

Existence and stability results on a class of Non Linear Schrödinger Equations in bounded domains with Dirichlet boundary conditions

Marco Ghimenti¹, Dimitrios Kandilakis², Manolis Magiropoulos³

¹Dipartimento di Matematica,
Università di Pisa,
Largo Buonarroti 1/c,
56127 Pisa, Italy
marco.ghimenti@dma.unipi.it

²School of Architectural Engineering
Technical University of Crete,
73100 Chania, Greece
dimkand@gmail.com

³Technological Educational Institute of Crete
Department of Electrical Engineering
71500 Heraklion, Crete, Greece
mageir@staff.teicrete.gr

Abstract

Existence of solution and L^2 and H^1 localization results on a class of Non Linear Schrödinger type equations with a bounded nonlinearity are obtained, for a bounded domain and with Dirichlet boundary conditions. The kind of stability under discussion shows that the corresponding solution exhibits features of a solitary wave type.

Keywords: Non Linear Schrödinger Equation, stability of solutions, solitary wave.

2010 Mathematics Subject Classification: 35Q55, 37K45.

The first author is partial supported by G.N.A.M.P.A.

1 Introduction

We study the existence, stability and localization of soliton type solutions for the Non Linear Schrödinger Equation (briefly, NLSE) in the semiclassical limit (that is for $h \rightarrow 0^+$), for a bounded domain with Dirichlet boundary conditions.

In our framework, the problem takes the form

$$\begin{aligned} ih \frac{\partial \psi}{\partial t} &= -\frac{h^2}{2} \Delta \psi + \frac{1}{2h^\alpha} W'(|\psi|) \frac{\psi}{|\psi|} + V(x) \psi, \quad \psi \in C^1(\mathbb{R}_0^+, H_0^1(\Omega, \mathbb{C})) \\ \psi(0, x) &= \phi_h(x), \quad x \in \Omega, \end{aligned} \tag{1}$$

$$\psi(t, x) = 0 \text{ on } \mathbb{R}_0^+ \times \partial\Omega,$$

$\Omega \subset \mathbb{R}^N$ being open and bounded, $N \geq 3$, $\alpha > 0$, where $\phi_h(x) \in H_0^1(\Omega)$ is a suitable initial datum, and V is an external potential. Conditions for the nonlinear term W and the potential V are to be precised and discussed in the following sections.

The NLSE in the presence of a potential is largely present in literature. In particular, it has been extensively studied the effect of the potential V on the existence and the profile of a stationary solution, that is a solution of the form $\psi(t, x) = U(x)e^{-\frac{i}{h}\omega t}$, $\omega = \lambda/2$, where U solves the equation

$$-h^2 \Delta U + \frac{1}{h^\alpha} W'(U) + 2V(x)U = \lambda U.$$

The first attempt to this direction is the work of Floer and Weinstein [15], for the one dimensional cubic NLSE (with a generalization for higher dimensions and different nonlinearity in [21]) where, by means of a Lyapunov - Schmidt reduction, it is proved that, if V has a non degenerate minimum, then a stationary solution exists, and this solution has a peak located at this minimum. Del Pino and Felmer [14] showed that any (possibly degenerate) minimum of V generates a stationary solution. We also mention [1, 20], in which similar results are obtained with different techniques. Concerning global methods, in [24] Rabinowitz proved the existence of a stationary solution with a Mountain Pass argument. Later, Cingolani and Lazzo [13] proved that the Lusternik - Schnirelmann category of the minimal level of V gives a lower bound for the number of stationary solutions. The topological approach was also adopted in [2], where a more refined topological invariant is used, and in [9], where the presence of a negative potential allows the existence of a solution in the so called “zero mass” case.

Another interesting feature is the influence of the domain in the stationary NLSE, when $V = 0$. In this case, a single-peaked solution can be constructed. In [22], Ni and Wei showed that the least energy solution for the equation

$$-h^2 \Delta u + u = u^p, \quad u > 0 \text{ in } \Omega, \tag{2}$$

with homogeneous Dirichlet boundary condition, has a unique peak, located at a point P_h with $d(P_h, \partial\Omega) \xrightarrow{h \rightarrow 0^+} \max_{\Omega} d(P, \partial\Omega)$. Later, Wei [26] proved a result

that can be viewed as the converse of the forementioned theorem. Namely, the author showed that for any local maximum P of the distance from the boundary $\partial\Omega$, one can construct a single-peaked solution of (2) whose peak tends to P as $h \rightarrow 0^+$. The profile of the solution is, up to rescaling, close to the profile of the ground state solution of the limit problem

$$-\Delta u + u = u^p, \quad u > 0,$$

in the whole of \mathbb{R}^N . We also mention [10] in which the existence of a multi-peaked solution of (2) is proved.

In the present work we follow a different approach, incorporating and exploiting ideas found in [4, 6, 7], where the problem (1) had been studied for the whole of \mathbb{R}^N for both cases: $V = 0$ (existence and stability), and $V \neq 0$ (existence, stability and dynamics).

We want to point out two main differences between this paper and [4]. First, dealing with a bounded domain and a bounded nonlinearity gives us enough compactness to easily prove an orbital stability result. In particular our result is also true for positive nonlinearities, which is forbidden in the whole space by Pohozaev's Theorem.

This difference is less evident in what comes after, since, when dealing with the semiclassical limit, we have to face a limit problem in the whole space \mathbb{R}^N , which is the same of [4], so we have to reintroduce some hypothesis.

However, our orbital stability result could be used when studying other situations, e.g. fixing h and looking for solitons of prescribed, possibly large, L^2 norm (the so called large solitons). If one is interested in these topics, we recommend the nice paper by Noris, Tavares, Verzini [23], which deals with orbital stability for solitons with prescribed L^2 norms in bounded domains, and the references therein. We point out that in [23], the authors work with pure power nonlinearities, so they can have some compactness loss -for L^2 -critical and supercritical powers- even if the domain is bounded. In particular, for critical and supercritical powers they have orbital stability only for small L^2 norms.

The second point in which the bounded domain marks a difference with the paper [4], is pointed out in the Appendix. In fact, when trying to describe dynamics, immediately it appears a repulsive effect of the boundary. The soliton is affected by a force oriented with the inward normal to the boundary. Unfortunately, the value of the force depends on the $D^{1,2}$ norm of the solution on the boundary, so, at the moment, we were not able to give a quantitative estimate of this repulsive force, and further efforts are needed.

According to this line of thought, we have divided the present work into three sections and an appendix:

In Section 2, existence and orbital stability results are obtained for the case $V = 0$, by referring to the related eigenvalue problem

$$\begin{aligned} -\Delta U + W'(U) &= \lambda U, \text{ in } \Omega \\ U &\equiv 0, \text{ on } \partial\Omega, \end{aligned} \tag{3}$$

given that a solution $U(x)$ in $H_0^1(\Omega)$ of (3) results to a solution $\psi = U(x)e^{-\frac{i\lambda t}{2}}$ of (1), with initial condition $\psi(0, x) = U(x)$. These results are summarized in Proposition 6. One should notice that the relative proofs work without having to impose the usual restriction $2 < p < 2 + \frac{4}{N}$, by relaxing the restriction to $2 < p < 2^* = \frac{2N}{N-2}$ instead. It is the boundedness of the domain that allows us in doing so.

In Section 3, where we assume the presence of an external potential, our basic result, obtained by means of a rescaling procedure, is to prove L^2 localization in the sense that if we start with an initial datum close to a ground state solution U of

$$\begin{aligned} -h\Delta U + \frac{1}{h^{\alpha+1}}W'(U) &= \frac{\omega}{h^{\alpha+1}}U, \text{ in } \Omega \\ U &\equiv 0, \text{ on } \partial\Omega, \end{aligned} \tag{4}$$

the corresponding solution of (1) will keep its L^2 profile along the motion, provided that h is sufficiently small. Here and in what follows, the restriction $2 < p < 2 + \frac{4}{N}$ is imposed, since we need to face a limit problem in \mathbb{R}^N .

In Section 4, an H^1 modular localization result is obtained for the case $V \neq 0$, and for both cases: the unbounded and the bounded one. When we work on the whole of \mathbb{R}^N , we start with a ground state solution U_1 of the \mathbb{R}^N counterpart of (3), proving that a solution of the \mathbb{R}^N counterpart of (1), with initial condition close to U_1 , preserves its basic modular H^1 profile as time passes, in the sense that given $\varepsilon > 0$, for all $t \geq 0$, the ratio of the squared L^2 norm of $|\nabla u_h(t, x)|$ with respect to the complement of a suitable open ball over the squared L^2 norm of $|\nabla u_h(t, x)|$ with respect to the whole of \mathbb{R}^N is less than ε for h sufficiently small, where $u_h(t, x)$ is taken by the polar expression of $\psi(t, x)$, namely $\psi(t, x) = u_h(t, x)e^{is_h(t, x)}$. The bounded case is treated by exploiting ideas developed for the L^2 problem (Section 3).

Finally, and as we mentioned before, in the Appedix it is described an attempt to study dynamics in the frame of a bounded domain, where we encountered difficulties due to computational complications related to the action of ∇V on the motion as well as to the repulsive effect of the boundary.

2 The case $V = 0$

2.1 Existence

For simplicity of the exposition we assume $h = 1$. As it has been already said, the case $V = 0$ is related to problem (3), and a solution $u(x)$ in $H_0^1(\Omega)$ of (3) results to a solution $\psi = u(x)e^{-\frac{i\lambda t}{2}}$ of (1) with initial condition $\psi(0, x) = u(x)$.

Notice that a minimizer of

$$J(u) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx$$

on $S_\sigma = \left\{ u \in H_0^1(\Omega) : \|u\|_{L^2(\Omega)} = \sigma \right\}$, for some fixed $\sigma > 0$, is a solution of (3), for suitable λ . Thus we focus on the existence of such a minimizer. We impose on W the following conditions:

Condition 1 W is a C^1 , bounded and even map $\mathbb{R} \rightarrow \mathbb{R}$.

