

# SOME HYPERBOLIC THREE-MANIFOLDS THAT BOUND GEOMETRICALLY

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**ABSTRACT.** A closed connected hyperbolic  $n$ -manifold *bounds geometrically* if it is isometric to the geodesic boundary of a compact hyperbolic  $(n+1)$ -manifold. A. Reid and D. Long have shown by arithmetic methods the existence of infinitely many manifolds that bound geometrically in every dimension.

We construct here infinitely many explicit examples in dimension  $n = 3$  using right-angled dodecahedra and 120-cells and a simple colouring technique introduced by M. Davis and T. Januszkiewicz. Namely, for every  $k \geq 1$ , we build an orientable compact closed 3-manifold tessellated by  $16k$  right-angled dodecahedra that bounds a 4-manifold tessellated by  $32k$  right-angled 120-cells.

A notable feature of this family is that the ratio between the volumes of the 4-manifolds and their boundary components is constant and, in particular, bounded.

## 1. INTRODUCTION

The study of hyperbolic manifolds that bound geometrically dates back to the works of D. Long, A. Reid [11, 12] and B. Niemi [17], motivated by a preceding work of M. Gromov [7, 8] and a question by F. Farrell and S. Zdravkovska [4]. This question is also related to hyperbolic instantons, as described by J. Ratcliffe and S. Tschantz [19, 20]. In particular, the following problems are of particular interest:

**Question 1.1.** Which compact orientable hyperbolic  $n$ -manifold  $\mathcal{N}$  can represent the totally geodesic boundary of a compact orientable hyperbolic  $(n+1)$ -manifold  $\mathcal{M}$ ?

**Question 1.2.** Which compact orientable flat  $n$ -manifold  $\mathcal{N}$  can represent the cusp section of a single-cusped orientable hyperbolic  $(n+1)$ -manifold  $\mathcal{M}$ ?

Once there exist such manifolds  $\mathcal{N}$  and  $\mathcal{M}$ , we say that  $\mathcal{N}$  *bounds*  $\mathcal{M}$  *geometrically*. In this note, we shall concentrate on Question 1.1, devoted to compact geometric boundaries. The recent progress on Question 1.2, that involves cusp sections, is indicated by [10, 12, 14, 15]. However, this is still

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an open problem in dimensions  $\geq 5$ . On the other hand, by a result of M. Stover [21], an arithmetic orbifold in dimension  $\geq 30$  cannot have a single cusp.

In [11], D. Long and A. Reid have shown that many closed hyperbolic 3-manifolds do not bound geometrically: a necessary condition is that the eta invariant of the 3-manifold must be an integer. The first known closed hyperbolic 3-manifold that bounds geometrically was constructed by J. Ratcliffe and S. Tschantz in [19] and has volume of order 200.

Then, D. Long and A. Reid produced in [13], by arithmetic techniques, infinitely many orientable hyperbolic  $n$ -manifolds  $\mathcal{N}$  that bound geometrically an  $(n+1)$ -manifold  $\mathcal{M}$ , in every dimension  $n \geq 2$ . Every such manifold  $\mathcal{N}$  is obtained as a cover of some  $n$ -orbifold  $O_{\mathcal{N}}$  geodesically immersed in a suitable  $(n+1)$ -orbifold  $O_{\mathcal{M}}$ .

In this paper, we construct an explicit infinite family in dimension  $n = 3$ , via a similar covering technique where the roles of  $O_{\mathcal{N}}$  and  $O_{\mathcal{M}}$  are played by the right-angled dodecahedron  $\mathcal{D}$  and 120-cell  $\mathcal{Z}$ . These two compact Coxeter right-angled regular polytopes exist in  $\mathbb{H}^3$  and  $\mathbb{H}^4$  respectively, and the first is a facet of the second. The existence of suitable finite covers is guaranteed here by assigning appropriate *colourings* to their facets, following A. Vesnin [23, 24], M. Davis and T. Januszkiewicz [3], I. Izvestiev [9].

