matrix still satisfying equation (2), and such that*

$$A_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ |\omega| \delta_1 & |\zeta| \delta_1 \\ |\omega| \delta_2 & |\zeta| \delta_2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} \mathbf{0}^H & |\zeta| \delta_2 & |\zeta| \delta_1 \\ \mathbf{0}^H & |\omega| \delta_2 & |\omega| \delta_1 \end{bmatrix}, \quad (17)$$

meaning that  $\zeta$  and  $\omega$  can be supposed real non negative. Moreover  $|\delta_1| \geq |\delta_2|$ .

PROOF. If  $\mathbf{d} = \mathbf{f} = \mathbf{0}$  then  $A_{12}$  and  $A_{21}$  have the required form. Let  $\mathbf{d} \neq \mathbf{0}$ , and first assume  $n > 4$ , the case  $\mathbf{f} \neq \mathbf{0}$  can be treated analogously. From equation (4) we have

$$A_{11}^H A_{11} - A_{11} A_{11}^H = A_{12} A_{12}^H - A_{21}^H A_{21}.$$

Knowing that blocks  $A_{12}$  and  $A_{21}$  have the structure given by (7), we have

$$A_{11}^H A_{11} - A_{11} A_{11}^H = (|\omega|^2 + |\zeta|^2)(\mathbf{d}\mathbf{d}^H - \mathbf{f}\mathbf{f}^H) = \mathbf{d}\mathbf{d}^H - \mathbf{f}\mathbf{f}^H.$$

Consider now the matrix  $B = A_{11}^H A_{11} - A_{11} A_{11}^H$  which is Hermitian and has zero trace. Let  $U$  be the unitary matrix which diagonalizes  $B$ . Then there exists a real nonnegative value  $\beta$  such that

$$U^H B U = U^H (A_{11}^H A_{11} - A_{11} A_{11}^H) U = \left[ \begin{array}{c|cc} O & & O \\ \hline O & \beta & 0 \\ & 0 & -\beta \end{array} \right]. \quad (18)$$

Let us assume that  $A$  has been already unitarily transformed so that  $B$  is in the form (18).

Splitting  $\mathbf{d} = (\mathbf{d}_1; \mathbf{d}_2)$  and  $\mathbf{f} = (\mathbf{f}_1; \mathbf{f}_2)$  accordingly with the partition (18) we get

$$\begin{aligned} \mathbf{d}_1 \mathbf{d}_1^H - \mathbf{f}_1 \mathbf{f}_1^H &= 0 \\ \mathbf{d}_1 \mathbf{d}_2^H - \mathbf{f}_1 \mathbf{f}_2^H &= 0 \\ \mathbf{d}_2 \mathbf{d}_2^H - \mathbf{f}_2 \mathbf{f}_2^H &= \begin{bmatrix} \beta & 0 \\ 0 & -\beta \end{bmatrix}. \end{aligned} \quad (19)$$

If  $\beta = 0$ , applying Lemma 9, from equations (19) we have  $\mathbf{d} = \mu \mathbf{f}$ , with  $|\mu| = 1$ . We can construct an Householder-like transformation  $P$  such that

$$P \mathbf{d} = \gamma \begin{bmatrix} \mathbf{0} \\ 1 \\ 1 \end{bmatrix},$$

where  $|\gamma| = \|\mathbf{d}\|_2/\sqrt{2}$  and  $\gamma = |\gamma| \exp(i\theta)$ , with  $\theta = -\arg(\bar{d}_{n-3} + \bar{d}_{n-2})$ , being  $\mathbf{d}_2 = (d_{n-3}, d_{n-2})^T$ . Setting  $\mu = \exp(i\varphi)$ , we can take the unitary matrix  $Q$  as

$$Q = \begin{bmatrix} \exp(i(\varphi/2 - \theta)) P & \\ & I_2 \end{bmatrix},$$

which is such that

$$Q^H A Q = \left[ \begin{array}{cc|cc} & & & & 0 & 0 \\ & & & & \frac{\omega\delta}{\omega\delta} & \frac{\zeta\delta}{\zeta\delta} \\ & & & & \omega\delta & \zeta\delta \\ \hline & & & & & \\ O & & \zeta\delta & \zeta\delta & & \\ & & \omega\delta & \omega\delta & & A_{22} \end{array} \right], \quad (20)$$

where  $\delta = |\gamma| \exp(i\varphi/2)$ , and hence  $A_{12}$  and  $A_{21}$  have the desired structure.

If  $\beta \neq 0$ , assume that  $\mathbf{d}_1 \neq \mathbf{0}$ , we have from Lemma 9 that  $\mathbf{d}_2 = \mu \mathbf{f}_2$ . This is however a contradiction because otherwise  $\mathbf{d}_2 \mathbf{d}_2^H - \mathbf{f}_2 \mathbf{f}_2^H = (|\mu|^2 - 1) \mathbf{f}_2 \mathbf{f}_2^H$  will have rank at most one, which is not possible because  $\beta \neq 0$ . Hence  $\mathbf{d} = (\mathbf{0}; \mathbf{d}_2)$  and  $\mathbf{f} = (\mathbf{0}; \mathbf{f}_2)$ .

Setting  $\mathbf{d} = (\mathbf{0}; \Delta_1; \Delta_2)$ , and  $\mathbf{f} = (\mathbf{0}; \Phi_1; \Phi_2)$ , we need to prove that we can reduce  $A$  to a matrix where  $\bar{\Phi}_1 = \Delta_2$  and  $\bar{\Phi}_2 = \Delta_1$ .

Rewriting (19) we have

$$\begin{aligned}\Delta_1 \bar{\Delta}_1 - \Phi_1 \bar{\Phi}_1 &= \beta \\ \Delta_2 \bar{\Delta}_2 - \Phi_2 \bar{\Phi}_2 &= -\beta \\ \Delta_1 \bar{\Delta}_2 &= \Phi_1 \bar{\Phi}_2.\end{aligned}$$

Summing together the first two equations we have

$$|\Delta_1|^2 + |\Delta_2|^2 = |\Phi_1|^2 + |\Phi_2|^2$$

that, together with  $|\Delta_1||\Delta_2| = |\Phi_1||\Phi_2|$ , implies that we can either have  $|\Delta_1| = |\Phi_1|$  and  $|\Delta_2| = |\Phi_2|$  or  $|\Delta_1| = |\Phi_2|$  and  $|\Delta_2| = |\Phi_1|$ . However the first case is not acceptable since  $\beta \neq 0$ .