Condition 2 $|W'(s)| \leq c|s|^{p-1}$, $2 < p < 2^* = \frac{2N}{N-2}$, where c is a suitable positive constant.

Remark 3 Although it is intuitively quite clear the construction of such maps, an easy concrete example is furnished by choosing $W = \sin(s^p)$, for $s \geq 0$, and evenly expanding it on the whole of \mathbb{R} . We also stress the fact that a bounded W is related to the global well posedness results by Cazenave.

Notice that

$$-\infty < \mu = \inf_{u \in S_\sigma} J(u). \quad (5)$$

If $\{u_n\}$ is a minimizing sequence in S_σ for $J(u)$, that is $J(u_n) \rightarrow \mu$, it is evident that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, thus, up to a subsequence, $u_n \rightharpoonup \bar{u} \in H_0^1(\Omega)$, and $u_n \rightarrow \bar{u}$ in $L^2(\Omega)$. The latter implies $\|u_n\|_{L^2(\Omega)} \rightarrow \|\bar{u}\|_{L^2(\Omega)}$, thus $\bar{u} \in S_\sigma$.

Next, we obtain a similar result to Proposition 11 in [4],

Proposition 4 If $\{w_n\}$ is a minimizing sequence in S_σ for J , that is $J(w_n) \rightarrow \mu$, satisfying the constrained P - S condition, that is, there exists a real sequence λ_n of Lagrange multipliers such that

$$-\Delta w_n + W'(w_n) - \lambda_n w_n = \sigma_n \rightarrow 0, \quad (6)$$

then λ_n is bounded.

Proof. Since, as we saw above, w_n is bounded in $H_0^1(\Omega)$, (6) implies

$$\left| \int_{\Omega} \left(|\nabla w_n|^2 + W'(w_n)w_n - \lambda_n w_n^2 \right) dx \right| \leq \|\sigma_n\|_* \|w_n\|_{H_0^1(\Omega)} \rightarrow 0,$$

where by $\|\cdot\|_*$ is denoted the dual norm for $H_0^1(\Omega)$. We have

$$\begin{aligned} \int_{\Omega} \left(|\nabla w_n|^2 + W'(w_n)w_n - \lambda_n w_n^2 \right) dx &= \\ \int_{\Omega} \left(|\nabla w_n|^2 + 2W(w_n) - 2W(w_n) + W'(w_n)w_n - \lambda_n w_n^2 \right) dx &= \\ 2J(w_n) - \lambda_n \sigma^2 + \int_{\Omega} (W'(w_n)w_n - 2W(w_n)) dx &\rightarrow 0. \end{aligned}$$

Notice that $J(w_n)$ is bounded, and because of Condition 2,

$$\begin{aligned} \left| \int_{\Omega} (W'(w_n)w_n - 2W(w_n)) dx \right| &\leq \\ \int_{\Omega} |W'(w_n)w_n| dx + 2 \int_{\Omega} |W(w_n)| dx &\leq \\ c_1 \|w_n\|_{H_0^1(\Omega)}^p + 2kmeas(\Omega) &< +\infty, \end{aligned}$$

where k is an upper bound of $|W|$. Thus λ_n is bounded. ■

By Ekeland's principle, if $\{u_n\}$ is a minimizing sequence in S_σ for $J(u)$, we may assume that it satisfies the constrained P - S condition, that is, there exists a real sequence λ_n so that (6) holds. Because of Proposition 4, λ_n is bounded, and the following hold

$$\begin{aligned} \lambda_n &\rightarrow \lambda \\ u_n &\rightharpoonup u \text{ in } H_0^1(\Omega) \\ u_n &\rightarrow u \text{ in } L^p(\Omega) \text{ for } 1 \leq p < 2^*. \end{aligned}$$

We have already shown that $u \in S_\sigma$. Thus, $u \neq 0$. Next, we show that

$$-\Delta u + W'(u) = \lambda u. \quad (7)$$

To this end, if φ is a test function, combining the three considerations above with Condition 2, we have

$$\begin{aligned} \int_{\Omega} \nabla u_n \nabla \varphi dx &\rightarrow \int_{\Omega} \nabla u \nabla \varphi dx \\ \int_{\Omega} W'(u_n) \varphi dx &\rightarrow \int_{\Omega} W'(u) \varphi dx \\ \lambda_n \int_{\Omega} u_n \varphi dx &\rightarrow \lambda \int_{\Omega} u \varphi dx, \end{aligned}$$

implying (7).

Notice next that due to Condition 2, the Nemytskii operator

$$W : L^t(\Omega) \rightarrow L^1(\Omega), \quad 2 < t < 2^*,$$

is continuous, whereas $u_n \rightarrow u$ in $L^t(\Omega)$, for $2 < t < 2^*$. Thus,

$$\mu \leq J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} W(u) dx \leq \lim_{n \rightarrow \infty} J(u_n) = \mu,$$

proving that $J(u) = \mu$.

This completes the proof for the existence of a non trivial solution of (3), for suitable λ . In fact, the weak convergence $u_n \rightharpoonup u$ turns out to be a strong one: Since $J(u_n) - J(u)$, $\int_{\Omega} (W(u_n) - W(u)) dx \rightarrow 0$, we obtain $\|u_n\|_{H_0^1(\Omega)} \rightarrow$

$\|u\|_{H_0^1(\Omega)}$, thus proving that $u_n \rightarrow u$ in $H_0^1(\Omega)$. Since W has been assumed even, we may take a non trivial nonnegative solution of (3). By Harnack's inequality, this solution is strictly positive on Ω . We thus obtain a positive solution $\bar{u} \in S_\sigma$ for problem (3), for suitable λ . The wave function

$$\psi(t, x) = \bar{u}(x)e^{-i\omega t}, \quad \omega = \lambda/2 \quad (8)$$

is a stationary solution of (1), for $h = 1$, $V \equiv 0$, with initial condition $\phi(x) = \psi(0, x) = \bar{u}(x)$. Evidently, $-\bar{u}(x)e^{-i\omega t}$, $\omega = \lambda/2$, is a stationary solution of (1), too.

2.2 Stability

We turn next our attention to the stability of the stationary solution. To this end, we focus on the reduced form of (1),

$$\begin{aligned} 2i \frac{\partial \psi}{\partial t} &= -\Delta \psi + W'(|\psi|) \frac{\psi}{|\psi|} \text{ in } \mathbb{R}_0^+ \times \Omega, \\ \psi(0, x) &= \phi(x), \\ \psi(t, x) &= 0 \text{ on } \mathbb{R}_0^+ \times \partial\Omega, \end{aligned} \quad (9)$$

by taking, as it was mentioned above, $h = 1$. The different time slices $\psi_t(x)$ of each solution of (9), where such a solution may be understood as the time evolution of some initial condition $\psi_0(x)$, could be thought of as elements of a proper phase space $X \subset L^2(\Omega, \mathbb{C})$, with the set

$$\Gamma = \left\{ u(x)e^{i\theta}, \theta \in \mathbb{R}/2\pi\mathbb{Z}, u \in S_\sigma, J(u) = \mu = \inf_{w \in S_\sigma} J(w) \right\} \quad (10)$$

being an invariant (under evolution) manifold of X . Evidently, $\pm \bar{u}(x) \in \Gamma$.

To make the description of all this more clear, one should notice that if $\psi_{t_0}(x)$ is a time slice of a solution $\psi(t, x)$ of (9), the evolution map is defined by

$$U_t \psi_{t_0}(x) = \psi_{t_0+t}(x),$$

meaning that this time slice might be considered as the initial condition of the solution $\psi_1(t, x) = \psi(t + t_0, x)$. Now, if $u(x)e^{i\theta} \in \Gamma$, then u is a solution of (3), with suitable λ , and, at the same time, $u(x)e^{i\theta}$ is the initial condition of the solution $\psi(t, x) = u(x)e^{i(\theta - \lambda t/2)}$ of (9). Since $u(x)e^{i(\theta - \lambda t/2)} \in \Gamma$ for each $t \geq 0$, the invariance of Γ follows. We are going to prove *orbital* stability of $\bar{u}(x)e^{-i\omega t}$, $\omega = \lambda/2$, following the definition of orbital stability found in [12], meaning that Γ is stable in the following sense:

$\forall \varepsilon > 0, \exists \delta > 0$ such that if $\psi(t, x)$ is a solution of (9) satisfying

$$\inf_{\tilde{w} \in \Gamma} \|\psi(0, x) - \tilde{w}\|_{H_0^1(\Omega)} < \delta, \text{ then } \forall t \geq 0 \inf_{\tilde{w} \in \Gamma} \|\psi(t, x) - \tilde{w}\|_{H_0^1(\Omega)} < \varepsilon. \quad (11)$$

In Lemmas 21 and 22, we will see how the orbital stability result leads to a precise description of the modulus of a solution $\psi(t, x)$ starting from a suitable initial datum. It is the need to stretch the localization results described there that made us adopt the proof through polar decomposition that follows rather than the more straightforward approach in [12] (see also Remark 5).

Notice that Γ is bounded in $H_0^1(\Omega)$, since for each of its elements $\tilde{w} = w(x)e^{i\theta}$, $w(x)$ is a constrained minimizer of J , whereas W is bounded. Notice that we may take $w(x) > 0$.

