SEMICLASSICAL STATES FOR A STATIC SUPERCritical KLEIN-GORDON-MAXWELL-PROCA System ON A CLOSED RIEMANNIAN MANIFOLD

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Abstract. We establish the existence of semiclassical states for a nonlinear Klein-Gordon-Maxwell-Proca system in static form, with Proca mass 1, on a closed Riemannian manifold.

Our results include manifolds of arbitrary dimension and allow supercritical nonlinearities. In particular, we exhibit a large class of 3-dimensional manifolds on which the system has semiclassical solutions for every exponent \( p \in (2, \infty) \). The solutions we obtain concentrate at closed submanifolds of positive dimension as the singular perturbation parameter goes to zero.

1. Introduction

Let \((\mathcal{M}, g)\) be a closed (i.e. compact and without boundary) smooth Riemannian manifold of dimension \( m \geq 2 \). Given real numbers \( \varepsilon > 0, q > 0, \omega \in \mathbb{R} \) and \( p \in (2, \infty) \), and a real-valued \( C^1 \)-function \( \alpha \) such that \( \alpha(x) > \omega^2 \) on \( \mathcal{M} \), we consider the system

\[
\begin{cases}
-\varepsilon^2 \Delta_g u + \alpha(x)u = u^{p-1} + \omega^2(qv - 1)^2 u & \text{on } \mathcal{M}, \\
-\Delta_g v + (1 + q^2 u^2)v = qu^2 & \text{on } \mathcal{M}, \\
u, v > 0.
\end{cases}
\]

The space \( H^1_g(\mathcal{M}) \) is the completion of \( C^\infty(\mathcal{M}) \) with respect to the norm defined by

\[
||v||^2_g = \int_{\mathcal{M}} (|\nabla_g v|^2 + v^2) d\mu_g.
\]

Solutions to this system correspond to standing waves of a Klein-Gordon-Maxwell-Proca (KGMP) system in static form (i.e. one in which the external Proca field is time-independent) with Proca mass 1.

KGMP-systems are massive versions of the more classical electrostatic Klein-Gordon-Maxwell (KGM) systems: KGM-systems are KGMP-systems with Proca mass 0, i.e. the second equation in \((1.1)\) is replaced by

\[-\Delta_g v + q^2 u^2 v = qu^2.\]

Note that \( v = 1/q \) solves this last equation and reduces the KGM-system to a single Schrödinger equation in \( u \). So for the system on a closed manifold the Proca

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formalism is more interesting and more appropriate. We refer to [11] for a detailed discussion on KGMP-systems and their physical meaning.

For \( \varepsilon = 1 \) existence of solutions to system (1.1), which are stable with respect to the phase \( \omega \), was established by Druet and Hebey [7] and Hebey and Truong [10] for manifolds of dimension \( m = 3 \) and 4, and subcritical (2 < \( p < \frac{2m}{m-2} \)) or critical (\( p = \frac{2m}{m-2} \)) nonlinearities, under certain assumptions. For critical systems in dimension 3 Hebey and Wei [11] showed the existence of standing waves with multispark amplitudes, which are unstable with respect to the phase, i.e. they blow up with \( k \) singularities as the phase \( \omega \) approaches some phase \( \omega_0 \).

Here we are interested in semiclassical states, i.e. in solutions to system (1.1) for \( \varepsilon \) small. The existence of semiclassical states for similar systems in flat domains \( \Omega \) in \( \mathbb{R}^m \) has been investigated e.g. in [4, 5, 15]. On closed 3-dimensional manifolds, the existence of semiclassical states to system (1.1), which concentrate at a single point as \( \varepsilon \to 0 \), was established in [8] and [9] for subcritical exponents \( p \in (2, 6) \).

The results we present in this paper apply to manifolds of arbitrary dimension and include supercritical nonlinearities \( p > 2^*_m \), where \( 2^*_m := \frac{2m}{m-2} \) is the critical Sobolev exponent in dimension \( m \geq 3 \) and \( 2^*_2 := \infty \). In particular, we shall exhibit a large class of 3-dimensional manifolds on which the system (1.1) has semiclassical solutions for every exponent \( p \in (2, \infty) \). The solutions \( u \) we obtain concentrate at closed submanifolds of \( \mathfrak{M} \) of positive dimension. Moreover, for fixed \( \varepsilon \), they are stable with respect to the phase in the sense of [7].

Our approach consists in reducing system (1.1) to a system of a similar type on a manifold \( M \) of lower dimension but with the same exponent \( p \). This way, if \( n := \dim M < \dim \mathfrak{M} := m \) and \( p \in [2^*_m, 2^*_n) \), then \( p \) is subcritical for the new system but it is critical or supercritical for the original one. Moreover, solutions of the new system which concentrate at a point in \( M \) as \( \varepsilon \to 0 \) will give rise to solutions of the original system concentrating at a closed submanifold of \( \mathfrak{M} \) of dimension \( m - n \) as \( \varepsilon \to 0 \).

This approach was introduced by Ruf and Srikanth in [13], where a Hopf map is used to obtain the reduction. Reductions may also be performed by means of other maps which preserve the Laplace-Beltrami operator, or by considering warped products, or by a combination of both, see [3, 14] and the references therein. We describe these reductions in the following two subsections.

1.1. **Warped products.** If \((M, g)\) and \((N, h)\) are closed smooth Riemannian manifolds of dimensions \( n \) and \( k \) respectively, and \( f : M \to (0, \infty) \) is a \( C^1 \)-map, the **warped product** \( M \times_f N \) is the cartesian product \( M \times N \) equipped with the Riemannian metric \( g := g + f^2 h \).

For example, if \( M \) is a closed Riemannian submanifold of \( \mathbb{R}^\ell \times (0, \infty) \), then
\[
\mathfrak{M} := \{(y, z) \in \mathbb{R}^\ell \times \mathbb{R}^{k+1} : (y, |z|) \in M \},
\]
with the induced euclidian metric, is isometric to the warped product \( M \times_f \mathbb{S}^k \), where \( \mathbb{S}^k \) is the standard \( k \)-sphere and \( f(x_1, \ldots, x_{\ell+1}) = x_{\ell+1} \).

