

Decay and Scattering of Small Solutions of Pure Power NLS in \mathbb{R} with $p > 3$ and with a Potential

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Abstract

We prove decay and scattering of solutions of the nonlinear Schrödinger equation (NLS) in \mathbb{R} with pure power nonlinearity with exponent $3 < p < 5$ when the initial datum is small in Σ (bounded energy and variance) in the presence of a linear inhomogeneity represented by a linear potential that is a real-valued Schwarz function. We assume absence of discrete modes. The proof is analogous to the one for the translation-invariant equation. In particular, we find appropriate operators commuting with the linearization. © 2014 Wiley Periodicals, Inc.

1 Introduction

We consider

$$(1.1) \quad (i\partial_t + \Delta_V)u + \lambda|u|^{p-1}u = 0 \quad \text{for } t \geq 1, x \in \mathbb{R}, \text{ and } u(1) = u_0$$

with $\Delta_V := \Delta - V(x)$ and $\Delta := \partial_x^2$ and $\lambda \in \mathbb{R} \setminus \{0\}$. In this paper we focus on exponents $3 < p < 5$. V is a real-valued Schwartz function and Δ_V is taken without eigenvalues.

It is well known that for $2 \leq p < 5$ the initial value problem in (1.1) is globally well posed in $H^1(\mathbb{R})$. Our goal is to study the asymptotic behavior of solutions with initial data $u(1) = u_0$ of size ϵ in a suitable Sobolev norm, with ϵ sufficiently small. It is natural to ask whether such solutions are asymptotically free and satisfy

$$(1.2) \quad \|u(t)\|_{L^\infty(\mathbb{R})} \leq C_0 t^{-\frac{1}{2}} \epsilon,$$

that is, whether they have the decay rate of the solution to the linear Schrödinger equation.

We recall some of the results for $V = 0$. For spatial dimension d , McKean and Shatah [14] answered positively to our question for $1 + \frac{2}{d} < p < 1 + \frac{4}{d}$.

The case $p \geq 1 + \frac{4}{d}$ and $p < 1 + \frac{4}{d-1}$ for $d \geq 3$ was answered positively by W. Strauss [17], who proved that the zero solution is the only asymptotically free solution when $1 < p \leq 1 + \frac{2}{d}$ for $d \geq 2$ and when $1 < p \leq 2$ for $d = 1$ [16]. This result was extended to the case $1 < p \leq 3$ and $d = 1$ by J. Barab [1], using an idea of R. Glassey [11].

The exponent $p = 1 + \frac{2}{d}$ is critical and particularly interesting. The existence and the form of the scattering operator was obtained by Ozawa [15] for $d = 1$ and by Ginibre and Ozawa [10] for $d \geq 2$. The completeness of the scattering operator and the decay estimate were obtained by Hayashi and Naumkin [12]. Completeness of the scattering operator and decay estimate for all solutions, not only for small ones, for $d = 1$ and $\lambda < 0$ were obtained by Deift and Zhou [7]. See also [4, 5] for earlier references and [8] for a simpler proof. The result was extended to perturbations of the defocusing cubic NLS for $d = 1$ in [6]. For the focusing cubic NLS for $d = 1$, the pure radiation case, along with other cases reducible to the pure radiation one by means of Darboux transformations, was treated in [2], proceeding along the lines of [7].

Our goal in the present paper is to extend the result of McKean and Shatah [14] to the case $V \neq 0$ and $d = 1$, which to our knowledge is open. For V we assume the following hypothesis, where we refer to Section 4 for the definition of the transmission coefficient $T(\tau)$.

- (H) The potential V is a real-valued Schwartz function such that for the spectrum we have $\sigma(\Delta_V) = (-\infty, 0]$. Furthermore, V is generic; that is, the transmission coefficient $T(\tau)$ satisfies $T(0) = 0$.

We denote by Σ_s the Hilbert space defined as the closure of $C_0^\infty(\mathbb{R})$ functions with respect to the norm

$$\|u\|_{\Sigma_s}^2 := \|u\|_{H^s(\mathbb{R})}^2 + \||x|^s u\|_{L^2(\mathbb{R})}^2.$$

Our main result is the following:

THEOREM 1.1. *Assume that V satisfies (H), $s > \frac{1}{2}$, and $p > 3$. Then there exist constants $\epsilon_0 > 0$ and $C_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$ and $\|u(1)\|_{\Sigma_s} \leq \epsilon$, the solution to (1.1) satisfies the decay inequality (1.2) for $t \geq 1$. Furthermore, there exists $u_+ \in L^2(\mathbb{R})$ such that*

$$(1.3) \quad \lim_{t \rightarrow +\infty} \|u(t) - e^{it\Delta} u_+\|_{L^2(\mathbb{R})} = 0.$$

The hypothesis $\sigma(\Delta_V) = (-\infty, 0]$ is necessary since otherwise for any $s > \frac{1}{2}$ there are periodic solutions $u(t, x) = e^{i\lambda t} \phi_\lambda(x)$ of arbitrarily small Σ_s norm. The interesting case is for $p \in (3, 5)$ since the case $p \geq 5$ follows from [9, 19]. The case $V = 0$ is due to [14].

If $\sigma(\Delta_V) = (-\infty, 0]$, the existence of wave operators intertwining Δ_V and Δ and of Strichartz and dispersive estimates for $e^{it\Delta_V}$ is well known; see [9, 19, 20]. Such estimates are not sufficient to prove Theorem 1.1 even in the case $V = 0$.

The argument in McKean and Shatah [14] is based on the introduction of homogeneous $\dot{\mathcal{H}}^k(t)$ norms, defined by substituting the standard derivative $\frac{\partial}{\partial x_j}$ with operators $J_j(t)$; see Section 2. The authors [14] prove almost invariance of these norms and, by a form of the Sobolev embedding theorem, the dispersion (1.2). Such use of invariant norms goes back to the work on the wave equation by Klainerman; see, for example, [13].

The development of a theory of invariant norms in the case of non-translation-invariant equations such as (1.1) is an important technical problem. Here our main goal is to adapt the framework of [14] for $d = 1$ and to introduce appropriate surrogates $|J_V(t)|^s$ for the operators $|J(t)|^s$; see Section 2.

The operators $|J_V(t)|^s$ are used to define homogeneous spaces $\dot{\mathcal{H}}_V^s(t)$ that are then shown to be almost invariant.

The argument is more complicated than in [14] because of the presence of an additional commutator. But we can show that if Δ_V is generic in the sense of hypothesis (H), then the commutator can be treated by a bootstrap argument.

Another complication is that the $|J_V(t)|^s$ do not enjoy Leibniz-rule-type properties like $|J(t)|^s$, which play a key role in [14]. Nonetheless, we are able to treat $|J_V(t)|^s$ by switching from $|J_V(t)|^s$ to $|J(t)|^s$ by using the Leibniz rule for $|J(t)|^s$, and by going back to $|J_V(t)|^s$.

In the part of the argument on the Leibniz rule, an essential role is played by the observation that $\|\cdot\|_{\dot{\mathcal{H}}_V^s(t)} \approx \|\cdot\|_{\dot{\mathcal{H}}^s(t)}$ with fixed constants independent of t when $0 \leq s < \frac{1}{2}$. The proof of this equivalence is based on Paley-Littlewood decompositions associated to phase spaces both of Δ and Δ_V . We are able to prove this equivalence when the transmission coefficient $T(\tau)$ is such that either $T(0) = 0$ (the generic case) or $T(0) = 1$. Notice incidentally that the inclusion of this nongeneric case at least in this part of the paper is natural, since the fact that $T(0) = 1$ makes Δ_V more similar to Δ than the case when $T(0) = 0$ (recall that $T(0) = 1$ for Δ).

We now introduce some of the notation used later. Inequalities of type $A \lesssim B$ mean the existence of a constant $C > 0$ so that $A \leq CB$. Similarly, $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. The standard scalar product in $L^2 = L^2(\mathbb{R})$ will be denoted by $\langle \cdot, \cdot \rangle_{L^2}$. We use the notation L_x^p to mean $L^p(\mathbb{R})$. $L_t^p(X)$ stands for the L^p norm of functions with values in Banach space X . The homogeneous Sobolev space $\dot{\mathcal{H}}^s(\mathbb{R})$ (respectively, perturbed Sobolev space $\dot{\mathcal{H}}_V^s(\mathbb{R})$) for $s \geq 0$ is defined as the closure of $C_0^\infty(\mathbb{R})$ functions with respect to the norm

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L_x^2} \quad (\text{respectively, } \|(-\Delta + V)^{\frac{s}{2}} f\|_{L_x^2}).$$

These norms are used in two cases: functions depending only on x and functions depending on both t and x .

