The dynamics of a Bertrand duopoly with differentiated products: synchronization, intermittency and global dynamics.*

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Abstract

We study the dynamics of a duopoly game à la Bertrand with horizontal product differentiation as proposed by Zhang et al. (2009) by introducing opportune microeconomic foundations. The final model is described by a two-dimensional non-invertible discrete time dynamic system T. We show that synchronized dynamics occurs along the invariant diagonal being T symmetric; furthermore, we show that when considering the transverse stability, intermittency phenomena are exhibited. In addition, we discuss the transition from simple dynamics to complex dynamics and describe the structure of the attractor by using the critical lines technique. We also explain the global bifurcations causing a fractalization in the basin of attraction. Our results aim at demonstrating that an increase in either the degree of substitutability or complementarity between products of different varieties is a source of complexity in a duopoly with price competition.

Keywords: Synchronization; Contact bifurcations; Critical curves; Differentiated products; Duopoly; Price competition.

JEL Codes: C62; D43; L13

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1 Introduction

The present paper revisits the study of the dynamics of a nonlinear duopoly à la Bertrand (Bertrand 1883) with horizontal product differentiation analyzed by Zhang et al. (2009).[1]

Given the importance of product differentiation in the current industrial organization literature (see Singh and Vives, 1984), differently from Zhang et al. (2009), in this paper we develop a model with sound microeconomic foundations that determine the demand of differentiated products faced by each firm on the market. In this way, by using the degree of product differentiation as the key parameter some dynamic behaviors, which are of importance in both mathematics and economics, are discovered. From the former point of view, we establish that the unique interior fixed point of the two-dimensional system is locally stable if the degree of product differentiation is close to zero (i.e. products of both varieties tend to be independent). On the other hand, we show that if products tend to become homogeneous or complementary, then the Nash equilibrium is more likely to be destabilized, because the degree of horizontal product differentiation plays an ambiguous role on marginal profits and consequently on the reaction that every firm can adopt when setting the price to maximize profits in the future period.[2] From the latter point of view, therefore, policies aiming at reducing the degree of product differentiation tend to destabilize a duopolistic economy with price competition if players have limited information with regard to the profit function. Indeed, the burgeoning interest in nonlinear dynamic oligopolies (see, e.g., Dana and Montrucchio, 1986; Bischi et al., 2010; Rosser, 2010) has renewed the use of expectations formation mechanisms at all different from the paradigm defined by the rational expectations hypothesis. In fact, as claimed by Agliari et al. (2006, p. 527), "When one takes into account the fact that nonlinear dynamical systems can produce dynamic paths that are not so regular and predictable, one of the major arguments against adaptive expectations does not seem so strong.", because linear models represent an approximation of nonlinear models.[3]

The paper is organised as follows.

In Section 2 the Bertrand duopoly with horizontal differentiation is presented and the noninvertibility property is discussed. It plays a crucial role in the study of the topological structures of attracting sets of the map and related basins of attraction.

In Section 3 we determine the fixed points of the two-dimensional non-invertible map T. We also estabilish its invariant trapping sets and describe the dynamics occurring along the diagonal giving rise to synchronized trajectories. The question of synchronization in the long term is studied.

In Section 4 we deal with the case in which the invariant set on the diagonal loses transverse stability. We show how the attractor increases in complexity as the degree of substitutability or complementarity between products increases. We also study the global dynamics of T: the contact bifurcations related to tangiencies between critical curves and basin boundaries are described. They are responsible of changes in the basin structure while also causing a final bifurcation after which, through numerical simulations, it seems that any attractor at finite distance disappears.

Section 5 outlines the conclusions.

[1] For the notion of differentiated products see the original contributions by Hotelling (1929) and Chamberlin (1933).

[2] Of course, when products are perfect substitutes (homogeneous markets) the price equals the marginal cost. This result does not hold in Zhang et al. (2009).

[3] See also Kopel (1996).

2 The economic model

In order to describe the long-term behavior of a duopolistic market with price competition (Bertrand 1883) and horizontal product differentiation, we briefly present the economic setup leading to the final model studied in this paper. In particular, we explain the microeconomic foundations of the differentiated commodity setting and clarify the economic reasons why we assume specific demand and cost functions.

Consider an economy with two types of agents: firms and consumers.

The economy is bi-sectorial, i.e. there exists a competitive sector that produces the numeraire good $y \geq 0$, and a duopolistic sector with two firms, namely firm 1 and firm 2, each of which produces a differentiated good (or service), of variety 1 and variety 2, respectively. Denote with $p_i \geq 0$ and $q_i \geq 0$, $i = 1, 2$, the price and quantity of product of firm i, respectively.

Assume that there exists a continuum of identical consumers that have preferences towards $\underset{\sim}{q} = (q_1, q_2)$ and y. They are represented by a separable utility function $V(\underset{\sim}{q}, y) : \mathbb{R}^3_+ \to \mathbb{R}_+$ given by

$$V(\underset{\sim}{q}, y) = U(\underset{\sim}{q}) + y$$

where $U(\underset{\sim}{q}) : \mathbb{R}^2_+ \to \mathbb{R}_+$ is a twice differentiable function. The representative consumer maximizes $V(\underset{\sim}{q}, y)$ with respect to quantities $\underset{\sim}{q}$ subject to the budget constraint

$$p_1 q_1 + p_2 q_2 + y = M,$$

where $M \geq 0$ denotes the consumer's exogenously given income.

The representative consumer's optimization problem is then given by

$$\max_{q_1, q_2} U(q_1, q_2) - p_1 q_1 - p_2 q_2 + M.$$

By solving this problem, one yields the following inverse demand functions

$$p_i = \frac{\partial U}{\partial q_i} = p_i(\underset{\sim}{q}), \quad i = 1, 2 \tag{1}$$

representing the price of the goods and services produced by firm i as a function of the quantities produced by both firms.

Following Dixit (1979) and Singh and Vives (1984), we assume that the utility function is quadratic (leading to linear demand functions) and given by:

$$U(q_1, q_2) = a(q_1 + q_2) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2d q_1 q_2\right), \tag{2}$$

where $a > 0$ is a parameter that captures the size of the market demand and $-1 < d < 1$ represents the degree of horizontal product differentiation.

Observe that if $d = 0$, then the products of variety 1 and 2 are independent, that is each firm behaves as if it were a monopolist; if $d = 1$, then products 1 and 2 are perfect substitutes or, alternatively, homogeneous (in this case, the Nash equilibrium is such that the price equals the marginal cost); $0 < d < 1$ describes the case of imperfect substitutability. More precisely the degree of substitutability increases, or equivalently, the extent of product differentiation decreases as the parameter d raises. In contrast, a negative value of d implies that goods 1 and 2 are complements, while $d = -1$ reflects the case of perfect complementarity.

From equation (2) it follows that

$$\frac{\partial U}{\partial q_i} = a - q_i - dq_j, \quad i, j = 1, 2, i \neq j.$$

By substituting the previous formula in (1), the inverse downward sloping demand functions of products of the variety i are as follows:

$$p_i\left(q_i, q_j\right) = a - q_i - dq_j, \quad i, j = 1, 2, i \neq j. \tag{3}$$

By inverting the inverse demand functions, one obtains the direct demand functions describing the quantity of goods and services produced by firm i as a function of the prices of the products of both firms, i.e. $q_i = q_i\left(\underline{\mathrm{p}}\right)$.

From (3), the corresponding direct demand function of the products of variety 1 and variety 2 are given by:

$$q_1\left(p_1, q_2\right) = a - p_1 - dq_2 \tag{4}$$

and

$$q_2\left(p_2, q_1\right) = a - p_2 - dq_1. \tag{5}$$

By combining (4) and (5), we obtain:

$$q_1\left(p_1, p_2\right) = \frac{1}{1-d^2}\left[a\left(1-d\right) - p_1 + dp_2\right], \tag{6}$$

$$q_2\left(p_1, p_2\right) = \frac{1}{1-d^2}\left[a\left(1-d\right) - p_2 + dp_1\right]. \tag{7}$$

Following Correa-López and Naylor (2004), we assume that firm i produces goods and services of variety i through a production function with constant (marginal) returns to labor given by $q_i = L_i$, where $L_i \geq 0$ represents the labor force employed by firm i. In addition, the firms face the same (constant) average and marginal cost $w \geq 0$. Therefore, firm i's cost function is linear and given by:

$$C_i\left(q_i\right) = w\, L_i = w\, q_i \quad i = 1, 2. \tag{8}$$

Hence profits of firm i can be written as follows:

$$\pi_i = p_i q_i - w q_i = \left(p_i - w\right)\, q_i \quad i = 1, 2. \tag{9}$$

By substituting (6) and (7) into (9), profits of firm 1 and firm 2 are respectively given by:

$$\pi_1\left(p_1, p_2\right) = \frac{p_1 - w}{1-d^2}\left[a\left(1-d\right) - p_1 + dp_2\right], \tag{10}$$

and

$$\pi_2\left(p_1, p_2\right) = \frac{p_2 - w}{1-d^2}\left[a\left(1-d\right) - p_2 + dp_1\right]. \tag{11}$$