Let \( \pi_M : M \times_f N \to M \) be the projection. A straightforward computation gives the following result, cf. [2].

**Proposition 1.1.** Let \( \beta : M \to \mathbb{R} \) and \( \alpha = \beta \circ \pi_M \). Then \( u_\varepsilon, v_\varepsilon : M \to \mathbb{R} \) solve
\[
\begin{cases}
-\varepsilon^2 \text{div}_g \left( f^k(x) \nabla_g u \right) + f^k(x) \beta(x)u = f^k(x)u^{p-1} + \omega^2 f^k(x)(qv - 1)^2 u & \text{on } M, \\
-\text{div}_g \left( f^k(x) \nabla_g v \right) + f^k(x) \left(1 + qv^2\right) v = qf^k(x)u^2 & \text{on } M,
\end{cases}
\]
iff \( u_\varepsilon := u_\varepsilon \circ \pi_M, v_\varepsilon := v_\varepsilon \circ \pi_M : M \times f^2 N \to \mathbb{R} \) solve

\[
\begin{cases}
-\varepsilon^2 \Delta_g u + \alpha(x) u = u^{p-1} + \omega^2 (qv - 1)^2 u & \text{on } M \times f^2 N, \\
-\Delta_g v + (1 + qu^2) v = qu^2 & \text{on } M \times f^2 N.
\end{cases}
\]  

(1.3)

Proposition 1.2. Assume there exist \( \beta : M \to \mathbb{R} \) and \( \mu : M \to (0, \infty) \) such that \( \beta \circ \pi = \alpha \) and \( \mu \circ \pi = \lambda^2 \). Then \( u_\varepsilon, v_\varepsilon : M \to \mathbb{R} \) solve the system

\[
\begin{cases}
-\varepsilon^2 \Delta_g u + \frac{\beta(x)}{\mu(x)} u = \frac{1}{\mu(x)} u^{p-1} + \frac{\omega^2}{\mu(x)} (qv - 1)^2 u & \text{on } M, \\
-\Delta_g v + \frac{1}{\mu(x)} (1 + qu^2) v = \frac{\omega^2}{\mu(x)} u^2 & \text{on } M,
\end{cases}
\]  

(1.5)

iff \( u_\varepsilon := u_\varepsilon \circ \pi_M, v_\varepsilon := v_\varepsilon \circ \pi_M : \mathcal{M} \to \mathbb{R} \) solve the system

\[
\begin{cases}
-\varepsilon^2 \Delta_g u + \alpha(x) u = u^{p-1} + \omega^2 (qv - 1)^2 u & \text{on } \mathcal{M}, \\
-\Delta_g v + (1 + qu^2) v = qu^2 & \text{on } \mathcal{M}.
\end{cases}
\]  

(1.6)

Again, if \( p \in (2^*, 2^*_n) \), the system (1.5) is subcritical and the system (1.6) is supercritical and, if the functions \( u_\varepsilon \) concentrate at a point \( \xi_0 \in M \) as \( \varepsilon \to 0 \), the functions \( u_\varepsilon := u_\varepsilon \circ \pi_M \) concentrate at the \((m - n)\)-dimensional submanifold \( \pi^{-1}_M(\xi_0) \) of \( \mathcal{M} \) as \( \varepsilon \to 0 \).

1.3. The main result for the general system. Propositions 1.1 and 1.2 suggest studying a more general KGMP-system.

Let \( (M, g) \) be a closed Riemannian manifold of dimension \( n = 2 \) or \( 3 \), \( a, b, c \in \mathcal{C}^1(M, \mathbb{R}) \) be strictly positive functions, \( \varepsilon, q \in (0, \infty) \), \( p \in (2, 2^*_n) \), and \( \omega \in \mathbb{R} \) be such that \( \alpha(x) > \omega^2 b(x) \) on \( M \). We consider the subcritical system

\[
\begin{cases}
-\varepsilon^2 \text{div}_g (c(x) \nabla_g u) + a(x) u = b(x) u^{p-1} + b(x) \omega^2 (qv - 1)^2 u & \text{in } M, \\
-\text{div}_g (c(x) \nabla_g v) + b(x)(1 + qu^2)v = b(x)qu^2 & \text{in } M,
\end{cases}
\]  

(1.7)

\begin{align*}
u, v &> 0.
\end{align*}
Theorem 1.3. Let $K$ be a $C^1$-stable critical set of the function $\Gamma : M \to \mathbb{R}$ given by

$$\Gamma(x) := \frac{c(x)\overline{\nabla}_a(x)\overline{\nabla}_a - \overline{\nabla}_a}{b(x)^{\frac{2}{p-2}}}.$$ 

Then, for $\varepsilon$ small enough, the system (1.7) has a solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon$ concentrates at a point $\xi_0 \in K$ as $\varepsilon \to 0$.

Recall that $K$ is a $C^1$-stable critical set of a function $f \in C^1(M, \mathbb{R})$ if $K \subset \{x \in M : \nabla_g f(x) = 0\}$ and for any $\mu > 0$ there exists $\delta > 0$ such that, if $h \in C^1(M, \mathbb{R})$ with

$$\max_{d_g(x,K) < \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,$$

then $h$ has a critical point $x_0$ with $d_g(x_0, K) \leq \mu$. Here $d_g$ denotes the geodesic distance associated to the Riemannian metric $g$.

1.4. The main results for the KGMP-system. Theorem 1.3 together with Propositions 1.1 and 1.2 yields the following results.

Theorem 1.4. Let $\mathcal{M}$ be the warped product $M \times f^2 N$ of two closed Riemannian manifolds $(M, g)$ and $(N, h)$ with $n := \dim M = 2$ or 3. Set $k := \dim N$, and let $p \in (2, \infty)$ if $n = 2$ and $p \in (2, 6)$ if $n = 3$. Assume there exists $\beta \in C^1(M, \mathbb{R})$ such that $\alpha = \beta \circ \pi_M$ and let $K$ be a $C^1$-stable critical set for the function $\Gamma := f^4 \beta^{\frac{2}{1-p}} - \frac{2}{\overline{\nabla}_a}$ on $M$. Then, for $\varepsilon$ small enough, the KGMP-system (1.1) has a solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon$ concentrates at the submanifold $\pi^{-1}_M(\xi_0) \cong (N, f^2(\xi_0)h)$ for some $\xi_0 \in K$ as $\varepsilon \to 0$.