Suppose Γ is not stable. Then $\exists \varepsilon > 0$, and sequences $\delta_n \rightarrow 0^+$, $\psi_n(t, x)$ of solutions of (9), and $t_n \geq 0$ such that

$$\inf_{\tilde{w} \in \Gamma} \|\psi_n(0, x) - \tilde{w}\|_{H_0^1(\Omega)} < \delta_n, \quad \inf_{\tilde{w} \in \Gamma} \|\psi_n(t_n, x) - \tilde{w}\|_{H_0^1(\Omega)} \geq \varepsilon. \quad (12)$$

Notice that the first inequality of (12) implies

$$\inf_{\tilde{w} \in \Gamma} \|\psi_n(0, x) - \tilde{w}\|_{L^2(\Omega)} < C\delta_n \rightarrow 0,$$

where C is the Sobolev constant satisfying $\|\cdot\|_{L^2(\Omega)} \leq C\|\cdot\|_{H_0^1(\Omega)}$. Thus, we may obtain a sequence \tilde{w}_n in Γ , such that

$$\|\psi_n(0, x) - \tilde{w}_n\|_{L^2(\Omega)} \rightarrow 0. \quad (13)$$

We express now $\psi_n(t, x)$ in polar form, namely $\psi_n(t, x) = u_n(t, x)e^{is_n(t, x)}$, with $u_n(t, x) = |\psi_n(t, x)|$, $\forall t \geq 0$. Since $\|\tilde{w}_n\|_{L^2(\Omega)} = \sigma$, (13) implies that $u_n(0, x)$ is bounded in $L^2(\Omega)$, and at least up to a subsequence, still denoted by $u_n(0, x)$, $\|u_n(0, x)\|_{L^2(\Omega)} \rightarrow M \geq 0$. Rewriting (13) in its squared form, and taking into consideration that

$$\int_{\Omega} u_n(0, x)|w_n(x)|dx \leq \|u_n(0, x)\|_{L^2(\Omega)} \sigma,$$

we take

$$0 \geq M^2 - 2M\sigma + \sigma^2 = (M - \sigma)^2,$$

thus obtaining $M = \sigma$.

The polar form $\psi(t, x) = u(t, x)e^{is(t, x)}$, turns (9) into the system

$$\begin{aligned} -\frac{\Delta u}{2} + \frac{W'(u)}{2} + \left(\frac{\partial s}{\partial t} + \frac{1}{2}|\nabla s|^2\right)u &= 0 \\ \partial_t u^2 + \nabla \cdot (u^2 \nabla s) &= 0, \text{ in } \mathbb{R}_0^+ \times \Omega, \end{aligned} \quad (14)$$

$$u(0, x)e^{is(0, x)} = \phi(x), \quad u(t, x) = 0 \text{ on } \mathbb{R}_0^+ \times \partial\Omega,$$

with the two equations of (14) being the Euler - Lagrange equations of the action functional

$$A(u, s) = \frac{1}{4} \iint |\nabla u|^2 dxdt + \frac{1}{2} \iint W(u) dxdt + \frac{1}{2} \iint \left(\frac{\partial s}{\partial t} + \frac{1}{2}|\nabla s|^2\right) u^2 dxdt. \quad (15)$$

The total energy is given by

$$E(\psi) = E(u, s) = \int_{\Omega} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{2} u^2 |\nabla s|^2 + W(u) \right) dx, \quad (16)$$

that is,

$$E(\psi) = E(u, s) = J(u) + \frac{1}{2} \int_{\Omega} u^2 |\nabla s|^2 dx \quad (17)$$

Independence of time for the energy and for the charge imply that for a solution $\psi(t, x) = u(t, x)e^{is(t, x)}$ of (9), it holds

$$\frac{d}{dt} \int_{\Omega} u(t, x)^2 dx = 0 \quad (18)$$

$$\frac{d}{dt} E(u, s) = 0. \quad (19)$$

Equivalently, (18) and (19) can be expressed as

$$\|\psi(t, x)\|_{L^2(\Omega)} = \|\phi(x)\|_{L^2(\Omega)} \quad (20)$$

$$E(\psi(t, x)) = E(\phi(x)) \quad (21)$$

for all $t \geq 0$. Noteworthy, for stationary solution, (17) yields $E(\psi) = J(u)$.

Returning to the sequence $\psi_n(t, x)$ satisfying (12), we may assume, as we saw, that $\|u_n(0, x)\|_{L^2(\Omega)} \rightarrow \sigma$, that is $\|u_n(t, x)\|_{L^2(\Omega)} \rightarrow \sigma$, for $t \geq 0$, because of (20)). We want to show that $\{u_n\}_n$ is a minimizing sequence for the functional J on the constraint $\|u\|_{L^2(\Omega)} = \sigma$.

One should notice that the first inequality of (12), combined with the boundedness of Γ ensure that $\psi_n(0, x)$ is bounded in $H_0^1(\Omega)$. Since W is bounded, (17) ensures that $E(\psi_n(0, x))$ is bounded, and because of (21), $E(\psi_n(t, x))$ is bounded, for all n and all $t \geq 0$. In particular, $E(\psi_n(t_n, x))$ is bounded. A new application of (17), ensures now that $u_n(t_n, x)$ is bounded in $H_0^1(\Omega)$. The sequence $\hat{u}_n(t_n, x) = \alpha_n u_n(t_n, x)$, where $\alpha_n = \frac{\sigma}{\|u_n(t_n, x)\|_{L^2(\Omega)}}$, is in S_{σ} . We have, writing for simplicity u_n, \hat{u}_n instead of $u_n(t_n, x), \hat{u}_n(t_n, x)$, respectively, for suitable $l_n = l_n(x) \in (0, 1)$, and because of Condition 2,

$$\begin{aligned} |J(\hat{u}_n) - J(u_n)| &\leq \frac{1}{2} |\alpha_n^2 - 1| \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |W(\hat{u}_n) - W(u_n)| dx \\ &= \frac{1}{2} |\alpha_n^2 - 1| \int_{\Omega} |\nabla u_n|^2 dx \\ &\quad + |\alpha_n - 1| \int_{\Omega} |u_n W'(l_n u_n + (1 - l_n) \hat{u}_n)| dx \\ &\leq \frac{1}{2} |\alpha_n^2 - 1| \int_{\Omega} |\nabla u_n|^2 dx \\ &\quad + |\alpha_n - 1| \left\{ \int_{\Omega} c [l_n + (1 - l_n) \alpha_n]^{p-1} |u_n|^p dx \right\} \\ &\rightarrow 0, \end{aligned}$$

since in the right hand side of the last inequality, the two summands are products of a zero sequence by a bounded one. Thus, $J(\widehat{u}_n) - J(u_n) \rightarrow 0$. We return now to

$$\|\psi_n(0, x) - \widetilde{w}_n\|_{H_0^1(\Omega)} \rightarrow 0, \quad (22)$$

which, as a result of the triangle inequality combined with the boundedness of $\|\psi_n(0, x)\|_{H_0^1(\Omega)} + \|w_n(x)\|_{H_0^1(\Omega)}$, readily gives

$$\|\psi_n(0, x)\|_{H_0^1(\Omega)}^2 - \|w_n(x)\|_{H_0^1(\Omega)}^2 \rightarrow 0,$$

that is,

$$\int_{\Omega} \left[|\nabla u_n(0, x)|^2 + u_n^2(0, x) |\nabla s_n(0, x)|^2 - |\nabla w_n(x)|^2 \right] dx \rightarrow 0. \quad (23)$$

We claim that

$$\int_{\Omega} u_n^2(0, x) |\nabla s_n(0, x)|^2 dx \rightarrow 0. \quad (24)$$

If not so, up to a subsequence,

$$\int_{\Omega} \left[|\nabla u_n(0, x)|^2 - |\nabla w_n(x)|^2 \right] dx \rightarrow k < 0. \quad (25)$$

Combining L^1 convergence of $u_n(0, x) - w_n(x)$ to 0, with Condition 2, we have

$$\int_{\Omega} [W(u_n(0, x)) - W(w_n(x))] dx \rightarrow 0. \quad (26)$$

Now (25) and (26) give

$$J(u_n(0, x)) - J(w_n(x)) \rightarrow k/2 < 0. \quad (27)$$

However, as we have shown above,

$$J(\widehat{u}_n(0, x)) - J(u_n(0, x)) \rightarrow 0, \quad (28)$$

thus obtaining

$$J(\widehat{u}_n(0, x)) - J(w_n(x)) \rightarrow k/2 < 0, \quad (29)$$

an absurdity, since $\widehat{u}_n(0, x) \in S_{\sigma}$, and $w_n(x)$ is a S_{σ} minimizer of J . Thus (24) holds, and because of (23) and (26), we get

$$J(u_n(0, x)) \rightarrow \mu.$$

Thus,

$$E(\psi_n(0, x)) - E(\widetilde{w}_n) \rightarrow 0,$$

and by (21) we have

$$E(\psi_n(t_n, x)) - E(\widetilde{w}_n) \rightarrow 0. \quad (30)$$

From (30) we obtain

$$\int_{\Omega} u_n^2(t_n, x) |\nabla s_n(t_n, x)|^2 dx \rightarrow 0. \quad (31)$$

To see this, let us assume that this is not the case. Then, up to a subsequence,

$$\int_{\Omega} u_n^2(t_n, x) |\nabla s_n(t_n, x)|^2 dx \rightarrow \rho > 0.$$

Then (30) implies that

$$J(u_n(t_n, x)) - J(w_n(x)) \rightarrow -\rho < 0.$$

Since

$$J(\hat{u}_n(t_n, x)) - J(u_n(t_n, x)) \rightarrow 0,$$

we obtain

$$J(\hat{u}_n(t_n, x)) - J(w_n(x)) \rightarrow -\rho < 0,$$

which is absurd, for the same reason as above. That is, (31) holds, resulting to

$$J(u_n) = J(u_n(t_n, x)) \rightarrow \mu,$$

thus implying

$$J(\hat{u}_n(t_n, x)) \rightarrow \mu.$$

In other words, we may consider $u_n = u_n(t_n, x)$ as being a minimizing sequence in S_{σ} for $J(u)$. As such, by the previous discussion, $u_n(t_n, x) \rightarrow u'(x)$, with $u'(x) \in S_{\sigma}$, being a minimizer of $J(u)$. Now (31) ensures that $\|\psi_n(t_n, x)\|_{H_0^1(\Omega)} \rightarrow \|u'(x)\|_{H_0^1(\Omega)}$, and by lower semicontinuity of the norm we finally obtain that $\psi_n(t_n, x) \rightarrow u'(x)$, thus proving *orbital* stability of the stationary solution.

