LOW ENERGY SOLUTIONS FOR THE SEMICLASSICAL LIMIT OF SCHROEDINGER MAXWELL SYSTEMS

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ABSTRACT. We show that the number of solutions of Schroedinger Maxwell system on a smooth bounded domain $\Omega \subset \mathbb{R}^3$. depends on the topological properties of the domain. In particular we consider the Lusternik-Schnirelmann category and the Poincaré polynomial of the domain.

Dedicated to our friend Bernhard

1. INTRODUCTION

Given real numbers $q > 0$, $\omega > 0$ we consider the following Schroedinger Maxwell system on a smooth bounded domain $\Omega \subset \mathbb{R}^3$.

$$\begin{cases} -\varepsilon^2 \Delta u + u + \omega u v = |u|^{p-2} u & \text{in } \Omega \\ -\Delta v = q u^2 & \text{in } \Omega \\ u, v = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

This paper deals with the semiclassical limit of the system (1), i.e. it is concerned with the problem of finding solutions of (1) when the parameter ε is sufficiently small. This problem has some relevance for the understanding of a wide class of quantum phenomena. We are interested in the relation between the number of solutions of (1) and the topology of the bounded set Ω. In particular we consider the Lusternik Schnirelmann category $\operatorname{cat} \Omega$ of Ω in itself and its Poincaré polynomial $P_t(\Omega)$.

Our main results are the following.

Theorem 1. *Let $4 < p < 6$. For ε small enough there exist at least $\operatorname{cat}(\Omega)$ positive solutions of (1).*

Theorem 2. *Let $4 < p < 6$. Assume that for ε small enough all the solutions of problem (1) are non- degenerate. Then there are at least $2P_1(\Omega) - 1$ positive solutions.*

Schroedinger Maxwell systems recently received considerable attention from the mathematical community. In the pioneering paper [9] Benci and Fortunato studied system (1) when $\varepsilon = 1$ and without nonlinearity. Regarding the system in a semi-classical regime Ruiz [18] and D'Aprile-Wei [11] showed the existence of a family of radially symmetric solutions respectively for $\Omega = \mathbb{R}^3$ or a ball. D'Aprile-Wei [12] also proved the existence of clustered solutions in the case of a bounded domain Ω in $\mathbb{R}^3$.

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Recently, Siciliano [19] relates the number of solution with the topology of the set Ω when $\varepsilon = 1$, and the nonlinearity is a pure power with exponent p close to the critical exponent 6. Moreover, in the case $\varepsilon = 1$, many authors proved results of existence and non existence of solution of (1) in presence of a pure power nonlinearity $|u|^{p-2}u$, $2 < p < 6$ or more general nonlinearities [1, 2, 3, 4, 10, 14, 15, 17, 20].

In a forthcoming paper [13], we aim to use our approach to give an estimate on the number of low energy solutions for Klein Gordon Maxwell systems on a Riemannian manifold in terms of the topology of the manifold and some information on the profile of the low energy solutions.

In the following we always assume $4 < p < 6$.

2. Notations and definitions

In the following we use the following notations.

- $B(x,r)$ is the ball in $\mathbb{R}^3$ centered in x with radius r.
- The function $U(x)$ is the unique positive spherically symmetric function in $\mathbb{R}^3$ such that
$$-\Delta U + U = U^{p-1} \text{ in } \mathbb{R}^3$$
we remark that U and its first derivative decay exponentially at infinity.
- Given $\varepsilon > 0$ we define $U_\varepsilon(x) = U\left(\frac{x}{\varepsilon}\right)$.
- We denote by supp φ the support of the function φ.
- We define
$$m_\infty = \inf_{\int_{\mathbb{R}^3}|\nabla v|^2 + v^2 dx = |v|^p_{L^p(\mathbb{R}^3)}} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + v^2 dx - \frac{1}{p}|v|^p_{L^p(\mathbb{R}^3)}$$
- We also use the following notation for the different norms for $u \in H^1_g(M)$:
$$\|u\|^2_\varepsilon = \frac{1}{\varepsilon^3}\int_M \varepsilon^2|\nabla u|^2 + u^2 dx \qquad |u|^p_{\varepsilon,p} = \frac{1}{\varepsilon^3}\int_\Omega |u|^p dx$$
$$\|u\|^2_{H^1_0} = \int_\Omega |\nabla u|^2 dx \qquad |u|^p_p = \int_\Omega |u|^p dx$$
and we denote by H_ε the Hilbert space $H^1_0(\Omega)$ endowed with the $\|\cdot\|_\varepsilon$ norm.

Definition 3. Let X a topological space and consider a closed subset $A \subset X$. We say that A has category k relative to X ($\operatorname{cat}_M A = k$) if A is covered by k closed sets A_j, $j = 1, \dots, k$, which are contractible in X, and k is the minimum integer with this property. We simply denote $\operatorname{cat} X = \operatorname{cat}_X X$.

Remark 4. Let X_1 and X_2 be topological spaces. If $g_1 : X_1 \to X_2$ and $g_2 : X_2 \to X_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on X_1, then $\operatorname{cat} X_1 \le \operatorname{cat} X_2$.