By means of phase transformations we get  $\bar{\Phi}_1 = \Delta_2$  and  $\bar{\Phi}_2 = \Delta_1$ , proving that

$$A_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \Delta_1 & \zeta \Delta_1 \\ \omega \Delta_2 & \zeta \Delta_2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} \mathbf{0}^H & \zeta \Delta_2 & \zeta \Delta_1 \\ \mathbf{0}^H & \omega \Delta_2 & \omega \Delta_1 \end{bmatrix},$$

eventually with  $\Delta_1 = \Delta_2$  if  $\beta = 0$ . Let us prove that there exist  $\tilde{\omega}$  and  $\tilde{\zeta}$  real and nonnegative such that (17) holds.

Writing the (possibly complex) numbers  $\omega$  and  $\zeta$  in polar form, we have  $\omega = |\omega| \exp(i\theta_\omega)$ , and  $\zeta = |\zeta| \exp(i\theta_\zeta)$ . Let  $\phi = (\theta_\zeta - \theta_\omega)/2$ , and consider the phase matrix

$$S = \begin{bmatrix} I_{n-2} & & \\ & \exp(-i\phi) & \\ & & \exp(i\phi) \end{bmatrix},$$

then

$$SAS^{-1} = \begin{bmatrix} A_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

where

$$\tilde{A}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ |\omega| \delta_1 & |\zeta| \delta_1 \\ |\omega| \delta_2 & |\zeta| \delta_2 \end{bmatrix}, \quad \tilde{A}_{21} = \begin{bmatrix} \mathbf{0}^H & |\zeta| \delta_2 & |\zeta| \delta_1 \\ \mathbf{0}^H & |\omega| \delta_2 & |\omega| \delta_1 \end{bmatrix},$$

where  $\delta_1 = \Delta_1 \exp(i(\theta_\omega + \theta_\zeta)/2)$  and  $\delta_2 = \Delta_2 \exp(i(\theta_\omega + \theta_\zeta)/2)$ . The diagonal entries of block  $\tilde{A}_{22}$  are not modified by this phase transformation and equation (18) still holds. Finally, if  $|\delta_1| < |\delta_2|$ , the rows and columns with indices  $n-3$  and  $n-2$  can be permuted without destroying the structure properties used in this theorem.

If  $n = 3$  or  $n = 4$  the blocks  $A_{12}$  and  $A_{21}$  do not have the zeros in the first rows and columns, since they have respectively size  $1 \times 2$  (when  $n = 3$ ) and  $2 \times 2$  (if  $n = 4$ ).  $\square$

Theorem 1 holds only when  $A$  is not normal. However, it is in general possible that the leading principal  $(n-2) \times (n-2)$  minor is normal. Next Corollary gives necessary and sufficient condition for  $A_{11}$  to be normal.

**Corollary 1.** *Under the hypothesis of Theorem 2 the matrix  $A_{11}$  is normal if and only if  $|\delta_1| = |\delta_2|$ .*

PROOF. Assume  $A$  is reduced in the form described in Theorem 2. From (4) we have

$$A_{11}^H A_{11} - A_{11} A_{11}^H = -A_{21}^H A_{21} + A_{12} A_{12}^H. \quad (21)$$

Consider the right hand side of (21), using the equality (17), we have

$$A_{11}^H A_{11} - A_{11} A_{11}^H = \begin{bmatrix} O & & \\ & |\delta_1|^2 - |\delta_2|^2 & \\ & & |\delta_2|^2 - |\delta_1|^2 \end{bmatrix}.$$

Then we have that  $A_{11}$  is normal iff  $|\delta_1| = |\delta_2|$ .  $\square$

As underlined in Remark 2, not all the matrices satisfying equation (2) are almost normal. Next theorem gives necessary and sufficient conditions for a matrix  $A$  to belong to that class, meaning that we give conditions for the existence of a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$ . The theorem will be used in Section 6 for proving sufficient conditions.

**Theorem 3.** Let  $A$  be an  $n \times n$  complex matrix, with  $n \geq 3$ . Partition  $A$  as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{(n-2) \times (n-2)}.$$

If

- (a) Equation (2) holds with  $\alpha \neq 0$ ,
- (b)  $A_{12}$  and  $A_{21}$  are structured as in equation (7), with  $\omega$  and  $\zeta$  nonnegative reals such that  $\omega^2 + \zeta^2 = 1$ ,
- (c)  $A_{22} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix}$ .

Then  $A$  is almost normal (not normal) - meaning that there exists a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$  - if and only if  $\omega^2 b_{12} - \zeta^2 b_{21} \neq 0$ .

PROOF. If  $A$  is almost normal, not normal, then for Theorem 1 the rank-one matrix  $C$  is such that  $C = \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{pmatrix} (\mathbf{0}^H, \mathbf{y}_2^H)$  and  $\mathbf{x}_2 = \begin{bmatrix} \zeta \\ -\omega \end{bmatrix}$ ,  $\mathbf{y}_2^H = \kappa (\omega, -\zeta)$ . With such a choice

$$C A - A C = \left[ \begin{array}{c|cc} O & O & O \\ \hline O & \kappa(\omega^2 b_{12} - \zeta^2 b_{21}) & 0 \\ & 0 & -\kappa(\omega^2 b_{12} - \zeta^2 b_{21}) \end{array} \right] = \left[ \begin{array}{c|cc} O & O & O \\ \hline O & \alpha & 0 \\ & 0 & -\alpha \end{array} \right].$$

Since  $\alpha \neq 0$  ( $A$  is not normal) then  $(\omega^2 b_{12} - \zeta^2 b_{21}) \neq 0$ . Vice versa, if  $(\omega^2 b_{12} - \zeta^2 b_{21}) \neq 0$ ,  $\omega$  and  $\zeta$  being the coefficients in (7), a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$  is given by  $C = \begin{pmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{pmatrix} (\mathbf{0}^H, \mathbf{y}_2^H)$  with  $\mathbf{x}_2 = \begin{bmatrix} \zeta \\ -\omega \end{bmatrix}$ ,  $\mathbf{y}_2^H = \kappa (\omega, -\zeta)$ , and with

$$\kappa = \frac{\alpha}{\omega^2 b_{12} - \zeta^2 b_{21}}.$$

□

We are now ready to prove the main result of this paper. Notice that we do not require  $A$  to be almost normal, i.e. the existence of a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$  is not assumed. Next theorem states that any almost normal matrix (unitarily reduced to a form such that (2) holds) has a leading principal submatrix with the same almost normal structure.