A colouring determines a manifold covering, and the main factual observation is that a colouring of the dodecahedron  $\mathcal{D}$  can be enhanced in a suitable way to a colouring of the right-angled hyperbolic Coxeter 120-cell  $\mathcal{Z}$ . We produce in this way a degree-32 orientable cover of  $\mathcal{Z}$  that contains four copies of a non-orientable degree-8 cover of  $\mathcal{D}$ . By cutting along one such non-orientable geodesic submanifold we get a hyperbolic four-manifold  $\mathcal{M}_1$  with connected geodesic boundary  $\mathcal{M}_1 = \partial\mathcal{M}_1$ .

The colouring technique applied to a single polytope can produce only finitely many manifolds. To get infinitely many examples we assemble  $n$  copies of  $\mathcal{D}$  and  $\mathcal{Z}$  to get more complicated right-angled polytopes, to which the above construction easily extends. We finally obtain the following. Let  $V_{\mathcal{D}} \approx 4.3062\dots$  and  $V_{\mathcal{Z}} = \frac{34}{3}\pi^2$  be the volumes of  $\mathcal{D}$  and  $\mathcal{Z}$ , respectively.

**Theorem 1.3.** *For every  $n \geq 1$  there exists an orientable compact hyperbolic three-manifold  $\mathcal{N}_n$  of volume  $16nV_{\mathcal{D}}$  which bounds geometrically an orientable compact hyperbolic four-manifold  $\mathcal{M}_n$  of volume  $32nV_{\mathcal{Z}}$ .*

*The manifolds  $\mathcal{N}_n$  and  $\mathcal{M}_n$  are tessellated respectively by  $16n$  right-angled dodecahedra and  $32n$  right-angled 120-cells.*

An interesting feature of this construction is that it provides manifolds  $\mathcal{N}_n$  and  $\mathcal{M}_n$  of controlled volume. In particular, we deduce the following.

**Corollary 1.4.** *There are infinitely many hyperbolic three-manifolds  $\mathcal{N}$  that bound geometrically some hyperbolic  $\mathcal{M}$  with constant ratio:*

$$\frac{\text{Vol}(\mathcal{M})}{\text{Vol}(\mathcal{N})} = \frac{2V_{\mathcal{Z}}}{V_{\mathcal{D}}} < 53.$$

The manifold  $\mathcal{M}_1$  has volume  $16V_{\mathcal{D}} \approx 68.8992$  and is to our knowledge the smallest closed hyperbolic 3-manifold known to bound geometrically.

**Structure of the paper.** In Section 2 we introduce right-angled polytopes as orbifolds, and a simple colouring technique from [3] that produces manifold coverings of small degree. Then we show how this colouring technique passes easily from dimension  $n$  to  $n + 1$  and conversely, and may be used to produce  $n$ -manifolds that bound geometrically when  $n = 3$ . In Section 3 we assemble dodecahedra and 120-cells to produce the manifolds  $\mathcal{N}_k$  and  $\mathcal{M}_k$  of Theorem 1.3.

## 2. COLORINGS AND COVERS OF COXETER ORBIFOLDS

A right-angled hyperbolic polytope  $\mathcal{P} \subset \mathbb{H}^n$  may be interpreted as an orbifold with mirror boundary, where the mirrors correspond to its facets. As such an orbifold, it has a plenty of manifold coverings. A few of them may be constructed by colouring appropriately the facets of  $\mathcal{P}$  as shown in [3, 6].

**2.1. Colourings and manifold covers.** Let  $\mathcal{P} \subset \mathbb{H}^n$  be a convex compact right-angled polytope. Such objects exist only if  $2 \leq n \leq 4$ , see [18]; the two important basic examples we consider here are the right-angled dodecahedron  $\mathcal{D} \subset \mathbb{H}^3$  and the right-angled 120-cell  $\mathcal{Z} \subset \mathbb{H}^4$ .