Definition 5. Let X be any topological space and let $H_k(X)$ denotes its k-th homology group with coefficients in $\mathbb{Q}$. The Poincaré polynomial $P_t(X)$ of X is defined as the following power series in t
$$P_t(X) := \sum_{k\ge 0} (\dim H_k(X))\, t^k$$

Actually, if X is a compact space, we have that $\dim H_k(X) < \infty$ and this series is finite; in this case, $P_t(X)$ is a polynomial and not a formal series.

Remark 6. Let X and Y be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous operators such that $g \circ f$ is homotopic to the identity on X, then $P_t(Y) = P_t(X) + Z(t)$ where $Z(t)$ is a polynomial with non-negative coefficients.

These topological tools are classical and can be found, e.g., in [16] and in [5].

3. Preliminary results

Using an idea in a paper of Benci and Fortunato [9] we define the map $\psi : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by the equation

$$-\Delta\psi(u) = qu^2 \text{ in } \Omega \tag{2}$$

Lemma 7. *The map $\psi : H_0^1(\Omega) \to H_0^1(\Omega)$ is of class C^2 with derivatives*

$$\psi'(u)[\varphi] = i^*(2qu\varphi) \tag{3}$$

$$\psi''(u)[\varphi_1, \varphi_2] = i^*(2q\varphi_1\varphi_2) \tag{4}$$

where the operator $i_\varepsilon^* : L^{p'}, |\cdot|_{\varepsilon,p'} \to H_\varepsilon$ is the adjoint operator of the immersion operator $i_\varepsilon : H_\varepsilon \to L^p, |\cdot|_{\varepsilon,p}$.

Proof. The proof is standard. □

Lemma 8. *The map $T : H_0^1(\Omega) \to \mathbb{R}$ given by*

$$T(u) = \int_\Omega u^2\psi(u)dx$$

is a C^2 map and its first derivative is

$$T'(u)[\varphi] = 4\int_\Omega \varphi u\psi(u)dx.$$

Proof. The regularity is standard. The first derivative is

$$T'(u)[\varphi] = 2\int u\varphi\psi(u) + \int u^2\psi'(u)[\varphi].$$

By (3) and (2) we have

$$\begin{aligned} 2q\int u\varphi\psi(u) &= -\int \Delta(\psi'(u)[\varphi])\psi(u) = -\int \psi'(u)[\varphi]\Delta\psi(u) = \\ &= \int \psi'(u)[\varphi]qu^2 \end{aligned}$$

and the claim follows. □

At this point we consider the following functional $I_\varepsilon \in C^2(H_0^1(\Omega), \mathbb{R})$.

$$I_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 + \frac{\omega}{4}G_\varepsilon(u) - \frac{1}{p}|u^+|_{\varepsilon,p}^p \tag{5}$$

where

$$G_\varepsilon(u) = \frac{1}{\varepsilon^3}\int_\Omega u^2\psi(u)dx = \frac{1}{\varepsilon^3}T(u).$$

By Lemma 8 we have

$$I_\varepsilon'(u)[\varphi] = \frac{1}{\varepsilon^3}\int_\Omega \varepsilon^2\nabla u\nabla\varphi + u\varphi + \omega u\psi(u)\varphi - (u^+)^{p-1}\varphi$$

$$I_\varepsilon'(u)[u] = \|u\|_\varepsilon^2 + \omega G_\varepsilon(u) - |u^+|_{\varepsilon,p}^p$$

then if u is a critical points of the functional I_ε the pair of positive functions $(u, \psi(u))$ is a solution of (1).

4. Nehari Manifold

We define the following Nehari set

$$\mathcal{N}_\varepsilon = \left\{ u \in H_0^1(\Omega) \smallsetminus 0 \ : \ N_\varepsilon(u) := I'_\varepsilon(u)[u] = 0 \right\}$$

In this section we give an explicit proof of the main properties of the Nehari manifold, although standard, for the sake of completeness

Lemma 9. *$\mathcal{N}_\varepsilon$ is a C^2 manifold and* $\inf_{\mathcal{N}_\varepsilon} \|u\|_\varepsilon > 0$.

Proof. If $u \in \mathcal{N}_\varepsilon$, using that $N_\varepsilon(u) = 0$, and $p > 4$ we have

$$N'_\varepsilon(u)[u] = 2\|u\|_\varepsilon^2 + 4\omega G_\varepsilon(u) - p|u^+|_{\varepsilon,p} = (2-p)\|u\|_\varepsilon + (4-p)\omega G_\varepsilon(u) < 0$$

so $\mathcal{N}_\varepsilon$ is a C^2 manifold.