2 Definition of $|J_V(t)|^s$

In this section we assume $x \in \mathbb{R}^d$ with d a generic dimension, and we consider

$$(\mathbf{i}\partial_t + \Delta)u = 0.$$

Recall that the fundamental solution is given by $e^{\mathbf{i}t\Delta}(x, y) = \frac{\exp\mathbf{i}\frac{(x-y)^2}{4t}}{(4\pi\mathbf{i}t)^{d/2}}$ for $t > 0$. Consider the Fourier transform F and its inverse:

$$\begin{aligned} Ff(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathbf{i}x \cdot y} f(y) dy, \\ F^{-1}f(x) &= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathbf{i}x \cdot y} f(y) dy. \end{aligned} \tag{2.1}$$

We also introduce the dilation operator $D(t)\psi(x) = (2\mathbf{i}t)^{-d/2}\psi(x2t)$ and the multiplier operator $M(t)\psi(x) = \exp(\mathbf{i}x^2/4t)\psi(x)$. Then we have the following well-known formula:

$$e^{\mathbf{i}t\Delta} = M(t)D(t)F^{-1}M(t).$$

Let $g(x)$ be a function and denote by $g(q)$ the multiplier operator $g(q)\psi(x) := g(x)\psi(x)$. We set $p_j := \mathbf{i}\partial_{x_j}$ and $p = (p_1, \dots, p_d)$. More generally, set $g(p) := F^{-1}g(q)F$. The following identity is well-known:

$$e^{\mathbf{i}t\Delta}g(q)e^{-\mathbf{i}t\Delta} = M(t)g(2tp)M(-t) \tag{2.2}$$

for any $g(x)$. With an abuse of notation we will denote the operator $g(q)$ by $g(x)$. Notice that we have

$$\begin{aligned} [\mathbf{i}\partial_t + \Delta, e^{\mathbf{i}t\Delta}g(x)e^{-\mathbf{i}t\Delta}] &= \\ e^{\mathbf{i}t\Delta}[-\Delta, g(x)]e^{-\mathbf{i}t\Delta} + e^{\mathbf{i}t\Delta}[\Delta, g(x)]e^{-\mathbf{i}t\Delta} &= 0, \end{aligned}$$

so obviously the same commutation rule holds for the right-hand side of (2.2). In particular, for $g(x) = x_j$ we get on the right-hand side of (2.2) the operators

$$J_j = 2t\mathbf{i} \exp\left(\frac{\mathbf{i}x^2}{4t}\right)\partial_{x_j} \exp\left(-\frac{\mathbf{i}x^2}{4t}\right) = 2t\mathbf{i}\partial_{x_j} + x_j,$$

and we have

$$[\mathbf{i}\partial_t + \Delta, J_j] = 0.$$

We introduce for any $s \geq 0$ the following two operators:

$$|J(t)|^s := M(t)(-t^2\Delta)^{\frac{s}{2}}M(-t), \tag{2.3}$$

$$|J_V(t)|^s := M(t)(-t^2\Delta_V)^{\frac{s}{2}}M(-t). \tag{2.4}$$

3 Commutative Properties of $|J_V(t)|^s$

We start the section by establishing some useful commutator relations. In this section $x \in \mathbb{R}^d$ with d a generic dimension and $M(t) = e^{i|x|^2/4t}$.

LEMMA 3.1. *We have the following identities:*

$$[\mathbf{i}\partial_t, M(t)] = \frac{x^2}{4t^2}M(t), \quad [\mathbf{i}\partial_t, M(-t)] = -\frac{x^2}{4t^2}M(-t).$$

PROOF. A simple calculation gives

$$\mathbf{i}\partial_t M(t)f - M(t)\mathbf{i}\partial_t f = (\mathbf{i}\partial_t M(t))f = \frac{x^2}{4t^2}M(t)f.$$

The second relation can be verified similarly. □

Furthermore, we shall prove the following:

LEMMA 3.2. *We have*

$$\begin{aligned} [\Delta, M(t)] &= M(t) \left(\frac{\mathbf{i}d}{2t} - \frac{x^2}{4t^2} + \frac{\mathbf{i}x \cdot \nabla}{t} \right), \\ [\Delta, M(-t)] &= M(-t) \left(-\frac{\mathbf{i}d}{2t} - \frac{x^2}{4t^2} - \frac{\mathbf{i}x \cdot \nabla}{t} \right). \end{aligned}$$

PROOF. For the first relation we have

$$\begin{aligned} f &= f \Delta M(t) + 2\nabla M(t) \cdot \nabla f \\ &= M(t) \frac{\mathbf{i}d}{2t} f - M(t) \frac{x^2}{4t^2} f + M(t) \frac{\mathbf{i}x \cdot \nabla f}{t}. \end{aligned}$$

The second relation follows by taking complex conjugates. □

From Lemma 3.1 and Lemma 3.2 we get the following

LEMMA 3.3. *The following commutator relations hold:*

$$\begin{aligned} [\mathbf{i}\partial_t + \Delta, M(t)] &= M(t) \left(\frac{\mathbf{i}d}{2t} + \frac{\mathbf{i}x \cdot \nabla}{t} \right), \\ [\mathbf{i}\partial_t + \Delta, M(-t)] &= M(-t) \left(-\frac{\mathbf{i}d}{2t} - \frac{x^2}{2t^2} - \frac{\mathbf{i}x \cdot \nabla}{t} \right). \end{aligned}$$

PROOF. We shall check only the first relation, which follows directly from the above lemmas and

$$[\mathbf{i}\partial_t + \Delta, M(t)] = [\mathbf{i}\partial_t, M(t)] + [\Delta, M(t)]. \quad \square$$

LEMMA 3.4. *We have*

$$(3.1) \quad [\mathbf{i}\partial_t + \Delta_V, (-t^2 \Delta_V)^{\frac{s}{2}}] = \frac{\mathbf{i}s}{t} (-t^2 \Delta_V)^{\frac{s}{2}}.$$

PROOF. To prove (3.1) we shall use the fact that $(-\Delta_V)^{s/2}$ and Δ_V commute. Thus, we have

$$\begin{aligned} [\mathbf{i}\partial_t + \Delta_V, (-t^2 \Delta_V)^{\frac{s}{2}}]f &= [\mathbf{i}\partial_t, (-t^2 \Delta_V)^{\frac{s}{2}}]f + [\Delta_V, (-t^2 \Delta_V)^{\frac{s}{2}}]f \\ &= \mathbf{i}\partial_t((-t^2 \Delta_V)^{\frac{s}{2}})f = \frac{\mathbf{i}s}{t}(-t^2 \Delta_V)^{\frac{s}{2}}f. \quad \square \end{aligned}$$

Now we are ready to establish the main commutative property of the operator $|J_V(t)|^s$ with $s \geq 0$ defined in (2.4).

PROPOSITION 3.5. *We have the relation*

$$(3.2) \quad [\mathbf{i}\partial_t + \Delta_V, |J_V(t)|^s] = \mathbf{i}t^{s-1}M(t)A(s)M(-t)$$

where

$$A(s) := s(-\Delta_V)^{\frac{s}{2}} + [x \cdot \nabla, (-\Delta_V)^{\frac{s}{2}}].$$

PROOF. The proof relies on Lemmas 3.1 through 3.4 and the following commutator equalities:

$$[AB, C] = A[B, C] + [A, C]B, \quad [A, BC] = [A, B]C + B[A, C].$$

Indeed, we have

$$\begin{aligned} [\mathbf{i}\partial_t + \Delta_V, |J_V(t)|^s] &= [\mathbf{i}\partial_t + \Delta_V, M(t)(-t^2 \Delta_V)^{\frac{s}{2}}M(-t)] \\ &= [\mathbf{i}\partial_t + \Delta_V, M(t)](-t^2 \Delta_V)^{\frac{s}{2}}M(-t) \\ &\quad + M(t)[\mathbf{i}\partial_t + \Delta_V, (-t^2 \Delta_V)^{\frac{s}{2}}]M(-t) \\ &= \frac{\mathbf{i}d}{2t}|J_V(t)|^s + \frac{\mathbf{i}}{t}M(t)x \cdot \nabla(-t^2 \Delta_V)^{\frac{s}{2}}M(-t) \\ &\quad + M(t)[\mathbf{i}\partial_t + \Delta_V, (-t^2 \Delta_V)^{\frac{s}{2}}]M(-t) \\ &\quad + M(t)(-t^2 \Delta_V)^{\frac{s}{2}}[\mathbf{i}\partial_t + \Delta_V, M(-t)] \\ &= \frac{\mathbf{i}d}{2t}|J_V(t)|^s + \frac{\mathbf{i}}{t}M(t)x \cdot \nabla(-t^2 \Delta_V)^{\frac{s}{2}}M(-t) \\ &\quad + \frac{\mathbf{i}s}{t}|J_V(t)|^s + M(t)(-t^2 \Delta_V)^{\frac{s}{2}}M(-t) \left(-\frac{\mathbf{i}d}{2t} - \frac{x^2}{2t^2} - \frac{\mathbf{i}x \cdot \nabla}{t} \right) \\ &= \frac{\mathbf{i}s}{t}|J_V(t)|^s + \frac{\mathbf{i}}{t}M(t)x \cdot \nabla(-t^2 \Delta_V)^{\frac{s}{2}}M(-t) \\ &\quad - \frac{\mathbf{i}}{t}M(t)(-t^2 \Delta_V)^{\frac{s}{2}}M(-t)x \cdot \nabla - M(t)(-t^2 \Delta_V)^{\frac{s}{2}}\frac{x^2}{2t^2}M(-t) \\ &= \frac{\mathbf{i}s}{t}|J_V(t)|^s + \frac{\mathbf{i}}{t}M(t)[x \cdot \nabla, (-t^2 \Delta_V)^{\frac{s}{2}}]M(-t) \\ &\quad - M(t)(-t^2 \Delta_V)^{\frac{s}{2}}\frac{x^2}{2t^2}M(-t). \end{aligned}$$

Note that

$$\begin{aligned}
 & [x \cdot \nabla, (-t^2 \Delta_V)^{\frac{s}{2}} M(-t)] \\
 &= [x \cdot \nabla, (-t^2 \Delta_V)^{\frac{s}{2}}] M(-t) + (-t^2 \Delta_V)^{\frac{s}{2}} [x \cdot \nabla, M(-t)] \\
 (3.3) \quad &= [x \cdot \nabla, (-t^2 \Delta_V)^{\frac{s}{2}}] M(-t) - (-t^2 \Delta_V)^{\frac{s}{2}} \frac{\mathbf{i}x^2}{2t} M(-t),
 \end{aligned}$$

and hence we get

$$[\mathbf{i}\partial_t + \Delta_V, |J_V(t)|^s] = \frac{\mathbf{i}s}{t} |J_V(t)|^s + \frac{\mathbf{i}}{t} M(t) [x \cdot \nabla, (-t^2 \Delta_V)^{\frac{s}{2}}] M(-t).$$

The proof of (3.2) is completed. □

In the next lemma we shall assume $d = 1$.