We consider  $\mathcal{P}$  as an orbifold  $\mathbb{H}^n/\Gamma$ . The group  $\Gamma$  is a right-angled Coxeter group that may be presented as

$$\Gamma = \langle r_F \mid r_F^2, [r_F, r_{F'}] \rangle$$

where  $F$  varies over all the facets of  $\mathcal{P}$  and the pair  $F, F'$  varies over all the pairs of adjacent facets. The isometry  $r_F \in \text{Isom}(\mathbb{H}^n)$  is a reflection in the hyperplane containing  $F$ .

A right-angled polyhedron  $\mathcal{P}$  is *simple* [25, Theorem 1.8], which means that it looks combinatorially at every vertex  $v$  like the origin of an orthant in  $\mathbb{R}^n$ . In particular,  $v$  is the intersection of exactly  $n$  facets.

Let  $V$  be a finite-dimensional vector space over  $\mathbb{F}_2$ , thus isomorphic to  $\mathbb{F}_2^s$  for some  $s$ . A  $V$ -colouring (or simply, a colouring)  $\lambda$  is the assignment of a vector  $\lambda_F \in V$  to each facet  $F$  of  $\mathcal{P}$  (called its *colour*) such that the following holds: at every vertex  $v$ , the  $n$  colours assigned to the  $n$  adjacent facets around  $v$  are linearly independent vectors in  $V$ .

A colouring induces a group homomorphism  $\lambda: \Gamma \rightarrow V$ , defined by sending  $r_F$  to  $\lambda_F$  for every facet  $F$ . Its kernel  $\Gamma_\lambda = \ker \lambda$  is a subgroup of  $\Gamma$  which determines an orbifold  $M_\lambda = \mathbb{H}^n/\Gamma_\lambda$  covering  $\mathcal{P}$ .

**Proposition 2.1.** *The orbifold  $M_\lambda$  is a manifold.*

*Proof.* We follow [23, Lemma 1]. A torsion element in  $\Gamma$  fixes some face  $F$  of the tessellation of  $\mathbb{H}^n$  obtained by reflecting  $\mathcal{P}$  in its own facets. Up to conjugacy, we can suppose that  $F \subset \mathcal{P}$ . The stabilizer of  $F$  is generated by the reflections in the facets containing  $F$  (see [2, Theorem 12.3.4]) and

is hence mapped injectively into  $V$  by  $\lambda$ . Thus,  $\Gamma_\lambda$  is torsion-free and  $M_\lambda$  is a manifold.  $\square$

At each vertex  $v$  the colours  $\lambda_F$  of the  $n$  incident facets  $F$  are independent: therefore the image of  $\lambda$  has dimension at least  $n$  and the covering  $M_\lambda \rightarrow \mathcal{P}$  has degree  $|\Gamma : \Gamma_\lambda| \geq 2^n$ ; if the equality holds the manifold  $M_\lambda$  is called a *small cover* of  $\mathcal{P}$ . The manifold coverings of  $\mathcal{P}$  of smallest degree are precisely its small covers, see [6, Proposition 2.1].

We say that the colouring *spans*  $V$  if the vectors  $\lambda_F$  span  $V$  as  $F$  varies, which is equivalent to the map  $\lambda : \Gamma \rightarrow V$  being surjective. In that case the covering  $M_\lambda \rightarrow \mathcal{P}$  has degree  $|\Gamma : \Gamma_\lambda| = 2^{\dim V}$ .

**2.2.  $k$ -colourings.** Here, we give an example: recall that a  $k$ -colouring of a polytope is the assignment of a colour from the set  $\{1, \dots, k\}$  to each facet so that two adjacent facets have distinct colours. A  $k$ -colouring for  $\mathcal{P}$  produces an  $\mathbb{F}_2^k$ -colouring that spans  $\mathbb{F}_2^k$ : simply replace each colour  $i \in \{1, \dots, k\}$  with the element  $e_i$  of the canonical basis for  $\mathbb{F}_2^k$ .

**Example 2.2.** The dodecahedron has precisely one four-colouring, up to symmetries. This induces an  $\mathbb{F}_2^3$ -colouring (of lower dimension 3 rather than 4) on the hyperbolic right-angled dodecahedron  $\mathcal{D}$  described in [23] and hence a manifold covering having degree  $2^3 = 8$ .