Therefore, the marginal profits are obtained as follows:

$$\frac{\partial \pi_1\left(p_1, p_2\right)}{\partial p_1} = \frac{a\left(1-d\right) - 2p_1 + dp_2 + w}{1-d^2}, \tag{12}$$

$$\frac{\partial \pi_2\left(p_1, p_2\right)}{\partial p_2} = \frac{a\left(1-d\right) - 2p_2 + dp_1 + w}{1-d^2}. \tag{13}$$

Consider now a dynamic setting such that, at any time $t \in \mathbb{Z}_+$, the profit of firm i depends on the price set by both firms at time t. Equations (10) and (11) can then be written as follows:

$$\pi_{i,t}(p_{i,t}, p_{j,t}) = \frac{p_{i,t} - w}{1 - d^2} [a(1-d) - p_{i,t} + dp_{j,t}], \quad i, j = 1, 2, i \neq j$$

while equations (12) and (13) are given by:

$$\frac{\partial \pi_{i,t}(p_{i,t}, p_{j,t})}{\partial p_{i,t}} = \frac{a(1-d) - 2p_{i,t} + dp_{j,t} + w}{1 - d^2}, \quad i, j = 1, 2, i \neq j. \tag{14}$$

It is now important to specify the set of information of both players regarding profit functions (i.e., demand and cost functions), to determine the behavior of prices over time. If a player has complete knowledge of the profit function, then it will use some form of expectations about its rival's price decision (for example, naive, rational or adaptive expectations, or, alternatively, some weighted sum of previous choices) to set the price at time $t+1$. In contrast, if a player does not have complete knowledge of the profit function, it can use some form of local estimation (where *local* means at the current production state) of the marginal profit in order to follow the steepest (local) slope of the profit function. This because under the hypothesis of limited information, players are unable to completely solve the optimization problem $\max_{p_{i,t+1}} \pi_{i,t+1}\left(p_{i,t+1}, p^e_{j,t+1}\right)$ by considering expectations about the price that the competitor will choose for the next period, but they are able to get a correct estimate of their own local slope, i.e. the partial derivative of the profit function computed at the current state of production, so that firm i increases or decreases the price at time $t+1$ depending on whether its own marginal profits at time t are positive or negative.

Therefore, by assuming limited information and following Dixit (1986) and Bischi and Naimzada (1999), the adjustment mechanism of prices over time of the ith firm is described by:

$$p_{i,t+1} = p_{i,t} + \alpha_i p_{i,t} \frac{\partial \pi_{i,t}}{\partial p_{i,t}}, \tag{15}$$

where $\alpha_i > 0$ is a coefficient that captures the speed at which firm i adjusts its price with respect to the consequent marginal change in its profits.

Taking into account equation (15), the two-dimensional system that characterizes the dynamics of the Bertrand duopoly with horizontal differentiation is the following:

$$T := \begin{cases} p_{1,t+1} = p_{1,t} + \alpha p_{1,t} \frac{\partial \pi_{1,t}}{\partial p_{1,t}} \\ p_{2,t+1} = p_{2,t} + \alpha p_{2,t} \frac{\partial \pi_{2,t}}{\partial p_{2,t}} \end{cases}, \tag{16}$$

where, we set $\alpha_1 = \alpha_2 = \alpha$ in order to consider the case in which the speed of adjustment is equal for both firms (the case with $\alpha_1 \neq \alpha_2$ is left for further studies). By taking into account (14), system (16) becomes the following:

$$T := \begin{cases} p_{1,t+1} = p_{1,t} + \frac{\alpha p_{1,t}}{1-d^2} [a(1-d) - 2p_{1,t} + dp_{2,t} + w] \\ p_{2,t+1} = p_{2,t} + \frac{\alpha p_{2,t}}{1-d^2} [a(1-d) - 2p_{2,t} + dp_{1,t} + w] \end{cases}, \tag{17}$$

or equivalently

$$T : \begin{cases} x' = \left[1 + \frac{\alpha a}{1+d} + \frac{\alpha w}{1-d^2}\right] x - 2\frac{\alpha}{1-d^2} x^2 + \frac{\alpha d}{1-d^2} xy \\ y' = \left[1 + \frac{\alpha a}{1+d} + \frac{\alpha w}{1-d^2}\right] y - 2\frac{\alpha}{1-d^2} y^2 + \frac{\alpha d}{1-d^2} xy \end{cases}, \tag{18}$$

where $x' = p_{1,t+1}$, $x = p_{1,t}$, $y' = p_{2,t+1}$, and $y = p_{2,t}$, which is a two-dimensional dynamic system, continuous and differentiable in $\mathbb{R}^2_+$.

Consider now, for any given value of the speed of adjustment α, the role played by the (horizontal) differentiation parameter (d) on marginal profits. The extent and direction of the movement of future prices as d varies is ambiguous in the cases of both substitutability ($0 < d < 1$) and complementarity ($-1 < d < 0$). In the former (resp. latter) case, a reduction (resp. an increase) in the degree of substitutability (resp. complementarity) between products of variety 1 and variety 2, i.e., the parameter d moves towards 1 (resp. -1), directly reduces (resp. increases) the reaction of the ith firm, because it tends to reduce (resp. increase) its marginal profits, while also playing an indirect positive (resp. negative) effect through the reply of the rival (i.e., firm j) when d varies.

2.1 Noninvertibility property

Map (18) is a noninvertible map on $\mathbb{R}^2_+$. In fact, given the initial condition $(x(0), y(0)) \in \mathbb{R}^2_+$, the forward iteration of T defines a unique trajectory, while its backward iteration is not uniquely defined since a point belonging to $\mathbb{R}^2_+$ may have more than one preimage (or, equivalently, different initial conditions may be mapped into the same point under T)[4] so that two, or more, distinct points are mapped into the same point, i.e. rank-1 preimages $(x, y) = T^{-1}(x', y')$ may not exist or may not be unique.

By considering map T in equation (18), it can be noticed that rank-1 preimages of a given point $(x', y') \in \mathbb{R}^2_+$ are solutions of the fourth degree algebraic system

$$(x', y') = T(x, y),$$

and consequently it can have four or two real solutions or no real solution at all. As a consequence, T is of $Z_4 - Z_2 - Z_0$ type as $\mathbb{R}^2_+$ can be subdivided into regions whose points have 4, 2 or 0 preimages and the boundaries of such regions are characterized by the existence of at least two coincident (merging) preimages. Following the notation by Mira et al. (1996) and Abraham et al. (1997), we denote the critical curve of rank-1 by LC (it represents the locus of points having two or more coincident preimages) and the curve of merging preimages by LC_{-1}.

The locus LC_{-1} for a two dimensional continuous and differentiable map is given by the set of points such that the determinant of the Jacobian matrix is null. For system T the Jacobian matrix is given by

$$J(x, y) = \begin{pmatrix} 1 + \frac{\alpha[a(1-d)+w]}{1-d^2} - 4\frac{\alpha}{1-d^2}x + \frac{\alpha d}{1-d^2}y & \frac{\alpha d}{1-d^2}x \\ \frac{\alpha d}{1-d^2}y & 1 + \frac{\alpha[a(1-d)+w]}{1-d^2} - 4\frac{\alpha}{1-d^2}y + \frac{\alpha d}{1-d^2}x \end{pmatrix}. \tag{19}$$

so that its determinant vanishes in points $(x, y) \in \mathbb{R}^2_+$ such that

$$A^2 + BC(x^2 + y^2) + A(B + C)(x + y) + B^2xy = 0 \tag{20}$$

where $A = 1 + \frac{\alpha[a(1-d)+w]}{1-d^2}$, $B = -4\frac{\alpha}{1-d^2}$ and $C = \frac{\alpha d}{1-d^2}$.

The set of points such that condition (20) holds is given by the union of two branches namely $LC^{(a)}_{-1}$ and $LC^{(b)}_{-1}$ (see Figure 1 panel (a)). Curves LC given by $LC^{(a)} = T(LC^{(a)}_{-1})$ and $LC^{(b)} = T(LC^{(b)}_{-1})$ separate $\mathbb{R}^2_+$ into regions having a different number of preimages (see Figure 1 panel (b)).

[4]See Mira et al. (1996) for a description of the main properties of noninvertible maps of the plane.