**Theorem 4.** Let  $A$  be an  $n \times n$  complex matrix, with  $n \geq 6$ . Partition  $A$  as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{(n-2) \times (n-2)}.$$

If

- (a) Equation (2) holds with  $\alpha \neq 0$ ,
- (b)  $A_{12}$  and  $A_{21}$  are structured as in equation (17), with  $\omega$  and  $\zeta$  real nonnegative such that  $\omega^2 + \zeta^2 = 1$  and  $|\delta_1| > |\delta_2|$ ,
- (c)  $A_{22} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix}$ ,

then  $\omega \neq \zeta$  and  $A$  can be unitarily reduced to a matrix such that the block  $A_{11}$  has the form

$$A_{11} = \left[ \begin{array}{cc|cc} & & & O \\ & \hat{A}_{11} & & \hline & & \omega_1 \delta_1^{(1)} & \zeta_1 \delta_1^{(1)} \\ & & \omega_1 \delta_2^{(1)} & \zeta_1 \delta_2^{(1)} \\ \hline O & \zeta_1 \delta_2^{(1)} & \zeta_1 \delta_1^{(1)} & \hat{A}_{22} \\ & \omega_1 \delta_2^{(1)} & \omega_1 \delta_1^{(1)} & \end{array} \right], \quad (22)$$

and matrix block  $A_{11}$  satisfies (a), (c). Moreover the parameters of  $A_{11}$  are such that  $|\delta_1^{(1)}| \geq |\delta_2^{(1)}|$ ,  $\omega_1$  and  $\zeta_1$  are real non negative,  $\omega_1 > \zeta_1$  with  $\omega_1^2 + \zeta_1^2 = 1$ . In particular equation (2) for  $A_{11}$  becomes

$$A_{11}^H A_{11} - A_{11} A_{11}^H = \left[ \begin{array}{c|c} O & O \\ \hline O & \begin{matrix} \beta & 0 \\ 0 & -\beta \end{matrix} \end{array} \right], \quad \text{where } \beta > 0.$$

PROOF. Since  $n \geq 6$ , we can partition  $A_{11}$  as follows

$$A_{11} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \text{ where } \hat{A}_{11} \in \mathbb{C}^{(n-4) \times (n-4)}, \hat{A}_{22} \in \mathbb{C}^{2 \times 2}.$$

Rewriting equation (5) in block form we have

$$\begin{aligned} & \begin{bmatrix} \hat{A}_{11}^H & \hat{A}_{21}^H \\ \hat{A}_{12}^H & \hat{A}_{22}^H \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \delta_1 & \zeta \delta_1 \\ \omega \delta_2 & \zeta \delta_2 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \zeta \bar{\delta}_2 & \omega \bar{\delta}_2 \\ \zeta \bar{\delta}_1 & \omega \bar{\delta}_1 \end{bmatrix} A_{22} - \\ & - \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \zeta \bar{\delta}_2 & \omega \bar{\delta}_2 \\ \zeta \bar{\delta}_1 & \omega \bar{\delta}_1 \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \delta_1 & \zeta \delta_1 \\ \omega \delta_2 & \zeta \delta_2 \end{bmatrix} A_{22}^H = 0. \end{aligned} \quad (23)$$

Set  $\hat{A}_{12} = [\mathbf{g}, \mathbf{h}]$ , and  $\hat{A}_{21}^H = [\mathbf{r}, \mathbf{s}]$ , where  $\mathbf{g}, \mathbf{h}, \mathbf{r}$  and  $\mathbf{s}$  are complex vectors of length  $n - 4$ . Looking at the upper  $(n - 4)$  equations of (23) we get

$$\begin{aligned} \omega(\delta_1 \mathbf{r} + \delta_2 \mathbf{s}) - \zeta(\bar{\delta}_2 \mathbf{g} + \bar{\delta}_1 \mathbf{h}) &= 0 \\ \zeta(\delta_1 \mathbf{r} + \delta_2 \mathbf{s}) - \omega(\bar{\delta}_2 \mathbf{g} + \bar{\delta}_1 \mathbf{h}) &= 0. \end{aligned}$$

Writing the four equations of (23) involving the entries  $a_{ij}$  of  $\hat{A}_{22}$  and the entries  $b_{ij}$  of  $A_{22}$  we get

$$\omega \delta_1 \bar{a}_{11} + \omega \delta_2 \bar{a}_{21} - \zeta \bar{\delta}_1 a_{12} - \zeta \bar{\delta}_2 a_{11} = \omega \delta_1 \bar{b}_{11} - \omega \bar{\delta}_2 b_{21} + \zeta \delta_1 \bar{b}_{12} - \zeta \bar{\delta}_2 b_{11} \quad (24)$$

$$-\omega \bar{\delta}_1 a_{12} - \omega \bar{\delta}_2 a_{11} + \zeta \delta_1 \bar{a}_{11} + \zeta \delta_2 \bar{a}_{21} = \omega \delta_1 \bar{b}_{21} - \omega \bar{\delta}_2 b_{11} + \zeta \delta_1 \bar{b}_{11} - \zeta \bar{\delta}_2 b_{12} \quad (25)$$

$$\omega \delta_1 \bar{a}_{12} + \omega \delta_2 \bar{a}_{22} - \zeta \bar{\delta}_1 a_{22} - \zeta \bar{\delta}_2 a_{21} = \omega \delta_2 \bar{b}_{11} - \omega \bar{\delta}_1 b_{21} + \zeta \delta_2 \bar{b}_{12} - \zeta \bar{\delta}_1 b_{11} \quad (26)$$

$$-\omega \bar{\delta}_1 a_{22} - \omega \bar{\delta}_2 a_{21} + \zeta \delta_1 \bar{a}_{12} + \zeta \delta_2 \bar{a}_{22} = -\omega \bar{\delta}_1 b_{11} + \omega \delta_2 \bar{b}_{21} + \zeta \delta_2 \bar{b}_{11} - \zeta \bar{\delta}_1 b_{12} \quad (27)$$