**Example 2.3.** The 120-cell has a five-colouring (in fact, ten five-colourings up to symmetries [5]). Each produces a manifold covering of the hyperbolic right-angled 120-cell  $\mathcal{Z}$  of degree  $2^5 = 32$ .

**2.3. Orientable coverings.** The following lemma gives an orientability criterion analogous to that for small covers in [16] or Löbell manifolds in [23, Lemma 2]. Let  $\lambda$  be a  $V$ -colouring of a right-angled polytope  $\mathcal{P}$ .

**Lemma 2.4.** *Suppose  $\lambda$  spans  $V$ . The manifold  $M_\lambda$  is orientable if and only if, for some isomorphism  $V \cong \mathbb{F}_2^s$ , each colour  $\lambda_F$  has an odd number of 1's.*

*Proof.* Let  $\Gamma^+ \triangleleft \Gamma$  be the index two subgroup consisting of orientation-preserving isometries. Then  $\Gamma^+$  is the kernel of the homomorphism  $\phi : \Gamma \rightarrow \mathbb{F}_2$  that sends  $r_F$  to 1 for every facet  $F$ .

The manifold  $M_\lambda$  is orientable if and only if  $\Gamma_\lambda$  is contained in  $\Gamma^+$ , and this in turn holds if and only if there is a homomorphism  $\chi : V \rightarrow \mathbb{F}_2$  such that  $\phi = \chi \circ \lambda$ . The latter is equivalent to the existence of an isomorphism  $V \cong \mathbb{F}_2^s$  that transforms  $\lambda_F$  into a vector with an odd number of 1's for each  $F$ . Indeed, if such an isomorphism exists, then  $\chi$  can be taken to be the sum of the coordinates of a vector.

Conversely, suppose such an isomorphism does not exist. Since the vectors  $\lambda_F$  span  $V$  we may take some of them as a basis for  $V$  and write them as  $e_1 = (1, 0, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ ,  $\dots$ ,  $e_n = (0, \dots, 0, 0, 1)$ . By hypothesis, there exists a facet  $F$  such that  $\lambda_F$  has an even number of 1's.

Up to reordering, we may write  $\lambda_F = \sum_{i=1}^{2k} e_i$  for some  $k$ . Now we can see that the homomorphism  $\chi$  does not exist, since its existence would imply  $1 = \phi(r_F) = \sum_{i=1}^{2k} \phi(e_i) = \sum_{i=1}^{2k} 1 = 0$ .  $\square$

**Corollary 2.5.** *Let there be facets  $F$ ,  $F'$  and  $F''$  of  $\mathcal{P}$  such that  $\lambda_F + \lambda_{F'} + \lambda_{F''} = \mathbf{0}$ . Then  $\lambda$  is a non-orientable colouring.*

*Proof.* For a vector  $v = (v_1, v_2, \dots, v_s) \in \mathbb{F}_2^s$  let  $\epsilon(v) := \sum_{i=1}^s v_i$ . Suppose that  $\lambda$  is orientable, so there exists an isomorphism  $V \cong \mathbb{F}_2^s$ , such that each  $\lambda_F$  has an odd number of 1's. Then we arrive at a contradiction, since  $0 = \epsilon(\mathbf{0}) = \epsilon(\lambda_F + \lambda_{F'} + \lambda_{F''}) = \epsilon(\lambda_F) + \epsilon(\lambda_{F'}) + \epsilon(\lambda_{F''}) = 1$ .  $\square$

**Example 2.6.** The manifolds in Examples 2.2 and 2.3 are orientable.

**Example 2.7.** Consider the 25 small covers of the hyperbolic right-angled dodecahedron  $\mathcal{D}$  found by A. Garrison and R. Scott in [6]. The list is complete, up to isometries between the corresponding manifolds. Using the orientability criterion one sees immediately from [6, Table 1] that 24 of them are non-orientable and exactly 1 is orientable and corresponds to Example 2.2. Another example carried out in [6] is a small cover of the hyperbolic right-angled 120-cell  $\mathcal{Z}$ . This cover is again non-orientable. There is no classification of small covers of  $\mathcal{Z}$  known at present.

