ON THE EXISTENCE OF RAMIFIED ABELIAN COVERS

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ABSTRACT. Given a normal complete variety Y, distinct irreducible effective Weil divisors $D_1, \dots D_n$ of Y and positive integers $d_1, \dots d_n$, we spell out the conditions for the existence of an abelian cover $X \to Y$ branched with order d_i on D_i for $i = 1, \dots n$.

As an application, we prove that a Galois cover of a normal complete toric variety branched on the torus-invariant divisors is itself a toric variety.
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Dedicated to Alberto Conte on his 70^{th} *birthday.*

CONTENTS

1. Introduction

Given a projective variety Y and effective divisors $D_1, \dots D_n$ of Y, deciding whether there exists a Galois cover branched on $D_1, \dots D_n$ with given multiplicities is a very complicated question, which in the complex case is essentially equivalent to describing the finite quotients of the fundamental group of $Y \setminus (D_1 \cup \dots \cup D_n)$.

In Section 2 of this paper we answer this question for a normal variety Y in the case that the Galois group of the cover is abelian (Theorem 2.1), using the theory developed in [Par91] and [AP12]. In particular, we prove that when the class group $\mathrm{Cl}(Y)$ is torsion free, every abelian cover of Y branched on $D_1, \dots D_n$ with given multiplicities is the quotient of a maximal such cover, unique up to isomorphism.

In Section 3 we analyze the same question using toric geometry in the case when Y is a normal complete toric variety and $D_1, \dots D_n$ are invariant divisors and we obtain results that parallel those in Section 2 (Theorem 3.5). Combining the two approaches we are able to show that any cover of a normal complete toric variety branched on the invariant divisors is toric (Theorem 3.7).

Acknowledgments. We wish to thank Angelo Vistoli for useful discussions on the topic of this paper (cf. Remark 3.8).

Notation. G always denotes a finite group, almost always abelian, and $G^* := \mathrm{Hom}(G, \mathbb{K}^*)$ the group of characters; $o(g)$ is the order of the element $g \in G$ and $|H|$ is the cardinality of a subgroup $H < G$. We work over an algebraically closed field $\mathbb{K}$ whose characteristic does not divide the order of the finite abelian groups

we consider.
If A is an abelian group we write $A[d] := \{a \in A \mid da = 0\}$ (d an integer), $A^\vee := \mathrm{Hom}(A, \mathbb{Z})$ and we denote by $\mathrm{Tors}(A)$ the torsion subgroup of A.
The smooth part of a variety Y is denoted by Y_{sm}. The symbol $\equiv$ denotes linear equivalence of divisors. If Y is a normal variety we denote by $\mathrm{Cl}(Y)$ the group of classes, namely the group of Weil divisors up to linear equivalence.

2. Abelian covers

2.1. **The fundamental relations.** We quickly recall the theory of abelian covers (cf. [Par91], [AP12], and also [PT95]) in the most suitable form for the applications considered here.

There are slightly different definitions of abelian covers in the literature (see, for instance, [AP12] that treats also the non-normal case). Here we restrict our attention to the case of normal varieties, but we do not require that the covering map be flat; hence we define a cover as a finite morphism $\pi \colon X \to Y$ of normal varieties and we say that π is an abelian cover if it is a Galois morphism with abelian Galois group G (π is also called a "G-cover").

Recall that, as already stated in the Notations, throughout all the paper we assume that G has order not divisible by $\mathrm{char}\,\mathbb{K}$.

To every component D of the branch locus of π we associate the pair (H, ψ), where $H < G$ is the cyclic subgroup consisting of the elements of G that fix the preimage of D pointwise (the *inertia subgroup* of D) and ψ is the element of the character group H^* given by the natural representation of H on the normal space to the preimage of D at a general point (these definitions are well posed since G is abelian). It can be shown that ψ generates the group H^*.