We prove the second claim by contradiction. Take a sequence $\{u_n\}_n \in \mathcal{N}_\varepsilon$ with $\|u_n\|_\varepsilon \to 0$ while $n \to +\infty$. Thus, using that $N_\varepsilon(u) = 0$,

$$\|u_n\|_\varepsilon^2 + \omega G_\varepsilon(u_n) = |u_n^+|_{p,\varepsilon}^p \le C\|u_n\|_\varepsilon^p,$$

so

$$1 < 1 + \frac{\omega G_\varepsilon(u)}{\|u_n\|_\varepsilon} \le C\|u_n\|_\varepsilon^{p-2} \to 0$$

and this is a contradiction. □

Remark 10. If $u \in \mathcal{N}_\varepsilon$, then

$$\begin{aligned} I_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_\varepsilon^2 + \omega\left(\frac{1}{4} - \frac{1}{p}\right)G_\varepsilon(u) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)|u^+|_{p,\varepsilon}^p - \frac{\omega}{4}G_\varepsilon(u) \end{aligned}$$

Lemma 11. *It holds Palais-Smale condition for the functional I_ε on $\mathcal{N}_\varepsilon$.*

Proof. We start proving PS condition for I_ε. Let $\{u_n\}_n \in H_0^1(\Omega)$ such that

$$I_\varepsilon(u_n) \to c \qquad |I'_\varepsilon(u_n)[\varphi]| \le \sigma_n \|\varphi\|_\varepsilon \text{ where } \sigma_n \to 0$$

We prove that $\|u_n\|_\varepsilon$ is bounded. Suppose $\|u_n\|_\varepsilon \to \infty$. Then, by PS hypothesis

$$\frac{pI_\varepsilon(u_n) - I'_\varepsilon(u_n)[u_n]}{\|u_n\|_\varepsilon} = \left(\frac{p}{2} - 1\right)\|u_n\|_\varepsilon + \left(\frac{p}{4} - 1\right)\frac{G_\varepsilon(u_n)}{\|u_n\|_\varepsilon} \to 0$$

and this is a contradiction because $p > 4$.

At this point, up to subsequence $u_n \to u$ weakly in $H_0^1(\Omega)$ and strongly in $L^t(\Omega)$ for each $2 \le t < 6$. Since u_n is a PS sequence

$$u_n + \omega i_\varepsilon^*(\psi(u_n)u_n) - i_\varepsilon^*\left((u_n^+)^{p-1}\right) \to 0 \text{ in } H_0^1(\Omega)$$

we have only to prove that $i_\varepsilon^*(\psi(u_n)u_n) \to i_\varepsilon^*(\psi(u)u)$ in $H_0^1(\Omega)$, then we have to prove that

$$\psi(u_n)u_n \to \psi(u)u \text{ in } L^{t'}$$

We have $|\psi(u_n)u_n - \psi(u)u|_{\varepsilon,t'} \le |\psi(u)(u_n - u)|_{\varepsilon,t'} + |(\psi(u_n) - \psi(u))\, u_n|_{\varepsilon,t'}$. We get

$$\int_\Omega |\psi(u_n) - \psi(u)|^{\frac{t}{t-1}} |u_n|^{\frac{t}{t-1}} \le \left(\int_\Omega |\psi(u_n) - \psi(u)|^t\right)^{\frac{1}{t-1}} \left(\int_\Omega |u_n|^{\frac{t}{t-2}}\right)^{\frac{t-2}{t-1}} \to 0,$$

thus we can conclude easily.

Now we prove PS condition for the constrained functional. Let $\{u_n\}_n \in \mathcal{N}_\varepsilon$ such that

$$\begin{gathered} I_\varepsilon(u_n) \to c \\ |I'_\varepsilon(u_n)[\varphi] - \lambda_n N'(u_n)[\varphi]| \le \sigma_n \|\varphi\|_\varepsilon \quad \text{ with } \sigma_n \to 0 \end{gathered}$$

In particular $I'_\varepsilon(u_n)\left[\frac{u_n}{\|u_n\|_\varepsilon}\right] - \lambda_n N'(u_n)\left[\frac{u_n}{\|u_n\|_\varepsilon}\right] \to 0$. Then

$$\lambda_n \left\{ (p-2)\|u_n\|_\varepsilon + (p-4)\,\omega \frac{G_\varepsilon(u_n)}{\|u_n\|_\varepsilon} \right\} \to 0$$

thus $\lambda_n \to 0$ because $p > 4$. Since $N'(u_n) = u_n - i^*_\varepsilon(4\omega\psi(u_n)u_n) - p i^*_\varepsilon(|u_n^+|^{p-1})$ is bounded we obtain that $\{u_n\}_n$ is a PS sequence for the free functional I_ε, and we get the claim □

Lemma 12. *For all $w \in H^1_0(\Omega)$ such that $|w^+|_{\varepsilon,p} = 1$ there exists a unique positive number $t_\varepsilon = t_\varepsilon(w)$ such that $t_\varepsilon(w)w \in \mathcal{N}_\varepsilon$.*

Proof. We define, for $t > 0$

$$H(t) = I_\varepsilon(tw) = \frac{1}{2}t^2\|w\|^2_\varepsilon + \frac{t^4}{4}\omega G_\varepsilon(w) - \frac{t^p}{p}.$$

Thus

$$H'(t) = t\left(\|w\|^2_\varepsilon + t^2\omega G_\varepsilon(w) - t^{p-2}\right) \tag{6}$$

$$H''(t) = \|w\|^2_\varepsilon + 3t^2\omega G_\varepsilon(w) - (p-1)t^{p-2} \tag{7}$$