Remark 5 *As we claimed before, once $\psi(0, x)$ is sufficiently close to Γ , $\psi(t, x)$, $t \geq 0$, are all close to Γ . This implies that the modulus $|\psi(t, x)|$ is, for any $t \geq 0$, close to a (possibly different) ground state of the NLSE. This consequence is the key tool for proving two main results of the next section, Lemmas 21 and 22. Thus, the orbital stability in this context becomes a localization result for the moduli $|\psi(t, x)|$.*

We summarize the existence and stability results in the following:

Proposition 6 *The problem*

$$\begin{aligned} ih \frac{\partial \psi}{\partial t} &= -\frac{h^2}{2} \Delta \psi + \frac{1}{2h^\alpha} W'(|\psi|) \frac{\psi}{|\psi|} \text{ in } \mathbb{R}_0^+ \times \Omega, \\ \psi(0, x) &= \phi_h(x), \quad x \in \Omega, \\ \psi(t, x) &= 0 \text{ on } \mathbb{R}_0^+ \times \partial\Omega, \end{aligned} \quad (32)$$

where $h > 0$, and $\phi_h(x)$ being a suitable initial datum, admits a stationary solution $\psi(t, x)$ of the form $u_h(x)e^{-ikt}$. More concretely, $u_h(x)$ is obtained as a solution of the eigenvalue problem

$$\begin{aligned} -\frac{h}{2}\Delta u + \frac{1}{2h^{\alpha+1}}W'(u) &= ku, \text{ in } \Omega \\ u &\equiv 0, \text{ on } \partial\Omega, \end{aligned} \tag{33}$$

for suitable k . In addition, $\psi(t, x)$ is stable in the sense of (11).

We give next the definition of a *solitary wave* with respect to a bounded domain Ω . To this end, we have to define first the notion of the *barycenter* of a family of states $\psi_t(x)$, $t \geq 0$, whose members are obtained by the "time" evolution of an initial state $\psi_0(x)$, in the frame of a proper phase space $X \subset L^2(\Omega, \mathbb{C})$.

Definition 7 For $\psi_t(x)$, $t \geq 0$, as above, its barycenter, $q(t)$, is defined by the relation

$$q(t) = \frac{\int_{\Omega} x |\psi_t(x)|^2 dx}{\int_{\Omega} |\psi_t(x)|^2 dx}. \tag{34}$$

Remark 8 An analogous definition of the barycenter is given in [3], under the condition that it makes sense. In our case, the definition of $q(t)$ makes always sense, because Ω is bounded.

Remark 9 Notice that $q(t)$ does not belong to Ω necessarily, unless Ω has specific geometric features. For instance, convexity of Ω would ensure that $q(t) \in \Omega$ for all $t \geq 0$.

Definition 10 The state $\psi \equiv \psi_0(x)$ in the phase space $X \subset L^2(\Omega, \mathbb{C})$, is called a "solitary wave" in the frame of a dynamical system $U_t\psi \equiv \psi_t(x)$, $t \geq 0$, where $U : \mathbb{R}_0^+ \times X \rightarrow X$ is the evolution map, if: Given $\varepsilon > 0$, we may find $k(\varepsilon) > 0$ such that for each $t \geq 0$, there exists a neighborhood $V_{\varepsilon, t}$ of $q(t)$ with $\text{meas}[\Omega - (V_{\varepsilon, t} \cap \Omega)] \geq k(\varepsilon)$, and

$$\int_{\Omega} |\psi_t(x)|^2 dx - \int_{V_{\varepsilon, t} \cap \Omega} |\psi_t(x)|^2 dx < \varepsilon. \tag{35}$$

Remark 11 It is easy to see that the stationary solution of (32) is a solitary wave in the above sense: The barycenter in this case is fixed for all t , and one needs to suitably blow up a given neighborhood of it, in order to meet the requirements of the above definition.

3 L^2 localization for $V \neq 0$

For the rest of the exposition, we assume without loss of generality that $0 \in \Omega$. We also restrict p in Condition 2 so that $2 < p < 2 + \frac{4}{N}$, and we impose on W the additional condition:

Condition 12 $\exists s_0$ such that $W(s_0) < 0$,

and on V the following one:

Condition 13 $V(x) \in C^0(\overline{\Omega})$, is nonnegative.

Remark 14 *With these restrictions and additional conditions on the nonlinearity and the potential, problem (1) is globally well posed (see [11, Thm 3.3.1, and Thm 3.4.1]), with the energy and the mass remaining constant in time. Commenting especially on Condition 12, we notice that in the opposite case, there would exist no nontrivial stationary solution for the related \mathbb{R}^N problem, undermining thus the basic tool we will use in order to prove the main stability result. In the previous section, where we had posed weaker conditions on Ω , W , and p , we were led to nontrivial stationary solution without any particular comment on λ . It is trivial to see that for W having positive lower bound, λ is positive, too. Under the additional conditions imposed above on Ω , W , and p , we can prove, as we will see soon, that we may obtain a nontrivial solution of (3) sitting in S_σ for suitable σ , with $\lambda = \lambda(\sigma) < 0$, provided that Ω contains a suitably big open ball $B(0, r(\sigma))$ centered at 0 with radius $r(\sigma)$.*

3.1 Rescalings

We set $\beta = 1 + \frac{\alpha}{2}$. For $h < 1$, we define the inflated domain

$$\Omega_h = \{x \in \mathbb{R}^N : h^\beta x \in \Omega\}.$$

If v is a $H_0^1(\Omega_h)$ solution of the stationary problem

$$\begin{aligned} -\Delta u + W'(u) &= \omega u, \text{ in } \Omega_h \\ u &\equiv 0, \text{ on } \partial\Omega_h, \end{aligned} \tag{36}$$

then $v_h(x) = v(\frac{x}{h^\beta})$ is a $H_0^1(\Omega)$ solution of the stationary problem

$$\begin{aligned} -h\Delta u + \frac{1}{h^{\alpha+1}}W'(u) &= \frac{\omega}{h^{\alpha+1}}u, \text{ in } \Omega \\ u &\equiv 0, \text{ on } \partial\Omega. \end{aligned} \tag{37}$$

Furthermore, we define the functionals

$$\begin{aligned} C_h(u) &= \frac{1}{h^{N\beta}} \int_{\Omega} u^2(x) dx, \quad C_{\Omega_h}(u) = \int_{\Omega_h} u^2(x) dx, \\ J_h(u) &= \frac{1}{h^{N\beta}} \int_{\Omega} \left[\frac{h^2}{2} |\nabla u|^2 + W_h(u) \right] dx, \quad J_{\Omega_h}(u) = \int_{\Omega_h} \left[\frac{1}{2} |\nabla u|^2 + W(u) \right] dx, \end{aligned}$$

where $W_h(u) = \frac{1}{h^\alpha} W(u)$.

We have the following identities:

$$J_h(v_h) = h^{-\alpha} J_{\Omega_h}(v), \quad C_h(v_h) = C_{\Omega_h}(v). \quad (38)$$

We next define

$$m(h, \Omega) := \inf_{C_h=1} J_h.$$

Lemma 15 *For Ω_1, Ω_2 two bounded domains as described above, with $\Omega_1 \subset \Omega_2$, it holds $m(h, \Omega_2) \leq m(h, \Omega_1)$.*

Proof. It is straightforward, since $H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$ by extending a function in $H_0^1(\Omega_1)$ into a function in $H_0^1(\Omega_2)$ by zero on $\Omega_2 \setminus \Omega_1$. ■

Lemma 16 $m(h, \Omega) = h^{-\alpha} m(1, \Omega_h)$, where $m(1, \Omega_h) = \inf_{C_{\Omega_h}=1} J_{\Omega_h}$.

Proof. By rescaling. ■

Notice that the conditions satisfied by p, W correspond to the prerequisites for the existence result given in [4] to hold. Namely:

Lemma 17 *There exists some $\bar{\sigma} > 0$ such that for all $\sigma > \bar{\sigma}$ a positive minimizer $u_\sigma \in H^1(\mathbb{R}^N)$ exists for $J(u)$ over all $u \in H^1(\mathbb{R}^N)$, $\|u\|_{L^2(\mathbb{R}^N)} = \sigma$. In fact, $\|u_\sigma\|_{L^2(\mathbb{R}^N)} = \sigma$, $J(u_\sigma) < 0$, and u_σ is a solution of the \mathbb{R}^N version of (3), with $\lambda < 0$.*

We consider an increasing sequence $\{r_n\}$, with $r_n \rightarrow \infty$. Let

$$m_\sigma(B(0, r_n)) := \inf_{\|w\|_{L^2(B(0, r_n))} = \sigma} J(w).$$

Trivially, as Lemma 15 indicates, we see that

$$m_\sigma(B(0, r_1)) \geq m_\sigma(B(0, r_2)) \geq \dots \geq m_\sigma(\mathbb{R}^N) = \inf_{\|w\|_{L^2(\mathbb{R}^N)} = \sigma} J(w) > -\infty. \quad (39)$$

Lemma 18 $\lim_{n \rightarrow \infty} m_\sigma(B(0, r_n)) = m_\sigma(\mathbb{R}^N)$.

Proof. By Lemma 17, $m_\sigma(\mathbb{R}^N)$ is attained by some $\bar{u} \in H^1(\mathbb{R}^N)$, with $\|\bar{u}\|_{L^2(\mathbb{R}^N)} = \sigma$. Actually \bar{u} is radial ([12], Thm. II.1 and Rem. II.3).