**2.4. Induced colouring.** A facet  $F$  of a  $n$ -dimensional right-angled polytope  $\mathcal{P} \subset \mathbb{H}^n$  is a  $(n-1)$ -dimensional right-angled polytope. A  $V$ -colouring  $\lambda$  of  $\mathcal{P}$  induces a  $W$ -colouring  $\mu$  of  $F$  with  $W = V/\langle \lambda_F \rangle$ : simply assign to every face of  $F$  the colour of the facet of  $\mathcal{P}$  that is incident to it. The following lemma generalises [6, Proposition 2.3].

**Lemma 2.8.** *The manifold  $M_\mu$  is contained in  $M_\lambda$  as a totally geodesic sub-manifold, so that the cover  $M_\lambda \rightarrow \mathcal{P}$  restricts to the cover  $M_\mu \rightarrow F$ .*

*Proof.* Let  $\Gamma$  be the Coxeter group of  $\mathcal{P}$  and  $\mathbb{H}^{n-1} \subset \mathbb{H}^n$  be the hyperplane containing  $F$ . We regard  $F$  as the orbifold  $\mathbb{H}^{n-1}/\Gamma_F$  where  $\Gamma_F$  is the Coxeter group of  $F$ . The following natural diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle r_F \rangle & \longrightarrow & \Gamma \cap \text{Stab}(\mathbb{H}^{n-1}) & \xrightarrow{f} & \Gamma_F \longrightarrow 0 \\ & & & & \lambda \downarrow & & \downarrow \mu \\ & & & & V & \xrightarrow{\pi} & V/\langle \lambda_F \rangle \end{array}$$

The first line is an exact sequence. We deduce easily that  $f$  restricts to an isomorphism

$$f: \ker \lambda \cap \text{Stab}(\mathbb{H}^{n-1}) \longrightarrow \ker \mu.$$

Hence  $M_\mu = \mathbb{H}^{n-1}/\Gamma_{\lambda \cap \text{Stab}(\mathbb{H}^{n-1})}$  is naturally contained in  $M_\lambda = \mathbb{H}^n/\Gamma_\lambda$ .  $\square$

The pre-image of  $F$  in  $M_\lambda$  with respect to the regular covering  $M_\lambda \rightarrow \mathcal{P}$  consists of possibly several copies of  $M_\mu$ .

**2.5. Extended colouring.** Conversely, we can also extend a colouring from a facet to the whole polytope. We say that two colourings  $\lambda$  and  $\lambda'$  on  $\mathcal{P}$  are *equivalent* if they have isomorphic kernels  $\Gamma_\lambda \cong \Gamma_{\lambda'}$  (cf. the definition before [6, Proposition 2.4]).

**Proposition 2.9.** *Let  $F$  be a facet of a compact right-angled polytope  $\mathcal{P} \subset \mathbb{H}^n$ . Every colouring of  $F$  is equivalent to one induced by an orientable colouring of  $\mathcal{P}$ .*

*Proof.* Let  $\lambda$  be a  $V$ -colouring of the facet  $F$ . Fix an isomorphism  $V \cong \mathbb{F}_2^s$ . Define  $W = \mathbb{F}_2 \oplus \mathbb{F}_2^s \oplus \mathbb{F}_2^f$  where  $f$  is the number of facets of  $\mathcal{P}$  that are not adjacent to  $F$ . We define a  $W$ -colouring  $\mu$  of  $\mathcal{P}$  as follows:

- set  $\mu_F = (1, \mathbf{0}, \mathbf{0})$ ;
- set  $\mu_G = (\epsilon(\lambda_{G \cap F}) + 1, \lambda_{G \cap F}, \mathbf{0})$  where  $\epsilon(v) = \sum_{i=1}^s v_i$ , for every facet  $G$  adjacent to  $F$ ;
- set  $\mu_{G_i} = (\mathbf{0}, \mathbf{0}, e_i)$  for the remaining facets  $G_1, \dots, G_f$ .