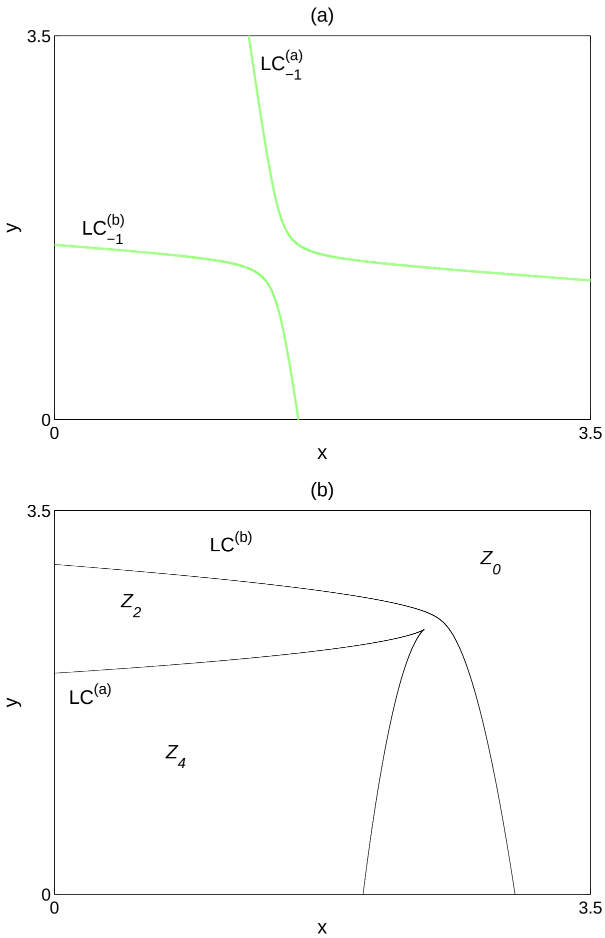

Figure 1: (a) Critical curves of rank-0, LC_{-1}, for system T and the following parameter values: $\alpha = 0.5, a = 3, w = 0.5, d = -0.395$. (b) Critical curves of rank-1, $LC = T(LC_{-1})$, for the same parameter values as in panel (a). These curves separate the plane into the regions Z_4, Z_2 and Z_0, whose points have different number of preimages.

The noninvertibility property plays a crucial role in the study of the global dynamics of system T. In particular it is useful in the investigation of the topological structures of the attracting sets of the map and of the related basins of attraction, as it will be shown later in this paper.

3 Fixed points, invariant sets and synchronization

3.1 Fixed points and local stability

Equilibria or fixed points of T are solutions of the following equation:

$$T(x,y) = (x,y).$$

Trivially the following proposition holds:

Proposition 1. *System T given by (18) admits four fixed points for all parameter values. They are given by:*

$$\begin{array}{l} E_0 = (0,0)\,, \quad E_1 = \left(0, \frac{1}{2}\left[a\left(1-d\right)+w\right]\right), \quad E_2 = \left(\frac{1}{2}\left[a\left(1-d\right)+w\right], 0\right), \\ \text{and } E^* = \left(\frac{a(1-d)+w}{2-d}, \frac{a(1-d)+w}{2-d}\right). \end{array} \tag{21}$$

Notice that fixed points E_0, E_1 and E_2 are located on the invariant coordinate axes and that E_1 and E_2 are in symmetrical positions with respect to the main diagonal

$$\Delta = \{(x,y) \in \mathbb{R}_+^2 : x = y\}.$$

On the other hand, $E^* \in \Delta$ is the unique interior Nash equilibrium of this model.

The previous evidence is due to the fact that, since the two players are identical, system T must remain the same under the exchange of the players, that is $T \circ S = S \circ T$ where $S : (x,y) \to (y,x)$ is the reflection through the diagonal Δ. We call this *symmetry property*. It is worth to recall also that, since map T is symmetric, then either an invariant set (attractors, basins of attraction, etc.) of the map is symmetric w.r.t. Δ, or its symmetric set is invariant as well.

By substituting out the equilibrium price x^* into the direct demand functions (equations (6) and (7)), and profit function (equation (9)), we obtain the equilibrium values of quantities and profits of both firms related to the unique interior fixed point, respectively, that is:

$$q^* = \frac{a-w}{(2-d)(1+d)}, \tag{22}$$

$$\pi^* = \frac{(a-w)^2(1-d)}{(2-d)^2(1+d)}. \tag{23}$$

From equation (22) it can easily be seen that $a > w$ must hold to ensure $q^* > 0$, while from equation (23) we observe that the case of perfect substitutability ($d = 1$) perfectly replicates the original result by Bertrand (1883), as prices of both firms equal the average and marginal cost w and profits are zero.

The local stability analysis of the four fixed points of T can be carried out by considering the Jacobian matrix associated with system T given by (19).

We observe that, since the eigenvalues of a diagonal or triangular matrix are given by the elements of the main diagonal, then E_0 is an unstable node while E_1 and E_2 can be both unstable nodes or saddle points (in this case the invariant axes are the stable manifolds of the saddle points).

About the local stability of the unique interior fixed point E^*, we recall that the Jacobian matrix associated with system T is given by (19). Since T is symmetric with respect to Δ, then its Jacobian matrix evaluated at a point on the diagonal Δ is of the kind

$$J(x,x) = \begin{pmatrix} J_1(x) & J_2(x) \\ J_2(x) & J_1(x) \end{pmatrix},$$

where

$$J_1 = 1 + \frac{\alpha[a(1-d)+w]}{1-d^2} + \frac{\alpha}{1-d^2}(d-4)x \tag{24}$$

and

$$J_2 = \frac{\alpha d}{1-d^2}x. \tag{25}$$

As a consequence, the eigenvalues of $J(x,x)$ are both real and they are given by:

$$\lambda_{\|}(x) = J_1(x) + J_2(x) = 1 + \frac{\alpha[a(1-d)+w]}{1-d^2} + 2\frac{\alpha}{1-d^2}(d-2)x$$

and

$$\lambda_{\perp}(x) = J_1(x) - J_2(x) = 1 + \frac{\alpha[a(1-d)+w]}{1-d^2} - 4\frac{\alpha}{1-d^2}x$$

while the corresponding eigenvectors are respectively given by $\underline{v}_{\|} = (1,1)$ and $\underline{v}_{\perp} = (1,-1)$.

The eigenvalues evaluated at the fixed point E^* are then given by

$$\lambda_{\|}(E^*) = 1 - \frac{\alpha[a(1-d)+w]}{1-d^2}$$

and

$$\lambda_{\perp}(E^*) = 1 - \frac{2+d}{2-d}\frac{\alpha[a(1-d)+w]}{1-d^2}$$

so that only the interior fixed point E^* can be attractive for suitable values of the parameters such that both $\lambda_{\|}(E^*)$ and $\lambda_{\perp}(E^*)$ belong to the set $(-1,1)$.

Before studying the stability of the interior fixed point and the bifurcations occurring as some parameters are changed, it is very important to observe that an attractor at finite distance of system (18) cannot be globally attracting in $\mathbb{R}^2_+$. To show this fact let $y(0) = kx(0)$ with $k \in [0,1]$. Then if $x(0) > 0$ is sufficiently large, the first iterate of T gives a negative value of x', that is $x(1) < 0$ (for the symmetry property the same behavior holds for y' while considering $x(0) = ky(0)$, $k \in [0,1]$, if $y(0) > 0$ is sufficiently large). This means that $(x(1), y(1))$ exits the set $\mathbb{R}^2_+$; furthermore all the subsequent iterates never enter $\mathbb{R}^2_+$. In fact, bytaking into account the second iterate, it can easily be shown that, if at least one component of the pair $(x(1), y(1))$ is negative, then $(x(2), y(2)) = T(x(1), y(1))$ has at least one negative component. Trivially, this fact implies that if $(x(i), y(i)) \notin \mathbb{R}^2_+$ then $T^t(x(i), y(i)) \notin \mathbb{R}^2_+$, $\forall t \in \mathbb{Z}_+$. In particular, numerical computations show that system (18) always generates unbounded trajectories if the initial condition $(x(0), y(0))$ belongs to a suitable neighborhood of infinity.

3.2 Invariant sets and synchronized trajectories

We now want to consider the invariant sets of system T and their stability. First of all, we observe that $T(x,0) = (x',0)$ and $T(0,y) = (0,y')$, i.e. each coordinate semiaxis is invariant for map (18) and consequently the dynamics of T on such lines are governed by the following one-dimensional map:

$$x' = \phi_0(x) = \left[1 + \frac{\alpha a}{1+d} + \frac{\alpha w}{1-d^2}\right] x - 2\frac{\alpha}{1-d^2}x^2. \tag{26}$$

It is easy to prove that $T(x,0) = T(0,x) = \phi_0$ and that map (26) is topologically conjugate to the standard logistic map $\omega' = \lambda\omega(1-\omega)$ through the linear transformation

$$x = \left[\frac{1-d^2}{2\alpha} + \frac{1}{2}[a(1-d)+w]\right]\omega, \tag{27}$$

where

$$\lambda = 1 + \frac{\alpha}{1-d^2}(a(1-d)+w).$$

Thus the dynamics generated by map T on the two invariant axes are completely known, as these can be obtained from those of the logistic map.

Observe now that also the diagonal Δ is invariant for T, i.e. $T(x,x) = (x',x')$; this means that equal initial conditions imply equal dynamic behavior forever. As a consequence $T(\Delta) \subseteq \Delta$ and the dynamics generated by T on the invariant submanifold Δ can be studied through the restriction of system T to the set Δ given by

$$T_\Delta = \phi(x) = \left[1 + \frac{\alpha a}{1+d} + \frac{\alpha w}{1-d^2}\right] x - \frac{\alpha}{1-d^2}(2-d)x^2, \tag{28}$$

that is topologically conjugate to the logistic map $z' = \mu z(1-z)$ with

$$\mu = \lambda = 1 + \frac{\alpha}{1-d^2}(a(1-d)+w), \tag{29}$$

by the linear transformation

$$x = \frac{1}{2-d}\left(\frac{1-d^2}{\alpha} + a(1-d) + w\right) z.$$

Thus also the dynamics generated by map T on the diagonal are completely known, since these can be obtained from those of the logistic map (see Devaney 2003).