Theorem 1.5. Assume there exist a closed Riemannian manifold $M$ with $n := \dim M = 2$ or 3 and a harmonic morphism $\pi : \mathcal{M} \to M$ whose dilation $\lambda$ is such that $\mu \circ \pi = \lambda^2$. Assume further that $\alpha = \beta \circ \pi$ with $\beta \in C^1(M, \mathbb{R})$. Let $p \in (2, \infty)$ if $n = 2$ and $p \in (2, 6)$ if $n = 3$, and let $K$ be a $C^1$-stable critical set for the function $\Gamma := \beta^{\frac{2}{1-p}} - \frac{2}{\mu \overline{\nabla}_a}$ on $M$. Then, for $\varepsilon$ small enough, the KGMP-system (1.3) has a solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon$ concentrates at the submanifold $\pi^{-1}(\xi_0)$ of $\mathcal{M}$ for some $\xi_0 \in K$ as $\varepsilon \to 0$.

This last result applies, in particular, to the standard 3-sphere $\mathcal{M} = \mathbb{S}^3$ and the real projective space $\mathcal{M} = \mathbb{R}P^3$ for all $p \in (2, \infty)$ with $\mu = \lambda \equiv 1$, see subsection 1.2.

The rest of the paper is devoted to the proof of Theorem 1.3. In section 2 we reduce the system to a single equation and give the outline of the proof of Theorem 1.3 which follows the well-known Lyapunov-Schmidt reduction procedure. In section 3 we establish the Lyapunov-Schmidt reduction and in section 4 we derive the expansion of the reduced energy functional. Section 5 is devoted to the proof of some technical results.

2. OUTLINE OF THE PROOF OF THEOREM 1.3

2.1. Reduction to a Single equation. First, we reduce the system to a single equation. To overcome the problems caused by the competition between $u$ and $v$, using an idea of Benci and Fortunato [2], we consider the map $\Psi : H^1_g(M) \to H^1_g(M)$ defined by the equation

$$(2.1) \quad - \text{div}_g(c(x)\nabla_g \Psi(u)) + b(x)(1 + q^2u^2)\Psi(u) = b(x)qu^2.$$
It follows from standard variational arguments that $\Psi$ is well-defined in $H^1_g(M)$.

Using the maximum principle and regularity theory it is not hard to prove that

$$0 < \Psi(u) < \frac{1}{q}$$

for all $u \in H^1_g(M)$.

For the proofs of the following two lemmas we refer to [7].

**Lemma 2.1.** The map $\Psi : H^1_g(M) \to H^1_g(M)$ is of class $C^1$, and its differential

$$V_u := \Psi'(u)$$

for every $h \in H^1_g(M)$. Moreover,

$$0 \leq \Psi'(u)[u] \leq \frac{2}{q}$$

for all $u \in H^1_g(M)$.

**Lemma 2.2.** The map $\Theta : H^1_g(M) \to \mathbb{R}$ given by

$$\Theta(u) := \frac{1}{2} \int_M b(x)(1 - q\Psi(u)) u^2 d\mu_g$$

is of class $C^1$ and

$$\Theta'(u)[h] = \int_M b(x)(1 - q\Psi(u))^2 u h d\mu_g$$

for all $u, h \in H^1_g(M)$.

Next, we introduce the functionals $I_\varepsilon, J_\varepsilon, G_\varepsilon : H^1_g(M) \to \mathbb{R}$ given by

$$I_\varepsilon(u) := J_\varepsilon(u) + \frac{\omega^2}{2} G_\varepsilon(u),$$

where

$$J_\varepsilon(u) := \frac{1}{2 \varepsilon^2} \int_M \left[ \varepsilon^2 c(x) |\nabla_g u|^2 + d(x) u^2 \right] d\mu_g - \frac{1}{p \varepsilon^2} \int_M b(x) (u^+)^p d\mu_g$$

with $d(x) := a(x) - \omega^2 b(x)$, and

$$G_\varepsilon(u) := \frac{q}{\varepsilon^2} \int_M b(x) \Psi(u) u^2 d\mu_g.$$

From Lemma 2.2 we deduce that

$$\frac{1}{2} G'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^2} \int_M b(x)[2 q \Psi(u) - q^2 \Psi^2(u)] u \varphi d\mu_g.$$

Hence,

$$I'_\varepsilon(u)[\varphi] = \frac{1}{\varepsilon^2} \int_M \varepsilon^2 c(x) |\nabla_g u| \nabla_g \varphi + a(x) u \varphi - b(x) (u^+)^{p-1} \varphi - b(x) \omega^2 (1 - q\Psi(u))^2 u \varphi d\mu_g.$$

Therefore, if $u$ is a critical point of the functional $I_\varepsilon$, then $u$ solves the problem

$$\begin{cases} 
-\varepsilon^2 \text{div}_g (c(x) \nabla_g u) + (a(x) - \omega^2 b(x)) u + \omega^2 q b(x) \Psi(u)(2 - q\Psi(u)) u = b(x) (u^+)^{p-1}, \\
   u \in H^1_g(M).
\end{cases}$$

If $u \neq 0$ by the maximum principle and regularity theory we have that $u > 0$. Thus the pair $(u, \Psi(u))$ is a solution of the system (1.7). This reduces the existence problem for the system (1.7) to showing that the functional $I_\varepsilon$ has a nontrivial critical point.
2.2. The limit problems. Theorem 1.3 concerns manifolds of dimensions 2 and 3. To simplify the exposition we shall treat in full detail only the case \( n = 2 \). Everything can be extended in a straightforward way to the case \( n = 3 \), except for the estimates in section 5. These estimates, however, were computed in the appendix of [9] for \( n = 3 \).