For each n , we may choose a $C^\infty(\mathbb{R}^N)$ real function χ_n satisfying

$$\begin{aligned} \chi_n &\equiv 1 \text{ if } |x| \leq r_n/2, \\ \chi_n &\equiv 0 \text{ if } |x| \geq r_n, \\ |\nabla \chi_n| &\leq 4/r_n. \end{aligned} \quad (40)$$

We define $w_n = \chi_n \bar{u}$. For suitable $t_n > 0$, we have $\|t_n w_n\|_{L^2(\mathbb{R}^N)} =$

$\|t_n w_n\|_{L^2(B(0, r_n))} = \sigma$. Setting $u_n = t_n w_n$, (39) yields

$$J(u_n) \geq m_\sigma(B(0, r_n)) \geq m_\sigma(\mathbb{R}^N). \quad (41)$$

We want to prove $\lim_{n \rightarrow \infty} J(u_n) = m_\sigma(\mathbb{R}^N)$, that will finish the proof. We have $w_n \rightarrow \bar{u}$ in $L^2(\mathbb{R}^N)$. Thus $t_n \rightarrow 1$, and $u_n \rightarrow \bar{u}$ in $L^2(\mathbb{R}^N)$. We also have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w_n - \nabla \bar{u}|^2 dx &= \int_{\mathbb{R}^N} |(\nabla \chi_n) \bar{u} + (\chi_n - 1) \nabla \bar{u}|^2 dx \\ &\leq 2 \int_{\mathbb{R}^N} |(\nabla \chi_n) \bar{u}|^2 dx + 2 \int_{\mathbb{R}^N} |(\chi_n - 1) \nabla \bar{u}|^2 dx \\ &\leq 32/r_n^2 \int_{|x| > r_n/2} |\bar{u}|^2 dx + 2 \int_{|x| > r_n/2} |\nabla \bar{u}|^2 dx \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. So $\nabla w_n \rightarrow \nabla \bar{u}$ in L^2 , and $\nabla u_n \rightarrow \nabla \bar{u}$ in L^2 too. Thus $u_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^N)$. This is combined with the continuity of the Nemytskii operator

$$W : L^t(\mathbb{R}^N) \rightarrow L^1(\mathbb{R}^N), \quad 2 < t < 2^*,$$

to ensure that

$$\int_{\mathbb{R}^N} W(u_n) dx \rightarrow \int_{\mathbb{R}^N} W(\bar{u}) dx.$$

Thus $\lim_{n \rightarrow \infty} J(u_n) = m_\sigma(\mathbb{R}^N)$, and the proof has been completed. ■

Remark 19 *The above Lemma makes clear the final assertion of Rem. 14: If Ω contains a suitably big open ball $B(0, r(\sigma))$ with $\bar{\sigma} < \sigma$, then $m_\sigma(1, \Omega)$ has to be negative, and so has to be the eigenvalue $\lambda = \lambda(\sigma)$ related to (3).*

Lemma 20 *For a sequence of positive numbers $h_k \rightarrow 0$, for $k \rightarrow \infty$, it holds*

$$\lim_{k \rightarrow \infty} m_\sigma(1, \Omega_{h_k}) = m_\sigma(\mathbb{R}^N),$$

where $m_\sigma(1, \Omega_{h_k}) = \inf_{C_{\Omega_{h_k}} = \sigma} J_{\Omega_{h_k}}$.

Proof. Combine Lemmas 15 and 18. ■

3.2 L^2 localization

To facilitate exposition, we make the harmless assumption that $\sigma = 1$, thus suppressing subindices in all involved infima m . We have the following:

Lemma 21 *For any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for any $u \in H_0^1(\Omega)$ with $C_h(u) = 1$, and $J_h(u) < m(h, \Omega) + \delta h^{-\alpha}$,*

$$\frac{1}{h^{N\beta}} \int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} u^2 dx < \varepsilon$$

to hold.

Proof. We argue by contradiction. Assuming the contrary, there exists $\varepsilon > 0$ such that for any $r > 0$ we may find sequences $\delta_n = \delta_n(r)$, $h_n = h_n(r) \rightarrow 0^+$, and a sequence $u_{h_n} = u_{h_n}(r) \in H_0^1(\Omega)$ with $C_{h_n}(u_{h_n}) = 1$, $J_{h_n}(u_{h_n}) < m(h_n, \Omega) + \delta_n h_n^{-\alpha}$ such that, for all open balls $B(q_r, h_n^\beta r) \subset \Omega$,

$$\int_{\Omega \setminus B(q_r, h_n^\beta r)} u_{h_n}^2 dx \geq \varepsilon h_n^{N\beta}$$

to hold. For each n , we now pass to the Ω_{h_n} counterpart of u_{h_n} , denoted by u_n , that is, $u_n(x) = u_{h_n}(h_n^\beta x)$. Combining (38) and Lemma 16, we have $C_{\Omega_{h_n}}(u_n) = 1$, $J_{\Omega_{h_n}}(u_n) < m(1, \Omega_{h_n}) + \delta_n$, and

$$\int_{\Omega_{h_n} \setminus B(h_n^{-\beta} q_r, r)} u_n^2 dx \geq \varepsilon, \quad (42)$$

According to Lemma 15 in [7], there is a $\delta > 0$, and an open ball $B(\hat{q}, R')$ in \mathbb{R}^N such that, for each w in $H^1(\mathbb{R}^N)$, with $\|w\|_{L^2(\mathbb{R}^N)} = 1$, $J(w) < m(\mathbb{R}^N) + \delta$,

$$\int_{\mathbb{R}^N \setminus B(\hat{q}, R')} w^2 dx < \varepsilon$$

to hold. Take now $\delta_n(R')$, $h_n(R')$ as above. Because of Lemma 20, for n big enough, we may ensure that $B(\hat{q}, R') \subset \Omega_{h_n(R')}$, and $m(1, \Omega_{h_n}) + \delta_n(R') < m(\mathbb{R}^N) + \delta$. Extending u_n by 0 outside $\Omega_{h_n(R')}$, we obtain a $H^1(\mathbb{R}^N)$ function meeting the requirements of Lemma 15 in [7], thus

$$\int_{\Omega_{h_n(R')} \setminus B(\hat{q}, R')} u_n^2 dx = \int_{\mathbb{R}^N \setminus B(\hat{q}, R')} u_n^2 dx < \varepsilon,$$

that contradicts (42). ■

Lemma 21 makes obvious the following:

Lemma 22 *For any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for a solution $\psi(t, x)$ of (1) with $C_h(|\psi(t, x)|) = 1$, and $J_h(|\psi(t, x)|) < m(h, \Omega) + \delta h^{-\alpha}$, for each $t \in \mathbb{R}_0^+$,*

$$\frac{1}{h^{N\beta}} \int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} |\psi(t, x)|^2 dx < \varepsilon$$

to hold.

The correlation of the solutions of the equations (36) and (37), combined with (38), ensure the existence of "ground state" solutions of (37), that is, solutions that are minimizers of $J_h(u)$, with u satisfying $C_h(u) = 1$.

We define next, the following set of admissible initial data, for given K , $h > 0$:

$$B_h^K = \left\{ \begin{array}{l} \psi(0, x) = u_h(0, x) e^{\frac{i}{h} s_h(0, x)} \\ \text{with } u_h(0, x) = (U + w)(x) \\ U \text{ is a ground state solution of (37), and } w \in H_0^1(\Omega) \text{ s.t.} \\ C_h(U + w) = 1, \text{ and } \|w\|_{H_0^1(\Omega)} < Kh^\alpha \\ \|\nabla s_h(0, x)\|_{L^\infty} \leq Kh^{N\beta/2} \\ \int_\Omega V(x) u_h^2(0, x) dx \leq Kh^{N\beta} \end{array} \right\} \quad (43)$$

We prove next the basic stability result.

Proposition 23 *Given $\varepsilon > 0$, there exists $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for a solution $\psi(t, x)$ of (1) with $C_h(|\psi(t, x)|) = 1$, and with initial data $\psi(0, x) \in B_h^K$, where K is a positive fixed number, it holds*

$$\frac{1}{h^{N\beta}} \int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} |\psi(t, x)|^2 dx < \varepsilon,$$

for any $t \in \mathbb{R}_0^+$.

Proof. Because of conservation of energy, we have

$$\begin{aligned} E(\psi(t, x)) &= E(\psi(0, x)) \\ &= h^{N\beta} J_h(u_h(0, x)) + \int_\Omega u_h^2(0, x) \left[\frac{|\nabla s_h(0, x)|^2}{2} + V(x) \right] dx \\ &\leq h^{N\beta} J_h(u_h(0, x)) + \frac{K^2}{2} h^{N\beta} + Kh^{N\beta} \\ &= h^{N\beta} J_h(u_h(0, x)) + Ch^{N\beta} \\ &= h^{N\beta} J_h(U + w) + Ch^{N\beta} \\ &\leq h^{N\beta} (m(h, \Omega) + C'Kh^\alpha + C) \\ &= h^{N\beta} [m(h, \Omega) + h^{-\alpha} (h^{2\alpha} C'K + Ch^\alpha)], \end{aligned}$$

since a Mean Value Theorem application ensures that $J_h(U + w) \leq m(h, \Omega) + C'Kh^\alpha$, for suitably small h . More precisely, since J_h is C^1 , we may find some $\eta \in (0, 1)$ so that

$$\begin{aligned} J_h(U + w) - J_h(U) &= J'_h(U + \eta w)[w] \\ &= \int_\Omega \nabla(U + \eta w) \nabla w dx \\ &\quad + \int_\Omega W'(U + \eta w) w dx. \end{aligned}$$

We have

$$\left| \int_\Omega \nabla(U + \eta w) \nabla w dx \right| \leq \|\nabla U\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} + \|\nabla w\|_{L^2(\Omega)}^2 \leq C_1 \|w\|_{H_0^1(\Omega)}, \quad (44)$$

since $\|w\|_{H_0^1(\Omega)} \leq 1$. Because of Condition 2,

$$\left| \int_{\Omega} W'(U + \eta w) w dx \right| \leq C_2 \|w\|_{L^p(\Omega)} \leq C_3 \|w\|_{H_0^1(\Omega)}. \quad (45)$$

