

# Fertility-related pensions and cyclical instability

Luciano Fanti • Luca Gori

**Abstract** In this paper, we study a general equilibrium model with overlapping generations, endogenous fertility and public pensions. By assuming Cobb-Douglas technology and logarithmic preferences, we show that the introduction of a fertility-related component in the pay-as-you-go pension scheme may destabilise the long-term equilibrium and cause endogenous fluctuations when individuals have static expectations. The possibility of cyclical instability increases (resp. reduces) when both the subjective discount factor and relative weight of individual fertility in PAYG pensions (resp. the parents' taste for children) increase(s). Interestingly, when public pensions are contingent on the individual number of children, the financing of small-sized benefits may cause the occurrence of a flip bifurcation, two-period cycles and cycles of a higher order. In addition, we show through numerical simulations that these results hold in a more general CIES-CES economy. Our findings identify a possible novel factor responsible for persistent deterministic fluctuations in a context of overlapping generations, while also representing a policy warning regarding the destabilising effects of fertility-related pension reforms, which are currently high in both the theoretical debate and the political agendas of several developed countries.

**Keywords** Endogenous fertility; Fertility-related pensions; Static expectations; OLG model

**JEL Classification** C62; H55; J14; J18; J26

## 1. Introduction

Social security is a pillar of the welfare state in several developed countries. It is essentially based on pay-as-you-go (PAYG) public pensions, i.e. those in work finance the pensions of those who are retired. The fertility crisis that is affecting an increasing number of countries (e.g., Germany, Italy, Japan and Spain), is threatening the viability of public pension budgets, as the number of young contributors is steadily falling and the number of old beneficiaries is steadily rising. This phenomenon is also being aggravated by reduced adult mortality (Fogel, 2004; Livi-Bacci, 2006). Motivated by concerns of ageing and below-replacement fertility on the existence of the PAYG system, pension reforms are currently high on the political agendas of several governments, especially in Europe (Boeri et al., 2001, 2002; Blinder and Krueger, 2004).

It has been suggested that families should be offered incentives to have more children so as to increase the ratio of economically active people to the total population, for instance

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through the public provision of child allowances (van Groezen et al, 2003; van Groezen and Meijdam, 2008), or, alternatively, by linking the size of the pension transfer to the individual number of children (Kolmar, 1997; Abio et al., 2004; Fenge and Meier, 2005, 2009; Cigno and Werding, 2007). This is the so-called fertility-related pension policy.<sup>1</sup>

Although fertility-related (henceforth FR) pensions already exist in some countries,<sup>2</sup> several economists and policy makers argue for a further extension. Sinn (2007, p. 10) argues for instance that the current fertility-related element in the German pension system is too low: “The pension system in Germany provides some relief for mothers who raise an additional child and work ten years after the birth. They receive, in terms of current value, 11,000 Euros as an additional pension. This is close to nothing.” With regard to policy proposals, Cigno et al. (2003) investigate the effectiveness of pension benefits contingent on the actual (or potential) earning capacity of the pensioner’s own children (see also Cigno, 2007; Cigno and Werding, 2007). This reform is also able to make operational the theoretical findings of Cigno et al. (2003, 2004), according to which it can help to avoid: (i) that parents substitute quantity for quality of children (Cigno et al., 2003), and (ii) the existence of moral hazard problems because parents’ investment (in terms of both time and money) on children is not observable by governments (Cigno et al. 2004).

Empirical evidence regarding the effects of FR cash benefits or tax breaks on fertility is controversial. In fact, as noted by Cigno (2010, p. 27), while in the short term it has been invariably found that FR cash benefits encourage fertility, they may solely represent a change in the timing of fertility behaviour (such as the trend that gave rise to a baby boom in the 1960s), rather than a change in completed cohort fertility. By contrast, the long-term effects appear to be small (as surveyed by Schultz, 2008). For instance, Cigno et al. (2002/2003) estimate the long-term elasticity of the Total Fertility Rate with respect to child benefit rate of only 0.23 for Germany.<sup>3</sup> More significantly perhaps, Cigno and Werding (2007), investigate what would have happened to the Total Fertility Rate from 1995 to the time of writing, and from that date to 2020, according to five alternative scenarios by exploiting the fact that the German public pension system became an FR-system in 1996. Their simulations seem to show that the same long-term fertility rates can be obtained either by drastically cutting back the pension coverage and increasing fertility subsidies, or by making pension benefits contingent on individual fertility. However, they argue that introducing FR pensions would not distort the quantity/quality mix of children.

With regard to the important issue of deterministic business cycles in a competitive economy, the idea that cyclical behaviours can occur in OLG models when individual have perfect foresight is well known in the economic literature (Grandmont, 1985). This topic has been investigated in the cases of both a one-good economy (Farmer, 1986; Reichlin, 1986; de Vilder, 1996) and a many-goods economy (Benhabib and Nishimura, 1979). However, this requires that production factors complement each other (i.e. the elasticity of capital-labour substitution should be lower than the Cobb-Douglas function), and/or consumption and

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<sup>1</sup> A relevant contribution on the theory of family taxation is Cigno (1986).

<sup>2</sup> As Cigno (2007, p. 39) claims: “examples of this are the *majoration de duree d’assurance pour enfants* in the French *Régime Général*, and the Swedish *extrapension for barn*. In 1986, the German government started crediting parents who withdraw from the labour market to look after a child with a notional pension contribution, *Kindererziehungszeiten*, originally set at 75 percent of average earnings, for up to one year. Later, this notional contribution was raised to 100 percent of average earnings, and extended to three years. Since 1996, however, the condition that the parent should actually give up work in order to qualify for the benefit has been removed, and *Kindererziehungszeiten* has become a fertility-related pension benefit just like the French and Swedish ones.”

<sup>3</sup> The economic reason for this finding is that: “doubling an already generous package of fertility-related cash benefits and tax breaks would eventually raise completed fertility by little over a quarter of a child per woman.” (Cigno, 2010, p. 27).

leisure are gross substitutes. In contrast, with static expectations trajectories can be oscillatory and non-convergent to the steady state (Michel and de la Croix, 2000, de la Croix and Michel, 2002; Chen et al., 2008; Fanti and Spataro, 2008), only when the inter-temporal elasticity of substitution in the utility function is higher than one (i.e., higher than in the case of Cobb-Douglas preferences).<sup>4</sup>

Over the last few decades there has been a growing body of literature dealing with the relationship between pensions, fertility, longevity and economic growth (e.g., Zhang et al., 2001, 2003; Pecchenino and Pollard, 2005). However, to the best of our knowledge, less attention has been devoted to studying the dynamic effects of financing public pensions in an OLG economy with endogenous fertility.

The aim of this paper is to examine the stability outcomes of the public provision of fertility-related PAYG pensions in a textbook OLG economy (Diamond, 1965) when fertility is endogenous. When individuals have static expectations, we show that the introduction of a component related to individual fertility in the pension formula may cause trajectories to be oscillatory and non-convergent for realistic parameters values. In fact, the negative effect of public pensions on capital accumulation is larger than in the case of traditional PAYG transfers, because individuals tend to increase the number of children they have (to get a larger benefit from pensions) and reduce saving. Furthermore, we extend the model to a Constant-Inter-temporal-Elasticity-of-Substitution (CIES) utility function, and a Constant-Elasticity-of-Substitution (CES) production function, and simulate how an increase in the relative weight of individual fertility in the pension formula affects stability outcomes for the case of Germany.

The contribution of this paper is twofold: (i) it investigates the dynamic properties of an OLG economy with endogenous fertility and fertility-related pensions; (ii) it shows a novel determinant of endogenous fluctuations, which emerge even with Cobb-Douglas utility and production functions.<sup>5</sup>

The policy implications of the findings are clear: to the extent that developed countries show a low preference for children and a high preference for future consumption (with static expectations), the often advocated reform for such countries of introducing (or extending) a fertility related element in the public pension formula may destabilise the economy, and also lead to chaotic dynamics.