Indeed, the map  $\mu$  is a colouring: the linear independence condition is satisfied at each vertex. Moreover, each vector  $\mu_F, \mu_G, \mu_{G_i}$  has an odd number of 1's. Finally, by construction  $\mu|_F$  is equivalent to  $\lambda$ .  $\square$

We call the colouring  $\mu$  an *extension* of  $\lambda$ .

**2.6. A more efficient extension.** Proposition 2.9 shows how to extend a  $V$ -colouring of a facet  $F$  to an orientable  $W$ -colouring of the polytope  $\mathcal{P}$ . The proof shows that the dimension of  $W$  can grow considerably during the process, since  $\dim W = \dim V + 1 + f$  where  $f$  is the number of facets of  $\mathcal{P}$  not adjacent to  $F$ .

We may use Proposition 2.9 with  $\mathcal{P}$  being the right-angled 120-cell and  $F$  its dodecahedral facet. However, in this case we could find examples where both  $V$  and  $W$  have smaller dimension via computer. The following was proved by using “Mathematica”.

**Proposition 2.10.** *Each of the 24 non-orientable  $\mathbb{F}_2^3$ -colourings of  $\mathcal{D}$  from [6, Table 1] is equivalent to one induced by an orientable  $\mathbb{F}_2^5$ -colouring of  $\mathcal{Z}$ .*

*Proof.* The Mathematica program code given in [22] takes a non-orientable  $\mathbb{F}_2^3$ -colouring of  $\mathcal{D}$  and produces an orientable  $\mathbb{F}_2^5$ -colouring of  $\mathcal{Z}$ , as required.

Each vector  $v = (v_1, v_2, \dots, v_s) \in \mathbb{F}_2^s$  is encoded by the binary number  $n_v = v_1 \cdot 2^0 + v_2 \cdot 2^1 + \dots + v_s \cdot 2^{s-1}$ . Let  $P_0 := \mathcal{D}$ , be the right-angled dodecahedron. We enumerate its faces exactly as shown in [6, Figure 3] and the corresponding 12-tuple of numbers encodes its colouring. Let  $P_i$ ,  $i = 1, \dots, 12$  be the dodecahedral facets incident to  $P_0$  at the respective faces  $F_i$ ,  $i = 1, \dots, 12$ . We start extending the colouring of  $P_0 := \mathcal{D}$  as follows:

- set  $\lambda_{P_0} = (0, 0, 0, 0, 1)$ ;
- if  $\mu_{F_i} = v = (v_1, v_2, v_3)$ , then  $\lambda_{P_i} := (v_1, v_2, v_3, 0, \epsilon(v) + 1)$ .

We obtain a 13-tuple, which is the initial segment of the colouring of  $\mathcal{Z}$ . Then the Mathematica code [22] attempts to produce an orientable 120-tuple, which encodes the entire colouring.  $\square$

We were not able to find any orientable  $\mathbb{F}_2^4$ -colouring of  $\mathcal{Z}$  extending a non-orientable  $\mathbb{F}_2^3$ -colouring of  $\mathcal{D}$ : our examples are not small covers.

Let now  $M_\lambda$  be the manifold obtained by one  $\mathbb{F}_2^5$ -colouring  $\lambda$  of  $\mathcal{Z}$ : it covers  $\mathcal{Z}$  with degree  $2^5 = 32$ . The colouring  $\lambda$  restricts to a non-orientable colouring  $\mu$  of the facet  $\mathcal{D}$ , which gives rise, by Lemma 2.8, to a non-orientable codimension-1 geodesic submanifold  $M_\mu \subset M_\lambda$ , covering  $\mathcal{D}$  with degree  $2^3 = 8$ .

