Delay-Tolerant Stochastic Algorithms for Parking Space Assignment

Arieh Schlote*, Christopher King†, Emanuele Crisostomi‡ and Robert Shorten§
*Hamilton Institute, National University of Ireland, Maynooth, Ireland
†Department of Mathematics, Northeastern University, Boston, MA 02115, USA
‡Department of Energy, Systems, Territory and Constructions Engineering, University of Pisa, Italy
§IBM Research, Ireland

***Abstract*—This paper introduces and illustrates some novel stochastic policies that assign parking spaces to cars looking for an available parking space. We analyse in detail both the main features of a single park, i.e., how a car could conveniently decide whether to try its luck at that parking lot or try elsewhere, and also the case when more parking lots are available, and how to choose the best one. We discuss the practical requirements of the proposed strategies in terms of infrastructure technology and vehicles' equipment and the mathematical properties of the proposed algorithms in terms of robustness against delays, stability and reliability. Preliminary results obtained from simulations are also provided to illustrate the feasibility and the potential of our stochastic assignment policies.**

I. Introduction

Finding a parking space in a densely populated area is a non-trivial challenge. Furthermore the unavailability of instantaneous parking causes significant damages, both economically and environmentally. People cruising for parking waste not only their own time, which they could spend working or for leisure, but also consume road capacity, burn fuel, and produce toxic emissions, thus contributing significantly to congestion, greenhouse gas emissions and pollution. It was recently reported that over one year in a small Los Angeles business district, cars cruising for parking burned 47,000 gallons of gasoline and produced 730 tons of carbon dioxide [14]. Further, the consulting firm McKinsey recently claimed that the average car owner in Paris spends four years of his life searching for parking spaces [4].

The parking assignment problem associated with electric vehicles becomes even more acute. Due to the limited range of these vehicles, the marginal cost of expending energy to search for spaces may, in some cities, be prohibitively high. Thus there is a real and compelling societal and economic need for parking guidance systems, and this need has given rise not only to interesting research questions, but also commercial opportunities of great potential. Indeed, already major companies are responding to these opportunities. Examples of commercial initiatives in this area include: SFPark (sfpark.org), parkatmyhouse.com, and BMWi (bmw-i.com), all of which are investing heavily in parking research and products within a smart cities context. In parallel, many researchers are also working on this topic.

II. Overview of prior work

Within the research community, the topic of parking has already attracted considerable interest.

Several authors, most notably [13] but also [2], [1], argue that the availability of free, or too cheap curb side parking spaces, incentivises drivers to cruise for a long time instead of using available off-street parking facilities for a fee. This has a negative impact on parking space availability, parking fee revenue, the time spent cruising for a parking space, as well as pollution levels and congestion. These works try to determine optimal pricing schemes that drive the system to an economically optimal state. Related work in this direction includes SFpark in which pricing mechanisms are used to regulate the number of free spaces in a given area at a certain level (for emergency situations), and [15] which focuses on understanding and modeling the behavioural side of parking. Note that this latter paper includes an extensive review in the area of parking.

A completely different approach is advocated in [6]. Here the parking problem is viewed as a dynamic resource allocation problem. Similarities to problems in communication networks are drawn, for which a host of tools and methods have been developed over the last decades. [6] proposes an online reservation system, where cars communicate their parking requirements and are assigned a parking space, which is then reserved and cannot be used by any other vehicle. A similar approach is proposed in [16], albeit with a different assignment routine, that allows the user to book a parking space in advance, and also allows the user to choose a price that he is willing to pay. The main focus of this paper is revenue maximisation, but it is also claimed that by finding the right number of different price segments and the correct prices, it is possible to achieve other goals, such as reducing traffic levels or ensuring some sorts of fairness between drivers from different social classes. It is also concluded that the optimal assignment strategy depends on the vehicle arrival process. It should be noted that [6] and [16] both require massive amounts of hardware to be distributed both to cars and car-parks, and potentially even to each parking space. Also compliance of all drivers with their scheme or a reliable and fast way of reservation enforcement is needed. This renders their solutions not viable at the present time. However, even without these problems, realising such a reservation system seems challenging. For example, determining the availability of a particular parking space is error prone, see [9]. Predicting a parking space's availability at the time that the customer arrives is even harder. On top of this it would be necessary to equip *all* cars and all car-parks with communication devices. Although equipping car-parks is certainly feasible, doing the same for cars will take a significant financial investment and perhaps regulatory impulses.

A more promising and technologically viable approach to improve parking has been proposed in [3] and further studied in [10]. The authors develop an approach in which car-parks are able to count the number of arriving and departing cars as well as the instantaneous occupancy, and communicate these numbers to participating cars. Cars in turn only have to be able to listen to broadcasts from the car-parks and are not required to communicate in the reverse direction. Their work yields an important technique, that allows cars to predict the likelihood of a parking space being available at the estimated time that the car will arrive there. This work uses ideas from queueing theory to predict the occupancy upon arrival, with car-parks being modelled as single server queues with a Poisson arrival process and exponentially distributed service times. It should be mentioned that this significant reduction in requirements by using a stochastic approach comes at the cost of certainty for the customers. The lack of a reservation system makes it possible that customers arrive to a fully occupied car-park. The main drawback of their approach is however, that ultimately the customers will want to use the information to make a decision whether to try their luck and drive to the car-park or to go somewhere else. Accordingly, there is feedback embedded in the system which needs to be taken into consideration; namely when drivers choose to drive to a car-park based on the predictions made, they then affect the arrival process - rendering the model and predictions no-longer valid. This feedback has been completely ignored by the authors. One of the goals of this paper is to investigate the effect of this feedback on the car-park occupancy prediction problem. In particular we aim at using ideas that have been employed in the context of urban pollution control to improve parking, see [12] for details.

III. Mathematical Aspects in Parking

Parking gives rise to a number of quite distinct mathematical problems, depending upon the perspective from which the problem is approached, the type of search being addressed, and the amount of infrastructure available to help find/allocate parking spaces.

(i) First, associated with each vehicle wishing to find a parking space are two basic costs. The first is the cost to the driver of searching for a parking space, while the second is the cost to the city of that same driver searching for a parking space. The first is usually a quality of service (QoS) issue based on, for example, the expected search time or the expected fuel consumption while looking for a space. The second cost could be based on emissions or pollutants being generated by the searching vehicle. Thus, while prioritising an electric vehicle over a large ICE based vehicle in assigning a parking space may make perfect sense in the context of rewarding responsible vehicle choices, it may be precisely the wrong assignment from the point of view of the municipality. Conflicts of this nature give rise to a number of questions with a game theoretic flavour in the parking space context.

(ii) Second, typically drivers may search for two distinct kinds of parking spaces. They may either choose to look for a space in a car-park, or they may search for on-street parking. The first gives rise to prediction type problems, where the driver, based on information concerning current occupancy (perhaps from a street information system), makes a decision based on the likelihood of a place being available when his/her vehicle arrives at the car-park. Problems of this kind are known to give rise to flapping (where two or more parking facilities take turns in being full and under-utilised) and highly localised congestion and pollution peaks [6] due to the fact that the majority of drivers are known to choose the car-park with the most available free spaces [6]. The second problem is a probabilistic routing problem. Drivers compete for spaces by following random routes chosen to maximise the expectation of finding a free parking space.

(iii) Third, one may categorise the parking problem according to the level of dedicated infrastructure that exists in support of the assignment problem. In some situations all vehicles and spaces may be instrumented, and in other situations we may only be able to place a probability on space availability. The first type of problem gives rise to optimisation based reservation systems where vehicles are assigned spaces based on optimality criteria. As we have already mentioned, problems of this kind are massively large scale, and give rise to certain inefficiencies. The second type of problem, typically arising in situations where drivers have access to the same information, gives rise to complex dynamic systems in which delays between drivers making a decision to opt for a car-park (parking space), and actually arriving at the location, leads to complications.

In this paper we consider the problem of guiding cars to a set of car-parks in a way that avoids localised congestion and pollution peaks. To solve this problem we assume instrumented car-parks (i.e. car-parks can estimate arrival and departure rates), and that this information can be broadcasted to vehicles. We do not assume that vehicles communicate directly with car-parks in order to make a reservation; rather vehicles must estimate the availability of a parking place based on the broadcasted information. Thus, the problem considered in this paper incorporates aspects of items (ii) and (iii) above.

Specifically, our objective in this paper is to consider the problem of assigning searching vehicles to car-parks where car-parks may broadcast to groups of searching vehicles, but where there is no direct communication from vehicles to the car-parks. In particular, we are interested in situations where broadcast information can be processed on-board (in GPS units for example) the vehicles to enable drivers to make decisions as to where to park. Thus, we have a problem where the effect of delays is present, and where the quality of service metric is the probability of cars arriving to the car-park when no spaces are available. In this context we shall consider two specific problems.