Our previous considerations enable us to completely understand the dynamics of the model on both the coordinate axes and the main diagonal.

More precisely, trajectories embedded into Δ, i.e. those characterized by $x = y$ for all t, are called *synchronized trajectories* (see Bischi and Gardini 2000 and Bischi et al. 1998). The phenomenon of synchronization arises when the diagonal is an invariant one-dimensional submanifold of $\mathbb{R}^2$. As we have proved, Δ is an invariant one-dimensional submanifold, since T is obtained by coupling two identical one-dimensional maps and synchronized trajectories are governed by ϕ (with regard to systems of coupled logistic maps see Bischi and Gardini 1998 and Bischi et al. 1998).

We now study the synchronized trajectories of system T, i.e. the properties of the sequences generated when both firms start from the same initial state, i.e. $x(0) = y(0)$. Let

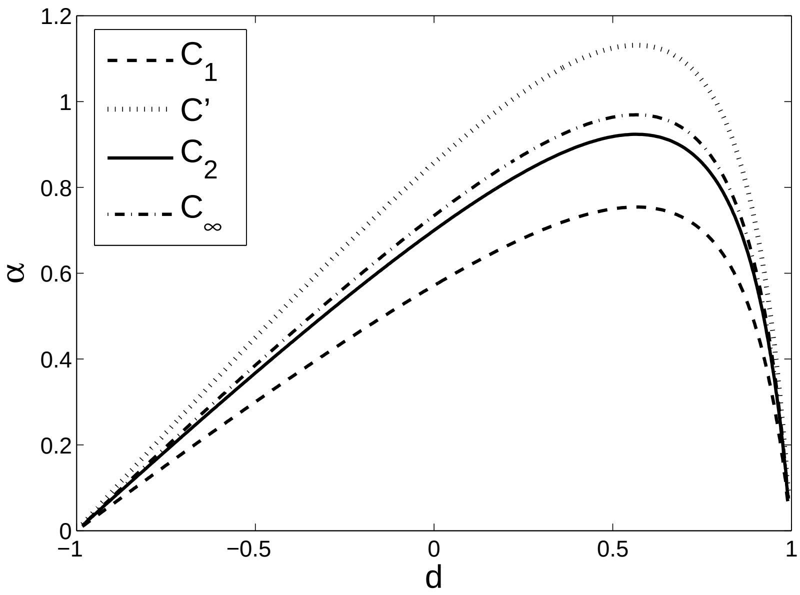

Figure 2: Bifurcation curves on the parameter plane (d, α) related to the bifurcation values of map ϕ for $a = 3$ and $w = 0.5$ being $x(0) = 0.3$.

$A \subset \Delta$ be the attractor of map ϕ. Then from the well-known properties of the logistic map (see Devaney, 2003) we can find suitable values of μ such that the synchronized trajectories converge to a cycle or a $m-$piece chaotic attractor etc.

Since we are interested in the role played by parameters d and α, in Figure 2 we depict different bifurcation curves in the parameter plane (d, α) corresponding to the set of points such that

$$\mu = 1 + \frac{\alpha}{1-d^2}(a(1-d)+w),$$

where μ is equal to some bifurcation values of the logistic map (μ_1, μ_2, μ_∞ and μ' respectively). The set $S^* = \{(d,\alpha) : 1 < \mu < 3\}$, given by points belonging to $\mathbb{R}^2_+$ that are located below the curve C_1, are such that z^* is stable (i.e. the synchronized trajectories converge to the unique interior fixed point), while the set $S' = \{(d,\alpha) : \mu > 4\}$, given by the points belonging to $\mathbb{R}^2_+$ that are located above the curve C', characterizes the parameter region corresponding to divergence. The region between curves C_1 and C_2 is associated with the existence of a two-period cycle to which the duopoly model converges if the firms start from the same initial condition; when passing from the points on C_2 to curve C_∞, the standard period doubling cascade occurs.

About the existence of divergent trajectories the following Proposition holds.

Proposition 2. *Let T be given by (18). Then, a threshold $\bar{\alpha}$ (or $\bar{a}$ or $\bar{w}$, respectively) does exist such that synchronized trajectories are divergent $\forall \alpha > \bar{\alpha}$ (or $a > \bar{a}$ or $w > \bar{w}$, respectively), given the other parameter values.*

Proof. This fact immediately follows when considering that the synchronized trajectories of T are those governed by ϕ, which behaves as the logistic map. Hence the generic trajectory diverges if $\mu > 4$, that is

$$\frac{\alpha}{1-d^2}(a(1-d)+w) > 3.$$

This relation holds if α (or a or w) is great enough. For instance if

$$\alpha > \frac{3(1-d^2)}{a(1-d)+w} = \bar{\alpha}.$$

□

The result proved in Proposition 2 is immediately confirmed when observing Figure 2, as the set S' is unbounded in $(-1,1)\times(0,+\infty)$. Observe also that, according to Proposition 2, a necessary condition for the attractor A on the main diagonal to exist is that parameters α, a and w are not too large.

Since we are wondering about the stability effects of horizontal product differentiation, in what follows we take d as the parameter of interest. As it is known from the existing literature on the dynamics of oligopoly models (see, amongst many others, Bischi and Naimzada, 1999; Agiza and Elsadany, 2003, 2004; Zhang et al., 2007; 2009; Tramontana, 2010), when at least one of the two players has limited information about profit functions, the higher the speed of adjustment α, the more likely the destabilization of the equilibrium of the map (see Proposition 2).

About the role played by d, from Figure 2 it is easy to observe that the positive fixed point z^* is stable for intermediate values of d if α is small enough (as set S^* contains points of the kind $(d,\alpha) : d \in (d_1,d_2)$ and $\alpha < \bar{\alpha}(d)$). This fact is proved in the following Proposition.

Proposition 3. *Let ϕ be given by (28).*
Then, a threshold value $\bar{N} > 2$ and $-1 < d_1 < d_2 < 1$ exist such that if $\alpha(a+w) < \bar{N}$ then z^* is a locally stable fixed point of ϕ for all $d \in (d_1,d_2)$.

Proof. Let ϕ be given by 28. Then z^* is locally stable if and only if

$$1+\frac{\alpha}{1-d^2}(a(1-d)+w) \in (1,3) \Leftrightarrow \frac{\alpha}{1-d^2}(a(1-d)+w) < 2 \Leftrightarrow$$

$$f(d) = -a\alpha d + \alpha(a+w) < 2(1-d^2) = g(d).$$

Recall that $d \in (-1,1)$. Function $g(d)$ is a concave parabola s.t. $g(-1) = g(1) = 0$ that has its maximum at $(0,2)$; function $f(d)$ is linear, decreasing, $f(1) = \alpha w > 0$ and it has a positive intercept given by $N = \alpha(a+w)$. Hence a value $\bar{N} > 2$ does exist such that f and g intersect each other at two points, namely d_1 and d_2 (being $-1 < d_1 < d_2 < 1$) for all $N < \bar{N}$. In such a case $f(d) < g(d)\ \forall d \in (d_1,d_2)$. □

From the previous Proposition the following Remark trivially holds.

Remark 4. *If $\alpha(a+w) < 2$ then z^* is a locally stable fixed point of ϕ for all $d \in (d_1,d_2)$, where $-1 < d_1 < d_2 < 1$, and $0 \in (d_1,d_2)$.*

According to Proposition 3, the local stability of the fixed point is necessarily associated with intermediate values of d, as it is shown in Figure 3. In particular, if Remark 4 holds, then z^* is locally stable for $d = 0$ and it loses stability via a period doubling bifurcation (the sequence of bifurcations related to the logistic map occurs) due to an increase in the degree of substitutability (resp. complementarity) between products, i.e. d moves from 0 to 1 (resp. to -1). This fact shows that synchronized dynamics increases in complexity while moving from the case of products of independent varieties to complementary or substitutable, while, at the limit cases ($d \to \pm 1$) no bounded dynamics occurs on Δ. In any case, if the generic trajectory does not diverge, then it is bounded inside an absorbing interval $I = [\phi^2(c_{-1}), \phi(c_{-1})]$ (being c_{-1} the critical point of ϕ) and it may consist in either a cycle or more complex set. The attractor A belongs to I, and I corresponds to a segment on the diagonal that is absorbing w.r.t. perturbation along Δ.

Similar arguments to that used to prove Proposition 3 can be taken into account to obtain the following result.

Proposition 5. *A threshold value $\bar{\bar{N}} > \bar{N}$ does exist such that if $\alpha(a + w) < \bar{\bar{N}}$, then ϕ admits an attractor $A \subseteq I$ for all $d \in (\bar{d}_1, \bar{d}_2)$, where $\bar{d}_1 \in (-1, d_1)$ and $\bar{d}_2 \in (d_2, 1)$.*

This result guarantees the existence of an attractor at finite distance of system T restricted to Δ for opportune parameter values such that $\alpha(a+w)$ is not too large (i.e. if the speed of adjustment α is sufficiently small for any given value of w and a). In addition, it is shown that A exists iff d belongs to a given neighborhood of the origin.