Combining (44) and (45), we obtain the desired inequality. Thus

$$\begin{aligned} J_h(u_h(t, x)) &= h^{-N\beta} \left\{ E(\psi(t, x)) - \int_{\Omega} u_h^2(t, x) \left[\frac{|\nabla s_h(t, x)|^2}{2} + V(x) \right] dx \right\} \\ &\leq [m(h, \Omega) + h^{-\alpha} (h^{2\alpha} C' K + Ch^{\alpha})], \end{aligned}$$

since $V(x) \geq 0$. We use now Lemma 22, by choosing h_0 small enough in order to ensure $h_0^{2\alpha} C' K + Ch_0^{\alpha} \leq \delta(\varepsilon)$, and the result follows. ■

Corollary 24 *Given $\varepsilon > 0$, if $q(t)$ is the barycenter of a solution $\psi(t, x)$ of (1) with $C_h(|\psi(t, x)|) = 1$, and with initial data $\psi(0, x) \in B_h^K$, where K is a positive fixed number, then for any $t \in \mathbb{R}_0^+$, $q(t) \in B(\widehat{q}_h, d\varepsilon + h^{\beta}R)$, where h and R are as in Proposition 23, and $d = \text{diam}(\Omega)$.*

Proof. One has for any $t \in \mathbb{R}_0^+$,

$$\begin{aligned} |\widehat{q}_h - q(t)| &= \left| \frac{\int_{\Omega} (\widehat{q}_h - x) |\psi(t, x)|^2 dx}{\int_{\Omega} |\psi(t, x)|^2 dx} \right| = h^{-N\beta} \left| \int_{\Omega} (\widehat{q}_h - x) |\psi(t, x)|^2 dx \right| \\ &\leq h^{-N\beta} \int_{\Omega} |\widehat{q}_h - x| |\psi(t, x)|^2 dx \\ &= h^{-N\beta} \left[\int_{\Omega \setminus B(\widehat{q}_h, h^{\beta}R)} |\widehat{q}_h - x| |\psi(t, x)|^2 dx \right. \\ &\quad \left. + \int_{B(\widehat{q}_h, h^{\beta}R)} |\widehat{q}_h - x| |\psi(t, x)|^2 dx \right] \\ &< h^{-N\beta} [dh^{N\beta}\varepsilon + h^{\beta}R h^{N\beta}] = d\varepsilon + h^{\beta}R, \end{aligned}$$

thus proving the claim.

Remark 25 *The above Corollary may be interpreted as follows: If we choose a pretty small $\varepsilon > 0$, and we make a consequent choice of small h , then $B(\widehat{q}_h, h^{\beta}R) \subset A_h = B(\widehat{q}_h, d\varepsilon + h^{\beta}R) \cap \Omega$, with $\text{meas}(\Omega - A_h) > 0$. In other words, we have found a neighborhood of $q(t)$, **for any** $t \in \mathbb{R}_0^+$, as in Def. 10, in the sense that a solution $\psi(t, x)$ of (1) with a perturbed initial state, meeting the requirements of Proposition 23, concentrates basically on this neighborhood of $q(t)$, exhibiting a behavior quite similar to a solitary wave.*

■

4 H^1 localization for $V \neq 0$

4.1 The case $\Omega = \mathbb{R}^N$

We start with the following assumption: The problem

$$ih \frac{\partial \psi}{\partial t} = -\frac{h^2}{2} \Delta \psi + \frac{1}{2h^{\alpha}} W'(|\psi|) \frac{\psi}{|\psi|} + V(x) \psi,$$

$$\psi(0, x) = \phi_h(x), \quad (46)$$

$$|\psi(t, x)| \rightarrow 0, \text{ as } |x| \rightarrow \infty,$$

considered as the \mathbb{R}^N analogue of (1) admits a unique solution

$$\psi \in C^0(\mathbb{R}, H^2(\mathbb{R}^N)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^N)) \quad (47)$$

(see [19], [11] or [18] for sufficient conditions.) We also impose on W, V further conditions, namely:

Condition 26 W is C^3 , with $|W''| \leq c|s|^{p-2}$ for some $c \geq 0$, $2 < p < 2 + \frac{4}{N}$

Condition 27 $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is a C^0 nonnegative function

In order to proceed, we will need the following lemma:

Lemma 28 For every $\varepsilon > 0$, there exists $R = R(\varepsilon) > 0$ such that for every ground state U , there exists $q(U) \in \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N \setminus B(q(U), R)} (|\nabla U|^2 + U^2) dx < \varepsilon$$

Proof. If we assume the contrary, then we may find $\varepsilon > 0$ such that we may have a sequence of pairs $(R_n > 0, U_n \text{ ground state})$ so that for each $q \in \mathbb{R}^N$

$$\int_{\mathbb{R}^N \setminus B(q(U), R_n)} (|\nabla U_n|^2 + U_n^2) dx \geq \varepsilon,$$

thus obtaining

$$\inf_{q \in \mathbb{R}^N} \int_{\mathbb{R}^N \setminus B(q(U), R_n)} (|\nabla U_n|^2 + U_n^2) dx \geq \varepsilon. \quad (48)$$

Then $\{U_n\}_n$ is a minimizing sequence, and by concentration compactness we know that $\{U_n\}_n$ is relatively compact up to a translation by $\{q_n\}_n \in \mathbb{R}^N$. Thus there exists a ground state U with $U_n(\cdot - q_n) \rightarrow U$ in $H^1(\mathbb{R}^N)$, and

$$\begin{aligned} & \inf_{q \in \mathbb{R}^N} \int_{\mathbb{R}^N \setminus B(q(U), R_n)} (|\nabla U_n|^2 + U_n^2) dx \\ & \leq \int_{\mathbb{R}^N \setminus B(-q_n, R_n)} (|\nabla U_n|^2 + U_n^2) dx \\ & = \int_{\mathbb{R}^N \setminus B(0, R_n)} (|\nabla U_n|^2 + U_n^2)(x - q_n) dx \\ & = \int_{\mathbb{R}^N \setminus B(0, R_n)} (|\nabla U|^2 + U^2) dx + o_n(1) = o_n(1), \end{aligned}$$

contradicting (48). ■

Lemma 29 For every $\varepsilon > 0$, there exist $\widehat{R} = \widehat{R}(\varepsilon) > 0$, $\delta = \delta(\varepsilon) > 0$ such that, for any $u \in J^{m+\delta} \cap S_\sigma$, we can find a point $\widehat{q} = \widehat{q}(u) \in \mathbb{R}^N$ such that

$$\frac{\int_{\mathbb{R}^N \setminus B(\widehat{q}, \widehat{R})} |\nabla u|^2(x) dx}{\int_{\mathbb{R}^N} |\nabla u|^2(x) dx} < \varepsilon, \quad (49)$$

where $m = m(\mathbb{R}^N)$ (see Lemma 21), $J^{m+\delta} = \{u \in H^1(\mathbb{R}^N) / J(u) < m + \delta\}$.

Proof. Exploiting Rel. (50) in [7], we obtain a point $\widehat{q} = \widehat{q}(u) \in \mathbb{R}^N$ and a radial ground state solution U such that

$$u(x) = U(x - \widehat{q}) + w, \text{ with } \|w\|_{H^1(\mathbb{R}^N)} \leq C\varepsilon, \quad (50)$$

where C is a constant not depending on U . According to the previous Lemma, we may find $R > 0$ and a point $q = q(U)$ such that

$$\int_{\mathbb{R}^N \setminus B(q, R)} (|\nabla U|^2 + U^2) dx < \frac{\sigma^2 C \varepsilon}{c_1^2}, \quad (51)$$

where c_1 is the Sobolev constant related to the embedding $H^1(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$. If we choose \widehat{R} big enough, then $B(q, R) \subset B(0, \widehat{R})$, resulting to

$$\int_{\mathbb{R}^N \setminus B(0, \widehat{R})} (|\nabla U|^2 + U^2) dx < \frac{\sigma^2 C \varepsilon}{c_1^2}. \quad (52)$$

We have

$$\frac{\int_{\mathbb{R}^N \setminus B(0, \widehat{R})} |\nabla U|^2(x) dx}{\int_{\mathbb{R}^N} |\nabla U|^2(x) dx} < \frac{c_1^2 \int_{\mathbb{R}^N \setminus B(0, \widehat{R})} (|\nabla U|^2 + U^2) dx}{\int_{\mathbb{R}^N} U^2 dx} < C\varepsilon. \quad (53)$$

Now

$$\begin{aligned} \int_{\mathbb{R}^N \setminus B(\widehat{q}, \widehat{R})} |\nabla u|^2(x) dx &< \int_{\mathbb{R}^N \setminus B(\widehat{q}, \widehat{R})} |\nabla U|^2(x - \widehat{q}) dx \\ &+ \int_{\mathbb{R}^N \setminus B(\widehat{q}, \widehat{R})} (|\nabla w|^2 + 2wU) dx \\ &= \int_{\mathbb{R}^N \setminus B(0, \widehat{R})} |\nabla U|^2(x) dx \\ &+ \int_{\mathbb{R}^N \setminus B(\widehat{q}, \widehat{R})} (|\nabla w|^2 + 2\nabla w \nabla U) dx. \end{aligned} \quad (54)$$

By (50), (53) and (54), we get the claim. One should notice that \widehat{R} does not depend on u , U . ■

Our main objective in this subsection is to prove an H^1 modular stability result of the solution of (46) with suitable initial data; more precisely, we prove that, for fixed $t \in \mathbb{R}_0^+$, this solution is a function on \mathbb{R}^N with one peak localized

in a ball with its center depending on t whereas its radius not. To this end, it is sufficient to assume that (46) admits global solutions $\psi(t, x) \in C(\mathbb{R}, H^1(\mathbb{R}^N))$ satisfying the conservation of the energy and of the L^2 norm.