Observe that  $\omega \neq \zeta$ . Otherwise, assume by contradiction that  $\omega = \zeta$ , then the left hand side of equations (24) and (25) are the same, and subtracting them we get

$$\delta_1(\bar{b}_{21} - \bar{b}_{12}) = \bar{\delta}_2(b_{12} - b_{21}). \quad (28)$$

From equation (2) on the block in position (2,2) we have

$$A_{12}^H A_{12} + A_{22}^H A_{22} - A_{21} A_{21}^H - A_{22} A_{22}^H = \begin{bmatrix} \alpha & \\ & -\alpha \end{bmatrix}.$$

Using the fact  $\omega^2 - \zeta^2 = 0$ , we have

$$|b_{21}|^2 - |b_{12}|^2 = \alpha \neq 0.$$

From the absolute value of (28) and observing that  $|\bar{b}_{21} - \bar{b}_{12}| = |b_{12} - b_{21}| \neq 0$ , we get  $|\delta_1| = |\delta_2|$  which contradicts the hypothesis because  $|\delta_1| > |\delta_2|$ .

Hence, because  $\omega \neq \zeta$ , with two linear combinations we get

$$(\omega^2 - \zeta^2)(\bar{\delta}_2 \mathbf{g} + \bar{\delta}_1 \mathbf{h}) = 0 \quad (29)$$

$$(\omega^2 - \zeta^2)(\delta_1 \mathbf{r} + \delta_2 \mathbf{s}) = 0 \quad (30)$$



Since  $\omega^2 - \zeta^2 \neq 0$ , the matrices  $\hat{A}_{12} = [\mathbf{g}, \mathbf{h}]$ , and  $\hat{A}_{21}^H = [\mathbf{r}, \mathbf{s}]$  have rank at most one. In particular, there exist two  $(n-4)$ -vectors  $\mathbf{d}^{(1)}$  and  $\mathbf{f}^{(1)}$  such that

$$\hat{A}_{12} = [\omega_1 \mathbf{d}^{(1)}, \zeta_1 \mathbf{d}^{(1)}], \quad \hat{A}_{21} = \begin{bmatrix} \zeta_1 \mathbf{f}^{(1)H} \\ \omega_1 \mathbf{f}^{(1)H} \end{bmatrix},$$

where  $\zeta_1 = \kappa \bar{\delta}_2$  and  $\omega_1 = -\kappa \bar{\delta}_1$  and  $\kappa$  is such that  $|\zeta_1|^2 + |\omega_1|^2 = 1$ . It follows that  $|\omega_1| > |\zeta_1|$ .

We need now to prove that the diagonal entries of  $\hat{A}_{22}$  are the same, that is  $a_{11} = a_{22}$ .

Performing some linear combinations on equations (24)-(27) we obtain

$$\begin{aligned} (\omega^2 - \zeta^2) \bar{\delta}_2 (a_{11} - a_{22}) &= 0 \\ (\omega^2 - \zeta^2) \bar{\delta}_1 (a_{11} - a_{22}) &= 0. \end{aligned}$$

Since we cannot have  $\delta_1 = \delta_2 = 0$ , we get  $a_{11} = a_{22}$  as claimed, proving property (c) for  $A_{11}$ .

Because of the structure of  $A_{12}$  and  $A_{21}$  described in (b) and using (4), we get

$$A_{11}^H A_{11} - A_{11} A_{11}^H = \begin{bmatrix} O & & \\ & |\delta_1|^2 - |\delta_2|^2 & \\ & & |\delta_2|^2 - |\delta_1|^2 \end{bmatrix},$$

proving (a) for  $A_{11}$ , and the property that  $\beta = |\delta_1|^2 - |\delta_2|^2$  is positive.

We note that matrix  $A_{11}$  satisfies the hypothesis of Theorem 2. Hence block  $\hat{A}_{12}$  and  $\hat{A}_{21}$  have the structure described in (22) with  $|\delta_1^{(1)}| \geq |\delta_2^{(1)}|$ . Note that the phase transformation which makes  $\omega_1$  and  $\zeta_1$  real nonnegative, modifies the external border, changing the polar part in the representation of  $\delta_1$  and  $\delta_2$ : in detail, setting

$$\delta_1 = |\delta_1| \exp(i\epsilon_1), \quad \delta_2 = |\delta_2| \exp(i\epsilon_2), \quad \kappa = |\kappa| \exp(i\epsilon_3),$$

the new value for  $\delta_1$  and  $\delta_2$  are respectively

$$|\delta_1| \exp(i(\epsilon_1 + \epsilon_2 - \epsilon_3)), \quad -|\delta_2| \exp(i(\epsilon_1 + \epsilon_2 - \epsilon_3)),$$

finding that  $\delta_2 |\delta_1| = -\delta_1 |\delta_2|$ . □

Theorem 4 does not consider the cases  $A$  is a matrix of size  $n < 6$ . However, if  $n = 3$ ,  $A_{11}$  collapses to a number and the thesis does not make any sense, also because hypothesis (b) is not true anymore. If  $n = 4$ , we do not have  $\delta_1^{(1)}$  and  $\delta_2^{(1)}$  because  $A_{11}$  is  $2 \times 2$  and then coincides with  $\hat{A}_{22}$ . If  $n = 5$  the structure of (22) simplifies,  $\hat{A}_{11}$  is just a number and  $|\delta_1^{(1)}| = |\delta_2^{(1)}|$ .

Note that if  $A$  is almost normal, and hence Theorem 4 can be applied, the matrix  $A_{11}$  is not, in general almost normal because matrix  $\hat{C}$  such that  $A_{11}^H A_{11} - A_{11} A_{11}^H = \hat{C} A_{11} - A_{11} \hat{C}$  might not exist. However if  $\omega_1^2 a_{12} - \zeta_1^2 a_{21} \neq 0$  then  $A_{11}$  is almost normal as well because of Theorem 3.