In this subsection we have studied the dynamics of synchronized trajectories, i.e. we have proved that, for all $\mu \leq 4$ system T admits an attractor $A \subseteq I$ on the diagonal. Obviously, if the firms start from the same initial condition and $\mu \leq 4$, then the duopoly model converges to A in the long term. Thus the fate of any synchronized trajectory (i.e. any trajectory starting from identical initial conditions) is known.

An important question arising is whether an attractor $A \subset \Delta$ of ϕ is also an attractor of the two-dimensional map T on the invariant submanifold Δ. In fact, it is of importance to know whether a duopoly with identical players starting from different initial conditions (i.e. $(x(0), y(0)) \notin \Delta$) evolves toward synchronization.

3.3 Synchronization

In this subsection we consider producers starting from different initial conditions and we study the mechanisms that can lead to the synchronization of trajectories. We recall that a trajectory starting out from Δ, that is with $x(0) \neq y(0)$, is said to synchronize if $|x(t) - y(t)| \to 0$ as $t \to +\infty$.

A natural question arising in our framework is whether identical competitors starting with different initial conditions synchronize in the long term. As an attracting set $A \subseteq I$ is stable w.r.t. perturbations along Δ, therefore in order to study the stability of A for T one has to consider the transverse stability (hence the stability of A w.r.t. perturbations transverse to Δ).

Obviously, at least partial synchronization is achieved when a transversely stable orbit exists on the diagonal (in the sense that it attracts points that do not belong to the diagonal itself). Such an attractor can also coexist with non synchronizing trajectories, as we will see later in this paper.

In order to see whether $I \subseteq \Delta$ is a trapping segment for system T, its transverse attractivity must be studied. For the fixed points and cycles embedded into the invariant line Δ, the stability conditions along Δ are the same as for the corresponding fixed points and cycles of the quadratic map $\phi(x)$. Therefore, in what follows we consider the transverse stability of invariants sets located on Δ.

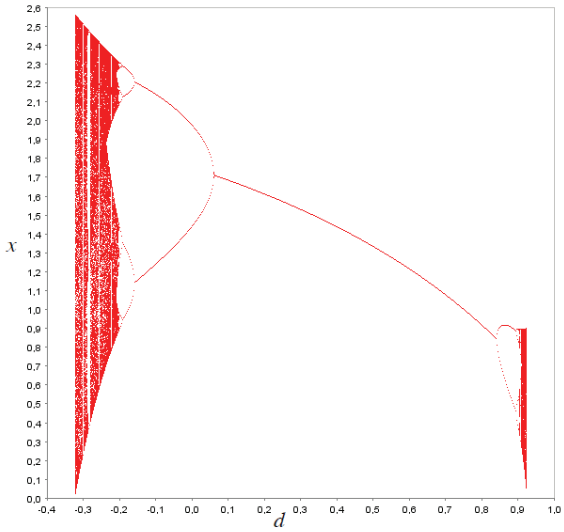

Figure 3: Bifurcation diagram of map ϕ w.r.t. d for $\alpha = 0.6$, $a = 3$ and $w = 0.5$ with $x(0) = 0.3$

Recall that for the fixed point E^* the transverse eigenvalue is given by:

$$\lambda_{\perp}(E^*) = 1 - \frac{2+d}{2-d}\frac{\alpha[a(1-d)+w]}{1-d^2}.$$

Hence E^* is transversally attracting iff

$$\frac{2+d}{2-d}\frac{\alpha[a(1-d)+w]}{1-d^2} \in (0,2), \tag{30}$$

and the following Proposition holds.

Proposition 6. *Let T be given by (18). Then, a threshold value $\bar{Q} > 0$ and $-1 < d_1^* < d_2^* < 1$ exist such that if $\alpha(a+w) < \bar{Q}$, then E^* is an asymptotically stable node for all $d \in (d_1^*, d_2^*)$.*

Proof. Let T be given by (18). Then E^* is a stable node if and only if condition (30) holds and Proposition 3 applies. Recall that Proposition 3 applies iff $\alpha(a+w) < \bar{N}$ and $d \in (d_1, d_2)$ (with $\bar{N} > 2$ and $-1 < d_1 < d_2 < 1$). On the other hand, condition (30) holds iff

$$f(d) = -a\alpha d + \alpha(a+w) < 2(1-d^2)\frac{2-d}{2+d} = G(d)$$

and $d \in (-1,1)$. After some algebra it can be observed that $G(d)$ is a continuous and differentiable function such that $G(-1) = G(1) = 0$, and it has a unique maximum point at $\bar{d} \in (-1,0)$. Since function $f(d)$ is linear, decreasing, such that $f(1) = w > 0$ and it has a positive intercept given by $P = \alpha(a+w)$, then a threshold $\bar{P}$ does exist such that f and G intersect each other at two points, namely d_1' and d_2' (with $-1 < d_1' < d_2' < 1$) for all $P < \bar{P}$. In such a case $f(d) < G(d)$ $\forall d \in (d_1', d_2')$. Let $\bar{Q} = \min\{\bar{N}, \bar{P}\}$ and $(d_1^*, d_2^*) = (d_1, d_2) \cap (d_1', d_2')$, then the statement is proved. □

From Proposition 6, it follows that the unique interior equilibrium is locally asymptotically stable if $\alpha(a+w)$ is not too large and parameter d assumes intermediate values. Obviously, in such a case the trajectories that start from a neighborhood of E^* synchronize, i.e. if firms start from different initial conditions, they will behave in the same way in the long term.

In addition, let $D = (d_1, d_2)$, $D' = (d_1', d_2')$ and $D^* = (d_1^*, d_2^*)$. As long as $d \in D^*$ then the trajectories synchronize, while it can easily be observed that E^* is a saddle for all $d \in (D \cup D') - D^*$ with unstable (or stable) set along Δ and stable (or unstable) set orthogonal to it, if $d \in (D' - D*)$ (or $d \in (D - D*)$) .

We now want to consider the case in which attractor A on the diagonal is different from the fixed point E^*. Recall that bounded dynamics on Δ occur if $\mu \in (1,4]$ corresponding to the condition

$$\frac{\alpha[a(1-d)+w]}{1-d^2} \in (0,3] \tag{31}$$

that is, a set $\bar{D} = (\bar{d}_1, \bar{d}_2)$ does exist such that if $d \in \bar{D}$ and $\alpha(a+w)$ is less enough, then ϕ admits an attractor $A \subseteq I$.

In order to study the transverse stability of A, recall that for an m-cycle $\{(x_1,x_1),...,(x_m,x_m)\}$ of T embedded into the invariant line Δ where synchronized dynamics take place and corresponding to the cycle $\{x_1,...,x_m\}$ of ϕ, the multipliers are given by

$$\lambda_{\parallel}^{(m)} = \prod_{i=1}^{m} \lambda_{\parallel}(x_i) \text{ and } \lambda_{\perp}^{(m)} = \prod_{i=1}^{m} \lambda_{\perp}(x_i),$$

where $\lambda_{\|}(x_i)$ and $\lambda_{\perp}(x_i)$ are the eigenvalues of the Jacobian matrix evaluated at a point (x_i, x_i) associated with eigenvectors parallel to Δ and with eigenvectors normal to Δ, respectively.

Consider now a more complex situation, i.e. A is a chaotic attractor on Δ. In such a case A includes infinitely many periodic orbits which are unstable in the direction along Δ. For any of these cycles the associated eigenvalues can be obtained. Making use of the eigenvalue $\lambda_{\perp}^{(m)}$ and of the transverse Lyapunov exponent, we can study the transverse stability of the chaotic attractor A, that is we can investigate whether initial conditions that are not synchronized give trajectories converging to it. We recall that, given a chaotic set $A \subset \Delta$, the *transverse Lyapunov exponent* is defined as

$$\Lambda_{\perp} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N} \ln |\lambda_{\perp}(x_n)| \tag{32}$$

where $x_0 \in A$ and x_n is a generic trajectory generated by ϕ. If A is an $m-$cycle then $\Lambda_{\perp} = \ln |\lambda_{\perp}^{(m)}|$ and the cycle is transversely stable if $\Lambda_{\perp} < 0$.

If x_0 belongs to a generic aperiodic trajectory embedded inside the chaotic set A, then $\Lambda_{\perp}$ is the *natural transverse Lyapunov exponent* $\Lambda_{\perp}^{n}$, where *natural* indicates that the exponent is computed for a typical trajectory taken in the chaotic attractor A (see Bischi and Gardini, 2000). Since infinitely many cycles, all unstable along the diagonal, are embedded inside the chaotic attractor A, a spectrum of transverse Lyapunov exponents can be defined by the inequality

$$\Lambda_{\perp}^{min} \leq ... \leq \Lambda_{\perp}^{n} \leq ... \leq \Lambda_{\perp}^{max}$$

and the natural transverse Lyapunov exponent represents a sort of *weighted balance* between the transversely repelling and transversely attracting cycles (see Bignami and Agliari, 2010). If all the cycles embedded in A are transversely stable, that is $\Lambda_{\perp}^{max} < 0$, then A is asymptotically stable in the Lyapunov sense, for the two dimensional map T; nevertheless it may occur that some cycles embedded in the chaotic set A become transversely unstable, that is $\Lambda_{\perp}^{max} > 0$, while $\Lambda_{\perp}^{n} < 0$. In such a case A is not stable in the Lyapunov sense but it is a stable attractor in the Milnor sense. If a Milnor attractor of T exists then some transversely repelling trajectories can be embedded into a chaotic set which is attracting only *on average* (see Bischi et al. 1998). In addition, such transversely repelling trajectories can be re-injected toward Δ so that their behavior is characterized by some bursts far from the diagonal, before the synchronization or before converging to a different attractor. This situation is called *on-off* intermittency.