The paper is organised as follows. Section 2 describes the model. Section 3 studies the local dynamic properties of the fixed point and presents numerical simulations of endogenous fluctuations. Section 4 outlines the conclusions.

## 2. The model

### 2.1. Government

The government redistributes resources across generations by using PAYG pensions that are partially or totally linked with the individual number of children. At time  $t$ , therefore, current workers finance pensions for current pensioners, and the fertility-related PAYG (henceforth FR-PAYG) pension accounting rule (per pensioner) reads as follows:

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<sup>4</sup> A relevant mathematical reference on period-doubling and period-halving bifurcations is Nusse and Yorke (1988), while interesting economic applications, in different contexts to ours, include Hommes (1991; 1994), Zhang (1999), Rosser (2001) and Antoci and Sodini (2009).

<sup>5</sup> Although a number of countries have introduced some forms of FR-pensions for 15-20 years, it is difficult to check whether they have already brought such economies into the region of non-convergent trajectories (as shown later in this paper). This is because our findings hold in a stylised economy, and several other reasons contribute to explain the economic cycles in real economies. In any case, the study of both the empirical evidence and case-studies regarding stability outcomes is beyond the scope of this paper.

$$p_t = \theta w_t [(1 - \omega) \bar{n}_{t-1} + \omega n_{t-1}], \quad (1)$$

the left-hand side ( $p_t$ ) being the pension expenditure, and the right-hand side the tax receipts. In particular,  $w_t$  is the wage earned by the young workers at time  $t$ ,  $0 < \theta < 1$  is the constant contribution rate, and  $0 \leq \omega \leq 1$  captures the weight of individual fertility ( $n_{t-1}$ ) with respect to average fertility ( $\bar{n}_{t-1}$ ) in the PAYG system (Kolmar, 1997; Fenge and von Weizsäcker, 2010). When  $\omega = 0$ , a traditional PAYG scheme is in place. When  $\omega = 1$  pensions are completely contingent on individual fertility. Following Fenge and Meier (2005),  $\omega$  is called “the child factor”.

## 2.2. Individuals

Consider a general equilibrium OLG closed economy populated by identical individuals of measure  $N_t$  per generation. Time is discrete and indexed by  $t = 0, 1, 2, \dots$ . The life of a typical agent is divided into childhood and adulthood. During childhood, he/she does not make economic decisions. During adulthood, he/she works and raises children and retires when old.

Young members of generation  $t$  are endowed with one unit of labour, which is inelastically supplied to firms. They receive a competitive wage  $w_t$  per unit of labour. This income is used for consumption, saving, childcare and to finance pensions through taxation. Raising children is costly for young parents. The amount of resources needed to care for a child is  $q w_t$ , where  $0 < q < 1$  is the percentage of working income devoted to child rearing (Wigger, 1999, Boldrin and Jones, 2002). Therefore, the budget constraint of a young individual of generation  $t$  reads as follows:

$$c_{1,t} + s_t + q w_t n_t = w_t (1 - \theta), \quad (2)$$

i.e. the disposable income is divided into material consumption when young ( $c_{1,t}$ ), saving ( $s_t$ ) and the cost of bearing  $n_t$  children. When old, individuals retire and live with the amount of resources saved when young, plus interests accrued from time  $t$  to time  $t+1$  at the rate  $r_{t+1}^e$ , and realised pension benefits (i.e., realised consumption depends on realised factor prices). The budget constraint at time  $t+1$  of an individual of generation  $t$  is given by:

$$c_{2,t+1} = R_{t+1}^e s_t + \theta w_{t+1}^e [(1 - \omega) \bar{n}_t + \omega n_t], \quad (3)$$

where  $c_{2,t+1}$  represents material consumption when old.

The individual representative of generation  $t$  draws utility from consumption when young, consumption when old, and the number of children that he/she has (Eckstein and Wolpin, 1985; Eckstein et al., 1988; Galor and Weil, 1996).<sup>6</sup> By assuming Cobb-Douglas preferences, the individual lifetime utility function can be written as  $V_t = c_{1,t} c_{2,t+1}^\beta n_t^\phi$ . Through the usual logarithmic transformation, we then get:

$$U_t = \ln(c_{1,t}) + \beta \ln(c_{2,t+1}) + \phi \ln(n_t), \quad (4)$$

By taking factor prices and expected factor prices, the contribution rate and the average fertility rate as given, the representative individual chooses fertility and life-cycle

<sup>6</sup> This way of modelling fertility is often called a weak form of altruism towards children (Zhang and Zhang, 1998). We note that assuming children as a consumption good characterises developed countries much better than developing countries. In the latter, in fact, the old-age security hypothesis (Bental, 1989) is also an important determinant of the demand for children. In addition, the quantity-quality trade-off contributes in explaining the behaviour of individual fertility (Becker and Lewis, 1973). We thank an anonymous reviewer for pointing this out.

consumption to maximise utility function (4) subject to the budget constraints (2) and (3), where  $0 < \beta < 1$  is the subjective discount factor and  $\phi > 0$  captures the parents' relative desire to have children.<sup>7</sup> The first order conditions for an interior solution are given by:

$$\frac{c_{2,t+1}}{c_{1,t}} \cdot \frac{1}{\beta} = R_{t+1}^e, \quad (5)$$

$$\frac{c_{1,t}}{n_t} \cdot \phi = q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}. \quad (6)$$

Eq. (5) equates the marginal rate of substitution between consumption when young and consumption when old to the interest factor. Eq. (6) equates the marginal rate of substitution between consumption when young and the number of children to the expected marginal cost of raising an extra child. This cost is given by the difference between the resources needed to take care of a child, minus the present value of the expected pension benefit weighted by the child factor. The higher the child factor, the lower the expected net marginal cost of raising an additional child. If  $\omega = 0$  (pure PAYG pensions), then the cost of children is determined as a share of the parent's working income. If  $0 < \omega \leq 1$  (FR-PAYG pensions), there is a positive effect due to a transfer of resources from the young generation to the old generation, which causes a reduction in the total cost of children.

Combining Eqs. (5) and (6) with the individual budget constraints, gives the following demand for children and saving:

$$n_t = \frac{\phi w_t (1 - \theta)}{(1 + \beta + \phi) q w_t - [(1 + \beta) \omega + \phi] \theta \frac{w_{t+1}^e}{R_{t+1}^e}}, \quad (7)$$

$$s_t = \frac{w_t (1 - \theta) [\beta q w_t - (\beta \omega + \phi) \theta \frac{w_{t+1}^e}{R_{t+1}^e}]}{(1 + \beta + \phi) q w_t - [(1 + \beta) \omega + \phi] \theta \frac{w_{t+1}^e}{R_{t+1}^e}}. \quad (8)$$

Eq. (7) determines the individual fertility in a partial equilibrium context. An increase in the child factor causes a positive effect due to a transfer of resources from young people to old people, which reduces the marginal cost of children and increases fertility ( $\partial n_t / \partial \omega > 0$ ). Eq. (8) determines saving in a partial equilibrium context. It reveals that an increase in the child factor plays a twofold role: (a) it tends to reduce saving because individuals expect a higher pension when the number of children increases (this is represented by an increase in the *public pension component* in saving – the second term in brackets in the numerator of (8));<sup>8</sup> (b) it tends to increase saving because a larger value of the child factor makes it more convenient to substitute consumption when young with children at time  $t$  (i.e., it reduces the denominator of (8)). However, the final (partial equilibrium) effect of an increase in the child factor on saving is negative ( $\partial s_t / \partial \omega < 0$ ).

### 2.3. Production and equilibrium

<sup>7</sup> With a similar characterisation of preferences, Zhang et al. (2001, p.501) note that “The taste for the number of children is in the range  $0.8 \leq \eta \leq 1.5$ ;  $\eta > 1.5$  seems unlikely since such a high value means that individuals care about the number of children far more than own consumption and the welfare of children.”