If we fix a primitive d-th root ζ of 1, where d is the exponent of the group G, then a pair (H, ψ) as above is determined by the generator $g \in H$ such that $\psi(g) = \zeta^{\frac{d}{o(g)}}$. We follow this convention and attach to every component D_i of the branch locus of π a nonzero element $g_i \in G$.

If π is flat, which is always the case when Y is smooth, the sheaf $\pi_*\mathcal{O}_X$ decomposes under the G-action as $\oplus_{\chi \in G^*} L_\chi^{-1}$, where the L_χ are line bundles ($L_1 = \mathcal{O}_Y$) and G acts on L_χ^{-1} via the character χ.

Given $\chi \in G^*$ and $g \in G$, we denote by $\overline{\chi}(g)$ the smallest non-negative integer a such that $\chi(g) = \zeta^{\frac{ad}{o(g)}}$. The main result of [Par91] is that the L_χ, D_i (the *building data* of π) satisfy the following *fundamental relations*:

$$L_\chi + L_{\chi'} \equiv L_{\chi+\chi'} + \sum_{i=1}^{n} \varepsilon^i_{\chi,\chi'} D_i \qquad \forall \chi, \chi' \in G^* \tag{2.1}$$

where $\varepsilon^i_{\chi,\chi'} = \lfloor \frac{\overline{\chi}(g_i) + \overline{\chi'}(g_i)}{o(g_i)} \rfloor$. (Notice that the coefficients $\varepsilon^i_{\chi,\chi'}$ are equal either to 0 or to 1). Conversely, distinct irreducible divisors D_i and line bundles L_χ satisfying (2.1) are the building data of a flat (normal) G-cover $X \to Y$; in addition, if $h^0(\mathcal{O}_Y) = 1$ then $X \to Y$ is uniquely determined up to isomorphism of G-covers.

If we fix characters $\chi_1, \dots \chi_r \in G^*$ such that G^* is the direct sum of the subgroups generated by the χ_j, and we set $L_j := L_{\chi_j}$, $m_j := o(\chi_j)$, then the solutions of the fundamental relations (2.1) are in one-one correspondence with the solutions of the following *reduced fundamental relations*:

$$m_j L_j \equiv \sum_{i=1}^{n} \frac{m_j \overline{\chi_j}(g_i)}{d_i} D_i, \qquad j = 1, \dots r \tag{2.2}$$

As before, denote by d the exponent of G; notice that if $\mathrm{Pic}(Y)[d] = 0$, then for fixed (D_i, g_i), $i = 1, \dots n$, the solution of (2.2) is unique, hence the *branch data* (D_i, g_i) determine the cover.

In order to deal with the case when Y is normal but not smooth, we observe first that the cover $X \to Y$ can be recovered from its restriction $X' \to Y_{\mathrm{sm}}$ to the smooth locus by taking the integral closure of Y in the extension $\mathbb{K}(X') \supset \mathbb{K}(Y)$. Observe then that, since the complement $Y \setminus Y_{\mathrm{sm}}$ of the smooth part has codimension > 1, we have $h^0(\mathcal{O}_{Y_{\mathrm{sm}}}) = h^0(\mathcal{O}_Y) = 1$, and thus the cover $X' \to Y_{\mathrm{sm}}$ is determined by the building data L_χ, D_i. Using the identification $\mathrm{Pic}(Y_{\mathrm{sm}}) = \mathrm{Cl}(Y_{\mathrm{sm}}) = \mathrm{Cl}(Y)$, we can regard the L_χ as elements of $\mathrm{Cl}(Y)$ and, taking the closure, the D_i as Weil divisors on Y, and we can interpret the fundamental relations as equalities in $\mathrm{Cl}(Y)$. In this sense, if Y is normal variety with $h^0(\mathcal{O}_Y) = 1$, then the G-covers $X \to Y$ are determined by the building data up to isomorphism.