LEMMA 3.6. *Assume $d = 1$ and let $A(s)$ be the operator that appears in (3.2) with $s < 2$. Then for a fixed constant C_s we have the inequality*

$$(3.4) \quad \|A(s)f\|_{L_x^1} \leq C_s \|f\|_{L_x^\infty}.$$

We postpone the proof of Lemma 3.6 to Section 7.

4 Spectral Theory for Δ_V

From now on we shall always work in the space dimension $d = 1$.

In this section we remind the reader of some classical material needed later. Recall that the Jost functions are solutions $f_\pm(x, \tau) = e^{\pm \mathbf{i}\tau x} m_\pm(x, \tau)$ of $-\Delta_V u = \tau^2 u$ with

$$\lim_{x \rightarrow +\infty} m_+(x, \tau) = 1 = \lim_{x \rightarrow -\infty} m_-(x, \tau).$$

We set $x^+ := \max\{0, x\}$, $x^- := \max\{0, -x\}$, and $\langle x \rangle := \sqrt{1 + x^2}$. We will denote by $L^{p,s}$ the space with norm

$$(4.1) \quad \|u\|_{L^{p,s}} = \|\langle x \rangle^s f\|_{L_x^p}.$$

The following lemma is well known:

LEMMA 4.1. *For $V \in \mathcal{S}(\mathbb{R})$ we have $m_\pm \in C^\infty(\mathbb{R}^2, \mathbb{C})$. There exist constants $C_1 = C_1(\|V\|_{L^{1,1}})$ and $C_2 = C_2(\|V\|_{L^{1,2}})$ such that*

$$(4.2) \quad |m_\pm(x, \tau) - 1| \leq C_1 \langle x^\mp \rangle \langle \tau \rangle^{-1} \left| \int_x^{\pm\infty} \langle y \rangle |V(y)| dy \right|,$$

$$(4.3) \quad |\partial_\tau m_\pm(x, \tau)| \leq C_2 (1 + x^2).$$

See lemma 1 in [3, p. 130]. The regularity follows by iterating the argument.

The transmission coefficient $T(\tau)$ and the reflection coefficients $R_\pm(\tau)$ are defined by the formula

$$(4.4) \quad T(\tau) m_\mp(x, \tau) = R_\pm(\tau) e^{\pm 2\mathbf{i}\tau x} m_\pm(x, \tau) + m_\pm(x, -\tau).$$

From [3] and [20] we have the following lemma:

LEMMA 4.2. For $V \in \mathcal{S}(\mathbb{R})$ we have $T, R_{\pm} \in C^{\infty}(\mathbb{R})$. Moreover,

$$(4.5) \quad |T(\tau) - 1| + |R_{\pm}(\tau)| \leq C \langle \tau \rangle^{-1} \quad \text{for } C = C(\|V\|_{L^{1,1}}),$$

$$(4.6) \quad |T(\tau)|^2 + |R_{\pm}(\tau)|^2 = 1,$$

$$(4.7) \quad \left| \frac{d}{d\tau} T(\tau) \right| + \left| \frac{d}{d\tau} R_{\pm}(\tau) \right| \leq C \quad \text{for } C = C(\|V\|_{L^{1,3}}).$$

In particular, (4.6) and (4.7) follow from [3, sec. 3] and (4.5) follows from theorem 2.3 in [20].

Set now $\Psi(x, \tau) = T(\tau)f_+(x, \tau)$ for $\tau \geq 0$ and $\Psi(x, \tau) = T(-\tau)f_-(x, -\tau)$ for $\tau \leq 0$. Then the distorted Fourier transform associated to Δ_V is defined by

$$(4.8) \quad F_V f(\tau) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \Psi(x, \tau) f(x) dx,$$

and we have the inverse formula

$$(4.9) \quad f(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \overline{\Psi(x, \tau)} F_V f(\tau) d\tau.$$

Our first application of this theory is the following lemma:

LEMMA 4.3. Let $V \in \mathcal{S}(\mathbb{R})$ and $\sigma(\Delta_V) = (-\infty, 0]$; then for any $s > \frac{1}{2}$ there exists a fixed C such that

$$(4.10) \quad \|f\|_{L_x^\infty} \leq C \|f\|_{L_x^2}^{1-1/(2s)} \|f\|_{\dot{H}_V^s}^{1/(2s)}.$$

PROOF. We claim that $\|f\|_{L_x^\infty} \leq c_0 \|F_V f\|_{L_x^1}$ for a fixed $c_0 = c_0(V)$. Assuming the claim, we have

$$\begin{aligned} \|F_V f\|_{L_x^1} &\leq \|F_V f\|_{L^2(|\xi| \leq \kappa)} \sqrt{2\kappa}^{\frac{1}{2}} + \| |\xi|^s F_V f \|_{L^2(|\xi| \geq \kappa)} \| |\xi|^{-s} \|_{L^2(|\xi| \geq \kappa)} \\ &\leq \sqrt{2\kappa}^{\frac{1}{2}} \|f\|_{L_x^2} + C_s \kappa^{\frac{1}{2}-s} \|f\|_{\dot{H}_V^s} \quad \text{with } C_s := \sqrt{\frac{2}{2s-1}}. \end{aligned}$$

For $\kappa = (2^{-1/2} C_s \|f\|_{\dot{H}_V^s})^{1/s} \|f\|_{L_x^2}^{-1/s}$ the last two terms are equal and we get (4.10).

We now prove $\|f\|_{L_x^\infty} \leq c_0 \|F_V f\|_{L_x^1}$. By (4.9) it suffices to prove $|\Psi(x, \tau)| \leq C_0$ for fixed C_0 . It is not restrictive to assume $x > 0$. Then for $\tau \geq 0$ we get the bound by $\Psi(x, \tau) = T(\tau)f_+(x, \tau)$ and Lemmas 4.1 and 4.2. Similarly for $\tau < 0$ we get a similar bound by

$$\Psi(x, \tau) = T(-\tau)f_-(x, -\tau) = R_+(-\tau)f_+(x, -\tau) + f_+(x, \tau). \quad \square$$

Consider now a function $u(t, x)$. By Lemma 4.3 we have for $s > \frac{1}{2}$:

$$(4.11) \quad \begin{aligned} \|u(t, \cdot)\|_{L_x^\infty} &\leq C \|M(-t)u(t, \cdot)\|_{L_x^2}^{1-1/(2s)} \|M(-t)u(t, \cdot)\|_{\dot{H}_V^s}^{1/(2s)} \\ &= \frac{C}{\sqrt{t}} \|u(t, \cdot)\|_{L_x^2}^{1-1/(2s)} \| |J_V(t)|^s u(t, \cdot) \|_{L_x^2}^{1/(2s)}. \end{aligned}$$

5 Proof of Theorem 1.1

Using the notation of Proposition 3.5 we have the following equation:

$$(5.1) \quad (i\partial_t + \Delta_V)|J_V|^s u - i t^{s-1} M(t)A(s)M(-t)u + \lambda |J_V|^s F = 0$$

with $F = |u|^{p-1}u$. Let $0 < s < 2$. Then by Strichartz estimates, which follow by [20], there are fixed C'_s and C such that

$$(5.2) \quad \begin{aligned} &\| |J_V|^s u \|_{L^\infty((1,T),L_x^2)} \\ &\leq C \| |J_V|^s(1)u \|_{L_x^2} + C'_s \| t^{s-1} A(s)M(-t)u \|_{L^{4/3}((1,T),L_x^1)} \\ &\quad + C \| |J_V|^s F \|_{L^1((1,T),L_x^2)}. \end{aligned}$$

By combining Lemma 3.6, (4.11), and the conservation of charge we get for every $\delta > 0$ a constant $M(\delta)$ such that

$$\begin{aligned} \| t^{s-1} A(s)M(-t)u \|_{L_t^{4/3}L_x^1} &\leq C_s \| t^{s-1} u \|_{L_x^\infty} \|_{L_t^{4/3}} \\ &\leq D_s \| t^{s-\frac{3}{2}} \|_{L_t^{4/3}} \| u(1) \|_{L_x^2}^{1-1/(2s)} \| |J_V|^s u \|_{L_t^\infty L_x^2}^{1/(2s)} \\ &\leq M(\delta) \| u(1) \|_{L_x^2} + \delta \| |J_V|^s u \|_{L_t^\infty L_x^2}, \end{aligned}$$

where we have considered $s < \frac{3}{4}$ so that $t^{s-3/2} \in L^{4/3}(1, \infty)$. Inserting this estimate in (5.2) we conclude

$$\begin{aligned} \| |J_V|^s u \|_{L^\infty((1,T),L_x^2)} &\leq C \| |J_V|^s u(1) \|_{L_x^2} + C_s \| u(1) \|_{L_x^2} \\ &\quad + C_s \| |J_V|^s F \|_{L^1((1,T),L_x^2)}. \end{aligned}$$

We shall use the following result:

LEMMA 5.1. *We have*

$$(5.3) \quad \| |J_V|^s f \|_{L_x^2} \sim \| J^s f \|_{L_x^2} \quad \text{for } 0 \leq s < \frac{1}{2}.$$

For $s \in (\frac{1}{2}, 1)$ and any $\varepsilon \in (0, \frac{1}{2})$ we have

$$(5.4) \quad \| |J_V|^s f \|_{L_x^2} \leq C t^{s+\varepsilon-\frac{1}{2}} (\| |J|^{\frac{1}{2}-\varepsilon} f \|_{L_x^2} + \| |J|^s f \|_{L_x^2}),$$

$$(5.5) \quad \| |J|^s f \|_{L_x^2} \leq C t^{s+\varepsilon-\frac{1}{2}} (\| |J_V|^{\frac{1}{2}-\varepsilon} f \|_{L_x^2} + \| |J_V|^s f \|_{L_x^2}).$$