In complete analogy to [13, Lemma 3.2], by cutting  $M_\lambda$  along  $M_\mu$  we get a compact orientable hyperbolic manifold tessellated by 32 right-angled 120-cells having a geodesic boundary isometric to a connected manifold  $\widetilde{M}_\mu$  that double-covers  $M_\mu$  and is hence tessellated by  $2 \cdot 2^3 = 16$  right-angled dodecahedra.

To prove Theorem 1.3 it only remains to extend this argument from one 120-cell to an appropriate assembling of  $n \geq 2$  distinct 120-cells.

### 3. ASSEMBLING RIGHT-ANGLED DODECAHEDRA AND 120-CELLS

Here, we assemble right-angled dodecahedra and 120-cells in order to construct more complicated convex compact right-angled convex compact polytopes.

**3.1. Connected sum of polytopes.** Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two right-angled polytopes in  $\mathbb{H}^n$ . If there is an isometry between two facets of  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , we may use it to glue them: the result is a new right-angled polytope in  $\mathbb{H}^n$  which we call a *connected sum* of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  along these facets.

**3.2. Assembling.** An *assembling* of right-angled dodecahedra (or 120-cells) is a right-angled polytope constructed from a finite sequence

$$\mathcal{P}_1 \# \mathcal{P}_2 \# \mathcal{P}_3 \# \dots \# \mathcal{P}_k$$

of connected sums performed from the left to the right, where each  $\mathcal{P}_i$  is a right-angled dodecahedron (or 120-cell).

**Lemma 3.1.** *An assembling of  $k$  right-angled dodecahedra is a facet of an assembling of  $k$  right-angled 120-cells.*

*Proof.* Consider  $\mathbb{H}^3$  inside  $\mathbb{H}^4$  as a geodesic hyperplane. Consider  $\mathcal{D} \subset \mathbb{H}^3$  as a facet of  $\mathcal{Z} \subset \mathbb{H}^4$ . Every time we attach a new copy  $\mathcal{D}_i$  of  $\mathcal{D}$  in  $\mathbb{H}^3$ , we correspondingly attach a new copy  $\mathcal{Z}_i$  of  $\mathcal{Z}$  having  $\mathcal{D}_i$  as a facet.  $\square$

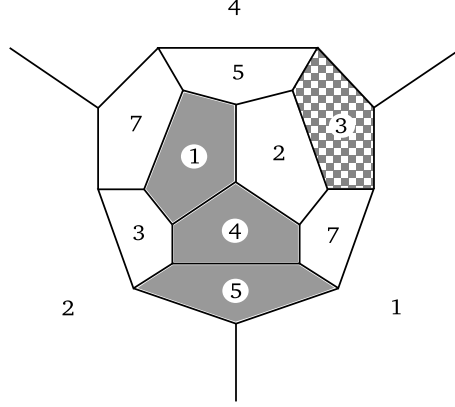


FIGURE 1. A non-orientable colouring of  $\mathcal{D}$  from [6, Table 1].  
Face colours are encoded by binary numbers

**3.3. Proof of Theorem 1.3.** We have described all the ingredients necessary to prove Theorem 1.3.

By Proposition 2.10, pick an orientable  $\mathbb{F}_2^5$ -colouring of the right-angled 120-cell  $\mathcal{Z}$  that induces a non-orientable  $\mathbb{F}_2^3$ -colouring of one dodecahedral facet  $\mathcal{D}$ .

Then assemble  $n$  copies of  $\mathcal{D}$  as

$$\mathcal{P} = \mathcal{D}_1 \# \mathcal{D}_2 \# \dots \# \mathcal{D}_n$$

and consider, as in Lemma 3.1, the resulting right-angled polyhedron  $\mathcal{P}$  as a facet of a right-angled polytope  $\mathcal{Q}$  made of  $n$  copies of the right-angled 120-cell, each having a  $\mathcal{D}_i$  as a facet.