We say that an abelian cover $\pi\colon X \to Y$ is *totally ramified* if the inertia subgroups of the divisorial components of the branch locus of π generate G, or, equivalently, if π does not factorize through a cover $X' \to Y$ that is étale over Y_{sm}. We observe that a totally ramified cover is necessarily connected; conversely, equations (2.2) imply that if G is an abelian group of exponent d and Y is a variety such that $\mathrm{Cl}(Y)[d] = 0$, then any connected G-cover of Y is totally ramified.

2.2. **The maximal cover.** Let Y be a complete normal variety, let $D_1, \dots D_n$ be distinct irreducible effective divisors of Y and let $d_1, \dots d_n$ be positive integers (it is convenient to allow the possibility that $d_i = 1$ for some i). We set $d := \mathrm{lcm}(d_1, \dots d_n)$.

We say that a Galois cover $\pi\colon X \to Y$ is *branched on* $D_1, \dots D_n$ *with orders* $d_1, \dots d_n$ if:

- the divisorial part of the branch locus of π is contained in $\sum_i D_i$;
- the ramification order of π over D_i is equal to d_i.

Let $\eta\colon \widetilde{Y} \to Y$ be a resolution of the singularities and set $\mathrm{N}(Y) := \mathrm{Cl}(Y)/\eta_*\mathrm{Pic}^0(\widetilde{Y})$. Since the map $\eta_*\colon \mathrm{Pic}(\widetilde{Y}) = \mathrm{Cl}(\widetilde{Y}) \to \mathrm{Cl}(Y)$ is surjective, $\mathrm{N}(Y)$ is a quotient of the Néron-Severi group $\mathrm{NS}(\widetilde{Y})$, hence it is finitely generated. It follows that $\eta_*\mathrm{Pic}^0(\widetilde{Y})$ is the largest divisible subgroup of $\mathrm{Cl}(Y)$ and therefore $\mathrm{N}(Y)$ does not depend on the choice of the resolution of Y (this is easily checked also by a geometrical argument). The group $\mathrm{Cl}(Y)^\vee$ coincides with $\mathrm{N}(Y)^\vee$, hence it is a finitely generated free abelian group of rank equal to the rank of $\mathrm{N}(Y)$.

Consider the map $\mathbb{Z}^n \to \mathrm{Cl}(Y)$ that maps the i-th canonical generator to the class of D_i, let $\phi\colon \mathrm{Cl}(Y)^\vee \to \oplus_{i=1}^n \mathbb{Z}_{d_i}$ be the map obtained by composing the dual map $\mathrm{Cl}(Y)^\vee \to (\mathbb{Z}^n)^\vee$ with $(\mathbb{Z}^n)^\vee = \mathbb{Z}^n \to \oplus_{i=1}^n \mathbb{Z}_{d_i}$ and let K_{min} be the image of ϕ. Let G_{max} be the abelian group defined by the exact sequence:

$$0 \to K_{\mathrm{min}} \to \oplus_{i=1}^n \mathbb{Z}_{d_i} \to G_{\mathrm{max}} \to 0. \tag{2.3}$$

Then we have the following:

Theorem 2.1. *Let Y be a normal variety with $h^0(\mathcal{O}_Y) = 1$, let $D_1, \dots D_n$ be distinct irreducible effective divisors, let $d_1, \dots d_n$ be positive integers and set $d := \operatorname{lcm}(d_1, \dots d_n)$. Then:*

(1) *If $X \to Y$ is a totally ramified G-cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$, then:*
  (a) *the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i descends to a surjection $G_{\max} \twoheadrightarrow G$;*
  (b) *the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for every $i = 1, \dots n$.*

(2) *If the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$ and $\mathrm{N}(Y)[d] = 0$, then there exists a maximal totally ramified abelian cover $X_{\max} \to Y$ branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$; the Galois group of $X_{\max} \to Y$ is equal to $G_{\max}$.*

(3) *If the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$ and $\mathrm{Cl}(Y)[d] = 0$, then the cover $X_{\max} \to Y$ is unique up to isomorphism of $G_{\max}$-covers and every totally ramified abelian cover $X \to Y$ branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ is a quotient of $X_{\max}$ by a subgroup of $G_{\max}$.*