Henceforth, we assume that \( \dim M = 2 \). We fix \( r > 0 \) smaller than the injectivity radius of \( M \). We identify the tangent space of \( M \) at \( \xi \) with \( \mathbb{R}^2 \) and denote by \( B(x, r) \) the ball in \( \mathbb{R}^2 \) centered at \( x \) of radius \( r \) and by \( B_\rho(\xi, r) \) the ball in \( M \) centered at \( \xi \) of radius \( r \), with respect to the distance induced by the Riemannian metric \( g \).

The exponential map \( \exp_\xi : B(0, r) \to B_\rho(\xi, r) \) provides local coordinates on \( M \), which are called normal coordinates. We denote by \( g_\xi \) the Riemannian metric at \( \xi \) given in normal coordinates by the matrix \((g_{ij})\). We denote the inverse matrix by \((g^{ij}(z)) := (g_{ij}(z))^{-1}\) and write \(|g_\xi(z)| := \det(g_{ij}(z))\). Then, we have that

\[
\tag{2.6}
g^{ij}(\varepsilon z) = \delta_{ij} + \frac{\varepsilon^2}{2} \sum_{r,k=1}^{n} \frac{\partial^2 g^{ij}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = \delta_{ij} + o(\varepsilon),
\]

\[
\tag{2.7}
|g(\varepsilon z)|^{\frac{1}{2}} = 1 - \frac{\varepsilon^2}{4} \sum_{i,r,k=1}^{n} \frac{\partial^2 g^{ii}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3 |z|^3) = 1 + o(\varepsilon).
\]

Here \( \delta_{ij} \) denotes the Kronecker symbol.

For \( p \in (2, \infty) \) and \( \xi \in M \), set

\[
A(\xi) := \frac{a(\xi)}{c(\xi)}, \quad B(\xi) := \frac{b(\xi)}{c(\xi)}, \quad \gamma(\xi) := \left( \frac{a(\xi)}{b(\xi)} \right)^{\frac{p}{p-2}}.
\]

We consider the problem

\[
-c(\xi)\Delta V + a(\xi)V = b(\xi)V^{p-1}, \quad V \in H^1(\mathbb{R}^2),
\]

and denote by \( V^{\xi} \) its unique positive spherically symmetric solution. This problem is equivalent to

\[
-\Delta V + A(\xi)V = B(\xi)V^{p-1}, \quad V \in H^1(\mathbb{R}^2).
\]

The function \( V^{\xi} \) and its derivatives decay exponentially at infinity. \( V^{\xi} \) can be written as

\[
V^{\xi}(z) = \gamma(\xi)U(\sqrt{A(\xi)}z),
\]

where \( U \) is the unique positive spherically symmetric solution to

\[
-\Delta U + U = U^{p-1}, \quad U \in H^1(\mathbb{R}^2).
\]

For \( \xi \in M \) and \( \varepsilon > 0 \) we define \( W_{\varepsilon, \xi} \in H^1_0(M) \) by

\[
W_{\varepsilon, \xi}(x) := \begin{cases} 
V^{\xi} \left( \frac{1}{\varepsilon} \exp^{-1}(x) \right) \chi \left( \exp^{-1}(x) \right) & \text{if } x \in B_\rho(\xi, r), \\
0 & \text{otherwise},
\end{cases}
\]

where \( \chi \in C^\infty(\mathbb{R}^n) \) is a radial cut-off function such that \( \chi(z) = 1 \) if \( |z| \leq r/2 \) and \( \chi(z) = 0 \) if \( |z| \geq r \). Setting \( V_\varepsilon(z) := V \left( \frac{z}{\varepsilon} \right) \) and \( y := \exp^{-1}_\xi x \) we have that

\[
W_{\varepsilon, \xi}(\exp_\xi(y)) = V^{\xi} \left( \frac{y}{\varepsilon} \right) \chi(y) = V^{\xi}(y)\chi(y),
\]

so the function \( W_{\varepsilon, \xi} \) is simply the function \( V^{\xi} \) rescaled, cut off and read in normal coordinates at \( \xi \) in \( M \).
Similarly, for \( i = 1, 2 \) we define
\[
Z_{\epsilon, \xi}^i(x) = \begin{cases} 
\psi_\xi^i \left( \frac{1}{\epsilon} \exp_\xi^{-1}(x) \right) & \text{if } x \in B_g(\xi, r), \\
0 & \text{otherwise,}
\end{cases}
\]
where
\[
\psi_\xi^i(\eta) = \frac{\partial}{\partial \eta_i} V_\xi(\eta) = \gamma(\xi) \sqrt{A(\xi)} \frac{\partial U}{\partial \eta_i} (\sqrt{A(\xi)} \eta).
\]
The functions \( \psi_\xi^i \) are solutions of the linearized equation
\[
-\Delta \psi + A(\xi) \psi = (p - 1)B(\xi) (V_\xi)^{p-2} \psi \quad \text{in } \mathbb{R}^2.
\]

**Proposition 2.3.** There is a positive constant \( C \) such that
\[
\int_{Z_{\epsilon, \xi}^h, Z_{\epsilon, \xi}^k} = C\delta_{hk} + o(1),
\]
as \( \epsilon \to 0 \).

**Proof.** From the Taylor expansions of \( g^{ij}(\epsilon z), |g(\epsilon z)|^{\frac{1}{2}}, a(\exp_\xi(\epsilon z)) \) and \( c(\exp_\xi(\epsilon z)) \) we obtain
\[
\int_{B(0, r/\epsilon)} \sum_{i,j} c(\exp_\xi(\epsilon z)) g^{ij}_\xi(\epsilon z) \frac{\partial}{\partial z_i} (\psi_\xi^j(\epsilon z) \gamma(\epsilon z)) \frac{\partial}{\partial z_j} (\psi_\xi^i(\epsilon z) \gamma(\epsilon z)) |g_\xi(\epsilon z)|^{\frac{1}{2}} d\mu_g
\]

as \( \epsilon \to 0 \).