<sup>8</sup> We denote the first (resp. the second) addendum in brackets in the numerator of Eq. (8) as the *public pension component* (*private component*) in saving.

Identical firms act competitively on the market. The technology of production at time  $t$  combines capital ( $K_t$ ) and labour ( $L_t = N_t$ ) to produce output ( $Y_t$ ) in such a way that  $Y_t = AK_t^\alpha L_t^{1-\alpha}$ , where  $A > 0$  and  $0 < \alpha < 1$ . By assuming a full depreciation of capital and knowing that output is sold at the unit price, profit maximisation yields:

$$R_t = \alpha A k_t^{\alpha-1}, \quad (9)$$

$$w_t = (1 - \alpha) A k_t^\alpha, \quad (10)$$

where  $k_t := K_t / N_t$  is capital per worker.

Given the government pension budget (1) and knowing that  $N_{t+1} = n_t N_t$ , the market-clearing condition in the capital market can be expressed as follows:

$$n_t k_{t+1} = s_t. \quad (11)$$

Combining Eqs. (7), (8) and (11), equilibrium implies:

$$k_{t+1} = \frac{\beta}{\phi} q w_t - \frac{\beta \omega + \phi}{\phi} \theta \frac{w_{t+1}^e}{R_{t+1}^e}. \quad (12)$$

The existence of a fertility-related component in PAYG pensions has two important effects on capital accumulation: (i) it makes the crowding out effect of public pensions on saving stronger than under the traditional PAYG system ( $\omega = 0$ ); it introduces a potential destabilising role with regard to the subjective discount factor ( $\beta$ ) and parents' taste for children ( $\phi$ ).

Before examining the dynamic analysis, some clarifications on the relationship between individual fertility and the size of pension benefits are necessary. In these models individuals are atomistic and thus they do not take the rate of fertility of the others into account when choosing the number of children. This is clear from the pension formula (1), where a difference is made between individual fertility and average fertility. Agents are thus unable to coordinate their fertility decisions. Otherwise they should be considered "ultra-rational", which seems to be an unusual and "clearly unrealistic" (see Cigno, 1995, p. 171) assumption. In addition, relaxing the assumption of atomistic individuals would imply that the standard PAYG scheme would always coincide with a PAYG scheme that was completely contingent on the individual number of children (Cigno, 1995).

We now focus on the dynamics of capital. It is usual in OLG models to investigate how the path of capital accumulation evolves, depending on whether individuals have either rational or static expectations regarding factor prices (Michel and de la Croix, 2000; de la Croix and Michel, 2002). Although static expectations represent a special condition in which all individuals are naïve, it may be in line with the same *raison d'être* for the provision of public pensions which, according to Samuelson (1975),<sup>9</sup> should compensate for an agent's shortsightedness. Although static expectations are recognised as appropriate in this class of models, some caveats are now necessary. Since a single period generally consists of almost thirty years in models with overlapping generations, people are frequently informed (e.g. by statistical offices) on movements of factor prices (which are dampened in an open economy), fertility rates and expected pension payments. Hence they have plenty of time to adapt their saving behaviour.

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<sup>9</sup> "Many social security systems, like the New Deal U.S. system, may be deemed most valuable precisely because the myopia ignored by the present models does in fact prevail. People live miserably in old age because they do not realize when young what are the consequences of their private saving habits. So by democratic fiat, they paternalistically impose on themselves a within-life pattern of consumption that favors old age at the expense of young." (Samuelson, 1975, p. 543).

*Rational expectations.* The expected factor prices depend on the stock of capital at time  $t+1$ , that is:

$$\begin{cases} R_{t+1}^e = \alpha A k_{t+1}^{\alpha-1} \\ w_{t+1}^e = (1-\alpha) A k_{t+1}^\alpha \end{cases} \quad (13)$$

Combining Eqs. (10), (12) and (13), the dynamics of capital is driven by the following nonlinear first order difference equation:

$$k_{t+1} = \frac{q \beta \alpha (1-\alpha) A}{\alpha \phi + \theta(1-\alpha)(\beta \omega + \phi)} k_t^\alpha := G(k_t). \quad (14)$$

Steady-state implies  $k_{t+1} = k_t = k^*$ . Then,

$$k^* = \left[ \frac{q \beta \alpha (1-\alpha) A}{\alpha \phi + \theta(1-\alpha)(\beta \omega + \phi)} \right]^{\frac{1}{1-\alpha}}. \quad (15)$$

*Static expectations.* The expected factor prices depend on the stock of capital at time  $t+1$ , that is:

$$\begin{cases} R_{t+1}^e = \alpha A k_t^{\alpha-1} \\ w_{t+1}^e = (1-\alpha) A k_t^\alpha \end{cases} \quad (16)$$

Then, from Eqs. (10), (12) and (16), the dynamics of capital is now given by:

$$k_{t+1} = \frac{\beta}{\phi} q (1-\alpha) A k_t^\alpha - \theta \cdot \frac{\beta \omega + \phi}{\phi} \cdot \frac{1-\alpha}{\alpha} k_t := J(k_t). \quad (17)$$

The fixed point is still determined by Eq. (15), Michel and de la Croix (2000).

We now study the (local) stability properties of  $k^*$  when individuals have static expectations (Section 3) and rational expectations (Appendix).

### 3. Local stability with static expectations

Define

$$\underline{\theta} = \underline{\theta}(\alpha, \beta, \phi, \omega) := \frac{\alpha^2}{(1-\alpha)^2} \cdot \frac{\phi}{\beta \omega + \phi}, \quad (18)$$

$$\bar{\theta} = \bar{\theta}(\alpha, \beta, \phi, \omega) := \frac{\alpha(1+\alpha)}{(1-\alpha)^2} \cdot \frac{\phi}{\beta \omega + \phi} = \underline{\theta} \cdot \frac{1+\alpha}{\alpha}, \quad (19)$$

$$\alpha_1 = \alpha_1(\beta, \phi, \omega) := \frac{1}{\beta \omega} [\beta \omega + \phi - \sqrt{\phi(\beta \omega + \phi)}], \quad 1/2 < \alpha_1 < 1, \quad (20)$$

$$\alpha_3 = \alpha_3(\beta, \phi, \omega) := \frac{1}{2\beta \omega} [2\beta \omega + 3\phi - \sqrt{\phi(8\beta \omega + 9\phi)}], \quad 1/3 < \alpha_3 < \alpha_1. \quad (21)$$

as two threshold values of the contribution rate and two threshold values of the output elasticity of capital, respectively. Then, from Eqs. (15) and (17), the following proposition holds.

**Proposition 1.** *In an OLG economy with FR-PAYG pensions and static expectations, the dynamics of capital is the following.*

(1) Let  $0 < \alpha < \alpha_3$  hold. Then  $\underline{\theta} < \bar{\theta} < 1$ , and:

(1.1) if  $0 < \theta < \underline{\theta}$ , trajectories are monotonic and convergent to  $k^*$ ;

(1.2) if  $\underline{\theta} < \theta < \bar{\theta}$ , trajectories are oscillatory and convergent to  $k^*$ ;

(1.3) if  $\theta = \bar{\theta}$ , a flip bifurcation occurs;

- (1.4) if  $\bar{\theta} < \theta < 1$ , trajectories are oscillatory and non-convergent to  $k^*$ .
- (2) Let  $\alpha_3 < \alpha < \alpha_1$  hold. Then  $\underline{\theta} < 1$ ,  $\bar{\theta} > 1$ , and:
- (2.1) if  $0 < \theta < \underline{\theta}$ , trajectories are monotonic and convergent to  $k^*$ ;
- (2.2) if  $\underline{\theta} < \theta < 1$ , trajectories are oscillatory and convergent to  $k^*$ .
- (3) Let  $\alpha_1 < \alpha < 1$  hold. Then  $\bar{\theta} > \underline{\theta} > 1$ , and trajectories are monotonic and convergent to  $k^*$  for any  $0 < \theta < 1$ .