Proof. Let $H_1, \dots H_t \in \mathrm{N}(Y)$ be elements whose classes are free generators of the abelian group $\mathrm{N}(Y)/\operatorname{Tors}(\mathrm{N}(Y))$, and write:

$$D_i = \sum_{j=1}^{t} a_{ij} H_j \mod \operatorname{Tors}(\mathrm{N}(Y)), \qquad j = 1, \dots t \tag{2.4}$$

Hence, the subgroup $K_{\min}$ of $\oplus_{i=1}^n \mathbb{Z}_{d_i}$ is generated by the elements $z_j := (a_{1j}, \dots a_{nj})$, for $j = 1, \dots t$.

Let $X \to Y$ be a totally ramified G-cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ and let (D_i, g_i) be its branch data. Consider the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$ that maps $1 \in \mathbb{Z}_{d_i}$ to g_i: this map is surjective, by the assumption that $X \to Y$ is totally ramified, and its restriction to $\mathbb{Z}_{d_i}$ is injective for $i = 1, \dots n$, since the cover is branched on D_i with order d_i. If we denote by K the kernel of $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$, to prove (1) it suffices to show that $K \supseteq K_{\min}$. Dually, this is equivalent to showing that $G^* \subseteq K_{\min}^\perp \subset \oplus_{i=1}^n (\mathbb{Z}_{d_i})^*$. Let $\psi_i \in (\mathbb{Z}_{d_i})^*$ be the generator that maps $1 \in \mathbb{Z}_{d_i}$ to $\zeta^{\frac{d}{d_i}}$ and write $\chi \in G^*$ as $(\psi_1^{b_1}, \dots \psi_n^{b_n})$, with $0 \le b_i < d_i$; if $o(\chi) = m$ then (2.2) gives $mL_\chi \equiv \sum_{i=1}^n \frac{mb_i}{d_i} D_i$. Plugging (2.4) in this equation we obtain that $\sum_{i=1}^n \frac{b_i a_{ij}}{d_i}$ is an integer for $j = 1, \dots t$, namely $\chi \in K_{\min}^\perp$.

(2) Let $\chi_1, \dots \chi_r$ be a basis of $G_{\max}^*$ and, as above, for $s = 1, \dots r$ write $\chi_s = (\psi_1^{b_{s1}}, \dots \psi_n^{b_{sn}})$, with $0 \le b_{si} < d_i$. Since by assumption $\mathrm{N}(Y)[d] = 0$, by the proof of (1) the elements $\sum_{j=1}^t (\sum_{i=1}^n \frac{b_{si} a_{ij}}{d_i}) H_j$, $s = 1, \dots r$, can be lifted to solutions $\overline{L_s} \in \mathrm{N}(Y)$ of the reduced fundamental relations (2.2) for a $G_{\max}$-cover with branch data (D_i, g_i), where $g_i \in G$ is the image of $1 \in \mathbb{Z}_{d_i}$. Since the kernel of $\mathrm{Cl}(Y) \to N(Y)$ is a divisible group, it is possible to lift the $\overline{L_s}$ to solutions $L_s \in \mathrm{Cl}(Y)$. We let $X_{\max} \to Y$ be the $G_{\max}$-cover determined by these solutions. It is a totally ramified cover since the map $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G_{\max}$ is surjective by the definition of $G_{\max}$.

(3) Since $\mathrm{Cl}(Y)[d] = 0$, any G-cover such that the exponent of G is a divisor of d is determined uniquely by the branch data; in particular, this holds for the cover $X_{\max} \to Y$ in (2) and for every intermediate cover $X_{\max}/H \to Y$, where $H < G_{\max}$. The claim now follows by (1). $\square$