Given $K > 0$, $h > 0$, we define the following set of admissible data:

$$B_{K,h} = \left\{ \begin{array}{l} \psi(0, x) = u_h(0, x) e^{\frac{i}{h} s_h(0, x)} \\ \text{with } u_h(0, x) = (U + w) \left(\frac{x-q}{h^\beta} \right) \\ q \in \mathbb{R}^N, U \text{ is a ground state solution, and } w \in H^1(\mathbb{R}^N) \text{ s.t.} \\ \|U + w\|_{L^2} = \|U\|_{L^2} = \sigma, \text{ and } \|w\|_{H^1} < Kh^\alpha \\ \|\nabla s_h(0, x)\|_{L^\infty} \leq K \text{ for all } h \\ \int_{\mathbb{R}^N} V(x) u_h^2(0, x) dx \leq Kh^{N\beta-2\alpha} \end{array} \right\}. \quad (55)$$

We next study the rescaling properties of the internal energy

$$\tilde{J}_h(u) = \int_{\mathbb{R}^N} \left(\frac{h^2}{2} |\nabla u|^2 + \frac{1}{h^\alpha} W(u) \right) dx,$$

and of the L^2 norm of a function $u(x)$ having the form

$$u(x) = v \left(\frac{x}{h^\beta} \right),$$

with $\beta = 1 + \frac{\alpha}{2}$. We have

$$\|u\|_{L^2}^2 = \int_{\mathbb{R}^N} v \left(\frac{x}{h^\beta} \right)^2 dx = h^{N\beta} \int_{\mathbb{R}^N} v(\xi)^2 d\xi = h^{N\beta} \|v\|_{L^2}^2,$$

and

$$\begin{aligned} \tilde{J}_h(u) &= \int_{\mathbb{R}^N} \left[\frac{h^2}{2} \left| \nabla_x v \left(\frac{x}{h^\beta} \right) \right|^2 + \frac{1}{h^\alpha} W \left(v \left(\frac{x}{h^\beta} \right) \right) \right] dx \\ &= \int_{\mathbb{R}^N} \left[\frac{h^{(N-2)\beta+2}}{2} |\nabla_\xi v(\xi)|^2 + h^{N\beta-\alpha} W(v(\xi)) \right] d\xi \\ &= h^{N\beta-\alpha} \int_{\mathbb{R}^N} \frac{1}{2} \left[|\nabla_\xi v(\xi)|^2 + W(v(\xi)) \right] d\xi = h^{N\beta-\alpha} J(v). \end{aligned}$$

We can now describe the concentration properties of the modulus of the solution of (46).

Lemma 30 *For any $\varepsilon > 0$, there exist positive numbers $\delta = \delta(\varepsilon)$, $\widehat{R} = \widehat{R}(\varepsilon)$ such that: for any $\psi(t, x)$ that solves (46), with $|\psi(t, h^\beta x)| \in J^{m+\delta} \cap S_\sigma$, for all t , there exists a map $\widehat{q}_h : \mathbb{R}_0^+ \rightarrow \mathbb{R}^N$ for which*

$$\frac{\int_{\mathbb{R}^N \setminus B(\widehat{q}_h(t), \widehat{R})} |\nabla u_h(t, x)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_h(t, x)|^2 dx} < \varepsilon. \quad (56)$$

Proof. For fixed h and t , we set $v(\xi) = |\psi(t, h^\beta \xi)|$. By Lemma 29, there exist $\delta > 0$, $\widehat{R} > 0$ and $\bar{q} = \bar{q}(v)$ such that: if $|\psi(t, h^\beta x)| \in J^{m+\delta} \cap S_\sigma$, then

$$\varepsilon > \frac{\int_{\mathbb{R}^N \setminus B(\bar{q}, \widehat{R})} |\nabla_\xi v(\xi)|^2 d\xi}{\int_{\mathbb{R}^N} |\nabla_\xi v(\xi)|^2 d\xi}.$$

By a change of variables, we obtain

$$\begin{aligned} \varepsilon &> \frac{\int_{\mathbb{R}^N \setminus B(\bar{q}, \widehat{R})} |\nabla_\xi v(\xi)|^2 d\xi}{\int_{\mathbb{R}^N} |\nabla_\xi v(\xi)|^2 d\xi} = \frac{\int_{\mathbb{R}^N \setminus B(h^\beta \bar{q}, h^\beta \widehat{R})} |\nabla u_h(t, x)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_h(t, x)|^2 dx} \\ &> \frac{\int_{\mathbb{R}^N \setminus B(h^\beta \bar{q}, \widehat{R})} |\nabla u_h(t, x)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_h(t, x)|^2 dx}, \text{ since } h < 1. \end{aligned}$$

Setting $\widehat{q}_h(t) = h^\beta \bar{q}$, we complete the proof. Notice that $\widehat{q}_h(t)$ depends on ε , and ψ , while \widehat{R} depends only on ε . ■

Proposition 31 *Let $V \in L_{loc}^\infty$. For every $\varepsilon > 0$, there exists $\widehat{R} > 0$ and $h_0 > 0$ such that, for any $\psi(t, x)$ that solves (46), with initial data $\psi(0, x) \in B_{K, h}$, where $h < h_0$, and for any t , there exists $\widehat{q}_h(t) \in \mathbb{R}^N$, for which*

$$\frac{1}{\|\nabla u_h(t, x)\|_{L^2}^2} \int_{\mathbb{R}^N \setminus B(\widehat{q}_h(t), h^\beta \widehat{R})} |\nabla u_h(t, x)|^2 dx < \varepsilon. \quad (57)$$

Proof. By the conservation law, the energy $E_h(\psi(t, x))$ is constant with respect to t . Then we have

$$\begin{aligned} E_h(\psi(t, x)) &= E_h(\psi(0, x)) \\ &= \widetilde{J}_h(u_h(0, x)) + \int_{\mathbb{R}^N} u_h^2(0, x) \left[\frac{|\nabla s_h(0, x)|^2}{2} + V(x) \right] dx \\ &\leq \widetilde{J}_h(u_h(0, x)) + \frac{K}{2} \sigma^2 h^{N\beta} + K h^{N\beta} \\ &\quad h^{N\beta-\alpha} J(U + w) + h^{N\beta} C, \end{aligned}$$

where C is a suitable constant. By rescaling, and using that $\psi(0, x) \in B_{K, h}$, and that $\|w\|_{H^1} < K h^\alpha$ implies $J(U + w) < m + K h^\alpha$ (see the proof of Proposition 23), we obtain

$$\begin{aligned} E_h(\psi(t, x)) &= h^{N\beta-\alpha} J(U + w) + h^{N\beta} C \\ &< h^{N\beta-\alpha} (m + K h^\alpha) + h^{N\beta} C \\ &= h^{N\beta-\alpha} (m + K h^\alpha + h^\alpha C) = h^{N\beta-\alpha} (m + h^\alpha C_1), \end{aligned}$$

where we have set $C_1 = K + C$. Thus

$$\begin{aligned} \widetilde{J}_h(u_h(t, x)) &= E_h(\psi(t, x)) - \int_{\mathbb{R}^N} u_h^2(t, x) \left[\frac{|\nabla s_h(t, x)|^2}{2} + V(x) \right] dx \\ &< h^{N\beta-\alpha} (m + h^\alpha C_1), \end{aligned} \quad (58)$$

since $V \geq 0$. By rescaling the inequality (58), we get

$$J(u_h(t, h^\beta x)) < m + h^\alpha C_1.$$

So, for h sufficiently small, we may apply Lemma 30, and get the claim. ■

4.2 The case Ω is bounded

The case where Ω is bounded is easily treated, once one makes the crucial remark that Lemma 29 has to replace Lemma 15 in [7], that it was used in the proof of Lemma 21. The rest of the proofs in the consequent Lemmas 32 and 33 and of the final Proposition 34 follow precisely the pattern of the proofs for Lemmas 21, 22, and of Proposition 23, respectively. For completeness, we give below the precise statements, where we have assumed for simplicity, as in the L^2 case, that $\sigma = 1$.

Lemma 32 *For any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for any $u \in H_0^1(\Omega)$ with $C_h(u) = 1$, and $J_h(u) < m(h, \Omega) + \delta h^{-\alpha}$,*

$$\frac{\int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} |\nabla u|^2 dx}{\int_{\Omega} |\nabla u|^2 dx} < \varepsilon$$

to hold.

Lemma 33 *For any $\varepsilon > 0$, there exist $\delta = \delta(\varepsilon)$, $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for a solution $\psi(t, x)$ of (1) with $C_h(|\psi(t, x)|) = 1$, and $J_h(|\psi(t, x)|) < m(h, \Omega) + \delta h^{-\alpha}$, for each $t \in \mathbb{R}_0^+$,*

$$\frac{\int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} |\nabla u(t, x)|^2 dx}{\int_{\Omega} |\nabla u(t, x)|^2 dx} < \varepsilon$$

to hold, where $u(t, x) = |\psi(t, x)|$.

Proposition 34 *Given $\varepsilon > 0$, there exists $h_0 = h_0(\varepsilon) > 0$, and $R = R(\varepsilon) > 0$ such that, for any $0 < h < h_0(\varepsilon)$, there is an open ball $B(\hat{q}_h, h^\beta R) \subset \Omega$ so that for a solution $\psi(t, x)$ of (1) with $C_h(|\psi(t, x)|) = 1$, and with initial data $\psi(0, x) \in B_h^K$, where K is a positive fixed number, it holds*

$$\frac{\int_{\Omega \setminus B(\hat{q}_h, h^\beta R)} |\nabla u(t, x)|^2 dx}{\int_{\Omega} |\nabla u(t, x)|^2 dx} < \varepsilon,$$

for any $t \in \mathbb{R}_0^+$, where $u(t, x) = |\psi(t, x)|$.

5 Appendix

In order to gain some control over the dynamics of the problem, that is, to try to formulate Newton's equation describing the motion of the barycenter for a fixed value of h , one needs to express suitably $\ddot{q}(t)$. To this end, a further assumption on W is made, namely that $W(0) = 0$. For the sake of simplicity, we fix $h = 1$, $\alpha = 1$ in what follows, the general case being straightforward. Also, we suppose that a solution $\psi(t, x)$ is sufficiently smooth in order to make the integration by parts meaningful. Given this, the general case can be proved with minor technical efforts. Finally, we use the Einstein convention on the summation indices.