Next, we compute the derivatives of \( W_{\epsilon, \xi} \) with respect to \( \xi \) in normal coordinates. Fix \( \xi_0 \in M \). We write the points \( \xi \in B_g(\xi_0, r) \) as
\[
\xi = \xi(y) = \exp_{\xi_0}(y) \quad \text{with } y \in B(0, r).
\]
We define
\[
\mathcal{E}(y, x) = \exp^{-1}_{\xi_0}(x) = \exp^{-1}_{\exp_{\xi_0}(y)}(x),
\]
where \( x \in B_g(\xi(y), r) \) and \( y \in B(0, r) \). Then we can write
\[
W_{\epsilon, \xi}(x) = \gamma(\xi(y)) U_\epsilon \left( \sqrt{A(\xi(y))} \exp^{-1}_{\xi(y)}(x) \right) \chi(\exp^{-1}_{\xi(y)}(x))
\]

where \( \tilde{A}(y) = A(\exp_{\xi_0}(y)) \) and \( \tilde{\gamma}(y) = \gamma(\exp_{\xi_0}(y)) \). Thus we have
\[
\frac{\partial}{\partial y_s} W_{\epsilon, \xi}(y) \bigg|_{y=0} = \left( \frac{\partial}{\partial y_s} \tilde{\gamma}(y) \right) \bigg|_{y=0} U_\epsilon \left( \frac{1}{\sqrt{\tilde{A}(y)}} \mathcal{E}(0, x) \right) \frac{\partial}{\partial y_s} \chi(\mathcal{E}(0, x))
\]

as \( \epsilon \to 0 \).
If \( x = \exp_{\xi_0} \varepsilon z, \xi_0 = \xi(0) \), then \( \mathcal{E}(0, x) = \varepsilon z \) and we have

\[
\frac{\partial}{\partial y_s} W_{\varepsilon, \xi}(y) \bigg|_{y=0} = \left( \frac{\partial}{\partial y_s} \tilde{\chi}(y) \right)_{y=0} U(\sqrt{A(0)} z) \chi(\varepsilon z) \\
+ \frac{\partial}{\partial \eta_k} \tilde{\chi}(\varepsilon z) \frac{\partial}{\partial y_s} \mathcal{E}_k(y, \exp_{\xi_0} \varepsilon z) \bigg|_{y=0} \\
+ \frac{\partial}{\partial \eta_k} \chi(\varepsilon z) \frac{\partial}{\partial y_s} U(\sqrt{A(0)} z) \frac{\partial}{\partial y_s} \mathcal{E}_k(y, \exp_{\xi_0} \varepsilon z) \bigg|_{y=0}.
\]

(2.8)

We also recall the following Taylor expansions:

\[
\frac{\partial}{\partial y_h} \mathcal{E}_k(0, \exp_{\xi_0} \varepsilon z) = -\delta_{kh} + O(\varepsilon^2 |z|^2).
\]

(2.9)

**2.3. Outline of the proof of Theorem 1.3.** Let \( H_\varepsilon \) denote the Hilbert space \( H^1_\varepsilon(M) \) equipped with the inner product

\[
\langle u, v \rangle_\varepsilon := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) \nabla_g u \nabla_g v \, d\mu_g + \int_M d(x) u v \, d\mu_g \right),
\]

which induces the norm

\[
\| u \|_\varepsilon^2 := \frac{1}{\varepsilon^2} \left( \varepsilon^2 \int_M c(x) |\nabla_g u|^2 \, d\mu_g + \int_M d(x) u^2 \, d\mu_g \right),
\]

with \( d(x) := a(x) - \omega^2 b(x) > 0 \). Similarly, let \( L^q_\varepsilon \) be the Banach space \( L^q_\varepsilon(M) \) with the norm

\[
|u|_{q,\varepsilon} := \left( \frac{1}{\varepsilon^2} \int_M |u|^q \, d\mu_g \right)^{1/q}.
\]

Since we are assuming that \( \dim M = 2 \), for each \( q \geq 2 \) the embedding \( H_\varepsilon \hookrightarrow L^q_\varepsilon \) is continuous. In fact, there is a positive constant \( C \), independent of \( \varepsilon \), such that

\[
|u|_{q,\varepsilon} \leq C \| u \|_\varepsilon \quad \forall u \in H_\varepsilon.
\]

(2.10)

Moreover, this embedding is compact.

Fix \( p \in (2, \infty) \). The adjoint operator \( i^*_\varepsilon : L^p_\varepsilon \rightarrow H_\varepsilon, p' := \frac{p}{p-1} \), to the embedding \( i_\varepsilon : H_\varepsilon \hookrightarrow L^p_\varepsilon \) is defined by

\[
u = i^*_\varepsilon(v) \iff \langle u, \varphi \rangle_\varepsilon = \frac{1}{\varepsilon^2} \int_M v \varphi \quad \forall \varphi \in H_\varepsilon \\
\iff -\varepsilon^2 \text{div}_g (c(x) \nabla_g u) + d(x) u = v, \quad u \in H^1_\varepsilon(M).
\]

One has that

\[
\| i^*_\varepsilon(v) \|_\varepsilon \leq C|v|_{p',\varepsilon} \quad \forall v \in L^p_\varepsilon,
\]

(2.11)

where the constant \( C \) does not depend on \( \varepsilon \).

Using the adjoint operator we can rewrite problem (2.5) as

\[
u = i^*_\varepsilon \left[ b(x)f(u) + \omega^2 b(x)g(u) \right], \quad u \in H_\varepsilon,
\]

(2.12)

where

\[
f(u) := (u^+)^{p-1} \quad \text{and} \quad g(u) := (q^2 \Psi^2(u) - 2q \Psi(u)) u.
\]

Let

\[
K_{\varepsilon, \xi} := \text{Span} \left\{ Z_{\varepsilon, \xi}', Z_{\varepsilon, \xi}'' \right\}
\]
and
\[ K_{\varepsilon, \xi}^\perp := \left\{ \phi \in H_{\varepsilon} : \langle \phi, Z_i^{\varepsilon, \xi} \rangle_{\varepsilon} = 0, \ i = 1, 2 \right\}. \]