Example 2.1. Take $Y = \mathbb{P}^{n-1}$ and let $D_1, \dots D_n$ be the coordinate hyperplanes. In this case the group $K_{\min}$ is generated by $(1, \dots 1) \in \oplus_{i=1}^n \mathbb{Z}_{d_i}$. Since any connected abelian cover of $\mathbb{P}^{n-1}$ is totally ramified, by Theorem 2.1 there exists a abelian cover of $\mathbb{P}^{n-1}$ branched over $D_1, \dots D_n$ with orders $d_1, \dots d_n$ iff d_i divides $lcm(d_1, \dots, \widehat{d_i}, \dots d_n)$ for every $i = 1, \dots n$. For $d_1 = \dots = d_n = d$, then $G_{\max} = \mathbb{Z}_d^n / < (1, \dots 1) >$ and $X_{\max} \to \mathbb{P}^{n-1}$ is the cover $\mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ defined by $[x_1, \dots x_n] \mapsto [x_1^d, \dots x_n^d]$.

In general, $X_{\max}$ is a weighted projective space $\mathbb{P}(\frac{d}{d_1}, \dots \frac{d}{d_n})$ and the cover is given by $[x_1, \dots x_n] \mapsto [x_1^{d_1}, \dots x_n^{d_n}]$.

3. Toric covers

Notations 3.1. Here, we fix the notations which are standard in toric geometry. A (complete normal) toric variety Y corresponds to a fan Σ living in the vector space $N \otimes \mathbb{R}$, where $N \cong \mathbb{Z}^s$. The dual lattice is $M = N^\vee$. The torus is $T = N \otimes \mathbb{C}^* = \operatorname{Hom}(M, \mathbb{C}^*)$.

The integral vectors $r_i \in N$ will denote the integral generators of the rays $\sigma_i \in \Sigma(1)$ of the fan Σ. They are in a bijection with the T-invariant Weil divisors D_i $(i = 1, \dots, n)$ on Y.

Definition 3.2. A *toric cover* $f\colon X \to Y$ is a finite morphism of toric varieties corresponding to the map of fans $F\colon (N', \Sigma') \to (N, \Sigma)$ such that:

(1) $N' \subseteq N$ is a sublattice of finite index, so that $N' \otimes \mathbb{R} = N \otimes \mathbb{R}$.
(2) $\Sigma' = \Sigma$.

The proof of the following lemma is immediate.

Lemma 3.3. *The morphism f has the following properties:*

(1) *It is equivariant with respect to the homomorphism of tori $T' \to T$.*
(2) *It is an abelian cover with Galois group $G = \ker(T' \to T) = N/N'$.*
(3) *It is ramified only along the boundary divisors D_i, with multiplicities $d_i \geq 1$ defined by the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$.*

Proposition 3.4. *Let Y be a complete toric variety such that $\operatorname{Cl}(Y)$ is torsion free, and $X \to Y$ be a toric cover. Then, with notations as above, there exists the* *following commutative diagram with exact rows and columns.*

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Cl}(Y)^\vee & \longrightarrow & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} d_i D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z}_{d_i} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & G & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

(Here the D_i^* are formal symbols denoting a basis of $\mathbb{Z}^n$). Moreover, each of the homomorphisms $\mathbb{Z}_{d_i} \to G$ is an embedding.

Proof. The third row appeared in Lemma 3.3, and the second row is the obvious one.

It is well known that the boundary divisors on a complete normal toric variety span the group $\mathrm{Cl}(Y)$, and that there exists the following short exact sequence of lattices:

$$0 \to M \longrightarrow \oplus_{i=1}^n \mathbb{Z} D_i \longrightarrow \mathrm{Cl}(Y) \to 0.$$

Since $\mathrm{Cl}(Y)$ is torsion free by assumption, this sequence is split and dualizing it one obtains the central column. Since $\oplus_{i=1}^n \mathbb{Z} D_i^* \to N$ is surjective, then so is $\oplus_{i=1}^n \mathbb{Z}_{d_i} \to G$. The group K is defined as the kernel of this map.

