On the Hilbert quasi-polynomials of non-standard graded rings

Let $R := \mathbb{K}[x_1, \ldots, x_k]$, I be an homogeneous ideal of R. The numerical function $H_{R/I} : \mathbb{N} \to \mathbb{N}$ defined by $H_{R/I}(d) := \dim_{\mathbb{K}}((R/I)_d)$ is called *Hilbert function of* R/I. It is eventually a polynomial, HP_R.

$R = \mathbb{Q}[x_1, \ldots, x_5]$, graded by $[1,1,1,1,1]$ (standard grading)

$$HP_R = 1/24x^4 + 5/12x^3 + 35/24x^2 + 25/12x + 1$$

R graded by $W = [1,2,3,4,6]$. $HP_R^W(n) = P_{n \bmod 12}(n)$

$$\begin{aligned}
HP_R^W = [P_0(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 1,\\
P_1(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456,\\
P_2(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 125/216,\\
P_3(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128,\\
P_4(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 20/27.\\
P_5(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456,\\
P_6(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 7/8,\\
P_7(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456,\\
P_8(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 19/27,\\
P_9(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128,\\
P_{10}(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 133/216,\\
P_{11}(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456]
\end{aligned}$$

- $d = lcm(1,2,3,4,6) = 12$, $\delta_3 = \delta_4 = 1$, $\delta_0 = 12$
- $\delta_2 = 2 = lcm(2)$ $(\{2,4,6\})$
- $\delta_1 = 6 = lcm(2,2,3,2)$ $(\{2,4\},\{2,6\},\{3,6\},\{4,6\})$