Summarizing, for any  $n \times n$  almost normal, not normal matrix  $A$ , with  $n > 4$ , partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{11} \in \mathbb{C}^{(n-2) \times (n-2)}, \quad A_{22} \in \mathbb{C}^{2 \times 2},$$

then either  $A_{11}$  is normal or we can unitarily reduce  $A$  to the following structure

$$A = \left[ \begin{array}{cc|cc|cc} & & & & O & & O \\ & & & & \hline & & & & \omega_1 \delta_1^{(1)} & \zeta_1 \delta_1^{(1)} & \\ & & & & \omega_1 \delta_2^{(1)} & \zeta_1 \delta_2^{(1)} & \\ \hline O & & \zeta_1 \delta_2^{(1)} & \zeta_1 \delta_1^{(1)} & & \hat{A}_{22} & \omega \delta_1 & \zeta \delta_1 \\ & & \omega_1 \delta_2^{(1)} & \omega_1 \delta_1^{(1)} & & & \omega \delta_2 & \zeta \delta_2 \\ \hline & & O & & \zeta \delta_2 & \zeta \delta_1 & & A_{22} \\ & & & & \omega \delta_2 & \omega \delta_1 & & \end{array} \right] \quad (31)$$

i.e., the blocks in position  $(1, 2)$  and  $(2, 1)$  of  $A_{11}$  have the same structure of the more external blocks and the entries of the internal blocks are related to parameters appearing in the more external blocks.

Previous theorem does not account for the case  $|\delta_1| = |\delta_2|$ . The case  $A_{11}$  is normal was however considered in Corollary 1. Next theorem gives additional necessary conditions on the parameters and on the diagonal form of  $A_{11}$ .

**Theorem 5.** *Let  $A$  be an  $n \times n$  complex matrix, with  $n > 4$ . Partition  $A$  as follows*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_{11} \in \mathbb{C}^{(n-2) \times (n-2)}.$$

*If*

- (a) Equation (2) holds with  $\alpha \neq 0$ ,
- (b)  $A_{12}$  and  $A_{21}$  are structured as in equation (17), with  $\omega$  and  $\zeta$  real nonnegative such that  $\omega^2 + \zeta^2 = 1$  and  $|\delta_1| = |\delta_2|$ ,
- (c)  $A_{22} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix}$ ,

*then the matrix  $A_{11}$  is normal. Moreover,  $A$  is unitarily reducible to a matrix such that*

- (1)  $\delta_1 = \delta_2 = \delta$ ,
- (2) and either  $\delta = 0$  (meaning that is  $A$  block diagonal), or the matrix  $\omega \delta A_{11}^H - \zeta \bar{\delta} A_{11}$  has as eigenvector  $\mathbf{e}_{n-3} + \mathbf{e}_{n-2}$  corresponding to an eigenvalue

$$\omega (\delta \bar{b}_{11} - \bar{\delta} b_{21}) + \zeta (\delta \bar{b}_{12} - \bar{\delta} b_{11}).$$

PROOF. For hypotheses (a), (b) and (c) we can apply Corollary 1 concluding that since  $|\delta_1| = |\delta_2|$  then  $A_{11}$  is normal.

Proceeding as in Theorem 2 we can unitarily reduce  $A$  to the form (20), obtaining a  $\delta = \delta_1 = \delta_2$ . In the case  $\delta = 0$  the matrix turns out to be block diagonal with  $A_{11}$  normal.

Assume  $\delta \neq 0$ . Using equality (5), we have

$$A_{11}^H \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \delta & \zeta \delta \\ \omega \delta & \zeta \delta \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \zeta \bar{\delta} & \omega \bar{\delta} \\ \zeta \bar{\delta} & \omega \bar{\delta} \end{bmatrix} A_{22} - A_{11} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \zeta \bar{\delta} & \omega \bar{\delta} \\ \zeta \bar{\delta} & \omega \bar{\delta} \end{bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \delta & \zeta \delta \\ \omega \delta & \zeta \delta \end{bmatrix} A_{22}^H = 0. \quad (32)$$

The thesis follows setting  $\mathbf{u} = \mathbf{e}_{n-3} + \mathbf{e}_{n-2}$  and rewriting the first column of previous equality as

$$(\omega \delta A_{11}^H - \zeta \bar{\delta} A_{11}) \mathbf{u} = (\omega (\delta \bar{b}_{11} - \bar{\delta} b_{21}) + \zeta (\delta \bar{b}_{12} - \bar{\delta} b_{11})) \mathbf{u}.$$

Observing that the second column of (32) gives the same relation above, we can conclude that  $\mathbf{e}_{n-3} + \mathbf{e}_{n-2}$  is an eigenvector of the matrix  $(\omega \delta A_{11}^H - \zeta \bar{\delta} A_{11})$  corresponding to the eigenvalue  $\omega (\delta \bar{b}_{11} - \bar{\delta} b_{21}) + \zeta (\delta \bar{b}_{12} - \bar{\delta} b_{11})$ . □

Summarizing, if the block  $A_{11}$  is normal we can have two situations.  $A$  is block diagonal, that is

$$A = \left[ \begin{array}{c|c} A_{11} & O \\ \hline O & A_{22} \end{array} \right], \text{ and moreover } A_{11} \text{ is normal}$$

otherwise, if  $A_{12}$  and  $A_{21}$  are not null,  $A$  can be unitarily reduced to the following structure

$$A = \left[ \begin{array}{cc|cc} & & & & O \\ & & & & \hline & & & & \omega \delta & \zeta \delta \\ & & & & \omega \delta & \zeta \delta \\ \hline O & & \zeta \delta & \zeta \delta & & \\ & & \omega \delta & \omega \delta & & A_{22} \end{array} \right].$$

Theorem 5 concludes the proof about the structure of almost normal matrices, showing that we can exploit a  $2 \times 2$  block tridiagonal structure of any almost normal matrix until we eventually meet a normal leading principal block. In that case, we stop.

### 3. Sufficient conditions

In this section we show how we can construct an almost normal matrix bordering any  $(n-2) \times (n-2)$  almost normal matrix with suitable vectors. The construction can be carried on inductively starting from a normal matrix, or from any  $1 \times 1$  or  $2 \times 2$  matrix.