Finally, the condition that $\mathbb{Z}_{d_i} \to G$ is injective is equivalent to the condition that the integral generator of $N' \cap \mathbb{R}_{\geq 0} r_i$ is $d_i r_i$, which holds by Lemma 3.3. □

Theorem 3.5. *Let Y be a complete toric variety such that $\mathrm{Cl}(Y)$ is torsion free, let $d_1, \dots d_n$ be positive integers and let $K_{\min}$ and $G_{\max}$ be defined as in sequence (2.3). Then:*

1. *There exists a toric cover branched on D_i of order d_i, $i = 1, \dots n$, iff the map $\mathbb{Z}_{d_i} \to G_{\max}$ is injective for $i = 1, \dots n$.*
2. *If condition (1) is satisfied, then among all the toric covers of Y ramified over the divisors D_i with multiplicities d_i there exists a maximal one $X_{\mathrm{Tmax}} \to Y$, with Galois group $G_{\max}$, such that any other toric cover $X \to Y$ with the same branching orders is a quotient $X = X_{\mathrm{Tmax}}/H$ by a subgroup $H < G_{\max}$.*

Proof. Let $X \to Y$ be a toric cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$, let N' be the corresponding sublattice of N and $G = N/N'$ the Galois group. Let $N'_{\min}$ be the subgroup of N generated by $d_i r_i$, $i = 1, \dots n$. By Lemma 3.3 one must have $N'_{\min} \subseteq N'$, hence the map $\mathbb{Z}_{d_i} \to N/N'_{\min}$ is injective since $\mathbb{Z}_{d_i} \to G = N/N'$ is injective by Proposition 3.4. We set $X_{\mathrm{Tmax}} \to Y$ to be the cover for $N'_{\min}$. Clearly, the cover for the lattice N' is a quotient of the cover for the lattice $N'_{\min}$ by the group $H = N'/N'_{\min}$.

Consider the second and third rows of the diagram of Proposition 3.4 as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies gives

$$\mathrm{Cl}(Y)^\vee \longrightarrow K \longrightarrow \mathrm{coker}(p') \longrightarrow 0$$

For $N' = N'_{\mathrm{min}}$, the map p' is surjective, hence $\mathrm{Cl}(Y)^\vee \to K$ is surjective too, and $K = K_{\mathrm{min}}$, $N/N'_{\mathrm{min}} = G_{\mathrm{max}}$.

Vice versa, suppose that in the following commutative diagram with exact row and columns each of the maps $\mathbb{Z}_{d_i} \to G_{\mathrm{max}}$ is injective.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Cl}(Y)^\vee & \xrightarrow{q} & K_{\mathrm{min}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} d_i D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z} D_i^* & \longrightarrow & \oplus_{i=1}^n \mathbb{Z}_{d_i} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle p} & & \downarrow & & \\
 & & & & N & & G_{\mathrm{max}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

We complete the first row on the left by adding $\ker(q)$. We have an induced homomorphism $\ker(q) \to \oplus \mathbb{Z} d_i D_i^*$, and we define N' to be its cokernel.

Now consider the completed first and second rows as a short exact sequence of 2-step complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. The associated long exact sequence of cohomologies says that $\ker(q) \to \oplus_{i=1}^n \mathbb{Z} d_i D_i^*$ is injective, and the sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow G_{\mathrm{max}} \longrightarrow 0$$

is exact. It follows that $N' = N'_{\mathrm{min}}$ and the toric morphism $(N'_{\mathrm{min}}, \Sigma) \to (N, \Sigma)$ is then the searched-for maximal abelian toric cover. □

Remark 3.6. Condition (1) in the statement of Theorem 3.5 can also be expressed by saying that for $i = 1, ..., n$ the element $d_i r_i \in N'_{\mathrm{min}}$ is primitive, where $N'_{\mathrm{min}} \subseteq N$ is the subgroup generated by all the $d_i r_i$.

We now combine the results of this section with those of §2 to obtain a structure result for Galois covers of toric varieties.