Every time we assemble a new copy of  $\mathcal{D}_i$ , we give  $\mathcal{D}_i$  the colouring of the adjacent dodecahedron, mirrored along the glued pentagonal face, and we do the same with each corresponding new 120-cell  $\mathcal{Z}_i$ . The resulting polytope  $\mathcal{Q}$  inherits an orientable  $\mathbb{F}_2^5$ -colouring  $\lambda$  that induces a non-orientable  $\mathbb{F}_2^3$ -colouring  $\mu$  of  $\mathcal{P}$ , if  $\mathcal{P}$  is assembled appropriately. Indeed, each of the 24 non-orientable colourings of  $\mathcal{D}$  has three faces  $F$ ,  $F'$  and  $F''$  satisfying the conditions of Corollary 2.5. Moreover, we can find a fourth face  $F^*$  which is disjoint from each of them. These properties are easily verifiable by using [6, Table 1]. Then, we start assembling  $\mathcal{D}_i$ 's by forming a connected sum along  $F^*$ . Then the colouring of the resulting polytope  $\mathcal{P}$  again satisfies Corollary 2.5 and hence is non-orientable, as required.

By cutting  $M_\lambda$  along  $M_\mu$  we get a compact orientable hyperbolic manifold tessellated by  $32n$  right-angled 120-cells having a geodesic boundary that is isometric to a connected manifold  $\widetilde{M}_\mu$  that double-covers  $M_\mu$  and is hence tessellated by  $2 \cdot 2^3 n = 16n$  right-angled dodecahedra.  $\square$

We conclude the paper by providing an example of the construction carried in the proof above.



**Example 3.2.** Let us choose a non-orientable colouring  $\mu$  of the dodecahedron  $\mathcal{D}$  from [6, Table 1], say the one having the maximal symmetry group  $(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_6$ . This is an  $\mathbb{F}_2^3$ -colouring depicted in Fig. 1, with face colours encoded by binary numbers in the decimal range  $1, \dots, 7$ . Here, the grey shaded faces of  $\mathcal{D}$  are exactly the faces  $F$ ,  $F'$  and  $F''$ , satisfying the conditions of Corollary 2.5. The face  $F^*$ , along which we take a connected sum in the proof of Theorem 1.3, has a checkerboard shading. Now, we take a connected sum of  $\mathcal{D}_1 := \mathcal{D}$  along  $F^*$  with its isometric copy  $\mathcal{D}_2$ , having the same colouring. Then we can choose a face of  $\mathcal{D}_1 \# \mathcal{D}_2$ , distinct from any of  $F$ ,  $F'$  or  $F''$  and continue assembling until we use all  $n$  given copies of  $\mathcal{D}$ , which produce a polyhedron  $\mathcal{P} = \mathcal{D}_1 \# \dots \# \mathcal{D}_n$ . The faces  $F$ ,  $F'$  and  $F''$  of  $\mathcal{D}_1$  still remain among those of  $\mathcal{P}$ . Thus, the colouring of  $\mathcal{P}$  is again non-orientable by Corollary 2.5.

Finally, by applying Proposition 2.9 and Lemma 3.1, we obtain a four-dimensional polytope  $\widetilde{\mathcal{P}}$  with an orientable  $\mathbb{F}_2^5$ -colouring, that induces a non-orientable colouring on one of its facets, which is isometric to  $\mathcal{P}$ . Indeed, the non-orientable  $\mathbb{F}_2^3$ -colouring of each  $\mathcal{D}_i$  can be extended by Proposition 2.9 to an orientable  $\mathbb{F}_2^5$ -colouring of a 120-cell  $\mathcal{Z}_i$ . Then, by Lemma 3.1,  $\mathcal{P} = \mathcal{D}_1 \# \dots \# \mathcal{D}_n$  is a facet of a polytope  $\widetilde{\mathcal{P}} = \mathcal{Z}_1 \# \dots \# \mathcal{Z}_n$ . Since the colourings of  $\mathcal{D}_i$ 's match under taking connected sums, the colourings of  $\mathcal{Z}_i$ 's also match. The polytope  $\widetilde{\mathcal{P}}$  gives rise to a covering manifold with totally geodesic boundary, as described in the proof of Theorem 1.3.

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