By (6) there exists $t_\varepsilon > 0$ such that $H'(t_\varepsilon)$. Moreover, by (6), (7) and because $p > 4$ we that $H''(t_\varepsilon) < 0$, so t_ε is unique. □

5. Main ingredient of the proof

We sketch the proof of Theorem 1. First of all, since the functional $I_\varepsilon \in C^2$ is bounded below and satisfies PS condition on the complete C^2 manifold $\mathcal{N}_\varepsilon$, we have, by well known results, that I_ε has at least cat I^d_ε critical points in the sublevel

$$I^d_\varepsilon = \left\{ u \in H^1 \ : \ I_\varepsilon(u) \le d \right\}.$$

We prove that, for ε and δ small enough, it holds

$$\operatorname{cat}\Omega \le \operatorname{cat}\left(\mathcal{N}_\varepsilon \cap I^{m_\infty+\delta}_\varepsilon\right)$$

where

$$m_\infty := \inf_{\mathcal{N}_\infty} \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 + v^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} |v|^p dx$$

$$\mathcal{N}_\infty = \left\{ v \in H^1(\mathbb{R}^3) \smallsetminus \{0\} \ : \ \int_{\mathbb{R}^3} |\nabla v|^2 + v^2 dx = \int_{\mathbb{R}^3} |v|^p dx \right\}.$$

To get the inequality $\operatorname{cat}\Omega \le \operatorname{cat}\left(\mathcal{N}_\varepsilon \cap I^{m_\infty+\delta}_\varepsilon\right)$ we build two continuous operators

$$\begin{aligned} \Phi_\varepsilon &: \ \Omega^- \to \mathcal{N}_\varepsilon \cap I^{m_\infty+\delta}_\varepsilon \\ \beta &: \ \mathcal{N}_\varepsilon \cap I^{m_\infty+\delta}_\varepsilon \to \Omega^+. \end{aligned}$$

where

$$\begin{gathered} \Omega^- = \{x \in \Omega \ : \ d(x,\partial\Omega) < r\} \\ \Omega^+ = \left\{x \in \mathbb{R}^3 \ : \ d(x,\partial\Omega) < r\right\} \end{gathered}$$

with r small enough so that $\operatorname{cat}(\Omega^-) = \operatorname{cat}(\Omega^+) = \operatorname{cat}(\Omega)$.

Following an idea in [7], we build these operators Φ_ε and β such that $\beta\circ\Phi_\varepsilon : \Omega^- \to \Omega^+$ is homotopic to the immersion $i:\Omega^-\to\Omega^+$. By the properties of Lusternik Schinerlmann category we have

$$\operatorname{cat}\Omega \le \operatorname{cat}\left(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}\right)$$

which ends the proof of Theorem 1.

Concerning Theorem 2, we can re-state classical results contained in [5, 8] in the following form.

Theorem 13. *Let I_ε be the functional (5) on $H^1(\Omega)$ and let K_ε be the set of its critical points. If all its critical points are non-degenerate then*

$$\sum_{u\in K_\varepsilon} t^{\mu(u)} = tP_t(\Omega) + t^2(P_t(\Omega)-1) + t(1+t)Q(t) \tag{8}$$

where Q(t) is a polynomial with non-negative integer coefficients and $\mu(u)$ is the Morse index of the critical point u.

By Remark 6 and by means of the maps Φ_ε and β we have that

$$P_t(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}) = P_t(\Omega) + Z(t) \tag{9}$$

where $Z(t)$ is a polynomial with non-negative coefficients. Provided that $\inf_\varepsilon m_\varepsilon =: \alpha > 0$, because $\lim\limits_{\varepsilon\to 0} m_\varepsilon = m_\infty$ (see 20) , we have the following relations [5, 8]

$$P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\alpha/2}) = tP_t(\mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}) \tag{10}$$

$$P_t(H_0^1(\Omega), I_\varepsilon^{m_\infty+\delta})) = t(P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\alpha/2}) - t) \tag{11}$$

$$\sum_{u\in K_\varepsilon} t^{\mu(u)} = P_t(H_0^1(\Omega), I_\varepsilon^{m_\infty+\delta})) + P_t(I_\varepsilon^{m_\infty+\delta}, I_\varepsilon^{\alpha/2}) + (1+t)\tilde{Q}(t) \tag{12}$$

where $\tilde{Q}(t)$ is a polynomial with non-negative integer coefficients. Hence, by (9), (10), (11), (12) we obtain (8). At this point, evaluating equation (8) for $t=1$ we obtain the claim of Theorem 2

6. The map Φ_ε

For every $\xi\in\Omega^-$ we define the function

$$W_{\xi,\varepsilon}(x) = U_\varepsilon(x-\xi)\chi(|x-\xi|)$$

where $\chi:\mathbb{R}^+\to\mathbb{R}^+$ where $\chi\equiv 1$ for $t\in[0,r/2)$, $\chi\equiv 0$ for $t>r$ and $|\chi'(t)|\le 2/r$.