Problem 1: Single Car-Park
First, we shall consider the problem of a single car-park, where a vehicle makes a choice to go to a car-park based on occupancy, i.e. the number of vehicles currently parked in the car-park, and then travels to this location, arriving some time later. This is a problem in the same vein as that studied by [10]. Our main contribution in this context is that we shall rigorously take into account the fact that the arrival process at the car-park and the decision of the individual drivers are coupled. In order to study the effect of this feedback on the occupancy prediction problem, we use a mix of queueing theory and ideas from the control theoretic study of communication networks. Our basic modelling assumption in solving this problem is that customers query the occupancy of a car park and decide whether to proceed to that car-park based on this information. In particular we assume that their willingness to proceed to the car-park is a non-increasing function of the occupancy at the time of their query. This assumption allows us to borrow ideas from the networking community to solve this problem. In particular, we adopt the Random Early Detection (RED) active queue management algorithm [5] to represent the customers' behaviour[1]*. Note that, the literature suggests that human behaviour with respect to travel mode choice and parking space choice is very complicated, see [15] and the references therein. However in a simple scenario with only one car-park, it is intuitively clear that an adaptive pricing scheme for the car-park will achieve any desired level of occupancy. We believe that the proposed price function is efficient and is also a good approximation of the willingness of people to risk going to the car-park. As we shall see, the use of a RED-like algorithm is very effective in this context. An important mathematical contribution of our work is that take into account the effect of delays due to travel.*

Problem 2: Multiple Car-Parks
The second problem that we shall study considers multiple car-parks. Our goal now is to avoid localised congestion and to balance searching vehicles amongst a number of car-parks so that occupancy is balanced. Algorithms of this nature were proposed in [7], [8] in the context of electric vehicles and balanced charging. We assume again that customers are informed of the occupancy of each car-park and choose which car-park to go to on the basis of this information. Our contribution here is to extend the literature on the charging framework to the parking case, and to give mathematical proofs that demonstrate convergence of our algorithms and "flapping free" behaviour. Note that by developing a decentralised solution for this problem one arrives at a situation, where car-parks can join and leave the system at will; namely we obtain a plug-and-play type solution that does not require any centralised infrastructure.

Thus, our main contributions in this paper are the following

1) We take feedback into account in the prediction of parking space availability in a single car-park.
2) In this context, we present an analysis to quantify stability issues that arise as a result of this feedback.
3) We then extend our approach to several car parks. Specifically, we propose a load balancing algorithm to balance demand across several car parks.
4) We then realise the balancing solution in a completely decentralised fashion.

This paper is organised as follows. In Section IV we give details on our approach in the single car-park scenario and provide analytic tools to determine its reliability. In Section V we extend our approach to a scenario with several car-parks and give a detailed analysis of the systems stability behaviour. A number of supporting and motivating simulations is given in Section VI. In Section VII we discuss commercial opportunities of our work and conclude the paper in Section VIII.

IV. Single car-park Model

We now describe problem 1. We consider a single car-park under the following assumptions.

- We assume that this car-park is instrumented so that its occupancy can be estimated.
- We assume that this information can be broadcasted to potential customers on a continuous basis.
- We assume that cars arrive and depart to/from the car-park according to two Poisson processes.

The Poisson arrival processes throughout the paper. The use of Poisson processes to model bursty traffic is well established.

[1]Alternatively this can be seen as a dynamic pricing scheme within the car-park, where the price to use the car-park is a non-decreasing function of the occupancy and cars make a decision to use a car-park based on the available price information. Different customers then may be willing to pay different parking fees to obtain a parking space.

Furthermore, the memory-less property of these processes eases analysis in our case. Recall that a process is described by a distribution that describes inter-arrival probabilities. In particular, if the expected time between two arrivals is $x > 0$ then the variance of this random time is x^2.

Objective : Our objective here is to develop algorithms which allow vehicle owners to make informed decisions as to whether a car parking space will be available at the car-park or not. Note, in this context the above assumptions are standard, see for example [3] and [10]. An important contribution of our work is that our approach takes into account the feedback between the decision making process of the driver and the arrival process at the car-park. Note also that previous studies on this topic have neglected the inherent feedback loop between the arrival and decision processes, thereby rendering results in those papers less useful than the results presented here [10].

The critical element in our modelling task is to determine the likelihood that a driver, upon receiving occupancy information from the car-park, will make the choice to travel to the car-park. We model this in a stochastic framework with a probability of travelling to the car-park that depends on the occupancy of the car-park at that time. As already mentioned, we assume that a reasonable way for people to make decisions of this nature is to drive to the car-park with a probability that is higher when the occupancy of that car-park is lower. Thus, given these facts, it seems reasonable to suggest the algorithm given in Algorithm IV.1 for making a decision as to whether or not to go to the car-park. This algorithm is based on RED [5] from internet congestion control. In RED a pricing signal is used to control queue occupancy; in our context, a probabilistic pricing signal is used to make suggestions based on car-park occupancy. The probability function used to define this algorithm is shown in Figure 1. Note, that the drivers will go to the car-park with probability 1 when the occupancy is low and will not go there when the occupancy is high. Note also, that this algorithm can be easily implemented using GPS devices or smart phones.

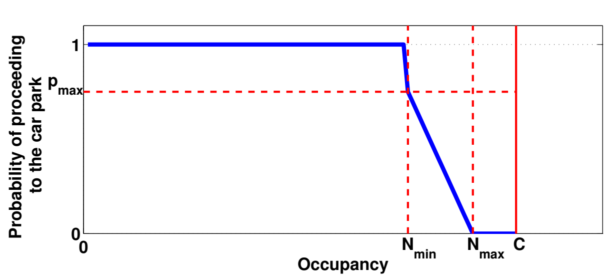


Fig. 1: The probability of proceeding to the car-park using the RED-approach.

Algorithm IV.1: SINGLE CAR-PARK()

comment: Executed by newly arriving car

$N \leftarrow$ occupancy of car-park

$$p \leftarrow \begin{cases} 1, & \text{if } N < N_{min}, \\ 0, & \text{if } N > N_{max}, \\ p_{max}\frac{N_{max}-N}{N_{max}-N_{min}}, & \text{otherwise,} \end{cases}$$

do Go to car-park with probability p.

In Algorithm IV.1 we use the parameters $0 \leq p_{max} \leq 1$, and $N_{min}, N_{max} \in \mathbb{N}$ with $N_{min} < N_{max} \leq C$, where C is the total capacity of the car-park. The occupancy of the car-park is broadcasted to all participating cars and will be updated in regular time intervals.

A. *Model I: Case of homogeneous delays*

If car i decides to go to the car-park, then we assume this takes a time τ_i (which can be expressed for example in seconds). In this section, to begin the analysis, we assume that $\tau_i = \tau$ is the same for all vehicles. For example we might assume that vehicles make a decision at a certain distance (measured in km, energy, or time depending on vehicle type) from the car-park. We now set the time between updates of the broadcasted occupancy information to be equal to τ. This yields discrete time steps $k = 0, 1, \ldots,$ where the k'th time interval is $[k\tau, (k+1)\tau]$. We further assume that a car which arrives to the car-park during a period when there are no free parking spaces will wait outside the car-park until space becomes available. Denote by $N(k)$ the number of cars parked in the car-park plus the number of cars waiting for a parking space at time $k\tau$. The evolution of $N(k)$ can be described as a difference equation of the form

$$N(k+1) = N(k) + A(k) - D(k), \tag{1}$$

where $A(k)$ is the number of cars that arrive to the car-park during the interval $[k\tau, (k+1)\tau]$ and $D(k)$ is the number of cars leaving from the car-park in that same interval. $D(k)$ takes values in $0, 1, \ldots, N(k)$ and we model it as a random variable with distribution depending on $N(k)$. In particular we assume that cars stay parked for a random time described by an exponentially distributed random variable with fixed rate $\mu > 0$. Note in particular that this assumption ensures that the evolution of the random variable is independent of all other cars and the occupancy process of the car-park. Finally, if we assume that cars stay, on average, much longer than the time between broadcasts, i.e. $\tau\mu \ll 1$, then the departure process, $D(k)$, changes slowly enough, so that we can approximate it as following a Poisson process that terminates once $N(k)$ car have left. Further this Poisson process has rate $\mu G(N(k))$, where $G(N(k)) = \min\{N(k), C\}$, where C is the car-park capacity. Accordingly, the distribution of $D(k)$ is described by

$$P(D(k) = t|N(k) = n) = e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!} \quad (2)$$

for all $n \in \mathbb{N}$ and all $t = 0, 1, \ldots, G(n) - 1$ and

$$\begin{aligned} &P(D(k) = G(n)|N(k) = G(n)) \quad (3)\\ =& \sum_{t=G(n)}^{\infty} e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!} \\ =& 1 - e^{-G(n)\mu\tau} \sum_{t=0}^{G(n)-1} \frac{(G(n)\mu\tau)^t}{t!}. \end{aligned}$$

As $\tau_i = \tau$ we know that all cars which arrive to the car-park in $[k\tau, (k+1)\tau]$ must have made the decision in the time interval $[(k-1)\tau, k\tau]$. It is important to note now that the arrival process of cars at the car-park is no longer a homogeneous Poisson process. It is however piecewise homogeneous, i.e. for all $k \geq 1$ the arrival process of cars to the car-park in the interval $[k\tau, (k+1)\tau]$ is homogeneous with rate $p(N(k-1))\gamma$. γ is the rate at which cars query the car-park occupancy in order to make a decision, and $p : \mathbb{N} \to [0, 1]$ is the probability function that was introduced in Algorithm IV.1. Clearly, the rate at which cars arrive at the car-park will be smaller than γ if some of the cars decide not to go to the car-park.