PROOF. (5.3) is a simple consequence of Corollary 6.7 in the next section, which states

$$(5.6) \quad \| (-\Delta)^{\frac{s}{2}} f \|_{L_x^2} \sim \| (-\Delta + V)^{\frac{s}{2}} f \|_{L_x^2} \quad \text{for } 0 < s < \frac{1}{2}.$$

To prove (5.4) (respectively, (5.5)) we use

$$\begin{aligned} \|\sqrt{-\Delta_V} f\|_{L_x^2}^2 &\leq \|\sqrt{-\Delta} f\|_{L_x^2}^2 + \|Vf^2\|_{L_x^1}, \\ \|Vf^2\|_{L_x^1} &\leq \|V\|_{L_x^{p'}} \|f\|_{L_x^{2p}}^2 \leq C \|(-\Delta)^{\frac{1}{4}-\frac{\delta}{2}} f\|_{L_x^2}^2 \\ &\text{for } \frac{1}{2p} = \frac{1}{2} - \left(\frac{1}{2} - \delta\right) = \delta \end{aligned}$$

(respectively, the inequalities with Δ_V and Δ interchanged: this will also use (5.6)). We thus obtain

$$\|\sqrt{-\Delta_V} f\|_{L_x^2} \leq C \|(-\Delta)^{\frac{1}{4}-\frac{\delta}{2}} (1 + (-\Delta)^{\frac{1}{4}+\frac{\delta}{2}}) f\|_{L_x^2}$$

(respectively, the inequality with Δ_V and Δ interchanged). Interpolation with (5.6) for $s = \frac{1}{2} - \delta$ yields

$$\begin{aligned} \|(-\Delta_V)^{\frac{s}{2}} f\|_{L_x^2} &\leq C \|(-\Delta)^{\frac{1}{4}-\frac{\delta}{2}} (1 + (-\Delta)^{\frac{s}{2}-\frac{1}{4}+\frac{\delta}{2}}) f\|_{L_x^2} \\ &\leq C (\|(-\Delta)^{\frac{1}{4}-\frac{\delta}{2}} f\|_{L_x^2} + \|(-\Delta)^{\frac{s}{2}} f\|_{L_x^2}) \end{aligned}$$

(respectively, the inequality with Δ_V and Δ interchanged). Multiplying this estimate by t^s and using again the fact that $M(t)$ is an L_x^2 bounded operator, we see that

$$\| |J_V|^s f \|_{L_x^2} \leq C (t^{s-\frac{1}{2}+\delta} \| |J|^{\frac{1}{2}-\delta} f \|_{L_x^2} + \| |J|^s f \|_{L_x^2}),$$

and for $\varepsilon = \delta$ we get (5.4) (respectively, (5.5)). □

By Lemma 5.1 we get

$$\| |J_V|^s u \|_{L^\infty((1,T),L_x^2)} \leq C_s \|u(1)\|_{\Sigma_s} + C_s \| |J_V|^s F \|_{L^1((1,T),L_x^2)},$$

since

$$\| |J|^s u(1) \|_{L_x^2} \leq C \|u(1)\|_{\Sigma_s}.$$

If we can show that for a fixed C for all T

$$(5.7) \quad \| |J_V|^s u \|_{L^\infty((1,T),L_x^2)} \leq C \|u(1)\|_{\Sigma_s},$$

then by (4.11) this will yield (1.2). Then scattering (1.3) will follow from (1.2) by a standard argument that we do not repeat.

By combining Lemma 5.1 with lemma 2.3 in [12], which states that

$$\| |J|^\gamma (|u|^{p-1}u) \|_{L_x^2} \leq C \|u\|_{L_x^\infty}^{p-1} \| |J|^\gamma u \|_{L_x^2} \quad \text{for } 0 \leq \gamma < 2 \text{ and } p \geq 3,$$

we have

$$\begin{aligned} &\| |J_V|^s (|u|^{p-1}u) \|_{L^1((1,t),L_x^2)} \\ &\leq C \| \langle t' \rangle^{s+\varepsilon-\frac{1}{2}} (\| |J|^{\frac{1}{2}-\varepsilon} (|u|^{p-1}u) \|_{L_x^2} + \| J^s (|u|^{p-1}u) \|_{L_x^2}) \|_{L^1(1,t)} \\ &\leq C' \| \langle t' \rangle^{s+\varepsilon-\frac{1}{2}} \|u\|_{L_x^\infty}^{p-1} (\| |J|^{\frac{1}{2}-\varepsilon} u \|_{L_x^2} + \| |J|^s u \|_{L_x^2}) \|_{L^1(1,t)} \\ &\leq C' \| \langle t' \rangle^{s+\varepsilon-\frac{1}{2}} \|u\|_{L_x^\infty}^{p-1} (\| |J_V|^{\frac{1}{2}-\varepsilon} u \|_{L_x^2} + \| |J|^s u \|_{L_x^2}) \|_{L^1(1,t)}. \end{aligned}$$

Again by Lemma 5.1 we can continue the estimate as follows:

$$\begin{aligned} \dots &\leq C' \|\langle t' \rangle^{2s+2\varepsilon-1} \|u\|_{L_x^\infty}^{p-1} (\| |J_V|^{\frac{1}{2}-\varepsilon} u \|_{L_x^2} + \| |J_V|^s u \|_{L_x^2}) \|_{L^1(1,t)} \\ &\leq C' \int_1^t \langle t' \rangle^{2(s+\varepsilon)-\frac{p+1}{2}} (\|u\|_{L_x^2}^{2s-1} \| |J_V|^s u \|_{L_x^2})^{\frac{p-1}{2s}} \\ &\quad \times (\| |J_V|^{\frac{1}{2}-\varepsilon} u \|_{L_x^2} + \| |J_V|^s u \|_{L_x^2}) dt' \end{aligned}$$

where in the last line we used (4.11).

Since $p > 3$ we can choose $s > \frac{1}{2}$ and $\varepsilon > 0$ such that $\frac{p+1}{2} - 2s - 2\varepsilon > 1$. Then

$$\begin{aligned} \| |J_V|^s (|u|^{p-1} u) \|_{L_t^1 L_x^2} &\leq \\ C_s \|u(1)\|_{L_x^2}^{(p-1)\frac{2s-1}{2s}} \| |J_V|^s u \|_{L_t^\infty L_x^2}^{\frac{p-1}{2s}} &(\| |J_V|^{\frac{1}{2}-\varepsilon} u \|_{L_t^\infty L_x^2} + \| |J_V|^s u \|_{L_t^\infty L_x^2}) \end{aligned}$$

on any interval $(1, t)$ with a constant C_s independent of t . Notice that the norm $\| |J_V|^{1/2-\varepsilon} u \|_{L_t^\infty L_x^2}$ can be bounded in terms of the other norms using interpolation; hence the proof of (5.7) follows by a standard continuity argument provided that we fix the constant $\epsilon_0 > 0$ in the statement of Theorem 1.1 sufficiently small.

6 Equivalence of Homogeneous Sobolev Norms

Along this section the functions $m_\pm(x, \tau)$, $f_\pm(x, \tau)$, $T(\tau)$, and $R_\pm(\tau)$ are the ones defined in Section 4. Also, the norm $\|V\|_{L^{p,q}}$ is the one defined in the same section. We consider for an appropriate cutoff $\varphi \in C_0^\infty(\mathbb{R}_+, [0, 1])$ a Paley-Littlewood partition of unity

$$1 = \sum_{j \in \mathbb{Z}} \varphi(t2^{-j}), \quad t > 0.$$

Then for any $s \in \mathbb{R}$ we have

$$\begin{aligned} \|(-\Delta_V)^{\frac{s}{2}} f\|_{L^2}^2 &\sim \sum_{j \in \mathbb{Z}} 2^{2js} \langle \varphi(2^{-j} \sqrt{-\Delta_V}) f, f \rangle_{L_x^2} \\ &\sim \sum_{j \in \mathbb{Z}} 2^{2js} \| \varphi(2^{-j} \sqrt{-\Delta_V}) f \|_{L_x^2}^2. \end{aligned}$$

We have the following result:

LEMMA 6.1. *Let V be a real valued Schwartz function such that $\sigma(\Delta_V) = (-\infty, 0]$ and $T(0)$ is equal to 0 or 1. Then for any pair of integer numbers $j, k \in \mathbb{Z}$ with $k \leq j$ and for any $f \in \mathcal{S}(\mathbb{R})$ such that*

$$(6.1) \quad \text{supp } \widehat{f}(\xi) \subseteq \{|\xi| \sim 2^k\},$$

the following inequality holds for $C_V = C(\|V\|_{L^{1,3}})$:

$$(6.2) \quad \langle \varphi(2^{-j} \sqrt{-\Delta_V}) f, f \rangle_{L_x^2} \leq C_V 2^{-|k-j|} \|f\|_{L_x^2}^2.$$

PROOF. For $\varphi(|\tau|) := \tau^2 \psi(|\tau|)$ we have

$$\begin{aligned} \langle \varphi(2^{-j} \sqrt{-\Delta_V}) f, f \rangle_{L_x^2} &= A_j(f) + B_j(f), \\ A_j(f) &:= -2^{-2j} \langle \psi(2^{-j} \sqrt{-\Delta_V}) f, \partial_x^2 f \rangle_{L_x^2}, \\ B_j(f) &:= 2^{-2j} \langle \psi(2^{-j} \sqrt{-\Delta_V}) f, Vf \rangle_{L_x^2}. \end{aligned}$$

It is straightforward that

$$(6.3) \quad \begin{aligned} |A_j(f)| &= 2^{-2j} \left| \langle \psi(2^{-j} \sqrt{-\Delta_V}) f, \partial_x^2 f \rangle_{L_x^2} \right| \\ &\leq 2^{-2j} \left\| \psi(2^{-j} \sqrt{-\Delta_V}) f \right\|_{L_x^2} \left\| \partial_x^2 f \right\|_{L_x^2} \leq C 2^{2k-2j} \|f\|_{L_x^2}^2. \end{aligned}$$

Notice that this constant C depends on the cutoff φ but not on V . This follows from the fact that the distorted Fourier transform (4.8) is an isometry.