We can define a map

$$\begin{aligned}\Phi_\varepsilon &: \Omega^- \to \mathcal{N}_\varepsilon\\ \Phi_\varepsilon(\xi) &= t_\varepsilon(W_{\xi,\varepsilon})W_{\xi,\varepsilon}\end{aligned}$$

Remark 14. We have that the following limits hold uniformly with respect to $\xi\in\Omega$

$$\begin{aligned}\|W_{\varepsilon,\xi}\|_\varepsilon &\to \|U\|_{H^1(\mathbb{R}^3)}\\ |W_{\varepsilon,\xi}|_{\varepsilon,t} &\to \|U\|_{L^t(\mathbb{R}^3)} \text{ for all } 2\le t\le 6\end{aligned}$$

Lemma 15. *There exists $\bar\varepsilon>0$ and a constant $c>0$ such that*

$$G_\varepsilon(W_{\varepsilon,\xi}) = \frac{1}{\varepsilon^3}\int_\Omega qW_{\varepsilon,\xi}^2(x)\psi(W_{\varepsilon,\xi})dx < c\varepsilon^2$$

Proof. It holds

$$\begin{aligned}\|\psi(W_{\varepsilon,\xi})\|^2_{H^1_0(\Omega)} &= \int_\Omega qW^2_{\varepsilon,\xi}(x)\psi(W_{\varepsilon,\xi})dx \le q\|\psi(W_{\varepsilon,\xi})\|_{L^6(\Omega)}\left(\int_\Omega W^{12/5}_{\varepsilon,\xi}dx\right)^{5/6}\\ &\le c\|\psi(W_{\varepsilon,\xi})\|_{H^1_0(\Omega)}\left(\frac{1}{\varepsilon^3}\int_\Omega W^{12/5}_{\varepsilon,\xi}dx\right)^{5/6}\varepsilon^{5/2}\end{aligned}$$

By Remark 14 we have that $\|\psi(W_{\varepsilon,\xi})\|_{H^1_0(\Omega)} \le \varepsilon^{5/2}$ and the claim follows by applying again Cauchy Schwartz inequality. □

Proposition 16. *For all $\varepsilon > 0$ the map Φ_ε is continuous. Moreover for any $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that, if $\varepsilon < \varepsilon_0$ then $I_\varepsilon(\Phi_\varepsilon(\xi)) < m_\infty + \delta$.*

Proof. It is easy to see that Φ_ε is continuous because $t_\varepsilon(w)$ depends continously on $w \in H^1_0$.

At this point we prove that $t_\varepsilon(W_{\varepsilon,\xi}) \to 1$ uniformly with respect to $\xi \in \Omega$. In fact, by Lemma 12 $t_\varepsilon(W_{\varepsilon,\xi})$ is the unique solution of

$$\|W_{\varepsilon,\xi}\|^2_\varepsilon + t^2\omega G_\varepsilon(W_{\varepsilon,\xi}) - t^{p-2}|W_{\varepsilon,\xi}|^p_{\varepsilon,p} = 0.$$

By Remark 14 and Lemma 15 we have the claim.

Now, we have

$$I_\varepsilon\left(t_\varepsilon(W_{\varepsilon,\xi})W_{\varepsilon,\xi}\right) = \left(\frac{1}{2}-\frac{1}{p}\right)\|W_{\varepsilon,\xi}\|^2_\varepsilon t^2_\varepsilon + \omega\left(\frac{1}{4}-\frac{1}{p}\right)t^4_\varepsilon G_\varepsilon(W_{\varepsilon,\xi})$$

Again, by Remark 14 and Lemma 15 we have

$$I_\varepsilon\left(t_\varepsilon(W_{\varepsilon,\xi})W_{\varepsilon,\xi}\right) \to \left(\frac{1}{2}-\frac{1}{p}\right)\|U\|^2_{H^1(\mathbb{R}^3)} = m_\infty$$

that concludes the proof. □

Remark 17. We set

$$m_\varepsilon = \inf_{\mathcal{N}_\varepsilon} I_\varepsilon.$$

By Proposition 16 we have that

$$\limsup_{\varepsilon\to 0} m_\varepsilon \le m_\infty. \tag{13}$$

7. The map β

For any $u \in \mathcal{N}_\varepsilon$ we can define a point $\beta(u) \in \mathbb{R}^3$ by

$$\beta(u) = \frac{\int_\Omega x|u^+|^p dx}{\int_\Omega |u^+|^p dx}.$$

The function β is well defined in $\mathcal{N}_\varepsilon$ because, if $u \in \mathcal{N}_\varepsilon$, then $u^+ \ne 0$.

We have to prove that, if $u \in \mathcal{N}_\varepsilon \cap I^{m_\infty+\delta}_\varepsilon$ then $\beta(u) \in \Omega^+$.

Let us consider partitions of Ω. For a given $\varepsilon > 0$ we say that a finite partition $\mathcal{P}_\varepsilon = \left\{P^\varepsilon_j\right\}_{j\in\Lambda_\varepsilon}$ of Ω is a "good" partition if: for any $j \in \Lambda_\varepsilon$ the set P^ε_j is closed; $P^\varepsilon_i \cap P^\varepsilon_j \subset \partial P^\varepsilon_i \cap \partial P^\varepsilon_j$ for any $i \ne j$; there exist $r_1(\varepsilon), r_2(\varepsilon) > 0$ such that there are points $q^\varepsilon_j \in P^\varepsilon_j$ for which $B(q^\varepsilon_j,\varepsilon) \subset P^\varepsilon_j \subset B(q^\varepsilon_j, r_2(\varepsilon)) \subset B_g(q^\varepsilon_j, r_1(\varepsilon))$, with $r_1(\varepsilon) \ge r_2(\varepsilon) \ge C\varepsilon$ for some positive constant C; lastly, there exists a finite number

$\nu \in \mathbb{N}$ such that every $x \in \Omega$ is contained in at most ν balls $B(q_j^\varepsilon, r_1(\varepsilon))$, where ν does not depends on ε.