**Proof.** By differentiating (17) with respect to  $k_t$  and using (15) we get:

$$J'(k^*) = \alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi}. \quad (22)$$

Monotonic and non-monotonic dynamics. From (22),  $J'(k^*) \stackrel{>}{<} 0$  implies:

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi} \stackrel{>}{<} 0 \Leftrightarrow \theta \stackrel{\leq}{>} \underline{\theta}, \quad (23)$$

where  $\theta = \underline{\theta}$  represents the value of the contribution rate below (resp. above) which trajectories are monotonic (resp. oscillatory). In addition,  $\underline{\theta} < 1$  (resp.  $\underline{\theta} > 1$ ) for any  $0 < \alpha < \alpha_1$  (resp.  $\alpha_1 < \alpha < 1$ ). Then,  $\underline{\theta} < 1$  if and only if  $\alpha < \alpha_1$  and  $\alpha > \alpha_2$ , where  $\alpha_1$  is defined by (20) and  $\alpha_2 = \alpha_2(\beta, \phi, \omega) := \frac{1}{\beta\omega} [\beta\omega + \phi + \sqrt{\phi(\beta\omega + \phi)}]$ . Since  $1/2 < \alpha_1 < 1$  and  $\alpha_2 > 1$ , the case where

$\alpha > \alpha_2$  can be ruled out. Now,  $\left. \frac{\partial k_{t+1}}{\partial k_t} \right|_{k_t=k^*} < 1$  gives:

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi} < 1 \Rightarrow \theta > -\frac{\alpha}{1-\alpha} \cdot \frac{\phi}{\beta\omega + \phi}. \quad (24)$$

Therefore, in the case of monotonic trajectories,  $0 < J'(k^*) < 1$  for any  $0 < \theta < 1$ .

Non-monotonic dynamics: stability analysis. The condition  $J'(k^*) \stackrel{>}{<} -1$  implies:

$$\alpha - \theta \frac{(1-\alpha)^2}{\alpha} \cdot \frac{\beta\omega + \phi}{\phi} \stackrel{>}{<} -1 \Rightarrow \theta \stackrel{\leq}{>} \bar{\theta}. \quad (25)$$

where  $\theta = \bar{\theta} > \underline{\theta}$ . Then, the fixed point is locally asymptotically stable if  $\theta < \bar{\theta}$ . It undergoes a flip bifurcation at  $\theta = \bar{\theta}$ , and a two-period cycle appears if  $\theta > \bar{\theta}$ .

In particular,  $\bar{\theta} < 1$  (resp.  $\bar{\theta} > 1$ ) for any  $0 < \alpha < \alpha_3$  (resp.  $\alpha_3 < \alpha < 1$ ). In addition,  $\bar{\theta} < 1$  if and only if  $\alpha < \alpha_3$  and  $\alpha > \alpha_4$ , where  $\alpha_3$  is defined by (21), and  $\alpha_4 = \alpha_4(\beta, \phi, \omega) := \frac{1}{2\beta\omega} [2\beta\omega + 3\phi + \sqrt{\phi(8\beta\omega + 9\phi)}]$ . Since  $1/3 < \alpha_3 < \alpha_1$  and  $\alpha_4 > 1$  for any  $\beta$ ,  $\phi$  and  $0 < \omega \leq 1$ , then the case where  $\alpha > \alpha_4$  can be ruled out. Therefore, (1) if  $0 < \alpha < \alpha_3$  then  $\underline{\theta} < \bar{\theta} < 1$  and:  $0 < J'(k^*) < 1$  for any  $0 < \theta < \underline{\theta}$ ,  $-1 < J'(k^*) < 0$  for any  $\underline{\theta} < \theta < \bar{\theta}$ ,  $J'(k^*) = -1$  if and only if  $\theta = \bar{\theta}$ , and  $J'(k^*) < -1$  for any  $\bar{\theta} < \theta < 1$ ; (2) if  $\alpha_3 < \alpha < \alpha_1$  then  $\underline{\theta} < 1$ ,  $\bar{\theta} > 1$  and:  $0 < J'(k^*) < 1$  for any  $0 < \theta < \underline{\theta}$ , and  $-1 < J'(k^*) < 0$  for any  $\underline{\theta} < \theta < 1$ ; (3) if  $\alpha_1 < \alpha < 1$  then  $\bar{\theta} > \underline{\theta} > 1$  and  $0 < J'(k^*) < 1$  for any  $0 < \theta < 1$ . **Q.E.D.**



Proposition 1 can easily be interpreted as follows: the stock of capital at time  $t+1$  is determined as saving divided by the number of children at time  $t$ . Therefore, the accumulation of capital depends on the difference between the private component and the public pension component in saving, both divided by the taste for children. With logarithmic preferences, the private component in saving only depends on the marginal willingness to save out of the wage income: it reflects the positive effect on capital accumulation of a larger wage due to an increase in the stock of capital at time  $t$  (i.e., individuals increase saving through this channel). In contrast, the public pension component in saving depends on both the pension benefit and the interest factor: it reflects the crowding out effect on capital accumulation (i.e. individuals expect a larger benefit from pensions and they then reduce saving through this channel). If the private component in saving dominates (resp. is dominated by) the public pension component, trajectories are monotonic (resp. oscillatory) and the fixed point is always locally asymptotically stable. This happens when the weight of labour in technology is sufficiently important and the contribution rate is small enough, which is when the relative weight of labour income is large and the relative weight of pension benefits is small.

We now perform a sensitivity analysis of the critical values of the contribution rate ( $\underline{\theta}$  and  $\bar{\theta}$ ) by comparing the pure PAYG pensions and FR-PAYG pensions. An analysis of (18) and (19) gives the following proposition.

**Proposition 2.** *The possibility of cyclical instability in an economy with FR-PAYG pensions is larger than with pure PAYG pensions. A rise in the distributive capital share,  $\alpha$ , monotonically reduces the possibility of cyclical instability, irrespective of the pension scheme. In addition, while with pure PAYG pensions a change in the subjective discount factor,  $\beta$ , and/or in the taste for children,  $\phi$ , is neutral with regard to stability outcomes, with FR-PAYG pensions an increase in (resp. a reduction in) in the child factor,  $\omega$ , (resp.  $\phi$ ) increases the possibility of cyclical instability.*

**Proof.** If  $\omega = 0$ , (18) collapses in  $\underline{\theta} = \underline{\theta}(\alpha) := \frac{\alpha^2}{(1-\alpha)^2}$ , so that  $\underline{\theta}(\alpha) < 1$  (resp.  $\underline{\theta}(\alpha) > 1$ ) for any  $0 < \alpha < 1/2$  (resp.  $1/2 < \alpha < 1$ ). Then from (18), when  $0 < \omega \leq 1$  the parametric region in the  $(\alpha, \theta)$  plane where trajectories are oscillatory is larger than when  $\omega = 0$ . This means that when  $0 < \omega \leq 1$ ,  $\underline{\theta}(\alpha, \beta, \phi, \omega)$  can be smaller than unity also when  $1/2 < \alpha < 1$ . If  $\omega = 0$ , then (19) boils down to  $\bar{\theta} = \bar{\theta}(\alpha) := \underline{\theta}(\alpha) \cdot \frac{1+\alpha}{\alpha}$ , so that  $\bar{\theta}(\alpha) < 1$  (resp.  $\bar{\theta}(\alpha) > 1$ ) for any  $0 < \alpha < 1/3$  (resp.  $1/3 < \alpha < 1$ ). Then from (19), when  $0 < \omega \leq 1$  the parametric region in the  $(\alpha, \theta)$  plane where trajectories are non-convergent is larger than when  $\omega = 0$ . Therefore, when  $0 < \omega \leq 1$ , the flip bifurcation value  $\bar{\theta}(\alpha, \beta, \phi, \omega)$  can be smaller than unity, also when  $1/3 < \alpha < 1$ . In addition, from (19) we get:

$$\frac{\partial \bar{\theta}}{\partial \alpha} = \frac{\phi(1+3\alpha)}{(1-\alpha)^3(\beta\omega + \phi)} > 0, \quad (26)$$

for any  $0 \leq \omega \leq 1$ , and

$$\frac{\partial \bar{\theta}}{\partial \omega} = -\frac{\alpha(1+\alpha)\phi\beta}{(1-\alpha)^2(\beta\omega + \phi)^2} < 0, \quad (27)$$

$$\frac{\partial \bar{\theta}}{\partial \beta} = -\frac{\alpha(1+\alpha)\phi\omega}{(1-\alpha)^2(\beta\omega + \phi)^2} < 0, \quad (28)$$

$$\frac{\partial \bar{\theta}}{\partial \phi} = \frac{\alpha(1+\alpha)\beta\omega}{(1-\alpha)^2(\beta\omega + \phi)^2} > 0, \quad (29)$$

for any  $0 < \omega \leq 1$ . **Q.E.D.**

Figures 1 and 2, and Table 1 illustrate Proposition 2. They show the different size of the parametric regions of monotonic stability and cyclical instability in the  $(\alpha, \theta)$  plane, under pure PAYG pensions (Figure 1) and FR-PAYG pensions (Figure 2).

If  $\omega = 0$ , oscillatory and non-convergent trajectories are possible only when  $\alpha < 1/3$ . In contrast, if  $0 < \omega \leq 1$ , the parametric region where oscillatory and non-convergent trajectories are possible is larger. This holds because of the destabilising effects played by both the subjective discount factor and parents' desire to have children introduced by the existence of a component contingent on individual fertility in the PAYG formula. Since  $\alpha_3(\beta, \phi, 1) > \alpha_3(\beta, \phi, \omega) > 1/3$ , fluctuations are more likely to occur when the weight of individual fertility in the PAYG system is large.

[FIGURES 1 AND 2 ABOUT HERE]

[TABLE 1 ABOUT HERE]

From Proposition 2 the following results with regard to the effects of preference parameters on (local) stability outcomes can be derived.

**Result 1.** *Given that the fertility rate is low because the parents' desire to have children ( $\phi$ ) is small,<sup>10</sup> the possibility of cyclical instability due the introduction of FR-PAYG pensions ( $0 < \omega \leq 1$ ) is larger than when fertility is high because the parents' desire to have children ( $\phi$ ) is large.*

**Result 2.** *When the subjective discount factor is large, the possibility of cyclical instability due to the introduction of FR-PAYG pensions ( $0 < \omega \leq 1$ ) is larger than when the subjective discount factor is low.*

Results 1 and 2 lead to a paradoxical consequence in terms of policy. The introduction of FR-PAYG pensions is essentially advocated in economies with low fertility rates in order to overcome the sustainability issue of public pensions. Our results imply that in economies where the parents' desire to have children is small, the possibility of destabilising the economy induced by a pension reform that makes the benefit contingent on individual fertility, is higher than when the parents' desire to have children is large. This is because a reduction in the taste for children reduces fertility and saving, while also increasing the weight of the public pension component in capital accumulation. The causal chain of this result is the following: (i) below-replacement fertility in developed countries is one of the most important causes for recommending the introduction of fertility-related pensions; (ii) one of the reasons why fertility is low in industrialised countries is that the parents' desire to have children is small. Since fertility-related pensions are introduced essentially as a stimulus to increase fertility (in order to keep public pension budget sustainable over time), then our findings imply that the possibility of destabilisation due to FR-PAYG pensions is plausible for several real economies, as shown in the numerical examples later in the paper.

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<sup>10</sup> We note that different values of  $\phi$  can be, broadly speaking, correlated with different levels of economic development. For instance, the so-called selfish lifestyle of individuals in developed countries has been retained as a reason for reducing the desire to have children in industrialised economies.

In addition, an increase in the subjective discount factor (*ceteris paribus*) implies that individuals want to smooth consumption over the retirement period and increase saving. However, it also causes an increase in the crowding-out effect of public pensions under a FR-PAYG system (thus acting as a destabilising device), while being neutral under a pure PAYG system. Therefore, in countries where the individual propensity to save is large and/or the parents' desire to have children is small, the introduction of a FR-PAYG scheme may cause cyclical instability and endogenous fluctuations, as shown in the next section.

### 3.1. Endogenous fluctuations: a numerical experiment

We now show through numerical simulations the emergence of endogenous fluctuations in an economy with FR-PAYG pensions. We take the following parameters:  $A = 10$ ,  $\alpha = 0.3$  (Gollin, 2002),  $\beta = 0.6$  (Žamac, 2007),  $\phi = 0.05$ ,  $q = 0.15$ ,  $\theta = 0.16$  (Feldstein, 2005; Cigno and Werding, 2007; Liikanen, 2007).<sup>11</sup> The values of  $\phi$  and  $q$  are “calibrated” to get a fertility rate close to the observed average values in several developed countries. In fact, with this parameter values the long-term fertility rate is almost 0.72 (i.e. 1.44 children per couple) when  $\omega = 1$ , and 0.47 (i.e. almost 1 child per couple) when  $\omega = 0.5$ .

Figures 3.a-3.c show the bifurcation diagrams for  $\theta$  with respect to three different values of the child factor (the initial condition is  $k_0 = 0.1$ ): pure PAYG pensions ( $\omega = 0$ ), mixed FR-PAYG pensions ( $\omega = 0.5$ ) and pure FR-PAYG pensions ( $\omega = 1$ ). The vertical axis shows the limit points of the equilibrium sequence of capital. When the contribution rate is sufficiently small, a unique limit point exists. When it increases, two-period cycles, cycles of a higher order and endogenous fluctuations are observed. As can easily be seen, the flip bifurcation value of the contribution rate dramatically reduces when the child factor is larger. In fact, under pure PAYG pensions, the fixed point undergoes a flip bifurcation at  $\theta = 0.795$ , while under pure FR-PAYG pensions, a flip bifurcation emerges at  $\theta = 0.061$ . This implies that the introduction of a component of pension benefits partially or totally contingent on individual fertility, exposes an economy to oscillations in macroeconomic and demographic variables for realistic values of the contribution rate.

Therefore, although fertility-related pensions are often advocated as a possible remedy against the peril of the sustainability of unfunded pensions in the near future (Sinn, 2007) as well as for allocation purposes (Abio et al., 2004), the shift from a pure PAYG system (Figure 3.a) to a PAYG system partially (Figure 3.b) or totally (Figure 3.c) linked to individual fertility may easily open the route to endogenous fluctuations.

[FIGURES 3.a, 3.b and 3.c ABOUT HERE]

## 4. The CIES-CES economy

This section extends our analysis by using more general utility and production functions. We assume that individual preferences over material consumption and the number of children are characterised by a Constant-Inter-temporal-Elasticity-of-Substitution (CIES) utility function, and goods and services are produced through a Constant-Elasticity-of-Substitution (CES) production function.