The system that we have described clearly allows for the undesirable situation where customers arrive to a full car-park and have to wait to gain entrance or leave to find parking at a different location. The following theorems guide the choice of algorithm parameters to ensure that this undesirable situation is a rare event.

To this end let $U(k)$ be the number of customers waiting outside the car park at the end of the interval $[k\tau, (k+1)\tau]$. $U(k)$ can be described by

$$U(k) = \max\{N(k) + A(k) - D(k) - C, 0\}. \quad (4)$$

We can now describe the probability of $U(k)$ being positive. Note that if $U(k)$ is positive then $N(k+1) > C \geq N_{max}$. According to Algorithm IV.1 all cars making a decision after time $(k+1)\tau$ will decide not to drive to the car-park until such time that the occupancy has dropped below N_{max}.

Theorem 1: Given $N(k-1) = m$ and $N(k) = n$ for some $n, m \in \mathbb{N}$ the probability that the number of customers waiting at time $(k+1)\tau$ is positive is given by

$$\begin{aligned} &1 - \sum_{l=0}^{C} e^{-\gamma p(m)\tau} \frac{(\gamma p(m)\tau)^l}{l!} \quad (5)\\ &+ \sum_{t=0}^{G(n)-1} e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!} \sum_{l=C-G(n)+t+1}^{C} e^{-\gamma p(m)\tau} \frac{(\gamma p(m)\tau)^l}{l!}. \end{aligned}$$

Proof:

The theorem gives the probability that $U(k)$ is positive given the values of $N(k)$ and $N(k-1)$. As $U(k)$ is a non-negative random variable

$$\begin{aligned} &P(U(k) > 0|N(k-1) = m, N(k) = n) \\ =& 1 - P(U(k) = 0|N(k-1) = m, N(k) = n). \quad (6) \end{aligned}$$

For fixed $k \in \mathbb{N}$ the number of vehicles that arrive to the car-park, $A(k)$, is described by a Poisson process with rate $\gamma p(N(k-1))$. Note that $A(k)$ and $D(k)$ are independent conditioned on $N(k-1)$ and $N(k)$. Let us use the following shorthand notation

$$P_{U,0} = P\left(U(k) = 0|N(k-1) = m, N(k) = n\right) \quad (7)$$

Hence for all $n, m \in \mathbb{N}$ according to Equation (4)

$$P_{U,0} = P(A(k) - D(k) \leq C - n|N(k-1) = m, N(k) = n),$$

where we have rearranged the terms in the inequality. $A(k)$ can only take the values $0, 1, \ldots, G(n)$ and hence

$$\begin{aligned} P_{U,0} =& \sum_{t=0}^{G(n)} P(D(k) = t|N(k) = n) \cdot \\ &\cdot P(A(k) \leq C - G(n) + t|N(k-1) = m \text{ and } D(k) = t) \end{aligned}$$

As $A(k)$ and $D(k)$ are independent conditioned on $N(k-1) = m$ and $N(k) = n$ we further obtain

$$\begin{aligned} P_{U,0} =& \sum_{t=0}^{G(n)} P(D(k) = t|N(k) = n) \cdot \\ &\cdot P(A(k) \leq C - G(n) + t|N(k-1) = m). \end{aligned}$$

We now use that $A(k)$ is Poisson with rate $\gamma p(m)$ and thus the probability of l cars arriving in τ seconds is given by $P(A(k) = l) = e^{-\gamma p(m)\tau} \frac{(\gamma p(m)\tau)^l}{l!}$ for all $l \in \mathbb{N}$. This together with Equations (2) and (3) then yields $P_{U,0} =$

$$\begin{aligned} =& \sum_{t=0}^{G(n)-1} e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!} \sum_{l=0}^{C-G(n)+t} e^{-\gamma p(m)\tau} \frac{(\gamma p(m)\tau)^l}{l!} \\ &+ \left(1 - \sum_{t=0}^{G(n)-1} e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!}\right) \sum_{l=0}^{C} e^{-\gamma p(m)\tau} \frac{(\gamma p(m)\tau)^l}{l!}, \end{aligned}$$

where we separated the case $t = G(n)$ from the rest of the sum over t. Rearranging yields the claim.

■

Comment : Theorem 1 gives a formula for calculating the probability of an overflow occurring at the car-park. Thus, it provides a tool to evaluate the performance of Algorithm IV.1 in a given scenario.

To give a qualitative idea of the order of magnitude of the probability that the number of arriving customers exceeds the available capacity in Theorem 1, Figure 2 shows such a probability for different values of γ and μ. In particular, we assume that the car-park has a capacity for 100 vehicles, $m = 80$ and $n = 90$, we choose parameters $N_{max} = 90$, $N_{min} = 75$ and $p_{max} = 0.75$ for the RED algorithm, τ equal to 5 minutes for all vehicles, and let the average time between queries (i.e., $1/\gamma$) vary between 10 and 30 seconds, and the average staying time between 0.5 and 1.5 hours (i.e., $1/\mu$). Clearly, as would obviously be expected, the most critical situations (i.e., highest probabilities of not finding a place) occur when cars arrive more frequently and stay for a longer period.

Figure 3 depicts the probability that the number of arriving customers exceeds the available capacity as a function of the car-parks occupancies at the present and one step back in the past, $m = N(k-1)$ and $n = N(k)$. Here we choose the parameters $C = 100$, $N_{min} = 75$, $N_{max} = 90$, $p_{max} = 0.75$, $\gamma = \frac{1}{20}$, $\mu = \frac{1}{3600}$ and τ is equal to 5 minutes. It can be seen that the probability of an overflow is quite low. In fact it is always 0, when $m \geq N_{max}$. The overflow probability is high only when n is close to C or $n > C$, and at the same time $m < N_{max}$. In this sense the figure is slightly misleading: Even though there are some situations that yield a significant probability of an overflow occurring at the next step, these situations themselves occur extremely rarely as they require a large number of cars to arrive during the k'th time interval.

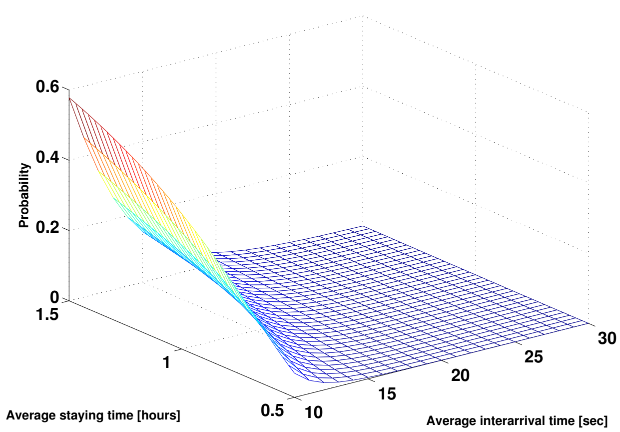


Fig. 2: Probability that the number of arriving customers exceeds the available capacity as a function of average time of arrival and average time of staying.

It should be noted that QoS measure given in Theorem 1 gives the probability of an overflow occurring at discrete time steps of length τ and disregards the probability that an overflow occurs and vanishes between the time steps. It thus underestimates the overflow probability. We now give a complementary result that gives an upper bound for the overflow probability.

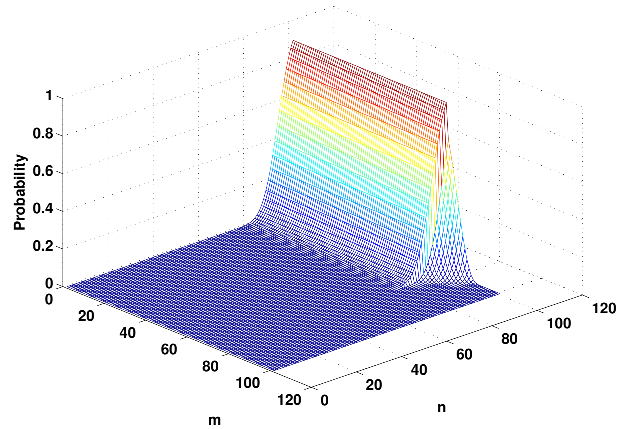


Fig. 3: Probability that the number of arriving customers exceeds the available capacity as a function of the car-parks occupancy.

Theorem 2: Given $N(k-1) = m$ and $N(k) = n$ for some $n, m \in \mathbb{N}$ the probability that at least one vehicle is rejected during the time interval $[k\tau, (k+1)\tau]$ is given by the last entry of the vector $\hat{\pi}_{k+1}$ computed according to

$$\hat{\pi}_{k+1}^{\top} = \pi_k^{\top} \exp(Q_k \tau), \tag{8}$$

where π_k is a column vector with a 1 in the $G(n)+1$ entry and 0 everywhere else, $\exp$ denotes the matrix exponential and Q_k is the $(C+1) \times (C+1)$ dimensional tri-diagonal matrix given by

$$\begin{bmatrix} -r & r & & & & \\ s & -(s+r) & r & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & s & -(s+r) & r & \\ & 0 & & s & -(s+r) & r \\ 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

with $r = \gamma p(m)$ and $s = G(n)\mu$.