Lemma 18. *There exists a constant $\gamma > 0$ such that, for any $\delta > 0$ and for any $\varepsilon < \varepsilon_0(\delta)$ as in Proposition 16, given any "good" partition $\mathcal{P}_\varepsilon = \left\{P_j^\varepsilon\right\}_j$ of the domain Ω and for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ there exists, for an index $\bar{j}$ a set $P_{\bar{j}}^\varepsilon$ such that*

$$\frac{1}{\varepsilon^3}\int_{P_{\bar{j}}^\varepsilon} |u^+|^p dx \geq \gamma.$$

Proof. Taking in account that $I'(u)[u] = 0$ we have

$$\begin{aligned} \|u\|_\varepsilon^2 &= |u^+|_{\varepsilon,p}^p - \frac{1}{\varepsilon^3}\int_\Omega \omega u^2\psi(u) \leq |u^+|_{\varepsilon,p}^p = \sum_j \frac{1}{\varepsilon^3}\int_{P_j} |u^+|^p \\ &= \sum_j |u_j^+|_{\varepsilon,p}^p = \sum_j |u_j^+|_{\varepsilon,p}^{p-2}|u_j^+|_{\varepsilon,p}^2 \leq \max_j \left\{|u_j^+|_{\varepsilon,p}^{p-2}\right\} \sum_j |u_j^+|_{\varepsilon,p}^2 \end{aligned}$$

where u_j^+ is the restriction of the function u^+ on the set P_j.

At this point, arguing as in [6, Lemma 5.3], we prove that there exists a constant $C > 0$ such that

$$\sum_j |u_j^+|_{\varepsilon,p}^2 \leq C\nu\|u^+\|_\varepsilon^2,$$

thus

$$\max_j \left\{|u_j^+|_{\varepsilon,p}^{p-2}\right\} \geq \frac{1}{C\nu}$$

that conludes the proof. □

Proposition 19. *For any $\eta \in (0,1)$ there exists $\delta_0 < m_\infty$ such that for any $\delta \in (0,\delta_0)$ and any $\varepsilon \in (0,\varepsilon_0(\delta))$ as in Proposition 16, for any function $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\infty+\delta}$ we can find a point $q = q(u) \in \Omega$ such that*

$$\frac{1}{\varepsilon^3}\int_{B(q,r/2)} (u^+)^p > (1-\eta)\frac{2p}{p-2}m_\infty.$$

Proof. First, we prove the proposition for $u \in \mathcal{N}_\varepsilon \cap I_\varepsilon^{m_\varepsilon+2\delta}$.

By contradiction, we assume that there exists $\eta \in (0,1)$ such that we can find two sequences of vanishing real number δ_k and ε_k and a sequence of functions $\{u_k\}_k$ such that $u_k \in \mathcal{N}_{\varepsilon_k}$,

(14)
$$m_{\varepsilon_k} \leq I_{\varepsilon_k}(u_k) = \left(\frac{1}{2}-\frac{1}{p}\right)\|u_k\|_{\varepsilon_k}^2 + \omega\left(\frac{1}{4}-\frac{1}{p}\right)G_{\varepsilon_k}(u_k) \leq m_{\varepsilon_k} + 2\delta_k \leq m_\infty + 3\delta_k$$

for k large enough (see Remark 17), and, for any $q \in \Omega$,

$$\frac{1}{\varepsilon_k^3}\int_{B(q,r/2)} (u_k^+)^p \leq (1-\eta)\frac{2p}{p-2}m_\infty.$$

By Ekeland principle and by definition of $\mathcal{N}_{\varepsilon_k}$ we can assume

$$\left|I'_{\varepsilon_k}(u_k)[\varphi]\right| \leq \sigma_k\|\varphi\|_{\varepsilon_k} \text{ where } \sigma_k \to 0. \tag{15}$$

By Lemma 18 there exists a set $P_k^{\varepsilon_k} \in \mathcal{P}_{\varepsilon_k}$ such that

$$\frac{1}{\varepsilon_k^3}\int_{P_k^{\varepsilon_k}} |u_k^+|^p dx \geq \gamma.$$

We choose a point $q_k \in P_k^{\varepsilon_k}$ and we define, for $z \in \Omega_{\varepsilon_k} := \frac{1}{\varepsilon_k}(\Omega - q_k)$

$$w_k(z) = u_k(\varepsilon_k z + q_k) = u_k(x).$$

We have that $w_k \in H_0^1(\Omega_{\varepsilon_k}) \subset H^1(\mathbb{R}^3)$. By equation (14) we have

$$\|w_k\|^2_{H^1(\mathbb{R}^3)} = \|u_k\|^2_{\varepsilon_k} \le C.$$

So $w_k \to w$ weakly in $H^1(\mathbb{R}^3)$ and strongly in $L^t_{\text{loc}}(\mathbb{R}^3)$.