**Theorem 6.** *Let  $A_{11}$  be an  $(n-2) \times (n-2)$  matrix such that either  $A_{11}$  is normal or is not normal but (2) holds (for  $A_{11}$ ). Assume  $A_{11}$  has the following structure*

$$A_{11} = \left[ \begin{array}{c|cc} & & & \\ & \hat{A}_{11} & & \\ & & & \\ \hline & & & O \\ O & \zeta_1 \delta_2^{(1)} & \zeta_1 \delta_1^{(1)} & \begin{array}{cc} \omega_1 \delta_1^{(1)} & \zeta_1 \delta_1^{(1)} \\ \omega_1 \delta_2^{(1)} & \zeta_1 \delta_2^{(1)} \end{array} \\ & \omega_1 \delta_2^{(1)} & \omega_1 \delta_1^{(1)} & \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{11} \end{array} \end{array} \right],$$

with  $|\delta_1^{(1)}| \geq |\delta_2^{(1)}|$ , and  $\omega^{(1)} > \zeta^{(1)}$ . Then we can border  $A_{11}$  in infinitely many ways to obtain an  $n \times n$  matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \text{where } A_{22} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix} \in \mathbb{C}^{2 \times 2},$$

such that (2) holds for  $A$  with  $\alpha \neq 0$ , and  $A$  has the structure described in (31). Moreover we can always choose the new parameters  $\delta_1, \delta_2, \omega, \zeta$  and the  $b_{ij}$ 's so that  $A$  is almost normal.

**PROOF.** Assume first that  $A_{11}$  is not normal and the eigenvalues of the difference matrix  $A_{11}^H A_{11} - A_{11} A_{11}^H$  are  $\beta, -\beta$ , with  $\beta > 0$ . We have to prove that if  $A_{12}$ ,  $A_{21}$  and  $A_{22}$  are constructed as in Theorem 4 then the matrix  $A$  has the desired properties. Let  $\omega$  and  $\zeta$  be non negative real numbers such that  $\omega^2 + \zeta^2 = 1$ , with  $\omega \neq \zeta$ . Set

$$A_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \omega \delta_1 & \zeta \delta_1 \\ \omega \delta_2 & \zeta \delta_2 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} \mathbf{0}^H & \zeta \delta_2 & \zeta \delta_1 \\ \mathbf{0}^H & \omega \delta_2 & \omega \delta_1 \end{bmatrix},$$

where  $\delta_1$  and  $\delta_2$  are such that (29) and (30) hold. This is achieved by setting  $\delta_1 = \mu \omega_1$  and  $\delta_2 = -\mu \zeta_1$ , for any complex constant  $\mu$  such that  $|\mu| = \sqrt{\beta}/(\omega_1^2 - \zeta_1^2)$ .

The submatrix  $A_{22}$  is determined by imposing the conditions (24)-(27) where the unknowns this time are the  $b_{ij}$ 's, since

$$\hat{A}_{22} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}$$

is given. Multiplying  $\zeta$  times equation (24) and  $\omega$  times equation (25), and then summing the two equations, we get

$$(\omega^2 - \zeta^2) (a_{12} \bar{\delta}_1 + a_{11} \bar{\delta}_2) = (b_{12} - b_{21}) \omega \zeta \bar{\delta}_2 + (\zeta^2 \bar{b}_{12} - \omega^2 \bar{b}_{21}) \delta_1 + b_{11} (\omega^2 - \zeta^2) \bar{\delta}_2.$$

Combining  $\omega$  times equation (24) and  $\zeta$  times equation (25) we have

$$(\omega^2 - \zeta^2) (\bar{a}_{11} \delta_1 + \bar{a}_{21} \delta_2) = (\bar{b}_{12} - \bar{b}_{21}) \omega \zeta \delta_1 + (\zeta^2 b_{12} - \omega^2 b_{21}) \bar{\delta}_2 + \bar{b}_{11} (\omega^2 - \zeta^2) \delta_1.$$

Performing the following changes of variables  $x_1 = b_{11}$ ,  $x_2 = \zeta^2 \bar{b}_{12} - \omega^2 \bar{b}_{21}$  and  $x_3 = b_{12} - b_{21}$ . We get the following linear system

$$\begin{bmatrix} (\omega^2 - \zeta^2) \bar{\delta}_2 & \delta_1 & \omega \zeta \bar{\delta}_2 \\ (\omega^2 - \zeta^2) \bar{\delta}_1 & \delta_2 & \omega \zeta \bar{\delta}_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (\omega^2 - \zeta^2) \begin{bmatrix} a_{12} \bar{\delta}_1 + a_{11} \bar{\delta}_2 \\ a_{21} \bar{\delta}_2 + a_{11} \bar{\delta}_1 \end{bmatrix}.$$

The above linear system is consistent and has infinitely many solutions of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k \begin{bmatrix} \omega \zeta \\ 0 \\ \zeta^2 - \omega^2 \end{bmatrix} + \frac{1}{|\delta_1|^2 - |\delta_2|^2} \begin{bmatrix} a_{11}(|\delta_1|^2 - |\delta_2|^2) + \delta_1 \bar{\delta}_2 a_{21} - \bar{\delta}_1 \delta_2 a_{12} \\ (\omega^2 - \zeta^2)(\delta_1^2 a_{12} - \bar{\delta}_2^2 a_{21}) \\ 0 \end{bmatrix}. \quad (33)$$

Then, taking  $b_{11} = x_1$ , the values  $b_{12}$  and  $b_{21}$  are the solutions of the linear system

$$\begin{bmatrix} \zeta^2 & -\omega^2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{21} \end{bmatrix} = \begin{bmatrix} \bar{x}_2 \\ x_3 \end{bmatrix},$$

obtaining

$$b_{12} = \frac{\bar{x}_2}{\zeta^2 - \omega^2} - k \omega^2, \quad b_{21} = \frac{\bar{x}_2}{\zeta^2 - \omega^2} - k \zeta^2. \quad (34)$$

To complete the proof we need to show that we can always choose  $k$  in such a way  $\omega^2 b_{12} - \zeta^2 b_{21} \neq 0$ . In Theorem 3 we showed that when the above quantity is different from zero, there exists a rank-one matrix  $C$  for which  $A$  is almost normal. Note that  $\omega^2 b_{12} - \zeta^2 b_{21} = x_3 - \bar{x}_2$ . Hence

$$\omega^2 b_{12} - \zeta^2 b_{21} = k(\zeta^2 - \omega^2) - \frac{1}{|\delta_1|^2 - |\delta_2|^2} (\omega^2 - \zeta^2)(\delta_1^2 \bar{a}_{12} - \delta_2^2 \bar{a}_{21}).$$

Since  $x_2$  does not depend on  $k$ ,  $|\delta_1| \neq |\delta_2|$  and  $\zeta^2 - \omega^2 \neq 0$ , we achieve our goal by choosing any  $k$  such that

$$k \neq \frac{-\delta_1^2 \bar{a}_{12} + \delta_2^2 \bar{a}_{21}}{|\delta_1|^2 - |\delta_2|^2}.$$