The next lemma in conjunction with (6.3) will complete the proof of Lemma 6.1.

LEMMA 6.2. *Assume the hypothesis of Lemma 6.1. Then there exists a fixed $C = C(\|V\|_{L^{1,3}})$ such that $|B_j(f)| \leq C 2^{-|k-j|} \|f\|_{L_x^2}^2$.*

PROOF. The first step in the proof is the following representation formula:

LEMMA 6.3. *We have*

$$(6.4) \quad \begin{aligned} &(\psi(2^{-j} \sqrt{-\Delta_V}) f)(x) \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} d\tau \psi(2^{-j} \tau) \\ &\times \left[T(\tau) m_+(x, \tau) \int_{y < x} m_-(y, \tau) e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. + T(-\tau) m_-(x, -\tau) \int_{y > x} m_+(y, -\tau) e^{i\tau(x-y)} f(y) dy \right]. \end{aligned}$$

PROOF. We recall the limiting absorption principle:

$$(6.5) \quad \begin{aligned} g(-\Delta_V)(x, y) &= \int_0^\infty g(\lambda) E_{a.c.}(d\lambda)(x, y), \\ E_{a.c.}(d\lambda)(x, y) &= \frac{1}{2\pi i} [R_{-\Delta_V}^+(x, y, \lambda) - R_{-\Delta_V}^-(x, y, \lambda)] d\lambda, \end{aligned}$$

where for $\lambda > 0$ and $x < y$ (for $x > y$ exchange x and y in the right-hand side)

$$(6.6) \quad R_{-\Delta_V}^\pm(x, y, \lambda) = \frac{f_-(x, \pm\sqrt{\lambda}) f_+(y, \pm\sqrt{\lambda})}{w(\pm\sqrt{\lambda})}$$

for the Wronskian

$$(6.7) \quad w(\tau) := (\partial_x f_+)(x, \tau) f_-(x, \tau) - f_+(x, \tau) \partial_x f_-(x, \tau).$$

Then for $x < y$ (for $x > y$ exchange x and y in the right-hand side)

$$\begin{aligned} g(-\Delta_V)(x, y) &= \int_0^\infty \tau g(\tau^2) \left[\frac{f_-(x, \tau) f_+(y, \tau)}{w(\tau)} - \frac{f_-(x, -\tau) f_+(y, -\tau)}{w(-\tau)} \right] \frac{d\tau}{\pi i} \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} T(\tau) g(\tau^2) f_-(x, \tau) f_+(y, \tau) d\tau, \end{aligned}$$

where we used the formula $\frac{1}{T(\tau)} = \frac{w(\tau)}{2i\tau}$; see [3, p. 144]. Therefore, making also a change of variable, we get

$$\begin{aligned} (6.8) \quad &g(-\Delta_V)f(x) \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} d\tau g(\tau^2) \left[T(\tau) f_+(x, \tau) \int_{-\infty}^x f_-(y, \tau) f(y) dy \right. \\ &\quad \left. + T(-\tau) f_-(x, -\tau) \int_x^\infty f_+(y, -\tau) f(y) dy \right]. \end{aligned}$$

For $g(\lambda) = \psi(2^{-j}\sqrt{\lambda})$ and $f_\pm(x, \xi) = e^{\pm i x \xi} m_\pm(x, \xi)$ we get Lemma 6.3. \square

We continue with the proof of Lemma 6.2 by writing

$$B_j(f) = B_j^{(1)}(f) + B_j^{(2)}(f)$$

with

$$\begin{aligned} (6.9) \quad &B_j^{(1)}(f) \\ &:= -\frac{1}{2\pi} 2^{-2j} \int_{\mathbb{R}} dx V(x) \overline{f(x)} \\ &\times \int_{\mathbb{R}} d\tau \psi(2^{-j}\tau) \left[T(\tau) m_+(x, \tau) \int_{y < x} (m_-(y, \tau) - 1) e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. + T(-\tau) m_-(x, -\tau) \int_{y > x} (m_+(y, -\tau) - 1) e^{i\tau(x-y)} f(y) dy \right] \end{aligned}$$

and

$$\begin{aligned} (6.10) \quad &B_j^{(2)}(f) := -\frac{1}{2\pi} 2^{-2j} \int_{\mathbb{R}} dx V(x) \overline{f(x)} \\ &\times \int_{\mathbb{R}} d\tau \psi(2^{-j}\tau) \left[T(\tau) m_+(x, \tau) \int_{y < x} e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. + T(-\tau) m_-(x, -\tau) \int_{y > x} e^{i\tau(x-y)} f(y) dy \right]. \end{aligned}$$

LEMMA 6.4. *Assume that f , j , and k are as in Lemma 6.1. Let V be a real-valued Schwartz function such that $\sigma(\Delta_V) = (-\infty, 0]$. We do not impose other hypotheses on V . Then, for fixed $C = C(\|V\|_{L^{1,3}})$, we have $|B_j^{(1)}(f)| \leq C2^{k-j}\|f\|_{L_x^2}^2$.*

PROOF. The inequality follows from the following:

$$(6.11) \quad |B_j^{(1)}(f)| \leq C2^{-j}\langle x \rangle^3 V\|_{L_x^1} \|f\|_{L_x^\infty}^2 \leq C'2^{k-j}\|f\|_{L_x^2}^2,$$

with $C = C(\|V\|_{L^{1,3}})$, and where we used the Bernstein inequality

$$(6.12) \quad \|f\|_{L_x^\infty} \lesssim 2^{\frac{k}{2}}\|f\|_{L_x^2}.$$

To prove the first inequality in (6.11), observe that the second line of (6.9) can be bounded by $C\langle x \rangle^3\|f\|_{L_x^\infty}$ with $C = C(\|V\|_{L^{1,1}})$ by using the following estimates, which follow from (4.2):

$$\begin{aligned} \int_{-\infty}^x |m_-(y, \tau) - 1| |f(y)| dy &\lesssim \|f\|_{L_x^\infty} \left(\int_{-\infty}^{x \wedge 0} \langle y \rangle^{-2} dy + \int_0^{x \vee 0} \langle y \rangle dy \right) \\ &\lesssim \langle x \rangle^2 \|f\|_{L_x^\infty} \end{aligned}$$

and

$$|m_+(x, \tau)| \lesssim \langle x \rangle.$$

Proceeding as above, the fourth line of (6.9) can be bounded by $C\langle x \rangle^3\|f\|_{L_x^\infty}$ with $C = C(\|V\|_{L^{1,1}})$ using estimates like

$$\begin{aligned} \int_x^\infty |m_+(y, -\tau) - 1| |f(y)| dy &\lesssim \|f\|_{L_x^\infty} \left(\int_{x \vee 0}^\infty \langle y \rangle^{-2} dy + \int_{x \wedge 0}^0 \langle y \rangle dy \right) \\ &\lesssim \langle x \rangle^2 \|f\|_{L_x^\infty} \end{aligned}$$

and

$$|m_-(x, \tau)| \lesssim \langle x \rangle.$$

This proves (6.11) and so also Lemma 6.4. □

LEMMA 6.5. *In addition to the hypotheses of Lemma 6.4, let us assume now that either $T(0) = 0$ or $T(0) = 1$. Then we have $|B_j^{(2)}(f)| \leq C2^{k-j}\|f\|_{L_x^2}^2$ for fixed $C = C(\|V\|_{L^{1,3}})$.*

PROOF. We use (4.4) and substitute

$$(6.13) \quad T(-\tau)m_-(x, -\tau) = R_+(-\tau)e^{-2i\tau x}m_+(x, -\tau) + m_+(x, \tau).$$

We then write

$$\begin{aligned} B_j^{(2)}(f) &= -\frac{1}{2\pi}2^{-2j} \int_{\mathbb{R}} dx V(x)\overline{f(x)} \\ &\quad \times \int_{\mathbb{R}} d\tau \psi(2^{-j}\tau) \left[T(\tau)m_+(x, \tau) \int_{y<x} e^{i\tau(x-y)} f(y)dy \right. \\ &\quad \quad \quad \left. + m_+(x, \tau) \int_{y>x} e^{i\tau(x-y)} f(y)dy \right. \\ &\quad \quad \quad \left. + R_+(-\tau)m_+(x, -\tau) \int_{y>x} e^{-i\tau(x+y)} f(y)dy \right]. \end{aligned}$$

Notice that Lemma 6.1 is elementary for $|k - j| \leq \kappa_0$ for any preassigned $\kappa_0 > 1$. So we will focus only on the case $k - j > \kappa_0$ with a fixed sufficiently large κ_0 . We write

$$(6.14) \quad \begin{aligned} \psi(2^{-j}\tau) \int_{y>x} e^{i\tau(x-y)} f(y)dy &= \psi(2^{-j}\tau)e^{i\tau x} \int_{\mathbb{R}} \overbrace{e^{-i\tau y} f(y)}^{\sqrt{2\pi}\widehat{f}(-\tau)} dy, \\ -\psi(2^{-j}\tau) \int_{y<x} e^{i\tau(x-y)} f(y)dy &= -\psi(2^{-j}\tau) \int_{y<x} e^{i\tau(x-y)} f(y)dy, \end{aligned}$$

because $\psi(2^{-j}\tau)\widehat{f}(-\tau) \equiv 0$ for $|j - k| > \kappa_0$ and κ_0 sufficiently large.