We set $\psi(u_k)(x) := \psi_k(x) = \psi_k(\varepsilon_k z + q_k) := \tilde{\psi}_k(z)$ where $x \in \Omega$ and $z \in \Omega_{\varepsilon_k}$. It is easy to verify that

$$-\Delta_z \tilde{\psi}_k(z) = \varepsilon_k^2 q w_k^2(z).$$

With abuse of language we set

$$\tilde{\psi}_k(z) = \psi(\varepsilon_k w_k).$$

Thus

$$\begin{aligned} I_{\varepsilon_k}(u_k) &= \frac{1}{2}\|u_k\|^2_{\varepsilon_k} - \frac{1}{p}|u_k^+|^p_{\varepsilon_k,p} + \frac{\omega}{4}\frac{1}{\varepsilon_k^3}\int_\Omega q u_k^2 \psi(u_k) = \\ &= \frac{1}{2}\|w_k\|^2_{H^1(\mathbb{R}^3)} - \frac{1}{p}\|w_k^+\|^p_{L^p(\mathbb{R}^3)} + \frac{\omega}{4}\int_{\Omega_{\varepsilon_k}} q w_k^2 \psi(\varepsilon_k w_k) = \\ &= \frac{1}{2}\|w_k\|^2_{H^1(\mathbb{R}^3)} - \frac{1}{p}\|w_k^+\|^p_{L^p(\mathbb{R}^3)} + \varepsilon_k^2\frac{\omega}{4}\int_{\mathbb{R}^3} q w_k^2 \psi(w_k) := E_{\varepsilon_k}(w_k) \end{aligned} \tag{16}$$

By definition of $E_{\varepsilon_k} : H^1(\mathbb{R}^3) \to \mathbb{R}$, we get $E_{\varepsilon_k}(w_k) \to m_\infty$.

Given any $\varphi \in C_0^\infty(\mathbb{R}^3)$ we set $\varphi(x) = \varphi(\varepsilon_k z + q_k) := \tilde{\varphi}_k(z)$. For k large enough we have that supp$\tilde{\varphi}_k \subset \Omega$ and, by (15), that $E'_{\varepsilon_k}(w_k)[\varphi] = I'_{\varepsilon_k}(u_k)[\tilde{\varphi}_k] \to 0$. Moreover, by definiton of E_{ε_k} and by Lemma 8 we have

$$\begin{aligned} E'_{\varepsilon_k}(w_k)[\varphi] &= \langle w_k, \varphi\rangle_{H^1(\mathbb{R}^3)} - \int_{\mathbb{R}^3} |w_k^+|^{p-1}\varphi + \omega\varepsilon_k^2 \int_{\mathbb{R}^3} q w_k \psi(w_k)\varphi + \\ &\to \langle w, \varphi\rangle_{H^1(\mathbb{R}^3)} - \int_{\mathbb{R}^3} |w^+|^{p-1}\varphi. \end{aligned}$$

Thus w is a weak solution of

$$-\Delta w + w = (w^+)^{p-1} \text{ on } \mathbb{R}^3.$$

By Lemma 18 and by the choice of q_k we have that $w \ne 0$, so $w > 0$.

Arguing as in (16), and using that $u_k \in \mathcal{N}_{\varepsilon_k}$ we have

$$\begin{aligned} I_{\varepsilon_k}(u_k) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_k\|^2_{\varepsilon_k} + \omega\left(\frac{1}{4} - \frac{1}{p}\right)\frac{1}{\varepsilon_k^3}\int_\Omega q u_k^2 \psi(u_k) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|w_k\|^2_{H^1(\mathbb{R}^3)} + \varepsilon_k^2\omega\left(\frac{1}{4} - \frac{1}{p}\right)\int_{\mathbb{R}^3} q w_k^2 \psi(w_k) \to m_\infty \end{aligned} \tag{17}$$

and

$$\begin{aligned} I_{\varepsilon_k}(u_k) &= \left(\frac{1}{2} - \frac{1}{p}\right)|u_k^+|^p_{p,\varepsilon_k} - \frac{\omega}{4}\frac{1}{\varepsilon_k^3}\int_\Omega q u_k^2 \psi(u_k) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)|w_k^+|^p_p - \varepsilon_k^2\frac{\omega}{4}\int_{\mathbb{R}^3} q w_k^2 \psi(w_k) \to m_\infty. \end{aligned} \tag{18}$$

So, by (17) we have that $\|w\|^2_{H^1(\mathbb{R}^3)} = \frac{2p}{p-2}m_\infty$ and that $\left(\frac{1}{2} - \frac{1}{p}\right)\|w_k\|^2_{H^1(\mathbb{R}^3)} \to m_\infty$ and we conclude that $w_k \to w$ strongly in $H^1(\mathbb{R}^3)$.