We will use the Lagrangian formalism. Equation (1) is the Euler-Lagrange equation relative to the following Lagrangian density \mathcal{L} :

$$\mathcal{L} = \operatorname{Re}(i\bar{\psi}\partial_t\psi) - \frac{1}{2}|\nabla\psi|^2 - W(|\psi|) - V(x)|\psi|^2$$

By Noether's theorem, there are continuity equations related to \mathcal{L} , which we will use to derive an equation for the motion. In particular, we are interested in the following continuity equations:

$$\frac{d}{dt}|\psi(t, x)|^2 = -\nabla \cdot \operatorname{Im}(\bar{\psi}\nabla\psi) \quad (59)$$

and

$$\frac{d}{dt}\operatorname{Im}(\bar{\psi}\nabla\psi) = -|\psi|^2\nabla V - \nabla \cdot T, \quad (60)$$

where T is the so called energy stress tensor and has the form

$$T_{jk} = \operatorname{Re}(\partial_{x_j}\psi\partial_{x_k}\bar{\psi}) - \delta_{jk}\left[\operatorname{Re}\left(\frac{1}{2}\bar{\psi}\Delta\psi\right) + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}W'(|\psi|)|\psi| + W(|\psi|)\right]$$

For an introduction to the Lagrangian formalism for equation (1) and continuity equations we refer to [5, 8, 16].

In the light of equation (59), and by divergence theorem, one has for $j = 1, \dots, N$,

$$\begin{aligned} \dot{q}_j(t) &= \frac{d}{dt} \int_{\Omega} x_j |\psi(t, x)|^2 dx = \int_{\Omega} x_j \partial_t(|\psi|^2) dx \\ &= - \int_{\Omega} x_j \nabla \cdot \operatorname{Im}(\bar{\psi}\nabla\psi) dx \\ &= - \int_{\Omega} \nabla \cdot [x_j \operatorname{Im}(\bar{\psi}\nabla\psi)] + \int_{\Omega} \operatorname{Im}(\bar{\psi}\partial_{x_j}\psi) \\ &= \int_{\Omega} \operatorname{Im}(\bar{\psi}\partial_{x_j}\psi), \end{aligned}$$

since $\psi(t, x) = 0$ on $\partial\Omega$. Thus we have the *momentum law*

$$\dot{q}(t) = \text{Im} \left(\int_{\Omega} \bar{\psi} \nabla \psi dx \right). \quad (61)$$

For the second derivative of the center of mass, we have, by (60) and by divergence theorem,

$$\begin{aligned} \ddot{q}_j(t) &= \frac{d}{dt} \int_{\Omega} \text{Im} (\bar{\psi} \partial_{x_j} \psi dx) = - \int_{\Omega} \partial_{x_k} T_{jk}(t, x) dx - \int_{\Omega} |\psi(t, x)|^2 \partial_{x_j} V(x) dx \\ &= \int_{\partial\Omega} T_{jk}(t, x) \cdot n_k d\sigma - \int_{\Omega} |\psi(t, x)|^2 \partial_{x_j} V(x) dx := I_1 + I_2, \end{aligned}$$

n being the inward normal to $\partial\Omega$.

Let us use the polar form $\psi(t, x) = u(t, x)e^{is(t, x)}$. Then

$$I_2 = - \int_{\Omega} u^2 \partial_{x_j} V(x) dx.$$

This appears to be a force term depending on the potential V . This, when the concentration parameter $h \rightarrow 0$, gives us the Newtonian law for the motion of a particle (see [7], where this approach is used in the whole space \mathbb{R}^N).

Since $u = 0$ on the boundary (and since $W(0) = 0$), the expression of T is simplified and the term I_1 becomes

$$\begin{aligned} I_1 &= \int_{\partial\Omega} T_{jk} n_k d\sigma = \int_{\partial\Omega} \left(\partial_{x_j} u \partial_{x_k} u - \frac{1}{4} \delta_{jk} \Delta(u^2) \right) n_k d\sigma \\ &= \int_{\partial\Omega} \left(\partial_{x_j} u \partial_{x_k} u - \frac{1}{2} \delta_{jk} |\nabla u|^2 \right) n_k d\sigma. \end{aligned}$$

Again, because $u = 0$ on the boundary, by implicit function theorem, we have that ∇u is orthogonal to $\partial\Omega$. In addition, we have by definition $u = |\psi| \geq 0$, so whenever $\nabla u \neq 0$, the inward pointing normal vector can be written as $n = \frac{\nabla u}{|\nabla u|}$. Thus

$$\begin{aligned} I_1 &= \int_{\partial\Omega} \left(\partial_{x_j} u \partial_{x_k} u - \frac{1}{2} \delta_{jk} |\nabla u|^2 \right) \frac{\partial_{x_k} u}{|\nabla u|} d\sigma \\ &= \int_{\partial\Omega} (\partial_{x_j} u |\nabla u| - \frac{1}{2} \partial_{x_j} u |\nabla u|) d\sigma = \frac{1}{2} \int_{\partial\Omega} \partial_{x_j} u |\nabla u| d\sigma = \\ &= \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 n_j d\sigma. \end{aligned}$$

Concluding, we have

$$\ddot{q}(t) = - \int_{\Omega} u^2 \nabla V(x) dx + \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 n d\sigma. \quad (62)$$

In the case of a bounded domain with Dirichlet boundary condition, it appears an extra term, which represents the centripetal force. Unfortunately, there are some obvious computational challenges concerning the last integral of (62), and we cannot give a simple expression of this term, when $h \rightarrow 0$. As it was said in the Introduction, these challenges call for further work on the dynamics of the solution of (1).

Acknowledgement M. Magiropoulos wishes to thank G. M. Kavoulakis and J. Smyrnakis for many helpful discussions. The authors want also to thank the referee for his useful remarks and comments.

References

- [1] A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rational Mech. Anal. 140 (1997), no. 3, 285-300
- [2] A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear Schrödinger equations with potentials*, Arch. Rational Mech. Anal. 159 (2001), no. 3, 253-271
- [3] J. Bellazzini, V. Benci, C. Bonanno, E. Sinibaldi, *Hylomorphic solitons in the nonlinear Klein-Gordon equation*, Dyn. Partial Differ. Eq. 6 (2009), 311-334
- [4] J. Bellazzini, V. Benci, M. Ghimenti, A.M. Micheletti, *On the existence of the fundamental eigenvalue of an elliptic problem in \mathbb{R}^N* , Adv. Nonlinear Stud., 7 (2007), 439-458
- [5] V. Benci, *Hylomorphic solitons*, Milan J. Math. 77 (2009), 271-332
- [6] V. Benci, M. Ghimenti, A.M. Micheletti, *The Nonlinear Schroedinger Equation: Existence, Stability and Dynamics of Solitons*, Progress in Variational Methods 11-29, Nankai Ser. Pure Appl. Math. Theoret. Phys., 7, World Sci. Publ., Hackensack, NJ, 2011
- [7] V. Benci, M. Ghimenti, A.M. Micheletti, *The nonlinear Schroedinger equation: Solitons dynamics*, J. Differential Equations, 249 (2010), 3312-3341
- [8] V. Benci, M. Ghimenti, A.M. Micheletti, *On the Dynamics of solitons in the nonlinear Schroedinger equation*, Archive for Rational Mechanics and Analysis 205 (2012), 467-492.
- [9] V. Benci, C.R. Grisanti, A.M. Micheletti, *Existence and non existence of the ground state solution for the nonlinear Schroedinger equations with $V(\infty) = 0$* , Topol. Methods Nonlinear Anal. 26 (2005), no. 2, 203-219
- [10] D. Cao, N.E. Dancer, E.S. Noussair, S. Yan, *On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems*, Discrete Cont. Dynam. Systems 2 (1996), no. 2, 221-236

- [11] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, vol.10, Courant Institute of Mathematical Sciences, New York, 2003
- [12] T. Cazenave, P.L. Lions, *Orbital Stability of Standing Waves for Some Nonlinear Schrödinger Equations*, Comm. Math. Phys. 85 (4) (1982), 549-561
- [13] S. Cingolani, M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. 10 (1997), no. 1, 1-13
- [14] M. del Pino, P.L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations 4 (1996), no. 2, 121-137
- [15] A. Floer, A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. 69 (1986), no. 3, 397-408
- [16] I.M. Gelfand, S.V. Fomin, *Calculus of Variations*, Prentice-Hall, Englewood-Cliffs, N.J., 1963
- [17] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*. Grundlehren, Vol. 224, Springer-Verlag, Berlin, 1983
- [18] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations. II. Scattering theory, general case*, J. Funct. Anal., 32 (1979), no. 1, 33-71
- [19] T. Kato, *Nonlinear Schrödinger equations*, Schrödinger Operators, Lecture Notes in Phys., vol. 345, Springer, Berlin, 1989, pp 218-263
- [20] Y.Y. Li, *On a singularly perturbed elliptic equation*, Adv. Differential Equations 2 (1997), no. 6, 955-980
- [21] Y-G. Oh, *Existence of semiclassical boundstates of nonlinear Schrödinger equations with potentials of the class $(V)_a$* , Comm. Partial Differential Equations 13 (1988), no. 12, 1499-1519
- [22] W-M. Ni, J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. 48 (1995), no. 7, 731-768
- [23] B. Noris, H. Tavares, G. Verzini, *Existence and orbital stability of the ground states with prescribed mass for the L^2 -critical and supercritical NLS on bounded domains*, Analysis & PDE, 8 (2015), 1807-1838
- [24] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. 43 (1992), no.2, 270-291

- [25] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), 149-162
- [26] J. Wei, *On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problem*, J. Differential Equations 129 (1996), no. 2, 315-333