By (6.14) we can write

$$\begin{aligned} B_j^{(2)}(f) &= -\frac{1}{2\pi}2^{-2j} \int_{\mathbb{R}} dx V(x)\overline{f(x)} \\ &\quad \times \int_{\mathbb{R}} d\tau \psi(2^{-j}\tau) \left[(T(\tau) - 1)m_+(x, \tau) \int_{y<x} e^{i\tau(x-y)} f(y)dy \right. \\ &\quad \quad \quad \left. + R_+(-\tau)m_+(x, -\tau) \int_{y>x} e^{-i\tau(x+y)} f(y)dy \right]. \end{aligned}$$

We rewrite the above as

$$\begin{aligned}
 & B_j^{(2)}(f) \\
 &= -\frac{1}{2\pi} 2^{-2j} \int_{\mathbb{R}} dx V(x) \overline{f(x)} \\
 (6.15) \quad & \times \int_{\mathbb{R}} d\tau \psi(2^{-j}\tau) \left\{ [T(\tau) - 1 - R_+(-\tau)] m_+(x, \tau) \int_{y < x} e^{i\tau(x-y)} f(y) dy \right. \\
 & \quad - R_+(-\tau) (e^{-i\tau x} m_+(x, -\tau) - e^{i\tau x} m_+(x, \tau)) \int_{y < x} e^{-i\tau y} f(y) dy \\
 & \quad \left. + R_+(-\tau) m_+(x, -\tau) e^{-i\tau x} \int_{\mathbb{R}} e^{-i\tau y} f(y) dy \right\}.
 \end{aligned}$$

The last factor is $\sqrt{2\pi} \widehat{f}(-\tau) = 0$ on the support of $\psi(2^{-j}\tau)$ as after (6.14). So the last line in (6.15) cancels out.

We now focus on the terms originating from the fourth line of (6.15). We will set $f_x(t) := f(t + x)$ and $Hf_x(\tau) := \int_{-\infty}^0 e^{-i\tau y} f(y + x) dy$. We have

$$Hg(\tau) = \int_{-\infty}^0 e^{-i\tau y} g(y) dy = \int_{\mathbb{R}} \widehat{\chi}_{(-\infty, 0]}(-\tau - \xi) \widehat{g}(\xi) d\xi = \widehat{\chi}_{(-\infty, 0]} * \widehat{g}(-\tau),$$

where here and below we use definition (2.1) of the Fourier transform.

We also have the relation $\widehat{\chi}_{(-\infty, 0]}(\tau) = -i(2\pi)^{-1/2}(\tau - i0)^{-1}$ [18, chap. 3, p. 206] and take into account the definition of the Fourier transform there. By the Sokhotskyi-Plemelj formula, $(\tau - i0)^{-1} = P.V. \frac{1}{\tau} + i\pi \delta(\tau)$. Then

$$\begin{aligned}
 Hg(\tau) &= \widehat{\chi}_{(-\infty, 0]} * \widehat{g}(\tau) = (2\pi)^{-\frac{1}{2}} (\pi \widehat{g}(-\tau) - i \mathcal{H}g(-\tau)), \\
 (6.16) \quad \mathcal{H}g(\tau) &:= \lim_{\epsilon \rightarrow 0^+} \int_{|\xi - \tau| \geq \epsilon} \frac{\widehat{g}(\xi)}{\xi - \tau} d\xi.
 \end{aligned}$$

By Lemma 4.1 we get

$$\begin{aligned}
 & |e^{-i\tau x} m_+(x, -\tau) - e^{i\tau x} m_+(x, \tau)| \\
 (6.17) \quad & \leq |e^{-2i\tau x} - 1| |m_+(x, \tau)| + |m_+(x, -\tau) - m_+(x, \tau)| \\
 & \leq (C_1 + C_2) \langle x \rangle^2 |\tau|,
 \end{aligned}$$

where the first term in the second line can be bounded by using (4.2) and the second term in the second line can be bounded using the mean value theorem and (4.3), and where $C_j = C(\|V\|_{L^{1,j}})$.

By (6.17) and by $|R_+(-\tau)| \leq C \langle \tau \rangle^{-1}$ with $C = C(\|V\|_{L^{1,1}})$, which follows from (4.5), the terms originating from the fourth line of (6.15) can be bounded by a constant $C = C(\|V\|_{L^{1,2}})$ times

$$(6.18) \quad 2^{-2j} \|f\|_{L_x^\infty} \int_{\mathbb{R}} dx |V(x)| \langle x \rangle^2 \int_{\mathbb{R}} d\tau |\psi(2^{-j}\tau)| |\tau| \langle \tau \rangle^{-1} |H \widehat{f}_x(\tau)|.$$

By $|j - k| > \kappa_0$, by $\widehat{f}_x(\tau) = e^{-i\tau x} \widehat{f}(\tau)$, and by (6.16), we get

$$\psi(2^{-j}\tau)|H\widehat{f}_x(\tau)| = \psi(2^{-j}\tau)|\mathcal{H}\widehat{f}_x(\tau)|.$$

We have then the upper bound

$$\begin{aligned} (6.18) &\leq 2^{-2j} \|f\|_{L_x^\infty} \|V\|_{L^{1,2}} \int_{|\tau|\sim 2^j} d\tau \frac{|\tau|}{\langle \tau \rangle^2} \int_{|\xi|\sim 2^k} \frac{|\widehat{f}(\xi)|}{|\tau - \xi|} d\xi \\ &\leq 2^{-j} \|f\|_{L_x^\infty} \|V\|_{L^{1,2}} \int_{|\xi|\sim 2^k} |\widehat{f}(\xi)| d\xi_{L_x^2} \\ &\leq C' 2^{\frac{k}{2}-j} \|f\|_{L_x^\infty} \|f\| \leq C 2^{k-j} \|f\|_{L_x^2}^2 \end{aligned}$$

where we used $|\tau - \xi| \approx |\tau|$ and where $C = C(\|V\|_{L^{1,2}})$.

Now we consider the contribution from the third line of (6.15). We assume

$$(6.19) \quad T(0) - 1 - R_+(0) = 0.$$

(6.19) occurs if $T(0) = 1$ (then $R_\pm(0) = 0$ by the identity (4.6)) and in the generic case $T(0) = 0$ (when $R_\pm(0) = -1$, see [3, p. 147], as can be seen by setting $\tau = 0$ in (4.4)). By (6.19) and (4.7) for the bound near $\tau = 0$ and by (4.5) for the bound away from 0, we get

$$|T(\tau) - 1 - R_+(-\tau)| \leq C \frac{|\tau|}{\langle \tau \rangle^2} \quad \text{with } C = C(\|V\|_{L^{1,3}}).$$

Then, by a similar argument to that for the fourth line of (6.15) we see that the contribution is bounded by $C 2^{k-j} \|f\|_{L_x^2}^2$ with $C = C(\|V\|_{L^{1,3}})$. □

Lemmas 6.4 and 6.5 together yield Lemma 6.2. □

The proof of Lemma 6.1 follows by combining (6.3) with Lemma 6.2. □

We remark that if $T(0) = \frac{2a}{1+a^2}$ with $a \neq 0$, then $R_+(0) = \frac{1-a^2}{1+a^2}$; see, for instance, [19, p. 512]. Then the right-hand side of (6.19) equals $2\frac{a-1}{1+a^2} \neq 0$ for $a \neq 1$, and our proof of Lemma 6.5 breaks down.

We have proved (6.2) for $k \leq j$. The next lemma shows that (6.2) also continues to hold for $k > j$.

LEMMA 6.6. *Let V be a real-valued Schwartz function with $\sigma(\Delta_V) = (-\infty, 0]$ and with $T(0)$ equal to 0 or 1. For any integer numbers $j, k \in \mathbb{Z}$ with $k > j$ and for any $f \in \mathcal{S}(\mathbb{R})$ satisfying (6.1), inequality (6.2) holds for a C_V of the same type.*

PROOF. The proof is similar to that of Lemma 6.1.

We have $f = \widetilde{\varphi}(2^{-k}\sqrt{-\Delta})f$ for some $\widetilde{\varphi} \in C_0^\infty(\mathbb{R}_+, [0, 1])$, and we have

$$\langle \varphi(2^{-j}\sqrt{-\Delta_V})f, f \rangle_{L_x^2} = -2^{-2k} \langle \varphi(2^{-j}\sqrt{-\Delta_V})f, \Delta \psi(2^{-k}\sqrt{-\Delta})f \rangle_{L_x^2}$$

with $\tau^2 \psi(\tau) = \tilde{\varphi}(\tau)$. Then we have

$$\begin{aligned} \langle \varphi(2^{-j} \sqrt{-\Delta_V}) f, f \rangle_{L_x^2} &= -2^{-2k} \langle \Delta_V \varphi(2^{-j} \sqrt{-\Delta_V}) f, \psi(2^{-k} \sqrt{-\Delta}) f \rangle_{L_x^2} \\ &\quad - 2^{-2k} \langle V \varphi(2^{-j} \sqrt{-\Delta_V}) f, \psi(2^{-k} \sqrt{-\Delta}) f \rangle_{L_x^2}. \end{aligned}$$

It is straightforward that, for a constant C independent of V ,

$$(6.20) \quad 2^{-2k} \left| \langle \Delta_V \varphi(2^{-j} \sqrt{-\Delta_V}) f, \psi(2^{-k} \sqrt{-\Delta}) f \rangle_{L_x^2} \right| \leq C 2^{2j-2k} \|f\|_{L_x^2}^2.$$

In what follows we prove the following for $C = C(\|V\|_{L^{1,3}})$, which with (6.20) yields Lemma 6.6:

$$(6.21) \quad 2^{-2k} \left| \langle V \varphi(2^{-j} \sqrt{-\Delta_V}) f, \psi(2^{-k} \sqrt{-\Delta}) f \rangle_{L_x^2} \right| \leq C 2^{j-k} \|f\|_{L_x^2}^2.$$

