On the Hilbert quasi-polynomials for non-standard graded rings

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Abstract

The Hilbert function, its generating function and the Hilbert polynomial of a graded ring $\mathbb{K}[x_1, \dots, x_k]$ have been extensively studied since the famous paper of Hilbert: *Ueber die Theorie der algebraischen Formen* ([Hilbert, 1890]). In particular the coefficients and the degree of the Hilbert polynomial play an important role in Algebraic Geometry.
If the ring graduation is non-standard, then its Hilbert function is not definitely equal to a polynomial but to a quasi-polynomial.
It turns out that a Hilbert quasi-polynomial P of degree n splits into a polynomial S of degree n and a lower degree quasi-polynomial T. We have completely determined the degree of T and the first few coefficients of P. Moreover, the quasi-polynomial T has a periodic structure that we have described.
We have also developed a software to compute effectively the Hilbert quasi-polynomial for any ring $\mathbb{K}[x_1, \dots, x_k]/I$.

Keywords: Non-standard graduations, Hilbert function, Hilbert quasi-polynomial.

Introduction

From now on, $\mathbb{K}$ will be a field and $R := \mathbb{K}[x_1, \dots, x_k]$ a polynomial ring over $\mathbb{K}$ graded by a weight vector $W := [d_1, \dots, d_k] \in \mathbb{N}_+^k$, that is

$$\deg(x_1^{\alpha_1} \cdots x_k^{\alpha_k}) := d_1\alpha_1 + \dots + d_k\alpha_k$$

and the component of R of degree n is given by

$$R_n := \{f \in R \mid \deg(m) = n \,\forall\, m \in Supp(f)\} \quad \forall\, n \in \mathbb{N}$$

If $W = [1, \dots, 1]$ the graduation is called **standard**.
Let $I \subset R$ be a homogeneous ideal. The **Hilbert function** $H_{R/I}^W : \mathbb{N} \to \mathbb{N}$ of R/I is defined by

$$H_{R/I}^W(n) := \dim_{\mathbb{K}}((R/I)_n)$$

and the Poincaré series of R/I is given by

$$HP_{R/I}^W(t) := \sum_{n \in \mathbb{N}} H_{R/I}^W(n) t^n \in \mathbb{N}[\![t]\!]$$

Let σ be a term-order. Thanks to Macaulay's lemma which asserts that $H_{R/I}(n) = H_{R/LT_\sigma(I)}(n)\ \forall n \in \mathbb{N}$, we can restrict to consider only monomial ideals.

Theorem 1 (Hilbert-Serre) *Let R be graded by $W = [d_1, \dots, d_k] \in \mathbb{N}_+^k$ and $I \subset R$ a homogeneous ideal. Then $HP_{R/I}(t)$ is a rational function, that is*

$$HP_{R/I}(t) = \frac{h(t)}{\prod_{i=1}^k (1 - t^{d_i})} \in \mathbb{Z}[\![t]\!]$$

Definition 2 *Let R be a polynomial standard graded ring and I a homogeneous ideal of R. Then there exists a polynomial $P_{R/I}(x) \in \mathbb{Q}[x]$ such that*

$$H_{R/I}(n) = P_{R/I}(n)\ \forall\, n \gg 0$$

*This polynomial is called **Hilbert polynomial of** R/I.*

Hilbert Quasi-Polynomials

From now on, $(R/I, W)$ stands for the polynomial ring R/I, where I is a homogeneous ideal of R and R is graded by $W := [d_1, \dots, d_k] \in \mathbb{N}_+^k$. Let $d := lcm(d_1, \dots, d_k)$.

Proposition 3 *There exists an unique set of d polynomials $P_{R/I}^W := \{P_0, \dots, P_{d-1}\}$ such that*

$$H_{R/I}^W(n) = P_i(n) \qquad \forall\, i \equiv n \mod d$$

*$P_{R/I}^W$ is called the **Hilbert quasi-polynomial associated to (R/I,W)** whose elements are the P_i's.*

Proposition 4 *All the elements P_i of $P_{R/I}^W$ are rational polynomials and*

- *If $I = (0) \Rightarrow \deg(P_i) = k - 1$*
- *If $I \neq (0) \Rightarrow \deg(P_i) \leq k - 2$*

Due to the following result, we can restrict our study to the simplest case $(R/I, W)$ where $I = (0)$ and W is such that $d = 1$.

Proposition 5 *Let $HP_{R/I}(t) = \dfrac{\sum_{j=0}^r a_j t^j}{\prod_{i=1}^k (1 - t^{d_i})}$ and let $W' := a \cdot W = [d'_1, \dots, d'_k]$ for some $a \in \mathbb{N}_+$.
Then it holds:*

- $P_{R/I}^W(n) = \sum_{j=0}^r a_j P_R^W(n - j)\ \forall\, n \gg 0$
- $P_R^{W'} = \{P'_0, \dots, P'_{ad-1}\}$ *is such that*

$$P'_i(x) = \begin{cases} 0 & \text{if } a \nmid i \\ P_{\frac{i}{a}}\left(\frac{x}{a}\right) & \text{if } a \mid i \end{cases}$$

The structure of the Hilbert quasi-polynomials has a sort of regularity.