Let us come back to the system T. In order to investigate the existence of a Milnor attractor A, we estimate the natural transverse Lyapunov exponent $\Lambda_{\perp}^{n}$, represented w.r.t. parameter d in Figure 4 (a). It is possible to observe that it can assume negative values; as an example, we consider $d = -0.3614$ at which $\Lambda_{\perp}^{n} < 0$ while $\Lambda_{\perp}^{max} > 0$ and the one dimensional map ϕ exhibits a 2-piece chaotic attractor. The unique attractor A at finite distance of system T belongs to the diagonal (see Figure 4 (b)), but a trajectory starting from an initial condition that does not belong to the diagonal has a long transient before converging to A (see Figure 4 (c)). In fact, by considering the difference $x(t) - y(t)$ for any t we can observe that the transient part of the trajectory is characterized by several bursts away from Δ. It then exhibits the typical on-off intermittency phenomenon. The whole trajectory starting from $x(0) = 1.5$ and $y(0) = 1$ is shown in Figure 4 (d).

The study of the geometrical properties of the critical lines may be used to estimate the maximum amplitude of the bursts, by obtaining the boundary of a compact trapping region of the phase space in which the on-off intermittency phenomena are confined.[5]

Following Mira et al. (1996), a practical procedure to obtain the boundary of the absorbing area in Figure 4 (e) is now described. Starting from a portion of LC_{-1} (taken in the interesting region and depicted in green), its images up to rank-6 are computed until the closed region is obtained. Since such a region is mapped into itself, then it is an absorbing area $\mathcal{A}$. The length of the initial segment has been taken by a trial and error method. Once an absorbing area is found, in order to see if it is invariant, the same procedure is repeated by taking only the portion $\gamma = \mathcal{A} \cap LC_{-1}$ as the starting segment. Since a natural m exists such that the union of the iterates of γ covers the whole boundary of $\mathcal{A}$, then $\mathcal{A}$ is invariant.

Following the procedure proposed by Bischi and Gardini 1998, the trapping region so determined is given by a minimal absorbing area including the Milnor attractor. In Figure 4 (e) the absorbing area for the case presented in (d) is depicted: observe that such a region contains the whole trajectory presented in panel (d). In order to check that the absorbing area obtained is the region in which on-off intermittency occurs, we have to verify that it is the smallest one including the Milnor attractor. A procedure that can be used is the so called *parameter mismatch* proposed in Bischi and Gardini (1998).

In our model the speed of adjustment α can be used as the parameter breaking the symmetry: in fact the existence of a minimal invariant absorbing area is a structurally stable property, persisting under a small perturbation of the parameters, even if such a perturbation breaks the symmetry. We introduce a slight difference in the speed of adjustment, i.e. we consider map T with $\alpha_1 \neq \alpha_2$ as in equation (15), more precisely we assume $\alpha_1 = 0.5$ and $\alpha_2 = 0.51$. Obviously the invariance of the diagonal is lost and the Milnor attractor as well, but the map T presents a strange attractor having the same shape as that of the absorbing area depicted in Figure 4 (e). This proves that the absorbing area previously obtained is minimal since it is completely covered by the attractor of T, and that the amplitude of the bursts arising in the trajectories of the map T can be estimated by considering the critical segments of critical curves and their iterations.

4 Global dynamics: basins and contact bifurcations

In this section we consider the situation in which the firms start from different initial states and we focus on the case in which an attractor exists on Δ but trajectories may not synchronize, for instance almost all initial conditions are locally repelled away from such an attractor. In this case, the generic trajectory starting from an interior point of $\mathbb{R}^2_+$ is attracted by a set Ω belonging to an absorbing area $\mathcal{A}$.

Recall Proposition 6 and consider the case in which $\alpha(a + w)$ is small enough. Then a set $D^* = (d_1^*, d_2^*)$ does exist such that if $d \in D^*$, then E^* is asymptotically stable. First, we want to describe the bifurcations occurring as d increases (resp. decreases) so that E^* is no longer a stable node. We fix all the parameter values but d at the following levels: $a = 3$, $\alpha = 0.5$ and $w = 0.5$. Observe that for the parameter constellation we have chosen, $\alpha(a + w) = 1.75 < 2$ so that $0 \in D^*$; furthermore $d_1' < d_1 < 0 < d_2' < d_2$. If $d = 0$, then E^* is a stable fixed point while as $d < 0$ decreases E^* becomes a saddle and Δ is transversally attracting (this is the reason why on-off intermittency occurs). On the other hand, as $d > 0$ increases and E^* becomes a saddle then the diagonal is its stable set.

[5]For the application of the critical curves technique in economic models see, amongst others, Bischi and Gardini (1998), Bischi and Lamantia (2002), Brianzoni et al. (2009).

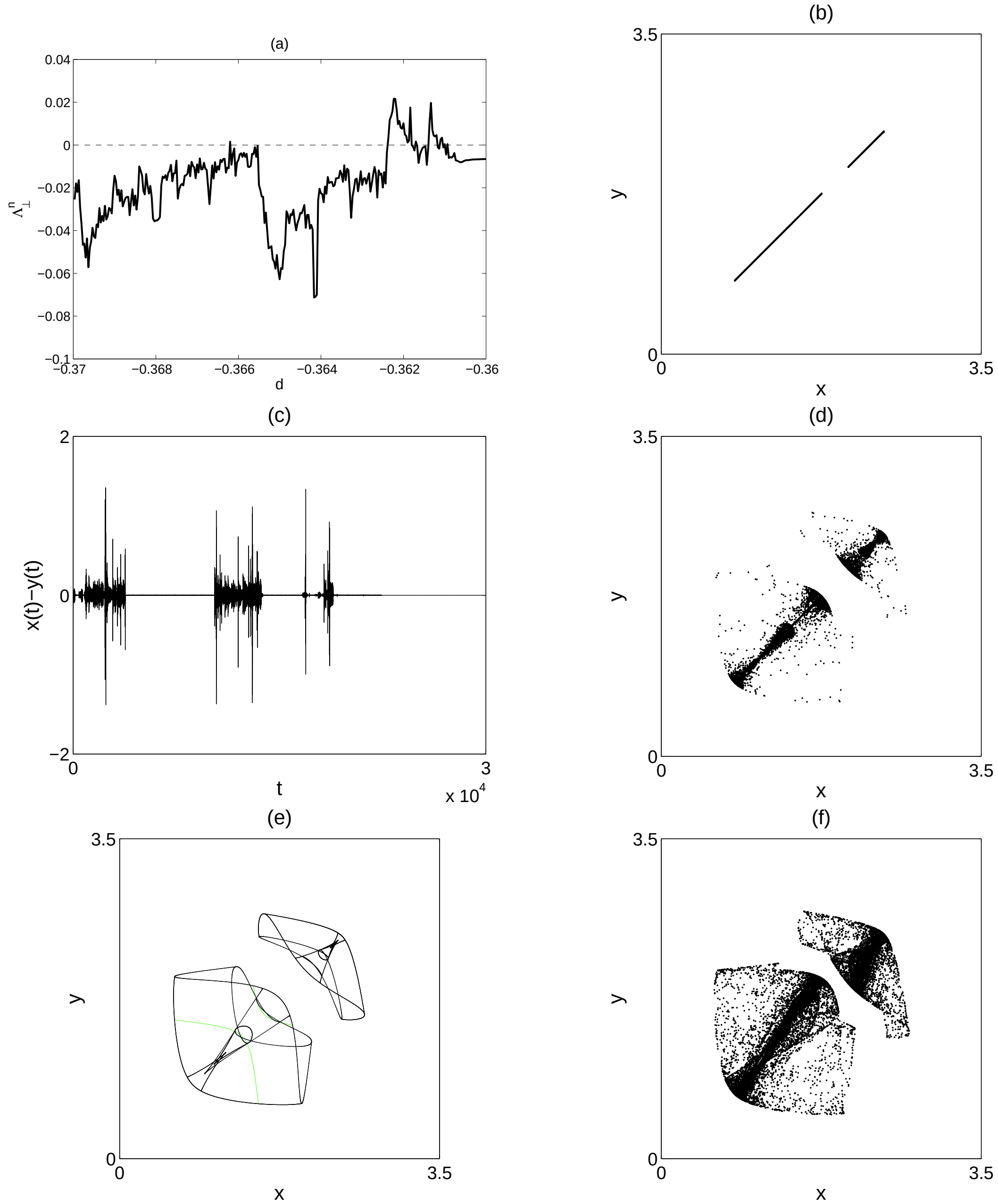

Figure 4: Parameter values $\alpha = 0.5$, $a = 3$ and $w = 0.5$. (a) The natural transverse Lyapunov exponent w.r.t. parameter d. (b) Two-piece chaotic attractor A of system T belonging to the diagonal for $d = -0.3614$. (c) Bursts away from the diagonal before synchronization for $d = -0.3614$, $x(0) = 1.5$ and $y(0) = 1$. (d) The whole trajectory starting from initial condition as in panel (c) and converging to the attractor in panel (b). (e) The minimal absorbing area in which on-off intermittency phenomenon occurs. Parameters and initial condition as in (c). (f) The introduction of parameter mismatch, which breaks the symmetry, makes it possible to check the absorbing area properties (we used $\alpha_1 = 0.5$ and $\alpha_2 = 0.51$). Indeed we obtain a strange attractor having the same shape as in (e).