Denote by $K(x, y)$ the integral kernel of $\varphi(2^{-j} \sqrt{-\Delta_V})$. Then, setting $g(\tau) = \varphi(2^{-j} \sqrt{\tau})$, from (6.8) we get

$$\begin{aligned} K(x, y) &\sim \chi_{x>y}(x, y) \int_{\mathbb{R}} \varphi(2^{-j} \tau) m_+(x, \tau) m_-(y, \tau) T(\tau) e^{i\tau(x-y)} \\ &\quad + \chi_{x<y}(x, y) \int_{\mathbb{R}} \varphi(2^{-j} \tau) m_-(x, -\tau) m_+(y, -\tau) T(-\tau) e^{i\tau(x-y)} d\tau \end{aligned}$$

with $\chi_{x \geq y}(x, y) = 1$ for $x \geq y$ and $\chi_{x \geq y}(x, y) = 0$ for $x \leq y$. Then the bound (6.20) is obtained, for $\Psi(x) = \psi(\frac{\sqrt{-\Delta}}{2^k}) f$, by bounding

$$(6.22) \quad \begin{aligned} &2^{-2k} \int_{\mathbb{R}} dx \overline{\Psi(x)} V(x) \int_{\mathbb{R}} d\tau \varphi(2^{-j} \tau) \\ &\times \left[T(\tau) m_+(x, \tau) \int_{y<x} m_-(y, \tau) e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. + T(-\tau) m_-(x, -\tau) \int_{y>x} m_+(y, -\tau) e^{i\tau(x-y)} f(y) dy \right]. \end{aligned}$$

We split (6.22) as $I_1 + I_2$ where

$$(6.23) \quad \begin{aligned} I_1 &:= 2^{-2k} \int_{\mathbb{R}} dx \overline{\Psi(x)} V(x) \int_{\mathbb{R}} d\tau \varphi(2^{-j} \tau) \\ &\times \left[T(\tau) m_+(x, \tau) \int_{y<x} (m_-(y, \tau) - 1) e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. + T(-\tau) m_-(x, -\tau) \int_{y>x} (m_+(y, -\tau) - 1) e^{i\tau(x-y)} f(y) dy \right] \end{aligned}$$

and

$$\begin{aligned}
 I_2 := & 2^{-2k} \int_{\mathbb{R}} dx \overline{\Psi(x)} V(x) \int_{\mathbb{R}} d\tau \varphi(2^{-j}\tau) \\
 (6.24) \quad & \times \left[T(\tau) m_+(x, \tau) \int_{y < x} e^{i\tau(x-y)} f(y) dy \right. \\
 & \left. + T(-\tau) m_-(x, -\tau) \int_{y > x} e^{i\tau(x-y)} f(y) dy \right].
 \end{aligned}$$

We start with I_1 and show for $C = C(\|V\|_{L^{1,3}})$ that

$$(6.25) \quad |I_1| \leq C 2^{j-k} \|f\|_{L_x^2}^2.$$

To prove (6.25) we focus for definiteness on the second line of (6.23) (the contribution from the third can be treated similarly). Then we have

$$\begin{aligned}
 & 2^{-2k} \int_{\mathbb{R}} dx |\Psi(x) V(x)| \int_{\mathbb{R}} d\tau |\varphi(2^{-j}\tau)| \text{second line of (6.23)} \\
 & \lesssim 2^{-2k} \int_{\mathbb{R}} dx |\Psi(x) V(x)| \langle x \rangle \\
 & \quad \times \int_{|\tau| \sim 2^j} d\tau \left(\int_{-\infty}^{0 \wedge x} \langle y \rangle^{-2} |f(y)| dy + \int_0^{x \vee 0} \langle y \rangle |f(y)| dy \right) \\
 & \leq C' 2^{j-2k} \|\Psi\|_{L_x^\infty} \|f\|_{L_x^\infty} \leq C'' 2^{j-k} \|\psi(2^{-k} \sqrt{-\Delta} V)\|_{L_x^2} \|f\|_{L_x^2} \\
 & \leq C 2^{j-k} \|f\|_{L_x^2}^2
 \end{aligned}$$

with constants $C(\|V\|_{L^{1,3}})$ and where we used Bernstein inequality (6.12).

We turn now to I_2 and show for $C = C(\|V\|_{L^{1,3}})$

$$(6.26) \quad |I_2| \leq C 2^{j-k} \|f\|_{L_x^2}^2.$$

We substitute (6.13) to get

$$\begin{aligned}
 I_2 = & 2^{-2k} \int_{\mathbb{R}} dx \overline{\Psi(x)} V(x) \\
 & \times \int_{\mathbb{R}} d\tau \varphi(2^{-j}\tau) \left[(T(\tau) - 1) m_+(x, \tau) \int_{y < x} e^{i\tau(x-y)} f(y) dy \right. \\
 & \left. + R_+(-\tau) m_+(x, -\tau) \int_{y > x} e^{-i\tau(x+y)} f(y) dy \right].
 \end{aligned}$$

We rewrite, proceeding as for (6.15),

$$(6.27) \quad \begin{aligned} I_2 &= 2^{-2k} \int_{\mathbb{R}} dx \overline{\Psi(x)} V(x) \\ &\times \int_{\mathbb{R}} d\tau \varphi(2^{-j}\tau) \left[(T(\tau) - 1 - R_+(-\tau)) m_+(x, \tau) \int_{y < x} e^{i\tau(x-y)} f(y) dy \right. \\ &\quad \left. - R_+(-\tau) (e^{-i\tau x} m_+(x, -\tau) - e^{i\tau x} m_+(x, \tau)) \int_{y < x} e^{-i\tau y} f(y) dy \right]. \end{aligned}$$

Then proceeding as in Lemma 6.5 we get for $C = C(\|V\|_{L^{1,3}})$

$$|I_2| \leq C 2^{-2k} \|\Psi\|_{L_x^\infty} \int_{\mathbb{R}} dx |V(x)| \langle x \rangle^2 \int d\tau |\varphi(2^{-j}\tau)| \frac{|\tau|}{\langle \tau \rangle^2} |H \widehat{f}_x(\tau)|$$

with $H \widehat{f}_x(\tau) := \int_{-\infty}^0 e^{-i\tau y} f(y+x) dy$ as before. From now on we focus only on $k - j > \kappa_0$ and we get

$$\begin{aligned} |I_2| &\leq C_1 2^{-2k} \|\Psi\|_{L_x^\infty} \|V\|_{L^{1,2}} \int_{|\tau| \sim 2^j} d\tau \frac{|\tau|}{\langle \tau \rangle^2} \int_{|\xi| \sim 2^k} \frac{|\widehat{f}(\xi)|}{|\tau - \xi|} d\xi \\ &\leq C_2 2^{2j-2k} \|\Psi\|_{L_x^\infty} 2^{-k} \int_{|\xi| \sim 2^k} |\widehat{f}(\xi)| d\xi \\ &\leq C_3 2^{2j-2k} \|f\|_{L_x^2} 2^{-\frac{k}{2}} \|\psi(2^{-k} \sqrt{-\Delta}) f\|_{L_x^\infty} \leq C 2^{2j-2k} \|f\|_{L_x^2}^2, \end{aligned}$$

where the constants are $C(\|V\|_{L^{1,3}})$. This completes the proof of (6.26) which, along with (6.25), yields (6.21) and completes the proof of Lemma 6.6. \square

From Lemma 6.6 and Lemma 6.1 we arrive at the following crucial result.

COROLLARY 6.7. *For $0 \leq s < \frac{1}{2}$ and for any $f \in C_0^\infty(\mathbb{R})$ we have*

$$\|(-\Delta)^{\frac{s}{2}} f\|_{L_x^2} \sim \|(-\Delta + V)^{\frac{s}{2}} f\|_{L_x^2}.$$

PROOF. The proof of \gtrsim is as follows (that of \lesssim is similar): We have

$$\begin{aligned} &\|(-\Delta + V)^{\frac{s}{2}} f\|_{L_x^2}^2 \\ &\sim \sum_{j,k,l \in \mathbb{Z}} 2^{2js} \left\langle \varphi\left(\frac{\sqrt{-\Delta} V}{2^j}\right) \varphi\left(\frac{\sqrt{-\Delta}}{2^k}\right) f, \varphi\left(\frac{\sqrt{-\Delta} V}{2^j}\right) \varphi\left(\frac{\sqrt{-\Delta}}{2^l}\right) f \right\rangle_{L_x^2} \\ &\leq C \sum_{j,k,l \in \mathbb{Z}} 2^{2js} 2^{-\frac{1}{2}|j-k| - \frac{1}{2}|j-l|} \left\| \varphi\left(\frac{\sqrt{-\Delta}}{2^k}\right) f \right\|_{L_x^2} \left\| \varphi\left(\frac{\sqrt{-\Delta}}{2^l}\right) f \right\|_{L_x^2} \\ &\leq C' \sum_{k \in \mathbb{Z}} 2^{2ks} \left\| \varphi\left(\frac{\sqrt{-\Delta}}{2^k}\right) f \right\|_{L_x^2}^2 \sim C' \|(-\Delta)^{\frac{s}{2}} f\|_{L_x^2}^2. \end{aligned}$$

Here we have used Young’s inequality and, for a fixed C ,

$$\begin{aligned}
 & 2^{-ls} \sum_{j,k} 2^{2js} 2^{-\frac{1}{2}|j-k|-\frac{1}{2}|j-l|} 2^{-ks} \\
 &= 2^{-ls} \sum_j 2^{2js} 2^{-\frac{1}{2}|j-l|} \left[2^{\frac{j}{2}} \sum_{k \geq j} 2^{-ks-\frac{k}{2}} + 2^{-\frac{j}{2}} \sum_{k \leq j-1} 2^{\frac{k}{2}-ks} \right] \\
 &\sim 2^{-ls} \sum_j 2^{2js} 2^{-\frac{1}{2}|j-l|} 2^{-js} \\
 &= 2^{\frac{l}{2}-ls} \sum_{j \geq l} 2^{js-\frac{1}{2}j} + 2^{-\frac{l}{2}-ls} \sum_{j \leq l-1} 2^{js+\frac{1}{2}j} \leq C. \quad \square
 \end{aligned}$$

Remark 6.8. The proof of Corollary 6.7 also continues to hold when from hypothesis (H) we drop the requirement that $\sigma(\Delta_V) = (-\infty, 0]$, but for f we require additionally $\langle f, \phi \rangle_{L^2} = 0$ for all eigenfunctions ϕ of Δ_V .