We denote the projections onto these subspaces by
\[ \Pi_{\varepsilon, \xi} : H_{\varepsilon} \to K_{\varepsilon, \xi} \quad \text{and} \quad \Pi_{\varepsilon, \xi}^\perp : H_{\varepsilon} \to K_{\varepsilon, \xi}^\perp. \]

We look for a solution of (2.5) of the form
\[ u_\varepsilon := W_{\varepsilon, \xi} + \phi \quad \text{with} \quad \phi \in K_{\varepsilon, \xi}^\perp. \]

This is equivalent to solving the pair of equations
\begin{align*}
(2.13) & \quad \Pi_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \phi - i \varepsilon \left[ b(x)f(W_{\varepsilon, \xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon, \xi} + \phi) \right] \right\} = 0, \\
(2.14) & \quad \Pi_{\varepsilon, \xi} \left\{ W_{\varepsilon, \xi} + \phi - i \varepsilon \left[ b(x)f(W_{\varepsilon, \xi} + \phi) + \omega^2 b(x)g(W_{\varepsilon, \xi} + \phi) \right] \right\} = 0.
\end{align*}

The first step of the proof of Theorem 1.3 is to solve equation (2.13). More precisely, for any fixed \( \xi \in M \) and \( \varepsilon \) small enough, we will show that there is a function \( \phi \in K_{\varepsilon, \xi}^\perp \) such that (2.13) holds. To do this we consider the linear operator
\[ L_{\varepsilon, \xi} : K_{\varepsilon, \xi}^\perp \to K_{\varepsilon, \xi}^\perp \]

For the proof of the following statement we refer to Lemma 4.1 of [3] (see also Proposition 3.1 of [12]).

**Proposition 2.4.** There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for every \( \varepsilon \in (0, \varepsilon_0) \), \( \xi \in M \) and \( \phi \in K_{\varepsilon, \xi}^\perp \),
\[ \| L_{\varepsilon, \xi}(\phi) \|_{\varepsilon} \geq C \| \phi \|_{\varepsilon}. \]

This result allows to use a contraction mapping argument to solve equation (2.13). The following statement is proved in section 3.

**Proposition 2.5.** There exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that, for each \( \xi \in M \) and each \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique \( \phi_{\varepsilon, \xi} \in K_{\varepsilon, \xi}^\perp \) which solves equation (2.13).

Moreover,
\[ \| \phi_{\varepsilon, \xi} \|_{\varepsilon} \leq C \varepsilon. \]

The map \( \xi \mapsto \phi_{\varepsilon, \xi} \) is a \( C^1 \)-map.

The second step is to solve equation (2.14). More precisely, for \( \varepsilon \) small enough we will find a point \( \xi \in M \) such that equation (2.14) is satisfied. To this end we introduce the reduced energy functional \( I_{\varepsilon} : M \to \mathbb{R} \) defined by
\[ I_{\varepsilon}(\xi) := I_{\varepsilon}(W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi}), \]

where \( I_{\varepsilon} \) is the variational functional defined in (2.1) whose critical points are the solutions to problem (2.5). It is easy to verify that \( \xi_{\varepsilon} \) is a critical point of \( I_{\varepsilon} \) if and only if the function \( u_{\varepsilon} = W_{\varepsilon, \xi} + \phi_{\varepsilon, \xi} \) is a critical point of \( I_{\varepsilon} \).

In Lemmas 4.1 and 4.2 we compute the asymptotic expansion of the reduced functional \( I_{\varepsilon} \) with respect to the parameter \( \varepsilon \). We prove the following result.

**Proposition 2.6.** The expansion
\[ \tilde{I}_{\varepsilon}(\xi) = C \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{n-2}{2}} b(\xi)^{\frac{n}{2}}}{b(\xi)^{\frac{n}{2}}} + o(1) = C \Gamma(\xi) + o(1), \]
holds true \( C^1 \)-uniformly with respect to \( \xi \) as \( \varepsilon \to 0 \), where \( C = \left( \frac{1}{\frac{1}{2}} - \frac{1}{p} \right) \int_{\mathbb{R}^n} U^p dz \).
Using the previous propositions we now prove Theorem 1.3.

**Proof of Theorem 1.3.** Since $K$ is a $C^1$-stable critical set for $\Gamma$, by Proposition 2.6 $I_{\varepsilon}$ has a critical point $\xi_{\varepsilon} \in M$ such that $d_{g}(\xi_{\varepsilon}, K) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, $u_{\varepsilon} = W_{\varepsilon, \xi_{\varepsilon}} + \phi_{\varepsilon, \xi_{\varepsilon}}$ is a solution of (2.3), and the pair $(u_{\varepsilon}, \Psi(u_{\varepsilon}))$ is a solution to the system (1.7) such that $u_{\varepsilon}$ concentrates at a point $\xi_0 \in K$ as $\varepsilon \rightarrow 0$. \hfill $\square$

3. The finite dimensional reduction

This section is devoted to the proof of Proposition 2.6. We denote by

\begin{equation}
\|u\|_{g}^2 := \int_{M} (|\nabla_{g} u|^2 + u^2) \, d\mu_{g} \quad \text{and} \quad |u|_{g,q} := \int_{M} |u|^q \, d\mu_{g}
\end{equation}

the standard norms in the spaces $H^1_{g}(M)$ and $L^q(M)$.