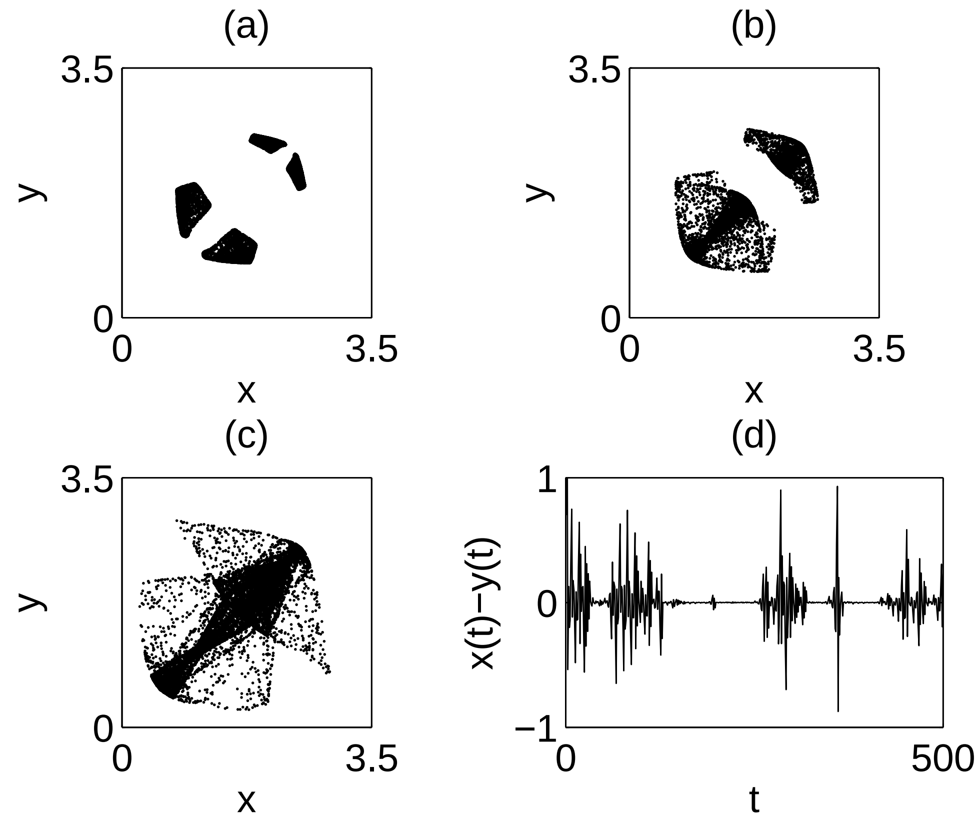

Figure 5: Attractor of T for different negative values of parameter d: (a) $d = -0.34$; (b) $d = -0.355$, (c) $d = -0.395$. (d) For $d = -0.395$, and initial condition $x(0) = 0.2$ and $y(0) = 0.3$, the evolution of $x(t) - y(t)$ versus time is depicted and the intermittency phenomenon is observed.

We now describe the change in the qualitative behavior of the system in these two cases.

Consider first the case in which $d < 0$ decreases (i.e. products tend to be complements) then as value d_1 (where $d_1 \simeq -0.14$) is crossed, a flip bifurcation occurs creating a two period cycle on Δ; as d still decreases a sequence of period-doubling bifurcations is observed on Δ that increases the complexity of the attractor of T. As long as $d \in (d_1', d_1)$ then Δ is transversally attracting and the trajectories synchronize. As d still decreases and the value d_1' is crossed the diagonal becomes a repelling set. In Figure 5 panels (a), (b) and (c) a 4-piece, 2-piece and one-piece chaotic attractor existing outside the diagonal are presented. Recall that these attractors coexist with the one on the diagonal. While looking at Figure 5 panel (c) it can be noticed that the density of the iterated points inside the chaotic area is mainly concentrated along the diagonal revealing the occurrence of *on-off intermittency* dynamics. In Figure 5 panel (d) the difference $x(t) - y(t)$ is presented versus time; it can be seen that $x(t) \simeq y(t)$ several time periods and that, in other periods, the state variables are distant from each other, so that it is very difficult to make predictions about their occurrence.

We now consider the case in which d is positive and increasing. As d crosses the value

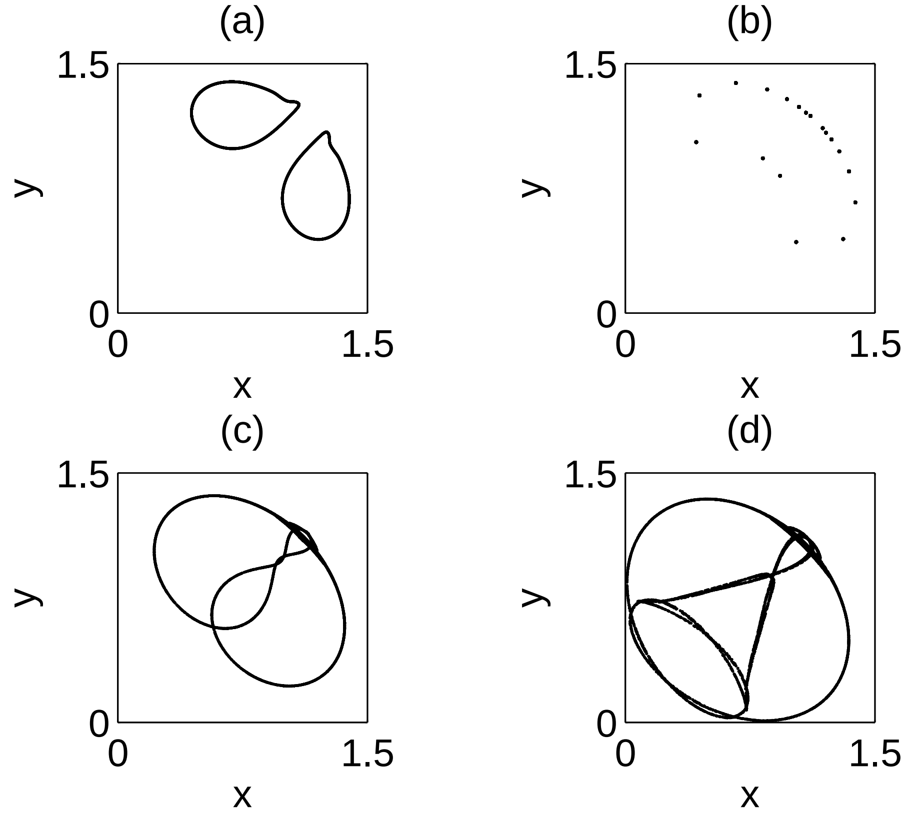

Figure 6: Attractor of T for different positive values of parameter d: (a) $d = 0.64$; (b) $d = 0.65$, (c) $d = 0.675$. (d) $d = 0.701$.

d_2' (where $d_2' \simeq 0.3596$) the fixed point E^* becomes a saddle and a two-period cycle is created outside the diagonal via a transverse flip bifurcation of E^*. The attractor Ω is given by a two-period cycle for $d \in (d_2', \tilde{d})$ (where $\tilde{d} \simeq 0.5875$). As d crosses $\tilde{d}$ this two-cycle undergoes a supercritical Neimark-Sacker bifurcation, leading to the existence of two cyclical attracting closed invariant curves[6] that arise around the unstable two-period cycle (see Figure 6 panel (a)). Then, after the Neimark-Sacker bifurcation, the generic trajectory exhibits a quasi-periodic or a periodic behavior. As d further increases, an $m-$period cycle is observed (see Figure 6 panel (b)), while in Figure 6 panels (c) and (d) we observe a one-piece chaotic attractor for greater values of d. The attractor presented in panel (c) may be due to a homoclinic bifurcation of the saddle E^* (a similar bifurcation occurs in Bignami and Agliari (2010)).

As it can be observed by comparing Figures 5 and 6, the structure of the attractor Ω existing outside the diagonal and coexisting with the one belonging to Δ increases in complexity as the degree of complementarity or substitutability of the two products increases; anyway, while in the case of negative values of d (complementarity) intermittency may be present, with positive values of d (substitutability) this phenomenon is ruled out.