Assume now  $A_{11}$  is normal. In accordance with Theorem 5, either we can choose  $\delta = 0$ , or the parameters  $\omega, \zeta$  and  $\delta$  should be chosen in such a way matrix  $\omega \delta A_{11}^H - \zeta \bar{\delta} A_{11}$  has eigenvector  $\mathbf{e}_{n-3} + \mathbf{e}_{n-2}$ . If we decide for  $\delta = 0$ , then we set  $A_{12} = O$  and  $A_{21} = O$ , and  $A$  is a block diagonal matrix  $A = \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}$ , with  $A_{22}$  any  $2 \times 2$  matrix with  $b_{11} = b_{22}$ . With a unitary transformation acting on the last two rows and columns of  $A$  we can always obtain a matrix satisfying (2). With the particular choice  $b_{22} = b_{11}$  and  $b_{12} \neq b_{21}$  we get an almost normal matrix and

$$C = \kappa \left[ \begin{array}{c|cc} O & & O \\ \hline & 1 & -1 \\ O & -1 & 1 \end{array} \right],$$

where  $\kappa(b_{12} - b_{21}) = |b_{21}|^2 - |b_{12}|^2$ .

In the case  $A_{11}$  is normal and  $\mathbf{u} = \mathbf{e}_{n-3} + \mathbf{e}_{n-2}$  is eigenvector of  $\omega \delta A_{11}^H - \zeta \bar{\delta} A_{11}$  we have infinitely many different choices for the entries  $b_{ij}$  of  $A_{22}$ . Let  $A_{11} = QDQ^H$ , with  $D$  diagonal and  $Q$  the unitary matrix whose columns are the eigenvectors. Denote by  $\mu_i$  the eigenvalues of  $A$ , which are the diagonal entries of  $D$ , and let  $\mu$  be the eigenvalue of  $A_{11}$  such that

$$\omega \delta \bar{\mu} - \zeta \bar{\delta} \mu = \omega(\delta \bar{b}_{11} - \bar{\delta} b_{21}) + \zeta(\delta \bar{b}_{12} - \bar{\delta} b_{11}), \quad (35)$$

knowing in advance that we need  $b_{11} = b_{22}$ . For the necessary conditions proven in Theorem 5, we have to find suitable  $b_{ij}$ 's such that equality (35) holds.

Reasoning similarly to what we did for the non normal case, we obtain the following linear equation

$$(\omega^2 - \zeta^2) \bar{\delta} \mu = (\omega^2 - \zeta^2) \bar{\delta} b_{11} + \delta(\zeta^2 \bar{b}_{12} - \omega^2 \bar{b}_{21}) + \omega \zeta \bar{\delta}(b_{12} - b_{21}). \quad (36)$$

With the same change of variables  $x_1 = b_{11}$ ,  $x_2 = \zeta^2 \bar{b}_{12} - \omega^2 \bar{b}_{21}$  and  $x_3 = b_{12} - b_{21}$ , we have that if  $\omega \neq \zeta$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} \omega \zeta \\ 0 \\ (\zeta^2 - \omega^2) \end{bmatrix} + k_2 \begin{bmatrix} \zeta^2 - \omega^2 \\ \bar{\delta}/\delta \\ 0 \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix},$$

where  $k_1$  and  $k_2$  are free complex parameters. In the case  $\omega = \zeta$  the above equation (36) simplifies and we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ \bar{\delta}/\delta \\ -2 \end{bmatrix},$$

where  $k_1$  is a free parameter,  $|k_2|$  is free but the angle  $\alpha_k$  of  $k_2$  is such that  $\alpha_k = \alpha_\delta \pm \pi/2$ . where  $\alpha_\delta$  is the angle of  $\delta$ .

Thus we have infinitely many solutions in the above systems depending on  $k_1$  and  $k_2$ , and hence infinitely many choices of the  $b_{ij}$ 's. Moreover, similarly to the not normal case, we can always choose  $k_1$  and  $k_2$  in such a way there exists a rank-one matrix  $C$  such that  $A^H A - A A^H = C A - A C$ . For example, possible values guaranteeing  $\omega^2 b_{12} - \zeta^2 b_{21} \neq 0$  are  $k_1 = 0$  and  $k_2 \neq 0$  (in the case  $\omega = \zeta$ ,  $k_2$  should still be such that  $\alpha_k = \alpha_\delta \pm \pi/2$ ).  $\square$

**Remark 3.** *If  $A_{11}$  is not normal then, for any two different values of  $k$ , the corresponding almost normal matrices differ only for the tailing  $2 \times 2$  principal minor  $A_{22}$ , and the difference of the two almost normal matrices is a multiple of matrix  $C$  itself.*

*In the case the principal minor  $A_{11}$  is normal, given an almost normal matrix obtained with parameters  $k_1$  and  $k_2$  we can choose a second almost normal matrix  $B$  differing from  $A$  only for the tailing  $2 \times 2$  principal block with parameters  $k'_1$  and the same  $k_2$  such that  $A - B = (k_1 - k'_1) C$ .*

Previous theorem gives the conditions for the recursive construction of an almost normal matrix. We now have to describe the structure of the basic block to start the recursion. We have two cases.  $A_{11}$  is normal, or it is a  $2 \times 2$  non normal matrix. If  $A_{11}$  is an odd-size normal matrix - eventually of size one - we can border it as described in Theorem 6 obtaining an odd-size almost normal matrix. If  $A_{11}$  has even size, we end up with an even-size almost normal matrix. Everything works well starting with a  $2 \times 2$  matrix as proven in next theorem.