Given $T>0$, by the definiton of w_k we get, for k large enough

$$\begin{aligned} |w_k^+|^p_{L^p(B(0,T))} &= \frac{1}{\varepsilon_k^3}\int_{B(q_k,\varepsilon_k T)}|u_k^+|^p dx \le \frac{1}{\varepsilon_k^3}\int_{B(q_k,r/2)}|u_k^+|^p dx \\ &\le (1-\eta)\,\frac{2p}{p-2}m_\infty. \end{aligned} \tag{19}$$

Then we have the contradiction. In fact, by (18) we have $\left(\frac{1}{2}-\frac{1}{p}\right)|w_k^+|_p^p \to m_\infty$ and this contradicts (19). At this point we have proved the claim for $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\varepsilon+2\delta}$. Now, by the thesis for $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\varepsilon+2\delta}$ and by (18) we have

$$I_{\varepsilon_k}(u_k)=\left(\frac{1}{2}-\frac{1}{p}\right)|u_k^+|^p_{p,\varepsilon_k}+O(\varepsilon^2)\ge(1-\eta)m_\infty+O(\varepsilon^2)$$

and, passing to the limit,

$$\liminf_{k\to\infty} m_{\varepsilon_k}\ge m_\infty.$$

This, combined by (13) gives us that

$$\lim_{\varepsilon\to 0} m_\varepsilon = m_\infty. \tag{20}$$

Hence, when ε,δ are small enough, $\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\infty+\delta}\subset\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\varepsilon+2\delta}$ and the general claim follows. □

Proposition 20. *There exists $\delta_0\in(0,m_\infty)$ such that for any $\delta\in(0,\delta_0)$ and any $\varepsilon\in(0,\varepsilon(\delta_0)$ (see Proposition 16), for every function $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\infty+\delta}$ it holds $\beta(u)\in\Omega^+$. Moreover the composition*

$$\beta\circ\Phi_\varepsilon:\Omega^-\to\Omega^+$$

is s homotopic to the immersion $i:\Omega^-\to\Omega^+$

Proof. By Proposition 19, for any function $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\infty+\delta}$, for any $\eta\in(0,1)$ and for ε,δ small enough, we can find a point $q=q(u)\in\Omega$ such that

$$\frac{1}{\varepsilon^3}\int_{B(q,r/2)}(u^+)^p>(1-\eta)\,\frac{2p}{p-2}m_\infty.$$

Moreover, since $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\infty+\delta}$ we have

$$I_\varepsilon(u)=\left(\frac{p-2}{2p}\right)|u^+|^p_{p,\varepsilon}-\frac{\omega}{4}\frac{1}{\varepsilon^3}\int_\Omega qu^2\psi(u)\le m_\infty+\delta.$$

Now, arguing as in Lemma 15 we have that

$$\|\psi(u)\|^2_{H^1(\Omega)}=q\int_\Omega\psi(u)u^2\le C\|\psi(u)\|_{H^1(\Omega)}\left(\int_\Omega u^{12/5}\right)^{5/6},$$

so $\|\psi(u)\|_{H^1(\Omega)}\le\left(\int_\Omega u^{12/5}\right)^{5/6}$, then

$$\begin{aligned}\frac{1}{\varepsilon^3}\int\psi(u)u^2 &\le \frac{1}{\varepsilon^3}\|\psi\|_{H^1(\Omega)}\left(\int_\Omega u^{12/5}\right)^{5/6}\le C\frac{1}{\varepsilon^3}\left(\int_\Omega u^{12/5}\right)^{5/3}\\ &\le C\varepsilon^2|u|^4_{12/5,\varepsilon}\le C\varepsilon^2\|u\|^4_\varepsilon\le C\varepsilon^2\end{aligned}$$

because $\|u\|_\varepsilon$ is bounded since $u\in\mathcal{N}_\varepsilon\cap I_\varepsilon^{m_\infty+\delta}$.

Hence, provided we choose $\varepsilon(\delta_0)$ small enough, we have

$$\left(\frac{p-2}{2p}\right)|u^+|_{p,\varepsilon}^p \le m_\infty + 2\delta_0.$$

So,

$$\frac{\frac{1}{\varepsilon^3}\int_{B(q,r/2)}(u^+)^p}{|u^+|_{p,\varepsilon}^p} > \frac{1-\eta}{1+2\delta_0/m_\infty}$$

Finally,

$$\begin{aligned}
|\beta(u)-q| &\le \frac{\left|\frac{1}{\varepsilon^3}\int_\Omega (x-q)(u^+)^p\right|}{|u^+|_{p,\varepsilon}^p} \\
&\le \frac{\left|\frac{1}{\varepsilon^3}\int_{B(q,r/2)} (x-q)(u^+)^p\right|}{|u^+|_{p,\varepsilon}^p} + \frac{\left|\frac{1}{\varepsilon^3}\int_{\Omega\smallsetminus B(q,r/2)} (x-q)(u^+)^p\right|}{|u^+|_{p,\varepsilon}^p} \\
&\le \frac{r}{2} + 2\mathrm{diam}(\Omega)\left(1-\frac{1-\eta}{1+2\delta_0/m_\infty}\right),
\end{aligned}$$

so, choosing η, δ_0 and $\varepsilon(\delta_0)$ small enough we proved the first claim. The second claim is standard. □

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