*Preferences.* The lifetime utility of the individual representative of generation  $t$  is given by the following CIES formulation:

<sup>11</sup> In particular, Cigno and Werding (2007, pp. 26–27) find that the contribution rate to the PAYG system is about 19.5 per cent and 16.45 per cent of gross wages in Germany and France, respectively.

$$U_t = \left(1 - \frac{1}{\sigma}\right)^{-1} c_{1,t}^{1-\frac{1}{\sigma}} + \beta \left(1 - \frac{1}{\sigma}\right)^{-1} c_{2,t+1}^{1-\frac{1}{\sigma}} + \phi \left(1 - \frac{1}{\sigma}\right)^{-1} n_t^{1-\frac{1}{\sigma}}, \quad (30)$$

where  $\sigma > 0$  ( $\sigma \neq 1$ ) is the (constant) inter-temporal elasticity of substitution (see de la Croix and Michel, 2002, for the case of exogenous fertility, and Spataro and Fanti, 2011, for the case of endogenous fertility). The maximisation of (30) with respect to  $n_t$  and  $s_t$  subject to the budget constraints (2) and (3), gives:

$$n_t = \frac{\phi^\sigma w_t (1 - \theta)}{\left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^\sigma \left[1 + \beta^\sigma (R_{t+1}^e)^{\sigma-1} + \phi^\sigma \left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^{1-\sigma}\right] - \phi^\sigma (1 - \omega) \theta \frac{w_{t+1}^e}{R_{t+1}^e}}, \quad (31)$$

$$s_t = \frac{w_t (1 - \theta) \left[ \beta^\sigma (R_{t+1}^e)^{\sigma-1} \left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^\sigma - \phi^\sigma \theta \frac{w_{t+1}^e}{R_{t+1}^e} \right]}{\left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^\sigma \left[1 + \beta^\sigma (R_{t+1}^e)^{\sigma-1} + \phi^\sigma \left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^{1-\sigma}\right] - \phi^\sigma (1 - \omega) \theta \frac{w_{t+1}^e}{R_{t+1}^e}}. \quad (32)$$

*Production and equilibrium.* At time  $t$  production takes place according to the CES technology

$$Y_t = A[aK_t^{-\rho} + (1-a)L_t^{-\rho}]^{\frac{-1}{\rho}}, \quad (33)$$

introduced by Arrow et al. (1961), where  $\rho > -1$  ( $\rho \neq 0$ ) is the degree of substitutability of the inputs,  $A > 0$  is a scale parameter and  $0 < a < 1$  measures the relative weight of capital in production and determines, for any given value of  $\rho$ , the “functional distribution of income” (Arrow et al., 1961, p. 230).<sup>12</sup> When  $-1 < \rho < 0$  (resp.  $\rho \rightarrow -1$ ), capital and labour are highly substitutable (resp. perfect substitutes). In contrast, when  $\rho > 0$  (resp.  $\rho \rightarrow +\infty$ ), capital and labour are poor substitutes (resp. perfect complements). The limit case  $\rho = 0$  represents the Cobb-Douglas production function. The elasticity of substitution is given by  $\varepsilon = 1/(1 + \rho)$ : it is larger (resp. smaller) than unity for any  $-1 < \rho < 0$  (resp.  $\rho > 0$ ). Consequently, (33) can also be written in per worker terms as follows:

$$y_t = A(a k_t^{-\rho} + 1 - a)^{\frac{-1}{\rho}}. \quad (34)$$

With CES technology and competitive markets, the interest factor and the wage rate are respectively given by following equations:

$$R_t = aA[a + (1-a)k_t^\rho]^{\frac{1+\rho}{\rho}}. \quad (35)$$

$$w_t = (1-a)A(a k_t^{-\rho} + 1 - a)^{\frac{1+\rho}{\rho}}, \quad (36)$$

The equilibrium condition in the capital market is still given by (11). Then, by combining (31), (32) and (11), we get:

$$k_{t+1} = \left(\frac{\beta}{\phi}\right)^\sigma (R_{t+1}^e)^{\sigma-1} \left(q w_t - \omega \theta \frac{w_{t+1}^e}{R_{t+1}^e}\right)^\sigma - \theta \frac{w_{t+1}^e}{R_{t+1}^e}. \quad (37)$$

By assuming static expectations and using (35)-(37), the dynamics of capital are now described by the following nonlinear difference equation:

<sup>12</sup> The parameter  $a$  in the CES function is a proxy of the distributive capital share  $\alpha$  in the limit case of Cobb-Douglas function.

$$k_{t+1} = \left( \frac{\beta}{\phi} \right)^\sigma (aA)^{\sigma-1} \frac{(ak_t^{-\rho} + 1 - a)^{\frac{(1-\sigma)(1+\rho)}{\rho}}}{k_t^{(\sigma-1)(1+\rho)}} \left[ q(1-a)A(ak_t^{-\rho} + 1 - a)^{\frac{-(1+\rho)}{\rho}} - \omega\theta \frac{1-\alpha}{\alpha} k_t^{1+\rho} \right]^\sigma - \theta \frac{1-\alpha}{\alpha} k_t^{1+\rho} \quad (38)$$

From (38), it is clear that closed-form expressions for the steady-state stock of capital cannot be derived. This means that in a CIES-CES economy, a flip bifurcation analysis cannot be developed explicitly. However, numerical experiments show that a unique flip bifurcation value of  $\theta$  exists, and the effects on stability outcomes of an increase in the child factor  $\omega$  are qualitatively the same as those studied in an economy with logarithmic preferences and Cobb-Douglas technology. In addition, the use of a CIES-CES model-economy with pensions introduces a role for the elasticity of substitution parameters on stability outcomes. In particular, *ceteris paribus*, (1) for any  $\rho > -1$  ( $\rho \neq 0$ ), an increase in  $\sigma$  plays a de-stabilising (resp. stabilising) role when  $\sigma > 1$  (resp.  $\sigma < 1$ ), i.e. it reduces (resp. increases) the flip bifurcation value of  $\theta$ . This is in line with Michel and de la Croix (2000) in a model without pensions, and (2) for any  $\sigma > 0$  ( $\sigma \neq 1$ ), an increase in the absolute value of  $\rho$  plays a de-stabilising (resp. stabilising) role when  $\rho > 0$  (resp.  $-1 < \rho < 0$ ), i.e. it reduces (resp. increases) the flip bifurcation value of  $\theta$ . Unlike Michel and de la Croix (2000), this last result is unique to our model with public pensions.

Below, we concentrate on the role played by the child factor on the local stability of the fixed point, rather than making a stability analysis of the elasticity of substitution parameters

Since elements of fertility-related pensions exist (amongst others) in Germany, we now simulate how an increase in the weight of the child factor affects long-term dynamics in the case of the German economy. We calibrate the values of preference, technology and policy parameters to match German macroeconomic and demographic data. Of course, we are aware that our model-economy is highly stylised. Therefore, numerical experiments are performed without claiming that the results can be interpreted as possible practical policy suggestions.

Discussions regarding the choice of parameter values are now needed. Empirical estimates of the inter-temporal elasticity of substitution ( $\sigma$ ) are generally very varied, stemming from the fact that microeconomic studies come to different conclusions than macroeconomic studies, which use aggregate data. On the one hand, there are estimates of  $\sigma$  very close to zero, i.e. Hall (1988) for the US, Blundell-Wignall et al. (1995) and Lund and Engsted (1996) for Germany, Girardin et al. (2000) for France, and Hyde and Cuthberston (2003) for both Germany and France. However, the omission of durable consumption goods following Hall's approach may cause a downward bias in the estimates (Ogaki and Reinhart, 1998). To avoid this problem, Márquez de la Cruz et al. (2007) employ (i) the real interest rate and (ii) the rate of return of the stock index for Germany (1970-2003), and thus find an inter-temporal elasticity of substitution of almost 1.6 in case (i) (resp. 1.3 in case (ii)) of Germany.<sup>13</sup> For our purposes, we use  $\sigma = 1.3$  for Germany.

The estimates of the elasticity of substitution in production ( $\rho$ ) are also controversial, ranging from values below or above unity. This essentially depends on whether time-series or cross-section analyses are performed. However, Lucas (1969) argues for the US that while cross-section studies support the Cobb-Douglas function, older time-series studies generally provided lower estimates of the elasticity of substitution. More recently Chirinko et al. (1999), with a cross-section analysis based on micro data, found low values of the elasticity of

<sup>13</sup> We note that estimates of  $\sigma$  included between these two values also exist. For instance, using the interest rate on saving deposits at statutory notice, Flaig (1988) found that  $\sigma$  ranges between 0.4 and 1.1 in Germany. However, these findings have been questioned by Hansen (1996).

substitution in production, ranging from 0.25 to 0.40, i.e. capital and labour tend to sufficiently complement each other. Therefore, with regard to technology parameters, we chose the scale parameter ( $A = 15$ ), the distributive parameter ( $a = 0.5$ )<sup>14</sup> and the degree of substitutability of the inputs ( $\rho = 0.55$ ),<sup>15</sup> to obtain a value of the output elasticity of capital around of 0.3.