**Theorem 7.** *Any  $2 \times 2$  matrix is almost normal.*

PROOF. We can assume  $A$  is in the Schur form, that is  $A$  is upper triangular,

$$A = \begin{bmatrix} \lambda_1 & \eta \\ 0 & \lambda_2 \end{bmatrix}.$$

We want to show that there exist a rank-one matrix  $C$  and a polynomial  $p(x) = ax + b$  with degree at most one, such that  $A^H - C = p(A)$ . Let  $\varepsilon, \varepsilon \neq 0$  such that  $\lambda_1 + \varepsilon \neq \lambda_2$ . Then there exist  $a$  and  $b$ , uniquely determined, such that

$$\begin{aligned} \bar{\lambda}_1 &= a\lambda_1 + b + \varepsilon a \\ \bar{\lambda}_2 &= a\lambda_2 + b - \frac{1}{\varepsilon}|\eta|^2. \end{aligned}$$

In fact

$$\det\left(\begin{bmatrix} \lambda_1 + \varepsilon & 1 \\ \lambda_2 & 1 \end{bmatrix}\right) = \lambda_1 + \varepsilon - \lambda_2 \neq 0.$$

Once we found  $a$  and  $b$  solutions of the linear system, we compute  $C = A^H - p(A)$  which is given by

$$C = \begin{bmatrix} \bar{\lambda}_1 - (a\lambda_1 + b) & -a\eta \\ \bar{\eta} & \bar{\lambda}_2 - (a\lambda_2 + b) \end{bmatrix},$$

whose determinant is zero, so that  $C$  has rank at most one.  $\square$

#### 4. Relationship with normal matrices

We already stressed that when adding a rank-one perturbation to a normal matrix we may end up with a matrix which is *not* almost normal because  $A^H A - A A^H$  can have rank up to four. However, next theorem shows that any almost normal matrix can be viewed as a *particular* rank-one correction of a *particular* normal matrix.

**Theorem 8.** *Let  $A$  be an almost normal matrix. Then there exists a normal matrix  $N$  and a constant  $h$  such that  $A = N + h C$ , where  $C$  is the same rank-one matrix such that  $A^H A - A A^H = C A - A C$ .*

PROOF. Let  $A$  be an almost normal matrix already in the condensed form described in (31). In this proof, for the sake of simplicity, we assume  $A_{11}$  is not normal; the case  $A_{11}$  normal is similar. In particular, as observed in Remark 3, we can choose infinitely many  $A^{(k)}$  that differ from  $A$  only for the lower right  $2 \times 2$  block, taking different values of  $k$  in the solution of system (33). Among the infinite possible  $A^{(k)}$  we prove that we can always find a particular  $\tilde{k}$  such that the matrix  $A^{(\tilde{k})}$  is normal.

In fact, an almost normal matrix, which is also normal should satisfy equations (4), (5) and (6) with  $\alpha = 0$ . Let  $A_{22}^{(k)} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix}$  the lower  $2 \times 2$  diagonal block. From (6), setting  $\alpha = 0$ , we have

$$|b_{21}|^2 - |b_{12}|^2 = (|\delta_1|^2 + |\delta_2|^2)(\zeta^2 - \omega^2).$$

Substituting in the previous equality the relations (34), where the dependence of  $b_{12}$  and  $b_{21}$  from  $k$  is explicit, we get

$$|k|^2 - k \frac{x_2}{(\zeta^2 - \omega^2)} - \bar{k} \frac{\bar{x}_2}{(\zeta^2 - \omega^2)} - (|\delta_1|^2 + |\delta_2|^2) = 0,$$

where  $x_2$  is independent of  $k$ . Setting  $c = x_2/(\zeta^2 - \omega^2)$ , a solution can be obtained expressing  $k$  and  $c$  in polar form, i.e.  $k = |k| \exp(i\theta_k)$ ,  $c = |c| \exp(i\phi)$ . We get

$$|k|^2 - 2|c||k| \cos(\theta_k + \phi) - (|\delta_1|^2 + |\delta_2|^2) = 0.$$

For any value of  $\theta_k$ , the previous equation in  $|k|$  has always two real roots, and at least one of the two is positive. For all these  $k$  we obtain a normal matrix. In particular, by choosing  $\theta_{\tilde{k}} = \pi/2 - \phi$ , and  $|\tilde{k}| = \sqrt{|\delta_1|^2 + |\delta_2|^2}$ , we have that the corresponding  $A^{(\tilde{k})}$  is normal and is a rank-one perturbation of  $A$ . With the choice of  $h = k - \tilde{k}$ , we get the thesis.  $\square$

#### 5. Conclusions

In this paper we studied almost normal matrices, that is, the class of matrices for which there exists a rank-one perturbation  $C$  such that  $A^H A - A A^H = C A - A C$ , and we showed how to represent them in a condensed form by means of unitary transformations. The condensed form allows to represent almost normal matrices with  $O(n)$  parameters compared with the  $O(n^2)$  representation of the general Hessenberg form.

Our approach is essentially theoretical and is related to previous condensed representations of normal matrices [7] and low-rank perturbations of normal matrices [12]. With a recursive argument, we showed how to explicitly construct almost normal matrices by bordering, with suitable entries, smaller matrices in the same class. We also showed how to determine whether a matrix  $A$ , such that  $A^H A - A A^H$  has rank two, is almost normal by looking at the structure of the two outer rows and columns and applying Theorems 1 and 3. The membership to the class can be verified with a finite procedure.

In [7] and in [3, 12, 20] the reduction to other condensed forms is obtained using generalized Krylov subspace methods. On the contrary, our reduction is obtained applying unitary transformations that diagonalize rank-two matrices, which is a finite procedure, and phase transformations, which guarantee that some of the parameters are real.

Another interesting problem is the design of  $QR$ -like algorithms that take advantage of the block tridiagonal structure, similarly to what has been done for other rank-structured matrices [4].

As emphasized in the introduction, a rank-one perturbation of a normal matrix is not, in general, in our class. We plan to study rank-one perturbations of normal matrices to see if they have some structure that can lead to computational advantages.

## Appendix

**Lemma 9.** *Let  $\mathbf{x}, \mathbf{y}, \mathbf{u}$  and  $\mathbf{v}$  be  $n$ -vectors such that*

$$\mathbf{x}\mathbf{y}^H = \mathbf{u}\mathbf{v}^H.$$

*If  $\mathbf{x} \neq 0$ , then there exists a constant  $\mu$ , such that  $\mathbf{y}^H = \mu \mathbf{v}^H$ .*

PROOF. By multiplying both sides of the equality by  $\mathbf{x}^H$  on the left. We get

$$(\mathbf{x}^H \mathbf{x})\mathbf{y}^H = (\mathbf{x}^H \mathbf{u})\mathbf{v}^H,$$

hence setting  $\mu = (\mathbf{x}^H \mathbf{u})/\|\mathbf{x}\|^2$  we get  $\mathbf{y}^H = \mu \mathbf{v}^H$ . □

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