[6] For a better description of bifurcations related to the creation and destruction of closed invariant curves see Aronson et al. (1982).

We now describe some global bifurcations that are responsible for a change in the structure of the basins of attraction of the attracting set Ω. As previously discussed, system T always admits divergent trajectories. Let $B(\infty)$ be the set of points generating diverging trajectories. Assume now that T admits a unique attractor at finite distance, namely Ω, and let $B(\Omega)$ be the basin of attraction of Ω (i.e. the set of initial conditions generating trajectories converging to Ω). Then, $B(\Omega) = Int(\mathbb{R}^2_+/B(\infty))$ being $Int(M)$ the interior points of set M.

In order to determine the basin of attraction of Ω, we observe that the boundary of $B(\Omega)$, namely $\partial B(\Omega)$, coincides with the boundary of $B(\infty)$, given by $\partial B(\infty)$. Hence in what follows we explain the procedure to obtain $\partial B(\infty)$ for system T; notice that a key role is played by the two invariant coordinate axes. Recall that the one dimensional restriction of T to the set of points $(x, 0)$ is given by ϕ_0 defined in (26) and that for the symmetry property the same map governs the dynamics along the other invariant coordinate axis. Since ϕ_0 is topologically conjugated to the logistic map $\omega' = \lambda\omega(1-\omega)$ through a linear transformation $x = h(\omega)$ given by (27) with $\lambda = \mu$ as defined in (29), hence system T admits bounded trajectories along the invariant axes if $\mu \in (1, 4]$ and the initial condition is taken inside the segment $I = [0, 0_{-1}]$, where 0_{-1} is the rank-1 preimage of the origin computed by using ϕ_0.

While computing the transverse eigenvalues associated to the points belonging to I, it is easy to verify that the transverse directions to the coordinate axes are always repelling, hence the two segments

$$I_1 = ([0, 0_{-1}], 0) \in \mathbb{R}^2 \text{ and } \mathrm{I}_2 = (0, [0, 0_{-1}]) \in \mathbb{R}^2, \tag{33}$$

representing respectively the intersections of set I with the two coordinate x-axis and y-axis, must belong to $\partial B(\infty)$ and, hence, all their preimages of any rank also belong to $\partial B(\infty)$. Following these arguments, the following Proposition determining the boundary of $B(\infty)$ (and hence the boundary of $B(\Omega)$) holds.

Proposition 7. *Let $\mu \in (1, 4]$ and I_1 and I_2 be the segments of the two coordinate axes defined in (33). Then*

$$\partial B(A) = \left(\bigcup_{n=0}^{\infty} T^{-n}(I_1)\right) \cup \left(\bigcup_{n=0}^{\infty} T^{-n}(I_2)\right).$$

Notice that the whole $\partial B(\Omega)$ is given by the union of the preimages of segments I_1 and I_2 (for further details see Bischi et al. 1998). In Figure 7 panel (a) we show the attractor of T and its own basin (the white region) for suitable values of the parameters such that $\mu \in (1, 4]$ and the differentiation parameter d is positive.

We also depict the critical curves LC; by using numerical simulations it can be shown that the curve $LC^{(b)}$ moves upwards as parameter d increases so that, given the other parameter values, a threshold value $\bar{d}$ does exist such that a contact bifurcation (i.e. a contact between a critical curve and the basin boundary) occurs.[7] At this parameter value $LC^{(b)}$ is tangent to $\partial B(\infty)$ at two symmetric points; at $\bar{d}$ a global bifurcation occurs causing the transformation of $B(\infty)$ from connected to non connected, i.e. it is given by an infinite sequence of non connected regions (or holes) inside $B(\Omega)$. This bifurcation is due to the fact that a portion of the basin $B(\infty)$ enters a region characterized by a higher number of

[7]For this kind of bifurcation see, amongst others, Abraham et al. (1997) and Mira et al. (1994).

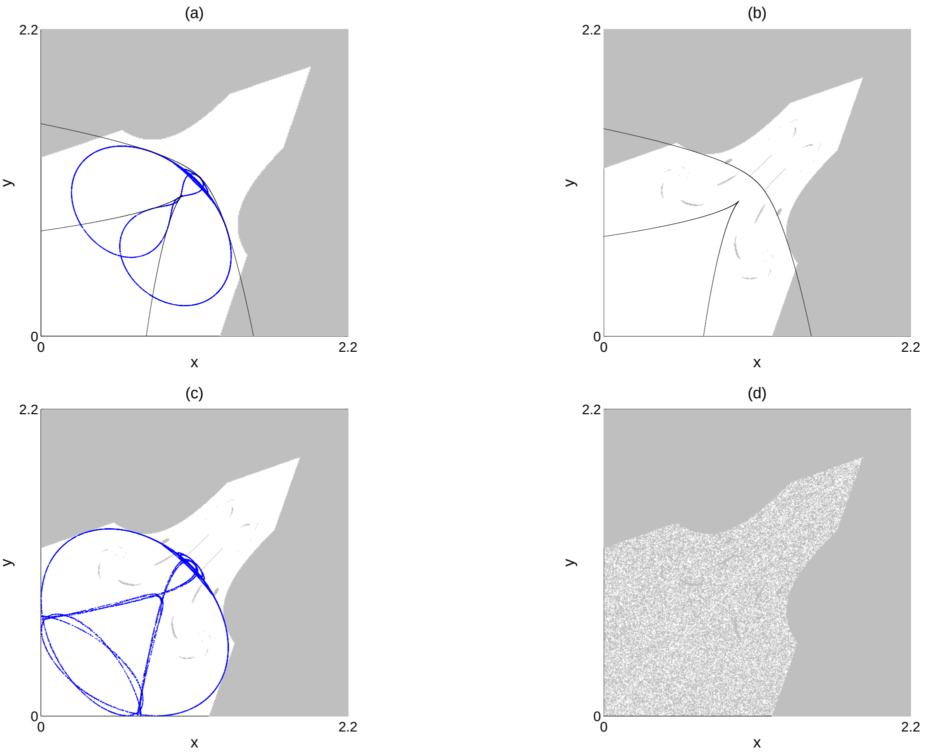

Figure 7: (a) Attractor of T and its basin for the following parameter values: $\alpha = 0.5, a = 3, w = 0.5, d = 0.675$. Curves LC are also depicted. (b) Basin of attraction of Ω (white region) and of infinity (gray region) for $d = 0.702$. After the global bifurcation, gray holes are depicted. Curves LC are also depicted showing the tangent contact with the basin boundary. (c) Attractor Ω and its basin in the same case of (b). (d) Basin of attractions for $d = 0.7021$.

primages (and hence the primages of any rank of such a portion also belong to $B(\infty)$). By using numerical computations, we obtain the value $\bar{d} \simeq 0.7019$ and in Figure 7 panel (b) we show the situation occurring immediately after such a global bifurcation.

Observe that the attractor Ω does not disappear after such a bifurcation (it is presented in Figure 7 panel (c)). This depends on the fact that the portion of curve LC involved in the contact bifurcation does not belong to the boundary of the absorbing area inside which the attractor Ω exists.

The invariant absorbing area is involved when the basin boundary has a contact with the boundary of the invariant absorbing area and this happens as parameter d increases. In Figure 7 panel (d) this situation is represented and the distribution of white and gray points appears quite complicated. The white points are associated with the ghost attractor surviving after the final bifurcation.

Finally, recall that when $\mu > 4$ then the generic trajectory starting from a point of the invariant diagonal is divergent. This case corresponds also to a global bifurcation for the two-dimensional system T, called final bifurcation, due to a contact between the bounded attractor Ω and the basin of infinity, causing the destruction of the attractor.

5 Conclusions and further developments

This paper has developed both local and global analyses in a nonlinear duopoly with price competition and horizontal product differentiation, by assuming players with limited information with regard to profit functions.

Analysis of both horizontal (quantity) and vertical (quality) differentiation is of importance in microeconomic literature (see Dixit (1979), Gabszewicz and Thisse (1979), Shaked and Sutton (1982), Singh and Vives (1984), Vives (1985), Motta (1993)).

The aim of this paper is to revisit the study of the dynamic properties of a nonlinear Bertrand duopoly, by taking the degree of horizontal product differentiation as a key parameter. This has led us to bring to light some interesting dynamic events such as synchronization, intermittency and global bifurcations. If the products of the two firms are independent from each other (i.e., every firm acts as if it were a monopolist in the market), then the unique interior fixed point is locally stable. In contrast with this, if every firm acts to increase the extent of product differentiation, the interior fixed point is no longer stable and attractors having a complex structure may appear. This holds because when products are substitutes (resp. complements), an increase in the degree of substitutability (resp. complementarity) induces firms to increase the price of their own product to capture advantages of competition with regard to their own profits (resp. joint profits) in the case of substitutability (resp. complementarity).

Further developments on the study of nonlinear duopolies with price competition are possible. In particular, the cost function can be non-linear or it can be endogenized by assuming the existence of imperfect labor markets (e.g., unions); a firm's objective different from pure profit maximization can also be considered through the study of managerial delegation schemes (e.g., sales, market share or relative profits delegation).

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