Proof: During the update epoch $[k\tau, (k+1)\tau]$ we can model the system as a continuous time Markov chain with $C+2$ states $0, 1, 2, \dots, C+1$, in which transitions from states N to $N+1$ happen with rate $\gamma p(m)$ and from $N+1$ to N with rate $G(n)\mu$ for all $N = 0, 1, \dots, C-1$. The state $C+1$ corresponds to the situation where at least one car arrives to the car park and cannot park. As we are interested in whether this state is reached during the regarded time interval or not, we may make it an absorbing state. Transitions from state C to $C+1$ thus occur with rate $\gamma p(m)$ while transitions from states $C+1$ to C occur with rate 0. The rate matrix of this chain is given by Q_k. π_k is the distribution of the chain at time $k\tau$, which is concentrated in the state $G(n)$. $\hat{\pi}_{k+1}^{\top}$ is the distribution of the states after time τ for our model starting in π_k and accordingly it is given by Equation (8); with the last entry corresponding to the probability of reaching state $C+1$. ■

Note that in Theorem 2 the vectors π_k and $\hat{\pi}_{k+1}$ give the probability of the system being in a certain state at times $k\tau$ and $(k+1)\tau$ respectively, i.e. the probability with which we observe a certain occupancy in the car park. As we know what the occupancy at time $k\tau$ is, the vector π_k is a unit vector, while $\hat{\pi}_{k+1}$ is the prediction our model allows on the distribution after the time τ.

Theorem 2 gives an upper bound to the car parks overflow probability. To give a quantitative idea of this bound, we refer to Figures 4 and 5, which were created in the same setup and with the same parameters as Figures 2 and 3 for Theorem 1. From visual inspection it seems that Figures 2 and 4, and 3 and 5 are practically the same. This indicates that the upper and lower bounds computed according to Theorems 1 and 2 are quite close, and thus they give practical insight into the dynamics of our proposed assignment scheme. However, the figures are not identical, as can be seen in Figure 6, where we compare Figures 2 and 4 for a fixed average staying time of an hour, and in Figure 7, where we compare Figures 3 and 5 for a fixed value of $N(k-1) = 75$. In both cases, the true probability has to lie between the lower and the upper bounds suggested by the aforementioned theorems.

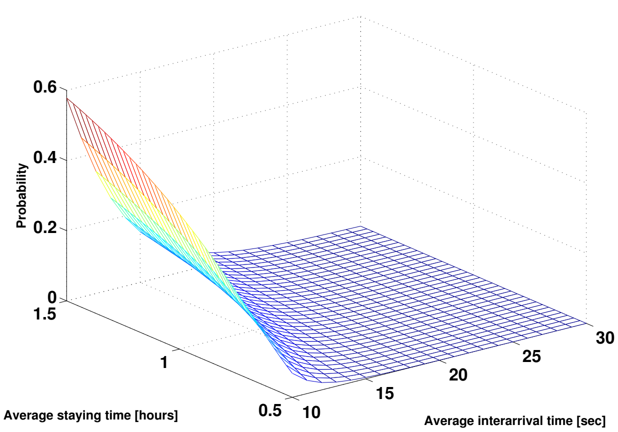


Fig. 4: Probability that the number of arriving customers exceeds the available capacity as a function of average time of arrival and average time of staying.

B. Model II: Heterogeneous Delays

We now relax the assumption on τ_i. Specifically, here we allow a different τ_i to be associated with each vehicle as follows. For car i we model the time τ_i between making a decision and arriving at the car-park as a random variable. We assume that τ_i is bounded for all n and uniformly distributed on $[0, T]$, for some $T \in \mathbb{R}_+$. We set the time between updates of the occupancy information to T. Then, on average, half of the cars making their decision in $[(k-1)T, kT]$ and half

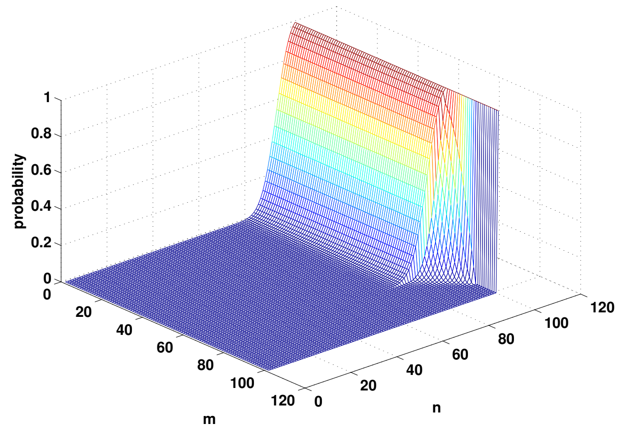


Fig. 5: Probability that the number of arriving customers exceeds the available capacity as a function of the car-parks occupancy.

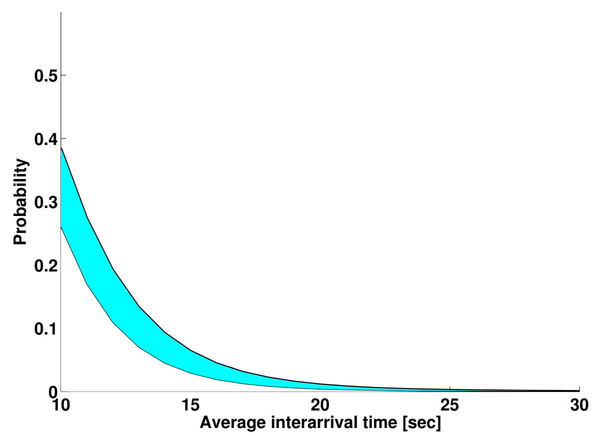


Fig. 6: Comparison of Figures 4 and 2 for an average staying time of one hour as functions of the average time between queries of vehicles.

of the cars making their decision in $[kT, (k+1)T]$ arrive to the car-park in $[kT, (k+1)T]$. Accordingly, we obtain a new equation for the number of the parked and waiting vehicles

$$N(k+1) = N(k) + A_1(k) + A_2(k) - D(k), \qquad (10)$$

where $A_1(k)$ is the arrival process of cars that make their decision to drive to the car-park in $[(k-1)T, kT]$ and $A_2(k)$ is the arrival process of cars that make their decision in $[kT, (k+1)T]$. As in the case with constant τ_i we are interested in the probability of customers arriving to a full car-park in this scenario. To this end again let $U(k)$ be the number of customers waiting outside the car-park at time $(k+1)T$.

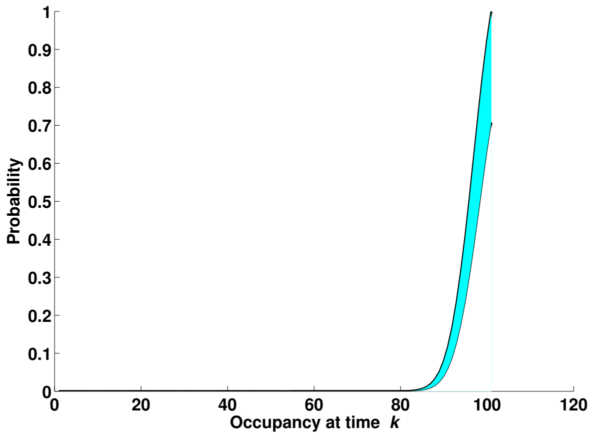


Fig. 7: Comparison of Figures 5 and 3 for $N(k-1) = 75$ as functions of $N(k)$.

Here it can be described by

$$U(k) = \max\{N(k) + A_1(k) + A_2(k) - D(k) - C, 0\}. \quad (11)$$

The following theorem quantifies the probability of the car-park being full at the end of the interval $[kT, (k+1)T]$.

Theorem 3: Given $N(k-1) = m$ and $N(k) = n$ for some $n, m \in \mathbb{N}$ the probability that the number of customers waiting at time $(k+1)T$ is positive is given by

$$1 - \sum_{l=0}^{C} e^{-\nu\tau} \frac{(\nu\tau)^l}{l!} + \sum_{t=0}^{G(n)-1} e^{-G(n)\mu\tau} \frac{(G(n)\mu\tau)^t}{t!} \sum_{l=C-G(n)+t+1}^{C} e^{-\nu\tau} \frac{(\nu\tau)^l}{l!}, \quad (12)$$

where we used the abbreviation $\nu = \frac{1}{2}\gamma(p(m) + p(n))$.

Proof: Here again we use that

$$\begin{aligned} &P(U(k) \geq 0 | N(k-1) = m, N(k) = n) \\ =&1 - P(U(k) = 0 | N(k-1) = m, N(k) = n) \end{aligned} \quad (13)$$

Due to τ_i being uniformly distributed on $[0, T]$ for all i, in the time interval $(kT, (k+1)T)$ the processes $A_1(k)$ and $A_2(k)$ are Poisson processes with rates $\frac{1}{2}\gamma p(m)$ and $\frac{1}{2}\gamma p(n)$ respectively, where again $p(\cdot)$ is the probability function introduced in Algorithm IV.1. Hence $A_1(k) + A_2(k)$ is again Poisson with rate $\frac{1}{2}\gamma(p(m)+p(n))$. The claim is now a direct corollary of Theorem 1.