Theorem 3.7. *Let Y be a normal complete toric variety and let $f\colon X \to Y$ be a connected cover such that the divisorial part of the branch locus of f is contained in the union of the invariant divisors $D_1, \dots D_n$.*
Then $\deg f$ *is not divisible by* $\mathrm{char}\, \mathbb{K}$ *and $f\colon X \to Y$ is a toric cover.*

Proof. Let $U \subset Y$ be the open orbit and let $X' \to U$ be the cover obtained by restricting f. Since U is smooth, by the assumptions and by purity of the branch locus, $X' \to U$ is étale. Let $X'' \to U$ be the Galois closure of $X' \to U$: the cover $X'' \to U$ is also étale, hence by [Mi, Prop. 1] it is, up to isomorphism, a homomorphism of tori. Since the kernel of an étale homomorphism of tori is reduced, it follows that $X'' \to U$ is an abelian cover such that $\operatorname{char} \mathbb{K}$ does not divide the order of the Galois group.

Moreover, the intermediate cover $X' \to U$ is also abelian (actually $X' = X''$). The cover $f \colon X \to Y$ is abelian, too, since X is the integral closure of Y in $\mathbb{K}(X')$. We denote by G the Galois group of f and by $d_1, \dots d_n$ the orders of ramification of $X \to Y$ on $D_1, \dots D_n$.
Assume first that $\mathrm{Cl}(Y)$ has no torsion, so that every connected abelian cover of Y is totally ramified (cf. §2). Then by Theorem 2.1 every connected abelian cover branched on $D_1, \dots D_n$ with orders $d_1, \dots d_n$ is a quotient of the maximal abelian cover $X_{\max} \to Y$ by a subgroup $H < G_{\max}$. In particular, this is true for the cover $X_{\mathrm{Tmax}} \to Y$ of Theorem 3.5. Since $X_{\max}$ and X_{Tmax} have the same Galois group it follows that $X_{\max} = X_{\mathrm{Tmax}}$. Hence $X \to Y$, being a quotient of X_{Tmax}, is a toric cover.

Consider now the general case. Recall that the group $\operatorname{Tors} \mathrm{Cl}(Y)$ is finite, isomorphic to $N/\langle r_i \rangle$, and the cover $Y' \to Y$ corresponding to $\operatorname{Tors} \mathrm{Cl}(Y)$ is toric, and one has $\operatorname{Tors} \mathrm{Cl}(Y') = 0$. Indeed, on a toric variety the group $\mathrm{Cl}(Y)$ is generated by the T-invariant Weil divisors D_i. Thus, $\mathrm{Cl}(Y)$ is the quotient of the free abelian group $\oplus \mathbb{Z} D_i$ of all T-invariant divisors modulo the subgroup M of principal T-invariant divisors. Thus, $\operatorname{Tors} \mathrm{Cl}(Y) \simeq M'/M$, where $M' \subset \oplus \mathbb{Q} D_i$ is the subgroup of $\mathbb{Q}$-linear functions on N taking integral values on the vectors r_i. Then $N' := M'^{\vee}$ is the subgroup of N generated by the r_i, and the cover $Y' \to Y$ is the cover corresponding to the map of fans $(N', \Sigma) \to (N, \Sigma)$. On Y' one has $N' = \langle r_i \rangle$, so $\operatorname{Tors} \mathrm{Cl}(Y') = 0$.

Let $X' \to Y'$ be a connected component of the pull back of $X \to Y$: it is an abelian cover branched on the invariant divisors of Y', hence by the first part of the proof it is toric. The map $X' \to Y$ is toric, since it is a composition of toric morphisms, hence the intermediate cover $X \to Y$ is also toric. □

Remark 3.8. The argument that shows that the map f is an abelian cover in the proof of Theorem 3.7 was suggested to us by Angelo Vistoli. He also remarked that it is possible to prove Theorem 3.6 in a more conceptual way by showing that the torus action on the cover $X' \to U$ of the open orbit of Y extends to X, in view of the properties of the integral closure. However our approach has the advantage of describing explicitly the fan/building data associated with the cover.

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