7 Proof of Lemma 3.6

LEMMA 7.1. For $V_1 = 2V + x \frac{d}{dx} V$, for $A(s)$ the operator in (3.2), and for $0 < s < 2$ we have for a constant $c(s)$

$$(7.1) \quad A(s) = c(s) \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V_1 (\tau - \Delta_V)^{-1} d\tau.$$

PROOF. Set $S := x \partial_x$. Recall the formula

$$(-\Delta_V)^{\frac{s}{2}} = c(s) (-\Delta_V) \int_0^\infty \tau^{\frac{s}{2}-1} (\tau - \Delta_V)^{-1} d\tau$$

for $0 < s < 2$ and $[c(s)]^{-1} = \int_0^\infty \tau^{\frac{s}{2}-1} (\tau + 1)^{-1} d\tau$. Then

$$(7.2) \quad A(s) = s(-\Delta_V)^{\frac{s}{2}} + c(s) \int_0^\infty \tau^{\frac{s}{2}-1} [S, -\Delta_V (\tau - \Delta_V)^{-1}] d\tau.$$

We have

$$\begin{aligned}
 & [S, -\Delta_V (\tau - \Delta_V)^{-1}] \\
 &= [S, -\Delta_V] (\tau - \Delta_V)^{-1} - \Delta_V [S, (\tau - \Delta_V)^{-1}] \\
 &= [S, -\Delta_V] (\tau - \Delta_V)^{-1} + \Delta_V (\tau - \Delta_V)^{-1} [S, -\Delta_V] (\tau - \Delta_V)^{-1}
 \end{aligned}$$

and also

$$[S, -\Delta_V] = [S, -\Delta] + [S, V] = 2\Delta + SV = 2(\Delta - V) + V_1 = 2\Delta_V + V_1.$$

Then we get

$$\begin{aligned}
 & [S, -\Delta_V(\tau - \Delta_V)^{-1}] \\
 &= 2\Delta_V(\tau - \Delta_V)^{-1} + 2\Delta_V^2(\tau - \Delta_V)^{-2} \\
 (7.3) \quad &+ V_1(\tau - \Delta_V)^{-1} + \Delta_V(\tau - \Delta_V)^{-1}V_1(\tau - \Delta_V)^{-1} \\
 &= 2\tau\Delta_V(\tau - \Delta_V)^{-2} + \tau(\tau - \Delta_V)^{-1}V_1(\tau - \Delta_V)^{-1}.
 \end{aligned}$$

Inserting this into (7.2) we get

$$\begin{aligned}
 (7.4) \quad A(s) &= s(-\Delta_V)^{\frac{s}{2}} + 2c(s) \int_0^\infty \tau^{\frac{s}{2}} \Delta_V(\tau - \Delta_V)^{-2} d\tau \\
 &+ c(s) \int_0^\infty \tau^{\frac{s}{2}} (\tau - \Delta_V)^{-1} V_1(\tau - \Delta_V)^{-1} d\tau.
 \end{aligned}$$

Then (7.1) follows from the fact that the first line of the right-hand side is 0: for $y > 0$ we have, integrating by parts,

$$-2c(s)y \int_0^\infty \tau^{\frac{s}{2}} (\tau + y)^{-2} d\tau = -2c(s)y \frac{s}{2} \int_0^\infty \tau^{\frac{s}{2}-1} (\tau + y)^{-1} d\tau = -sy^{\frac{s}{2}}. \quad \square$$

LEMMA 7.2. *Given hypothesis (H) there is a fixed $C = C(\|V\|_{L^{1,1}})$ such that for any $f \in \mathcal{S}(\mathbb{R})$ and at any $x \in \mathbb{R}$ we have*

$$(7.5) \quad |[(\tau - \Delta_V)^{-1} f](x)| \leq C \langle \tau \rangle^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-\sqrt{\tau}|x-y|} \langle y \rangle |f(y)| dy.$$

PROOF. Consider the Wronskian $w(\sqrt{\tau})$ defined in (6.7). Recall that, since $V \in \mathcal{S}(\mathbb{R})$, we have $w(\sqrt{\tau}) > 0$ for $\tau > 0$ and $w(\sqrt{\tau}) \sim \sqrt{\tau}$ as $\tau \rightarrow +\infty$. The hypothesis that $T(0) = 0$ implies that $w(0) > 0$.

We have

$$\begin{aligned}
 [(\tau - \Delta_V)^{-1} f](x) &= \int_{-\infty}^x \frac{m_+(x, \sqrt{\tau})m_-(y, \sqrt{\tau})}{w(\sqrt{\tau})} e^{-\sqrt{\tau}|x-y|} f(y) dy \\
 &+ \int_x^{+\infty} \frac{m_+(y, \sqrt{\tau})m_-(x, \sqrt{\tau})}{w(\sqrt{\tau})} e^{-\sqrt{\tau}|x-y|} f(y) dy.
 \end{aligned}$$

We will use $0 < w^{-1}(\sqrt{\tau}) < C_1 \langle \tau \rangle^{-1/2}$ for a fixed $C_1 = C(\|V\|_{L^{1,1}})$. Inequality (7.5) follows in elementary fashion by the following inequalities, where $C_2 = C(\|V\|_{L^{1,1}})$ is a fixed sufficiently large number:

- for $x \geq 0$ we have $|m_+(x, \sqrt{\tau})m_-(y, \sqrt{\tau})| \leq C_2 \langle y \rangle$,
- for $x \geq 0$ we have $|m_-(x, \sqrt{\tau})m_+(y, \sqrt{\tau})| \chi_{\mathbb{R}^+}(y - x) \leq C_2 \langle x \rangle \leq C_2 \langle y \rangle$,
- for $x < 0$ we have $|m_+(x, \sqrt{\tau})m_-(y, \sqrt{\tau})| \chi_{\mathbb{R}^+}(x - y) \leq C_2 \langle x \rangle \leq C_2 \langle y \rangle$,
- for $x < 0$ we have $|m_-(x, \sqrt{\tau})m_+(y, \sqrt{\tau})| \leq C_2 \langle y \rangle$. □

LEMMA 7.3. *Under hypothesis (H) there is a fixed C such that*

$$(7.6) \quad \|(\tau - \Delta_V)^{-1} V_1 (\tau - \Delta_V)^{-1} f\|_{L_x^1} \leq C \left(\|V\|_{L^{1,2}} + \left\| \frac{d}{dx} V \right\|_{L^{1,3}} \right) \tau^{-1} \langle \tau \rangle^{-1} \|f\|_{L_x^\infty}.$$

PROOF. We can factorize $V_1 = \langle x \rangle^{-2} V_2$ with $V_2 \in L^1(\mathbb{R})$. We have

$$\|(\tau - \Delta_V)^{-1} V_1 (\tau - \Delta_V)^{-1} f\|_{L_x^1} \leq \|(\tau - \Delta_V)^{-1} \langle x \rangle^{-1}\|_{L_x^1 \rightarrow L_x^1} \|V_2\|_{L_x^1} \|\langle x \rangle^{-1} (\tau - \Delta_V)^{-1}\|_{L_x^\infty \rightarrow L_x^\infty}.$$

and

$$\begin{aligned} \|(\tau - \Delta_V)^{-1} \langle \cdot \rangle^{-1} f\|_{L_x^1} &\leq C \langle \tau \rangle^{-\frac{1}{2}} \|e^{-\sqrt{\tau}|\cdot|} * (\langle \cdot \rangle |\langle \cdot \rangle^{-1} f(\cdot))\|_{L_x^1} \\ &\leq C' \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} \|f\|_{L_x^1}. \end{aligned}$$

The following bound with the same C' follows by duality:

$$\|\langle x \rangle^{-1} (\tau - \Delta_V)^{-1} f\|_{L_x^\infty} \leq C' \tau^{-\frac{1}{2}} \langle \tau \rangle^{-\frac{1}{2}} \|f\|_{L_x^\infty}.$$

Finally, by $V_2 = \langle x \rangle^2 (2V + xV')$ it follows that $\|V_2\|_{L^1} \lesssim \|V\|_{L^{1,2}} + \|\frac{d}{dx} V\|_{L^{1,3}}$. This yields inequality (7.6). \square

PROOF OF LEMMA 3.6. Inequality $\|A(s)f\|_{L_x^1} \leq C \|f\|_{L_x^\infty}$ for fixed $C > 0$ follows by Lemmas 7.1 and 7.3, which justify the following inequalities:

$$\begin{aligned} \|A(s)f\|_{L_x^1} &\leq c(s) \int_0^\infty \tau^{\frac{s}{2}} \|(\tau - \Delta_V)^{-1} V_1 (\tau - \Delta_V)^{-1} f\|_{L_x^1} d\tau \\ &\leq C' \|f\|_{L_x^\infty} \int_0^\infty \tau^{\frac{s}{2}-1} \langle \tau \rangle^{-1} d\tau \leq C \|f\|_{L_x^\infty} \end{aligned}$$

where the integral converges if $0 < s < 2$ and where $C = C(s, \|V\|_{L^{1,2}}, \|V'\|_{L^{1,3}})$. \square

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