■

Comment : As in Section IV-A, Theorem 3 gives a lower bound on the probability of an overflow occurring at the car-park.

Following the approach in Section IV-A we now obtain an upper bound on the overflow probability as a corollary.

Theorem 4: Given $N(k-1) = m$ and $N(k) = n$ for some $n, m \in \mathbb{N}$ the probability that at least one vehicle is rejected during the time interval $[k\tau, (k+1)\tau]$ is given by the last entry of the vector $\hat{\pi}_{k+1}$ computed according to

$$\hat{\pi}_{k+1}^\top = \pi_k^\top \exp(Q_k \tau), \quad (14)$$

where π_k is a column vector with a 1 in the $G(n)+1$ entry and 0 everywhere else, $\exp$ denotes the matrix exponential and Q_k is the $(C+1) \times (C+1)$ dimensional tri-diagonal matrix given by

$$\begin{bmatrix} -\nu & \nu & & & & \\ s & -(s+\nu) & \nu & & 0 & \\ & \ddots & \ddots & \ddots & & \\ & & s & -(s+\nu) & \nu & \\ & 0 & & s & -(s+\nu) & \nu \\ 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (15)$$

with $\nu = \frac{1}{2}\gamma(p(m) + p(n))$ and $s = G(n)\mu$.

C. Relaxing the Assumption on the Distribution of τ_i

τ_i being uniformly distributed on $[0, T]$ may not be a realistic assumption as some cars may be closer to the car-park than others upon deciding to find a parking space. The model can easily be extended to take this into account. In the following, we outline a procedure that shows that for any given distribution of τ_i, the process $A_1(k) + A_2(k)$ is still Poisson with rate $\gamma(\alpha p(m) + (1-\alpha)p(n))$, where $\alpha \in [0, 1]$ is a parameter determined by the distribution of τ_i. To see this it is helpful to consider an isolated update interval, say $[0, T]$. Assume that cars query the infrastructure and decide to drive to the car-park with rate $\tilde{\gamma} = \gamma p(N(0))$. This process generates an infinite number of vehicles, but we are only interested in A_1, the number of vehicles that arrive to the car-park before T. Let for $i \in \mathbb{N}$

$$a_i = \begin{cases} 1 & \text{, if car } i \text{ reaches the car-park before } T, \\ 0 & \text{, else.} \end{cases} \quad (16)$$

Now, the i'th car makes a decision at time t_i and then arrives at the car-park after a delay of τ_i. We will only assume that τ_i is a non-negative random variable which is independently and identically distributed for all $i \in \mathbb{N}$ with cumulative probability density function $F_\tau : [0, T] \to [0, 1]$. We assumed that t_i is generated from a Poisson process with rate $\tilde{\gamma}$, and hence it is distributed according to an Erlang distribution with parameters $(i, \tilde{\gamma})$ and thus its probability density function f_{t_i} is given by

$$f_{t_i}(x) = \frac{\tilde{\gamma}^i x^{i-1} e^{-\tilde{\gamma}x}}{(i-1)!}. \quad (17)$$

Accordingly, the probability that $a_i = 1$ is given by, $P(a_i = 1) = F_{t_i+\tau_i}(T)$, where $F_{t_i+\tau_i}$ is the cumulative probability density function of $t_i + \tau_i$, which can be computed according to

$$F_{t_i+\tau_i}(T) = \int_{-\infty}^{\infty} F_\tau(T-x) f_{t_i}(x)\ dx \tag{18}$$

$$= \int_0^T F_\tau(T-x) \frac{\tilde{\gamma}^i x^{i-1} e^{-\tilde{\gamma}x}}{(i-1)!}\ dx, \tag{19}$$

where we used Equation (17) and the fact that τ_i and t_i are positive random variables to change limits of integration. Using Equation (16), we obtain $A_1 = \sum_{i=1}^{\infty} a_i$ and using the linearity of the expectation operator, we can rewrite the expected number of cars that arrive to the car-park before T as

$$E[A_1] = E[\sum_{i=1}^{\infty} a_i] = \sum_{i=1}^{\infty} E[a_i] = \sum_{i=1}^{\infty} P(a_i = 1) \tag{20}$$

$$= \sum_{i=1}^{\infty} \int_0^T F_\tau(T-x) f_{t_i}(x)\ dx \tag{21}$$

$$= \sum_{i=1}^{\infty} \int_0^T F_\tau(T-x) \frac{\tilde{\gamma}^i x^{i-1} e^{-\tilde{\gamma}x}}{(i-1)!}\ dx. \tag{22}$$

According to Lebesgue's monotone convergence theorem, we may exchange summation and integration and obtain

$$E[A_1] = \int_0^T F_\tau(T-x) \tilde{\gamma} e^{-\tilde{\gamma}x} \underbrace{\sum_{i=0}^{\infty} \frac{(\tilde{\gamma}x)^i}{(i)!}}_{=e^{\tilde{\gamma}x}}\ dx \tag{23}$$

$$= \tilde{\gamma} \int_0^T F_\tau(T-x)\ dx \tag{24}$$

The remaining integral is independent of the Poisson arrival process and is further known to be equal to $(T - E[\tau]) \leq T$ and this yields

$$E[A_1] = \tilde{\gamma}(T - E[\tau]). \tag{25}$$

The total number of cars expected to make a decision in $[0, T]$ is $\tilde{\gamma}T$, hence a fraction of $\frac{T-E[\tau]}{T}$ arrives to the car-park in $[0, T]$ and the rest, i.e. a fraction of $\frac{E[\tau]}{T}$ arrives in the interval $[T, 2T]$.

V. Multiple car-parks

So far we have concentrated on a single car-park and cars could only decide to either go to the car-park or go somewhere else. Clearly "somewhere else" is most likely going to be another parking facility. In this section we investigate how our approach can be extended to the more realistic situation, where the vehicles's drivers have to make a decision between several parking facilities. To this end, we assume a situation, where a number of car parks are close together and the driver is not inconvenienced too much by having to go to anyone of them. In particular, vehicles make a decision to travel to a particular car-park based on Algorithm V.1. This can be viewed as an extension of Algorithm IV.1 to the multiple car-park case.

Algorithm V.1: Multiple car-parks()

comment: Executed by newly arriving car

for $j \leftarrow 1$ **to** n
  do $X_i \leftarrow$ number of free spaces in car-park i
for $j \leftarrow 1$ **to** n
  do $p_j \leftarrow \frac{X_j}{\sum_{i=1}^{L} X_i}$

  do Go to car-park j with probability p_j.

Objective : Our objective here is to develop algorithms that balance the demand on multiple car-parks in a plug-and-play manner. Balancing demand has the advantage that it avoids localised congestion and pollution peaks as not all cars make their way to a single car-park. Again, feedback between the arrival process and the decision process in individual vehicles is considered, as is the interaction between competing car-parks.

We now consider a region or zone with L parking lots. As cars arrive into the zone, they are each assigned to one of the available parking lots. We assume that this assignment occurs in a randomised way depending on the current number of free spaces in each lot. We also assume that each car proceeds to its assigned lot. The protocol is one-way, in the sense that information flows from the parking lots to the cars, but not in the reverse direction. Thus, as before, there is no system of reservation. Again, as before, there is also a delay between the time when a parking lot is assigned, and the time when the car arrives at the lot. Finally we also assume that cars leave the parking lots in a random fashion, in such a way that the total arrival rate on average is equal to the total departure rate, so that the system is in equilibrium.

The behaviour of the system is determined by the following factors: (1) the statistics of the arrival process for the cars, (2) the statistics of the departure process, (3) the assignment rule, (4) the delays between assignment and parking. We make the following assumptions:

(1) The arrival process is Poisson with rate λ
(2) Each car independently departs after an exponential parking time. Let $C_1, \ldots, C_L$ be the capacities of the parking lots, and let $X_1(t), \ldots, X_L(t)$ be the numbers of free spaces at time t. Then the probability that the next departure occurs from parking lot j at time t is

$$q_j(t) = \frac{C_j - X_j(t)}{\sum_{i=1}^{L} C_i - X_i(t)} \tag{26}$$

(3) Let $p_1(t), \dots, p_L(t)$ be the probabilities that an arrival at time t is assigned to lot $1, \dots, L$ respectively. Then we assume that the probabilities $p_j(t)$ are determined by the numbers $X_j(t)$, in some way that favours lots with more free spaces. For example, one particular rule is

$$p_j(t) = \frac{X_j(t)}{\sum_i X_i(t)} \tag{27}$$

(4) Each arrival experiences a delay τ which depends on its location and the location of the assigned parking lot, and perhaps also some exogenous factors causing randomness.

λ refers to the rate at which cars make a decision. Note that in this case this corresponds to the aggregate arrival rate at all car-parks.

In this case it possible to calculate the probability that the number of arriving customers exceeds capacity, in a manner similar to above. A more pressing issue in this case is whether the protocol balances the load, and whether flapping is avoided. Flapping is a manifestation of instability and occurs when car-parks take turns being full. Clearly, this situation should be avoided, and thus, the main question of interest now is to analyse the stability and fairness of the protocol, and to find the dependence on the number of parking lots, the number of available spaces, the arrival rate and the delays.