With regard to the preference parameters, the parents' desire to have children ( $\phi = 0.6$ ) and the subjective discount factor ( $\beta = 0.6$ ) are chosen to match both the fertility behaviour in Germany (we thus set the percentage of working income devoted to child rearing at  $q = 0.3$ ), which shows an average fertility rate of around 0.67, i.e. 1.35 children per couple (see Cigno and Werding, 2007, Figure 1.1, p. 2), and a household saving rate of almost 12 per cent. The evolution of household savings is very different in Germany from other European countries. The German saving rate decreased in 1985-2005, and is currently similar, for example, to the saving rate in France, which is around 11-12 per cent of disposable income (Hiebert, 2006, p. 1), which, in contrast, increased in the same period.

Finally, the contribution rate to the PAYG system is fixed at  $\theta = 0.195$  (Cigno and Werding, 2007).

For the set of parameters that resembles the German macroeconomic and demographic variables, the bifurcation diagram plotted in Figure 4 shows the limit points of the long-term sequence of capital when the child factor increases. The figure shows that the fixed point is locally asymptotically stable when  $\omega$  is low, it undergoes a flip bifurcation at  $\omega = \bar{\omega} = 0.31$ , and a two-period cycle then appears for values of  $\omega$  that range between 0.31 and (almost) 1. Cycles of a higher order and endogenous fluctuations can be observed when  $\omega > 1$ .<sup>16</sup> In addition, Figures 5.a and 5.b depict the time evolution of long-term fertility for the parameter values discussed above. The figures clearly show that when the child factor is small ( $\omega = 0.2$ ), fertility oscillates between 0.48 and 0.6 and approaches the long-term values (Figure 5.a). In contrast, when the child factor is larger ( $\omega = 0.5$ ), fertility oscillates permanently from 0.525 and 0.725 (Figure 5.b).

[FIGURE 4 ABOUT HERE]

[FIGURES 5.a and 5.b ABOUT HERE]

## 5. Conclusions

We have analysed the dynamics of an overlapping generations economy with endogenous fertility and fertility-related pay-as-you-go public pensions with both rational and static expectations. We have shown that introducing a component contingent on individual fertility in the pension formula, dramatically increases the possibility of cyclical instability when individuals have static expectations compared to pure PAYG pensions.

We also found that the use of fertility-related pensions, which are often advocated as a remedy against low fertility, act as a destabilising device especially when fertility is low because the parents' desire to have children is small and the subjective discount factor is high. Numerical simulations for realistic parameter values show that oscillations in macroeconomic and demographic variables can be a plausible scenario when FR-PAYG pensions are operative.

<sup>14</sup> See Bolt and van Els (2000).

<sup>15</sup> This assumes an elasticity of substitution in production of almost 0.65, i.e. capital and labour are complements.

<sup>16</sup> For a discussion on values of  $\omega > 1$  see Fenge and von Weizsäcker (2010), which defines the so-called *hyper-child* pension system. In contrast, the case  $\omega < 0$  defines *under-child* pension systems.

Our findings constitute a policy warning regarding the possible (cyclical) instability caused by the introduction of fertility-related elements in PAYG pensions. They also provide a further explanation of the occurrence of persistent (deterministic) cycles in economies with endogenous fertility.

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## Appendix. Rational expectations

**Proposition A.1.** *In an OLG economy with FR-PAYG pensions and rational expectations, trajectories are always monotonic and convergent to  $k^*$ .*

**Proof.** Differentiating (14) with respect to  $k_t$  and using (15) we get:

$$G'(k^*) = \alpha \frac{q \beta \alpha (1 - \alpha) A}{\alpha \phi + \theta (1 - \alpha) (\beta \omega + \phi)} (k^*)^{\alpha-1} = \alpha. \quad (\text{A1})$$

Therefore,  $0 < G'(k^*) < 1$  for any  $0 < \theta < 1$ . **Q.E.D.**

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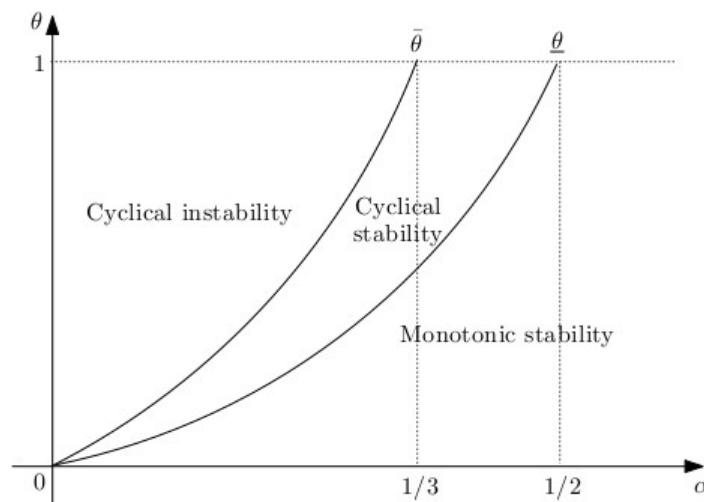
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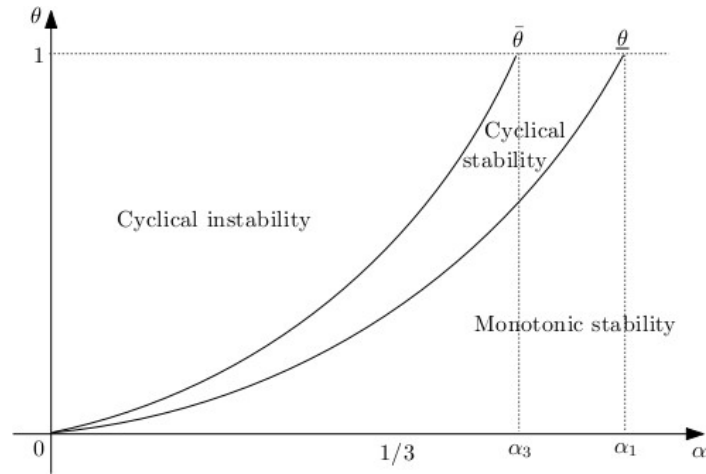
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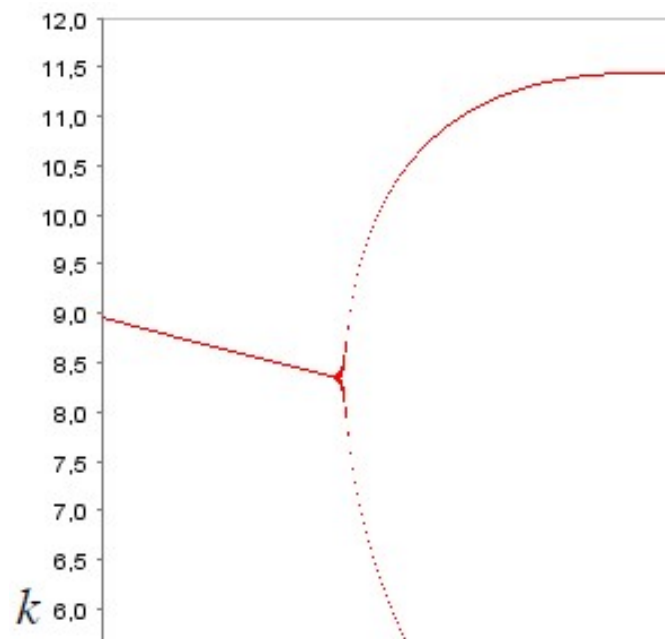
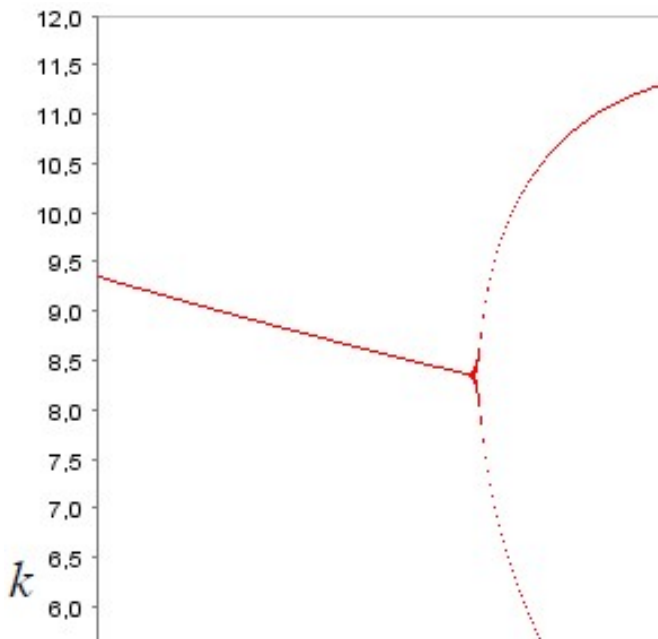
**Figure 1.** Case  $\omega = 0$ . Stability and instability regions in the  $(\alpha, \theta)$  plane.