Equation (2.13) is equivalent to

\begin{equation}
L_{\varepsilon, \xi}(\phi) = N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi},
\end{equation}

where

\begin{align*}
N_{\varepsilon, \xi}(\phi) &:= \Pi_{\varepsilon, \xi}^{-1} \{ i_{\varepsilon}^*[b(x) \, f(W_{\varepsilon, \xi} + \phi) - f(W_{\varepsilon, \xi})] \phi \}, \\
S_{\varepsilon, \xi}(\phi) &:= \omega^2 \Pi_{\varepsilon, \xi}^{-1} \{ i_{\varepsilon}^*[b(x)(g^2 \Psi^2(W_{\varepsilon, \xi} + \phi) - 2q\Psi(W_{\varepsilon, \xi} + \phi))(W_{\varepsilon, \xi} + \phi)] \}, \\
R_{\varepsilon, \xi} &:= \Pi_{\varepsilon, \xi}^{-1} \{ i_{\varepsilon}^*[b(x)f(W_{\varepsilon, \xi})] - W_{\varepsilon, \xi}\}.
\end{align*}

In order to solve equation (3.2) we will show that the operator $T_{\varepsilon, \xi} : K_{\varepsilon, \xi}^{-1} \rightarrow K_{\varepsilon, \xi}^{-1}$ defined by

\begin{equation}
T_{\varepsilon, \xi}(\phi) := L_{\varepsilon, \xi}^{-1}(N_{\varepsilon, \xi}(\phi) + S_{\varepsilon, \xi}(\phi) + R_{\varepsilon, \xi})
\end{equation}

has a fixed point. To this end we prove that $T_{\varepsilon, \xi}$ is a contraction mapping on suitable ball in $H_{\varepsilon}$. We start with an estimate for $R_{\varepsilon, \xi}$.

**Lemma 3.1.** There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\xi \in M$ and any $\varepsilon \in (0, \varepsilon_0)$, the inequality

\begin{equation}
\|R_{\varepsilon, \xi}\|_{g} \leq C \varepsilon
\end{equation}

holds true.

**Proof.** See Lemma 4.2 in [3]. \hfill $\square$

Next, we give an estimate for $N_{\varepsilon, \xi}(\phi)$.

**Lemma 3.2.** There exist $\varepsilon_0 > 0$, $C > 0$ and $\tilde{C} \in (0, 1)$ such that, for any $\xi \in M$, $\varepsilon \in (0, \varepsilon_0)$ and $R > 0$, the inequalities

\begin{align}
&\|N_{\varepsilon, \xi}(\phi)\|_{g} \leq C(\|\phi\|_{g}^2 + \|\phi\|_{g}^{p-1}), \\
&\|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_{g} \leq \tilde{C}\|\phi_1 - \phi_2\|_{g},
\end{align}

hold true for $\phi, \phi_1, \phi_2 \in \{ \phi \in H_{\varepsilon} : \|\phi\|_{g} \leq R \varepsilon \}$.

**Proof.** By direct computation we obtain

\begin{equation}
|f'(W_{\varepsilon, \xi} + v) - f'(W_{\varepsilon, \xi})| \leq \left\{
\begin{array}{ll}
C W_{\varepsilon, \xi}^{p-3}|v| & 2 < p < 3, \\
C(W_{\varepsilon, \xi}^{p-3}|v| + |v|^{p-2}) & p \geq 3.
\end{array}
\right.
\end{equation}

From the mean value theorem and inequality (2.11) we derive

\begin{equation}
\|N_{\varepsilon, \xi}(\phi_1) - N_{\varepsilon, \xi}(\phi_2)\|_{g} \leq C \|f'(W_{\varepsilon, \xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon, \xi})\|_{g, \varepsilon} \|\phi_1 - \phi_2\|_{g}.
\end{equation}
Using (3.5) we conclude that
\[ C |f'(W_{\varepsilon, \xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon, \xi})|_{\ell_{\varepsilon}} < 1 \]
provided \(\|\phi_1\|_\varepsilon\) and \(\|\phi_2\|_\varepsilon\) are small enough. The same estimates yield (3.3). \(\Box\)

Now we estimate \(S_{\varepsilon, \xi}(\phi)\).

**Lemma 3.3.** There exists \(\varepsilon_0 > 0\) and \(C > 0\) such that, for any \(\xi \in M, \varepsilon \in (0, \varepsilon_0)\) and \(R > 0\), the inequalities
\[
\|S_{\varepsilon, \xi}(\phi)\|_\varepsilon \leq C\varepsilon,
\]
\[
\|S_{\varepsilon, \xi}(\phi_1) - S_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq \varepsilon \|\phi_1 - \phi_2\|_\varepsilon,
\]
hold true for \(\phi, \phi_1, \phi_2 \in \{\phi \in H_\varepsilon : \|\phi\|_\varepsilon \leq R\varepsilon\}, \) where \(\varepsilon \to 0\) as \(\varepsilon \to 0\).

**Proof.** Let us prove (3.4). From the definition of \(i^\ast\) and inequality (2.11) we derive
\[
\|S_{\varepsilon, \xi}(\phi)\|_\varepsilon \leq C \left( \Psi^2 (W_{\varepsilon, \xi} + \phi) (W_{\varepsilon, \xi} + \phi) |_{p', \varepsilon} + |\Psi (W_{\varepsilon, \xi} + \phi) (W_{\varepsilon, \xi} + \phi) |_{p', \varepsilon} \right)
\]
\[
= : I_1 + I_2.
\]
For any \(t \in (2, \infty)\), setting \(s := \frac{tp}{p'}\) and \(\vartheta := \frac{2}{t} \in (1, 2)\) and applying Lemma 5.3 and Remark 5.2 we obtain
\[
I_2 \leq C \frac{1}{\varepsilon^2/p'} \left( \int_M |\Psi (W_{\varepsilon, \xi} + \phi)|^t d\mu_g \right)^{t/p'} \left( \int_M |W_{\varepsilon, \xi} + \phi|^s d\mu_g \right)^{s/p}
\]
\[
\leq C \frac{1}{\varepsilon^{2/p'}} \|\Psi (W_{\varepsilon, \xi} + \phi)\|_g \left( \varepsilon^2 \left( \frac{1}{\varepsilon^2} \int_M |W_{\varepsilon, \xi}|^p d\mu_g \right)^{p/p'} + |\phi|_{g, s} \right)
\]
\[
\leq C \left( \varepsilon^{\theta + \frac{2}{p'}} + \varepsilon^{\vartheta + 1 - \frac{2}{p'}} \right) = C \left( \varepsilon^{\theta + \frac{2}{p'}} + \varepsilon^{\vartheta + 1 - \frac{2}{p'}} \right)
\]

for all \(\|\phi\|_\varepsilon \leq R\varepsilon\). From this estimate we deduce that \(I_1 \leq C\varepsilon\) and, hence, (3.6) follows.