Comment : The assignment rule (Equation (27)) can be chosen to achieve a number of different objectives. For example they can be tuned to divert traffic from certain areas as may be necessary to mitigate congestion or pollution peaks or to reflect a pricing structure.

A. Analysis: the fluid model limit

It is challenging to analyse the stochastic model in full detail, so we begin with the analysis of a simplified deterministic model which describes the so-called fluid limit. This model should apply in the case where the arrival rate and the capacities of the parking lots are very large. In this limit the discrete model is replaced by a continuous model, and we can view the traffic as a fluid which flows into and out of the parking lots. Note that fluid models have been often employed to describe urban traffic, see for example [17]. The traffic enters and leaves the zone as a steady stream. The entering stream is split into L parts, which proceed to the L parking lots. The amount in each substream varies over time, depending on the available capacity at each lot. There is a delay before arrival at the parking lots. Each lot generates a departing stream, and these combine to form the outgoing stream. The evolution of this deterministic fluid model is described by a delay differential equation.

Let $C_1, \dots, C_L$ be the capacities of the parking lots, and let $X_1(t), \dots, X_L(t)$ be the amount of free space in each lot at time t. Note that $0 \leq X_j \leq C_j$, and that X_j is now a continuous random variable. We use the assignment rule (27) according to Algorithm V.1 and the departure rule (26). Thus the variables satisfy

$$\frac{dX_j(t)}{dt} = -\lambda\,\theta(X_j(t))\,\frac{X_j(t-\tau_j)}{\sum_i X_i(t-\tau_j)} + \lambda\,\frac{C_j - X_j(t)}{\sum_i (C_i - X_i(t))}$$

where τ_j is the delay associated with lot j, and where $\theta(X_j(t)) = 1$ for $X_j(t) > 0$ and 0 else. Note that λ is now the flow rate of the fluid limit and is in fact the same quantity that defines the Poisson arrival process (hence the use of the symbol λ). The factor $\theta(\cdot)$ enforces the condition that the solution to the delay differential satisfies $X_j(t) \geq 0$ for all t. If we now further assume that $X(t) > 0$, for all t, then we obtain:

$$\frac{d}{dt}\sum_j X_j(t) = 0$$

and hence the total number of available parking spaces is constant. Define this total to be

$$N = \sum_i X_i(t) \tag{28}$$

and also define the total capacity of the zone to be

$$C = \sum_i C_i. \tag{29}$$

Then still assuming that the variables X_j are always positive, we can use Equations (28) and (29) to obtain the equation

$$\frac{dX_j(t)}{dt} = -\frac{\lambda}{N}\,X_j(t-\tau_j) + \frac{\lambda}{C-N}\,(C_j - X_j(t)). \tag{30}$$

This is a delay differential equation. Note first that there is a constant solution, namely

$$X_j(t) = a_j = \frac{N}{C}\,C_j.$$

This is the 'reasonable' situation where the traffic is shared among the lots according to their capacities. However in the presence of delays it is not clear whether this solution is stable. To investigate stability we look for a solution of the form (under the usual assumptions on initial conditions of the delay differential equation)

$$X_j(t) = a_j + b_j e^{zt} \tag{31}$$

where z is a possibly complex parameter. Substituting into (30) gives the delay differential equation's characteristic equation

$$z = -\frac{\lambda}{N}\,e^{-z\tau_j} - \frac{\lambda}{C-N}. \tag{32}$$

According to [18, Chapter VII, Section 28, Theorem B], if all solutions to Equation (32) have negative real part this assures exponential stability of constant solution. We now investigate under which conditions on τ_j this property holds.

When $\tau_j = 0$, the solution is

$$z = -\frac{\lambda C}{N(C-N)}$$

which implies exponential stability of the constant solution.

For τ_j sufficiently small all solutions of (32) lie in the left half of the plane, and thus the constant solution (31) is still stable. However as τ_j increases, the solutions of (32) move toward the imaginary axis. Instability occurs when the first solution crosses the imaginary axis. Letting $z = x + iy$, this instability occurs when $x = 0$. In this case the Equation (32) becomes

$$iy = -\frac{\lambda}{N}\left(\cos(y\tau_j) - i\sin(y\tau_j)\right) - \frac{\lambda}{C-N} \tag{33}$$

which is equivalent to the two equations:

$$0 = -\frac{\lambda}{N}\cos(y\tau_j) - \frac{\lambda}{C-N} \tag{34}$$

$$y = \frac{\lambda}{N}\sin(y\tau_j) \tag{35}$$

The Equations (34) and (35) have no solution if

$$N > \frac{C}{2} \tag{36}$$

So if at least half the parking spaces are empty then the constant solution is stable.

Comment : This is a remarkable result. Given that there is enough free capacity, our approach yields a stable solution independent of the delay τ. It will be an objective of our future research to investigate how this result regarding the fluid limit carries over to the original system.

If $N \leq C/2$ then the solution is stable for τ sufficiently small. We now determine τ_{crit}, the precise threshold value where instability occurs. To this end, we can rearrange Equation (34) to

$$y\tau_j = \cos^{-1}\left(\frac{N}{C-N}\right). \tag{37}$$

Substituting this in Equation (35) yields

$$\begin{aligned} y &= \frac{\lambda}{N}\sin\left(\cos^{-1}\left(\frac{N}{C-N}\right)\right) \\ &= \frac{\lambda}{N}\sqrt{1-\left(\frac{N}{C-N}\right)^2}, \end{aligned} \tag{38}$$

where we used a standard trigonometric identity. Substituting y from Equation (38) into Equation (37) and rearranging yields

$$\tau_{crit} = \left(\frac{N}{\lambda}\right)\frac{\cos^{-1}\left(-\frac{N}{C-N}\right)}{\sqrt{1-\left(\frac{N}{C-N}\right)^2}} \tag{39}$$

If the condition (36) does not hold, and $\tau > \tau_{crit}$, then the constant solution is unstable, and some parking lot will completely empty or completely fill. Approximately, the condition for stability is

$$\lambda\tau_j \leq \frac{\pi N}{2} \tag{40}$$

Note that $\lambda\tau_j$ is the total number of arrivals during the interval between assignment of the parking lot and arrival at the parking. So the condition (40) says that the constant solution is stable if this total arrival number during a delay is less than the total number of available parking spaces.

VI. Simulations

In this section we present simulations to illustrate the efficacy of our algorithms. We use the open-source mobility simulator Sumo [11] together with Matlab. For this purpose, we designed a grid-like road network, depicted in Figure 8, which is artificial but similar to many planned cities. All traffic in Sumo consists of cars being routed between an origin and a destination street along the shortest path. To obtain the desired simulation results, we choose roads in the city that are supposed to contain a car park or are an entry or exit point for cars to the city; these virtual locations are not explicitly taken into account in Sumo, rather we use Matlab to keep track of car park occupancies and origin and destinations of vehicles. In practise we use Sumo to run the simulation until a new car arrives to the city and makes a decision or a previously parked car finishes its service and departs from the car park. Each of these events is generated in Matlab, which adds the new event to Sumo and consecutively starts a new Sumo simulation.

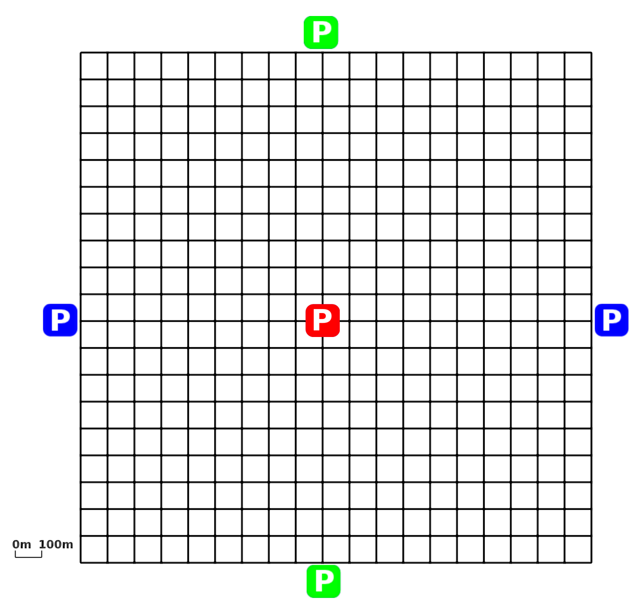


Fig. 8: Grid-like road network used for our simulations.

A. Single car-park scenario

In this section we show some results obtained from simulations of a scenario with a single car park, coloured in red, in the centre of our grid-like city depicted in Figure 8. It has capacity for 100 cars and is empty at the begin of our simulation. Over a time of 3 hours vehicles appear at random locations throughout the city. We model the arrival of new cars as a Poisson process with expected inter-arrival time of 10 seconds. Cars that decide to drive to the car park and find an available spot stay parked for an exponentially distributed random time with mean 20 minutes and then they disappear. We simulate two scenarios. In the first scenario occupancy information is not available to the cars and all cars decide to go to the car-park. In the second scenario cars have access to the occupancy information. The occupancy information is updated every 100 seconds. So not only do cars experience a delay between making a decision and arriving to the car-park, they are also using non-realtime data to make their decisions. In this scenario cars make their decision randomly according to Algorithm IV.1, with $N_{min} = 80$, $N_{max} = 95$ and $p_{max} = 0.75$. In Figure 9 the number of cars that are at the car park in both scenarios is depicted. We can see that in the scenario with feedback (green line) the car-parks occupancy stays below the capacity, so that no car arrives to a full car-park and there are always some spare spaces available. In the scenario without feedback (red line) on the other hand we can clearly see that that the available capacity of the car-park is not enough to satisfy the demand. A large number of cars arrives to a car-park with no free spaces. The drivers have to wait for someone to leave or have to find an alternative parking facility. This causes an unnecessary waste of time and fuel and contributes to congestion and pollution.