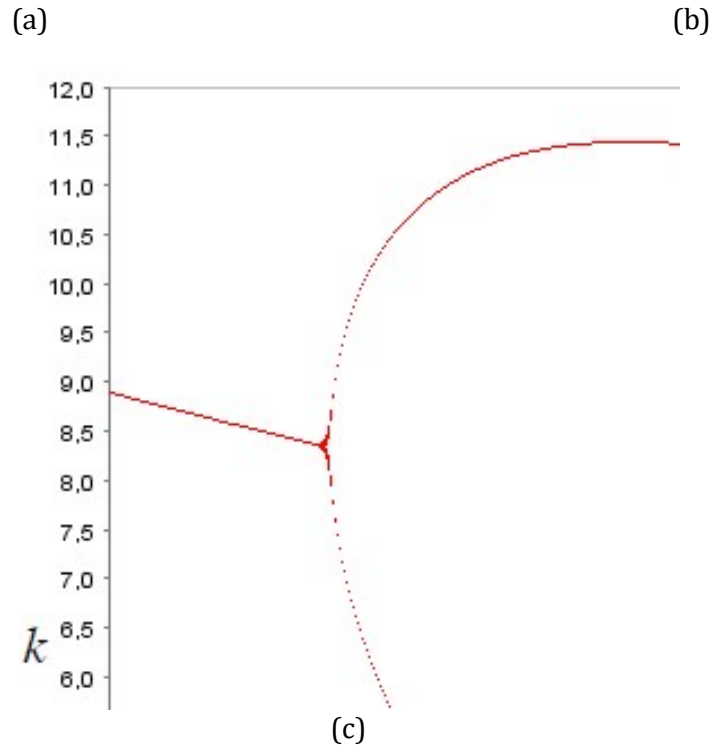


**Figure 2.** Case  $0 < \omega \leq 1$ . Stability and instability regions in the  $(\alpha, \theta)$  plane.

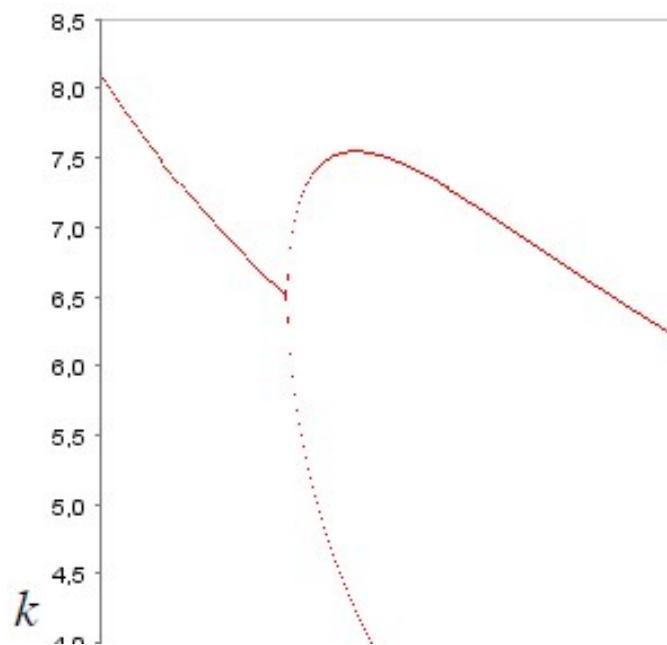
**Table 1.** Parametric regions of cyclical instability ( $0 < \bar{\theta} < 1$ ) under different PAYG systems.

Pure PAYG ( $\omega = 0$ )	Mixed FR-PAYG ( $0 < \omega < 1$ )	Pure FR-PAYG ( $\omega = 1$ )
$0 < \alpha < 1/3$	$0 < \alpha < \alpha_3(\beta, \phi, \omega)$	$0 < \alpha < \alpha_3(\beta, \phi, 1)$

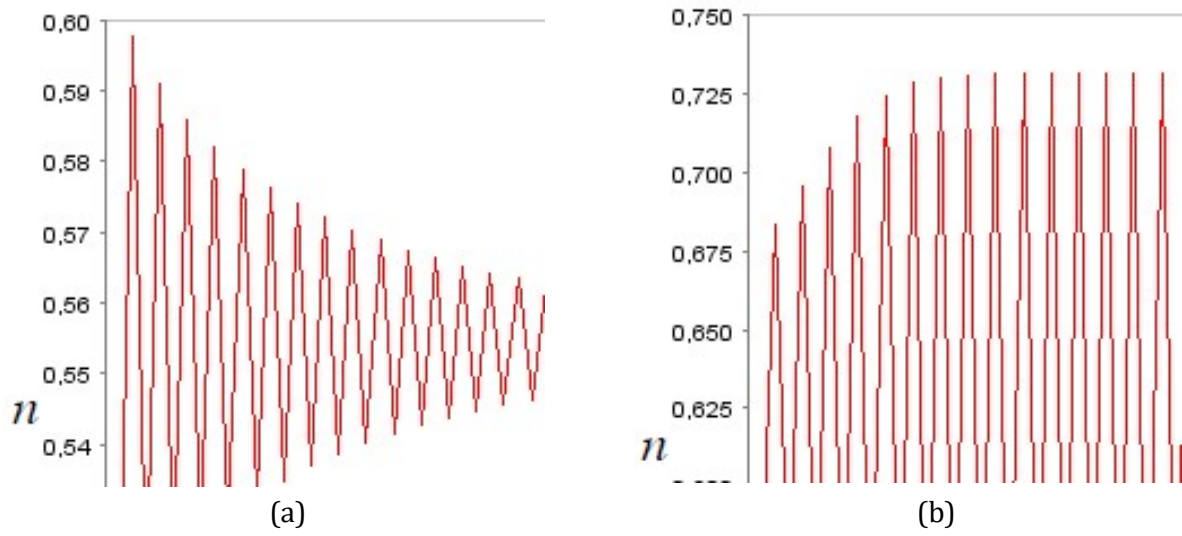




**Figure 3.** Bifurcation diagrams for  $\theta$ . (a)  $\omega = 0$  (pure PAYG):  $\bar{\theta} = 0.795$ ; (b)  $\omega = 0.5$  (mixed FR-PAYG):  $\bar{\theta} = 0.113$ ; (c)  $\omega = 1$  (pure FR-PAYG):  $\bar{\theta} = 0.061$ .



**Figure 4.** Bifurcation diagram for  $\omega$  ( $\bar{\omega} = 0.31$ ).



**Figure 5.** Oscillations in long-term fertility for different values of the child factor. (a)  $\omega = 0.2$ ; (b)  $\omega = 0.5$ .