Proposition 6 *It holds*

$$P_R^W(x) = S(x) + T(x)$$

where $S(x) \in \mathbb{Q}[x]$ (the fixed part) has degree $k - 1$, whereas $T(x)$ (the periodic part) is a rational quasi-polynomial of degree $\delta - 1$, where

$$\delta := max\{|I| \mid gcd(d_i)_{i \in I} \neq 1 \text{ and } I \subseteq \{1, \dots, k\}\}$$

In addition, the r^{th} coefficient of P_R^W has periodicity

$$\delta_r := lcm\ (\ gcd(d_i)_{i \in I} \mid |I| = r + 1, I \subseteq \{1, \dots, k\})$$

for $r = 0, \dots, k - 1$. Formally, if we denote the r^{th} coefficient of $P_i(x)$ by c_{ir}, therefore if $j = i + \delta_r \mod d$ then $c_{jr} = c_{ir}$.

The r^{th} coefficients for $r = \delta, \dots, k - 1$ are hence the same for all P_i. We devise a strategy to find the formulas for these coefficients and we compute the first two of them after the leading coefficient.

Proposition 7 *For each elements P_i of P_R^W it holds*

$$lc(P_i) = \frac{1}{(k-1)! \prod_{i=1}^k d_i}$$

Let $\delta \leq r$ and let c_r be the r^{th} coefficient of P_R^W. Then the following formulas hold

$$c_{k-2} = \frac{\sum_{i=1}^k d_i}{2 \cdot (k-2)! \cdot \prod_{i=1}^k d_i}$$

and

$$c_{k-3} = \frac{3\left(\sum_{i=1}^k d_i\right)^2 - \sum_{i=1}^k d_i^2}{24(k-3)! \prod_{i=1}^k d_i}$$

Numerical Examples

Example 8. Let $R = \mathbb{Q}[x_1, \dots, x_5]$ be graded by $W = [1, 2, 3, 4, 6]$. Then its Hilbert quasi-polynomial $P_R^W = \{P_0, \dots, P_{11}\}$ is given by:

$$\begin{aligned}
P_0(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 1\\
P_1(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456\\
P_2(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 125/216\\
P_3(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128\\
P_4(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 20/27\\
P_5(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456\\
P_6(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 1/2x + 7/8\\
P_7(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 43/108x + 1705/3456\\
P_8(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 25/54x + 19/27\\
P_9(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 5/12x + 75/128\\
P_{10}(x) &= 1/3456x^4 + 1/108x^3 + 5/48x^2 + 13/27x + 133/216\\
P_{11}(x) &= 1/3456x^4 + 1/108x^3 + 19/192x^2 + 41/108x + 1001/3456
\end{aligned}$$

We observe that the leading coefficient and c_3 are the same for all P_i, whereas c_2 has periodicity 2, c_1 has periodicity 6 and the constant term has periodicity 12. In fact, we have

- $\delta_3 = \delta_4 = 1$
- $\delta_2 = 2$ $(\{2, 4, 6\})$
- $\delta_1 = 6$ $(\{2, 4\}, \{2, 6\}, \{3, 6\}$ and $\{4, 6\})$
- $\delta_0 = 12$

Let us consider R/I with $I = (x^3, yz)$, then the Hilbert quasi-polynomial $P_{R/I}^W$ is given by

$$\begin{aligned}
P_0(x) &= 1/16x^2 + 1/2x + 1\\
P_1(x) &= 1/24x^2 + 1/3x + 5/8\\
P_2(x) &= 1/16x^2 + 1/2x + 3/4\\
P_3(x) &= 1/24x^2 + 1/3x + 5/8\\
P_4(x) &= 1/16x^2 + 1/2x + 1\\
P_5(x) &= 1/24x^2 + 1/3x + 7/24\\
P_6(x) &= 1/16x^2 + 1/2x + 3/4\\
P_7(x) &= 1/24x^2 + 1/3x + 5/8\\
P_8(x) &= 1/16x^2 + 1/2x + 1\\
P_9(x) &= 1/24x^2 + 1/3x + 5/8\\
P_{10}(x) &= 1/16x^2 + 1/2x + 3/4\\
P_{11}(x) &= 1/24x^2 + 1/3x + 7/24
\end{aligned}$$

It is easy to see that the coefficients of $P_{R/I}^W$ are somewhat periodic. The characterization of this periodicity is work on progress.

We have written Singular procedures to compute the Hilbert quasi-polynomial for rings $\mathbb{K}[x_1, \dots, x_k]/I$. These procedures can be downloaded from the website `www.dm.unipi.it/~caboara/Research/HilbertQP`

Conclusions and further work

We have produced a partial characterization of Hilbert quasi-polynomials for the $\mathbb{N}_+^k$-graded rings $\mathbb{K}[x_1, \dots, x_k]$. We want to complete this characterization. Specifically, we want to find the closed formulas for as many as possible coefficients of the Hilbert quasi-polynomial, periodic part included. Moreover, we want to extend our work to $\mathbb{N}_+^k$-graded quotient rings $\mathbb{K}[x_1, \dots, x_k]/I$. This will allow us to write more efficient procedures for the computation of Hilbert quasi-polynomials.

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