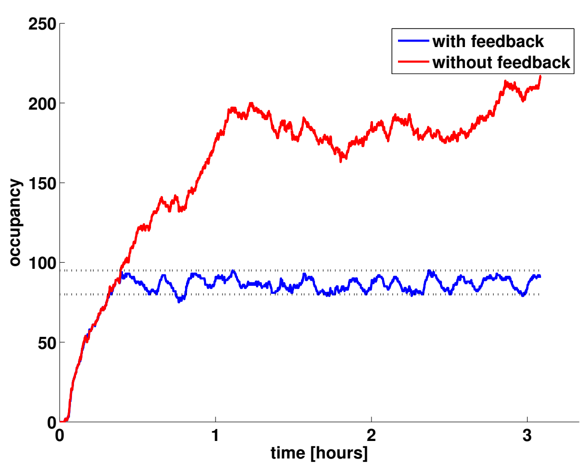


Fig. 9: Comparison of occupancy with (red line) and without (blue line) feedback. The dotted lines show the desired lower and upper occupancy of the car park.

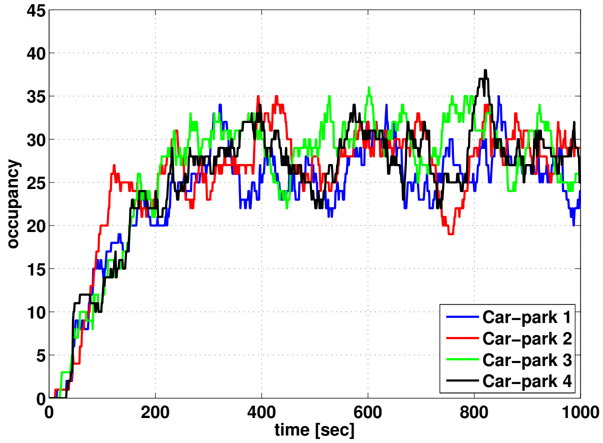
B. Multiple car-park scenario

We now present a simulation with several car parks. We regard 4 car parks in our grid-like city coloured in blue and green in Figure 8, with capacity for 40 cars each, distributed over the city as well as 4 main access roads. Over a time of about 3 hours 1000 cars arrive to the city according to a Poisson process with average inter-arrival time of 10 seconds and they arrive on each access road with the same probability. Upon arrival they query the occupancies of the car parks and choose one of them according to Algorithm V.1, then they drive to the chosen car park. Vehicles then stay parked for an exponentially distributed time with mean 20 minutes. The available car park occupancy information is updated only when cars arrive at or leave the car park, so no communication from the vehicles to the car parks is required. In Figure 10a we plot the occupancy of each car park at the time instants at which new cars arrive to the city. As we can see, our approach reasonably balances the occupancies.

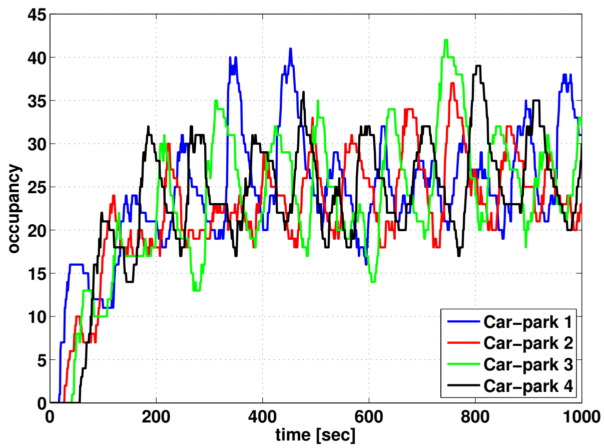
In Figure 10b we present the same result for a slightly different simulation, where we use a different assignment rule. Namely, we assign cars always to the car park with the lowest occupancy at the time of the vehicles query. Although this assignment rule intuitively seems reasonable, if it takes a long time for cars to drive to their car park and if many other vehicles arrive in this time, then its performance is poor. By comparing Figures 10a and 10b it can clearly be seen that our algorithm outperforms the deterministic assignment rule in the sense that cars are better balanced among the available car-parks. Specifically, our stochastic approach decreases the time averaged variance of the distribution of parked cars over all car-parks from 29.85 to 9.23.

C. Effectiveness of balancing strategies in the multiple car-park scenario

The objective of the previous simulations was to demonstrate the effectiveness of feedback strategies in terms of efficient usage of the infrastructures. We claim that balancing the vehicles among the available car-parks is an efficient way of utilising the available infrastructure efficiently. Clearly, this is not the only strategy. Another naïve strategy would be to simply associate a vehicle with the car-park that has more available places at that time instant that a request for parking is made. The objective of this simulation is to show that stochastically balancing the vehicles among the available car-parks is in fact a smart strategy that does outperform the naive deterministic strategy of associating vehicles with the emptiest car-park. For this purpose, we take the perspective of the users, and as a measure of effectiveness, we consider the percentage of unsatisfied users with respect to the overall number of users. Unsatisfied users are users that arrive to find a car-park full. To this end we ran a number of three hour simulations, each of them for a different value of the average time between the assignment of a driver to a car-park and the arrival to the car-park. These values range from 5 to 25 minutes. For



(a) Algorithm V.1.



(b) Deterministic assignment.

Fig. 10: Number of occupied spaces at each parking lot using Algorithm V.1 in Figure 10a and using the deterministic assignment rule in Figure 10b. Vehicles are clearly more balanced in 10a as the variance is lower.

each car a random number, uniformly distributed between -2 minutes and $+2$ minutes is added to the average delay. Figure 11 clearly shows that the percentage of unsatisfied users increases when the average travel time before getting to a car-park increases.

D. The benefit of load balancing to the user

The objective of balancing is to avoid localised congestion, pollution peaks, and to increase the probability of a given driver finding a space available. To do this, drivers are directed to a number of nearby car-parks. The cost of this strategy could sometimes be, increased driving time for individual drivers and some drivers being further away from their destination. Quantifying these effects in a very detailed manner is beyond the scope of the present paper. However, we give the following simulation to showcase the potential benefits of our approach

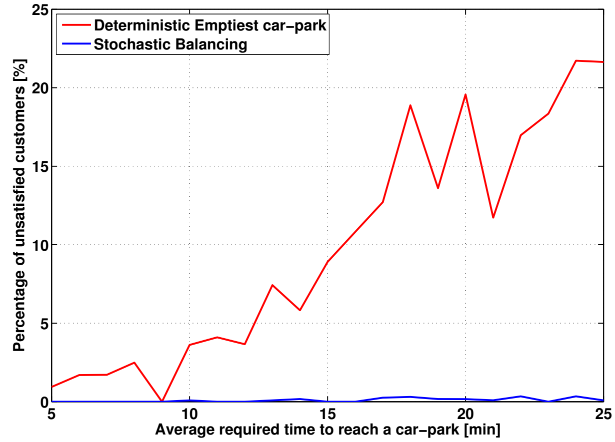


Fig. 11: The percentage of unsatisfied users increases if the naïve “emptiest car-park” strategy is used, when the distance from the car-park increases. On the other hand, the stochastic balancing strategy performs well even in case of long distances, thanks to the intrinsic feedback flavour in the strategy.

to the users, and to address in some manner the above concern. We consider again the grid-like network depicted in Figure 8. We assume there are the two car-parks coloured in blue, located at the left centre and the right centre of the map. Each of these has capacity for 100 cars and is initially empty. Now assume that there is an event happening close to the left car-park (car-park A), which 200 vehicles, uniformly distributed over the map, wish to attend. They all start their journey at the same time. We regard two scenarios: (i) All vehicles drive to the car-park A and the first 100 to arrive find a space, the rest has to drive to the right car-park (car-park B) from where the drivers will have to use public transport or walk to arrive at car-park A. (ii) All vehicles use our stochastic assignment rule and thus toss a fair coin to decide which car-park to go to. Once one of the car-parks is full, all later arriving vehicles have to drive to the other car-park. All drivers that end up in car-park B again have to use public transport to reach car-park A. In the first scenario 100 cars have to relocate from car-park A to car-park B, while in the second scenario this number is significantly smaller. In our simulation, only two cars had to relocate, a significant reduction in unnecessary travel, saving time and reducing congestion and pollution. Further, even without the additional relocation journeys, the second scenario is already superiour. Fig. 12 reports the travel times from the original position to the first car-park that is reached in both scenarios. It can clearly be seen that almost all travel times are higher in the first scenario. In fact the average travel time is 258 seconds in the first scenario and 174 seconds in the second scenario. This is due to local congestion around car-park A due to the mass of vehicles driving there. A short video showing the junction at which car-park A is located and the surrounding streets can be found at http://www.hamilton.ie/aschlote/sumo_movie.mov.

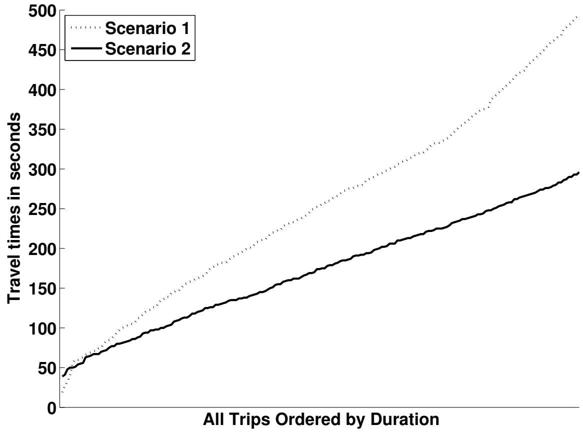


Fig. 12: All recorded travel times to the first car-park measured from both scenarios.

VII. Commercial opportunities

Before concluding we note that the multiple car-park algorithm gives rise to certain commercial opportunities. Office block car-parks are usually empty at certain times (evenings), and thus could compete for parking business. The plug-and-play nature of our algorithms, with appropriate H/W to count cars departing and arriving at car-parks, could enable such office blocks to compete for business during times in which they are not in office use. Similarly, there are frequently situations where there is local parking scarcity, but in the vicinity, there is parking availability. For example, university campuses are often in residential areas, which have lots of parking availability during working hours. Our plug-and-play system could also be used to integrate such spaces into the campus parking system during working hours.

VIII. Conclusions and future work

In this paper we propose and describe stochastic policies to associate cars with parking spaces. We first illustrate this problem from the perspective of a single car-park, where the main concern of the vehicle is whether it should be more convenient to go to that car-park, being aware that in the meanwhile it could get full, or to search for a place elsewhere. We then extend our approach to the scenario where several car-parks are available. In this case the interest is how to assign the vehicle to one particular parking lot. Differently from other works in the same area, we explicitly take into account several aspects of interest: the effect of feedback on the choice of the car-park; the benefits of stochastic assignment policies vs. more conventional deterministic strategies; the effect of delays between the communication of real-time occupancies and the moment when the cars in fact occupy the desired parking space. All such aspects have been tackled by using mathematical arguments, and have been illustrated by means of simulations. The proposed policy does not suffer from the known drawbacks of reservation strategies, where non-cooperative vehicles (i.e., vehicles that take a parking space without reservation) interfere with the rest of the framework. Also, we provided a bound on the delays (i.e., time to get to the car-park) that guarantees that the car assignment solution remains stable. Finally, the proposed policy can be realistically and efficiently implemented in practice to achieve the desired goal. In our simulations, we generally assumed that the vehicles followed the indications given by the infrastructure, but showed that the algorithm is robust even if this does not occur for all vehicles. From a practical point of view, simple pricing mechanisms can be employed to make the vehicles go to the assigned car-parks.

Currently, we are interested in validating the proposed algorithms beyond the simulation level. Mainly we would like to implement at the infrastructure level. We further wish to investigate the most convenient ways to communicate the relevant information to the vehicles (e.g., directly communicate the number of available places, or simply the probabilities, or only the chosen car-park).

References

[1] Anderson, S., and de Palma, A., *The economics of Pricing Parking*, Journal of Urban Economics, vol. 55, pp. 1–20, 2004.

[2] Arnott, R., *Spatial Competition between Parking Garages and Downtown Parking Policy*, Transport Policy, col. 13, pp. 458–469, 2006.

[3] Caliskan, M., Graupner, D., and Mauve, D., *Decentralised Discovery of Free Parking Places*, VANET '06 Proceedings of the 3rd international workshop on Vehicular ad hoc networks, pp. 30–39, New York, NY, USA, 2006 .

[4] *The smart city solution*, Elfrink, W., McKinsey Quarterly, 2012, available online 12 January 2013 http://www.mckinsey.com/features/government_designed_for_new_times/the_smart-city_solution.

[5] Floyd, S., and Jacobson, V., *Random Early Detection gateways for Congestion Avoidance*, IEEE/ACM Transactions on Networking, vol. 1, no. 4, pp. 397-413, 1993.

[6] Geng, Y., and Cassandras, C., *Dynamic Resource Allocation in Urban Settings: A "Smart Parking" Approach*, IEEE Symposium on Computer-Aided Control System Design, Denver, CO, USA, 2011.

[7] Häusler, F., Crisostomi, E., Schlote, A., Radusch, I., and Shorten, R., *Stochastically balanced parking and charging for fully electric and plug-in hybrid vehicles*, ACM/IEEE/IFAC/TRB International Conference on Connected Vehicles and Expo, Beijing, China, 2012.

[8] Häusler, F., Crisostomi, E., Schlote, A., Radusch, I., and Shorten, R., *Stochastic park-and-charge balancing for fully electric and plug-in hybrid vehicles*, IEEE Transactions on Intelligent Traffic Systems, vol. PP, no. 99, 2013.

[9] Idris, M., Leng, Y., Tamil, E., Noor, N., and Razak, Z., *car-park System: A Review of Smart Parking Systems and its Technology*, Information Technology Journal, vol. 8, no. 2, pp. 101-113, 2009.

[10] Klappenecker, A., Lee, H., and Welch, J., *Finding available Parking Spaces Made Easy*, Ad Hoc Networks, Available online 17 March 2012, http://www.sciencedirect.com/science/article/pii/S157087051200042X.

[11] Krajzewicz, D., Bonert, M., and Wagner, P., *The open source traffic simulation package SUMO*, RoboCup 2006 Infrastructure Simulation Competition, RoboCup 2006, Bremen, Germany, 2006

[12] Schlote, A., Häusler, F., Hecker, T., Bergmann, A., Crisostomi, E., Radusch, I., and Shorten R., *Cooperative regulation and trading of emissions using plug-in hybrid vehicles*, IEEE Transactions on Intelligent Transport Systems, vol. 14, no. 4, pp. 1572–1585, 2013.

[13] Shoup, D., *Cruising for Parking*, Transport Policy, vol. 13, pp. 479–486, 2006.

[14] Shoup, D., *Cruising for Parking*, ACCESS, no 30, Spring 2007, available online 12 January 2013, http://shoup.bol.ucla.edu/CruisingForParkingAccess.pdf.

[15] Kaplan, S., and Bekhor, S., *Exploring en-route parking type and parking-search route choice : decision making framework and survey design*, in Proceedings of the 2nd International Choice Modelling Conference, 2011.
[16] Teodorović, D., and Lučić, P., *Intelligent Parking Systems*, European Journal of Operational Research, no. 175, pp. 1666–1681, 2006.
[17] Garavello, M., and Piccoli B., *On fluido-dynamic models for urban traffic*, Networks and Heterogeneous media, vol. 4, pp. 107-116, 2009.
[18] Driver, R. D., *Ordinary and delay differential equations*, Applied Mathematical Sciences, Vol. 20, Springer-Verlag, New York-Heidelberg, 1977.

Arieh Schlote received his degree (Diplom) in mathematics from the University of Würzburg, Germany, in 2010. He is currently pursuing a Ph.D. degree at the Hamilton Institute, NUI Maynooth , where he works on mathematical modelling of road traffic networks. His research interests include stability theory of dynamical systems and matrix theory and their applications as well as the modelling and control of traffic systems.

Christopher King received his B.A. degrees in Mathematics and Physics at Trinity College Dublin, and his Ph.D. degree in Physics at Harvard University. He is currently Professor of Mathematics and Physics at Northeastern University in Boston, Massachusetts. His research interests include mathematical physics, quantum information theory and network analysis.

Emanuele Crisostomi received his Bachelor degree in Computer Science in 2002, his Master degree in Automatic Control in 2005 and his Ph.D. degree in Automatics, Robotics and Bioengineering in 2009 from University of Pisa. He is currently Assistant Professor of Electrical Engineering at the Department of Energy, Systems, Territory and Constructions Engineering at University of Pisa. His research interest include control and optimisation of large-scale systems, with applications to smart grids and green mobility networks.

Robert Shorten graduated from the University College Dublin (UCD) in 1990 with a First Class Honours B.E. degree in Electronic Engineering. He was awarded a Ph.D. in 1996, also from UCD, while based in at Daimler-Benz Research in Berlin, Germany. From 1993 to 1996 Prof. Shorten was the holder of a Marie Curie Fellowship at Daimler-Benz Research to conduct research in the area of smart gearbox systems. Following a brief spell at the Center for Systems Science, Yale University, working with Professor K. S. Narendra, Prof. Shorten returned to Ireland as the holder of a European Presidency Fellowship in 1997. Prof. Shorten is a co-founder of the Hamilton Institute at NUI Maynooth, where he was a full Professor until March 2013. He was also a Visiting Professor at TU Berlin from 2011-2012. Professor Shorten is currently a senior research manager at IBM Research Ireland. Prof. Shorten's research spans a number of areas. He has been active in computer networking, automotive research, collaborative mobility (including smart transportation and electric vehicles), as well basic control theory and linear algebra. His main field of theoretical research has been the study of hybrid dynamical systems.