Next, we prove (3.7). From inequality (2.11) we obtain that
\[
\|S_{\varepsilon, \xi}(\phi_1) - S_{\varepsilon, \xi}(\phi_2)\|_\varepsilon \leq C \left| \Psi (W_{\varepsilon, \xi} + \phi_1) - \Psi (W_{\varepsilon, \xi} + \phi_2) \right|_{p', \varepsilon}
\]
\[
+ C \left| \Psi^2 (W_{\varepsilon, \xi} + \phi_1) - \Psi^2 (W_{\varepsilon, \xi} + \phi_2) \right|_{p', \varepsilon}
\]
\[
+ C \left| \Psi (W_{\varepsilon, \xi} + \phi_1) - \Psi (W_{\varepsilon, \xi} + \phi_2) \phi_2 \right|_{p', \varepsilon}
\]
\[
+ C \left| \Psi^2 (W_{\varepsilon, \xi} + \phi_1) - \Psi^2 (W_{\varepsilon, \xi} + \phi_2) \phi_2 \right|_{p', \varepsilon}
\]
\[
= : I_1 + I_2 + I_3 + I_4.
\]
By Remark 5.2 and Lemma 5.4 with \(s := \frac{3}{2} t\), for some \(\theta \in (0, 1)\) we have that
\[
\begin{align*}
I_1' &\leq C \frac{1}{\varepsilon^2} \left( \int_M |\Psi' (W_{\varepsilon, \xi} + \theta \phi_1 + (1 - \theta) \phi_2) (\phi_1 - \phi_2)|^p \frac{d\mu_g}{\varepsilon^p} \right)^{\frac{1}{p}} \left( \int_M |W_{\varepsilon, \xi}|^{p'/p} \frac{d\mu_g}{\varepsilon^p} \right)^{\frac{p-1}{p}}
\leq C \frac{\varepsilon^{2(\theta - 1)}}{\varepsilon^2} \left( \varepsilon^{\theta + \frac{2}{p'}} + \|\phi_1\|_g + \|\phi_2\|_g \right)^{p'} \|\phi_1 - \phi_2\|_g
\leq C \varepsilon \|\phi_1 - \phi_2\|_\varepsilon^{p'},
\end{align*}
\]
\[
I_2' \leq C \frac{1}{\varepsilon^2} \left( \int_M |\Psi' (W_{\varepsilon, \xi} + \phi_1) (\phi_1 - \phi_2)|^p \frac{d\mu_g}{\varepsilon^p} \right)^{\frac{1}{p}} \left( \int_M |W_{\varepsilon, \xi}|^{p'/p} \frac{d\mu_g}{\varepsilon^p} \right)^{\frac{p-1}{p}}
\]
for \( \| \phi_1 \|_\epsilon, \| \phi_2 \|_\epsilon \leq R \epsilon \), with 
\( l_\epsilon := \epsilon^{\frac{(p-2)}{2}} \) → 0 as \( \epsilon \to 0 \). From the estimate of \( I_1 \), recalling that \( 0 \leq \Psi(u) \leq \frac{1}{q} \), we derive

\[
I_2' = \frac{1}{\epsilon^2} \int_M |\Psi(W_{e,\xi} + \phi_1) + \Psi(W_{e,\xi} + \phi_2)|^{p'} |\Psi(W_{e,\xi} + \phi_1) - \Psi(W_{e,\xi} + \phi_2)|^{p'} |W_{e,\xi}|^{p'} \\
\leq CI_1'.
\]

On the other hand, choosing \( \vartheta \in (1, 2) \) in Lemma 5.3 such that \( \vartheta p' > 2 \) and applying Lemma 5.4 with \( s := \frac{3}{2} \), we obtain

\[
I_3' \leq \frac{1}{\epsilon^2} \int_M |\Psi(W_{e,\xi} + \theta \phi_1 + (1 - \theta) \phi_2) (\phi_1 - \phi_2)|^{p'} |\phi_1|^{p'} \\
+ \frac{1}{\epsilon^2} \int_M |\Psi(W_{e,\xi} + \phi_2)|^{p'} |\phi_1 - \phi_2|^{p'} \\
\leq C \frac{1}{\epsilon^2} \left( \int_M |\Psi(W_{e,\xi} + \theta \phi_1 + (1 - \theta) \phi_2) (\phi_1 - \phi_2)|^p \right)^{\frac{p'}{p}} \left( \int_M |\phi_1 - \phi_2|^{p'} \right)^{\frac{p - p'}{p}} \\
+ C \frac{1}{\epsilon^2} \left( \int_M |\phi_1 - \phi_2|^p \right)^{\frac{p'}{p}} \left( \int_M |\Psi(W_{e,\xi} + \phi_2)|^{\frac{p'}{p}} \right)^{\frac{p - p'}{p}} \\
\leq C \frac{1}{\epsilon^2} \left( 4 \| \theta \phi_1 + (1 - \theta) \phi_2 \|_g \right)^{\frac{p'}{p}} \| \phi_1 - \phi_2 \|_{g}^{\frac{p'}{p}} \| \phi_1 \|_g^{\frac{p}{p}} \\
+ C \frac{\epsilon^{\vartheta p}}{\epsilon^2} (1 + \| \phi_1 - \phi_2 \|_g^2)^{\frac{p - p'}{2}} \| \phi_1 - \phi_2 \|_{g}^{\frac{p'}{p}} \\
\leq C \left( \frac{\epsilon^{\vartheta p}}{\epsilon^2} + \frac{\epsilon^{\vartheta p}}{\epsilon^2} \right) \| \phi_1 - \phi_2 \|_{g}^{\frac{p'}{p}} = l_\epsilon \| \phi_1 - \phi_2 \|_{g}^{\frac{p'}{p}}.
\]

for \( \| \phi_1 \|_\epsilon, \| \phi_2 \|_\epsilon \leq R \epsilon \), where \( l_\epsilon \to 0 \) as \( \epsilon \to 0 \).

Finally, from the estimate of \( I_2 \) we derive \( I_4' \leq CI_3' \). Collecting the previous estimates we obtain \( \| T_{e,\xi} (\phi) \|_\epsilon \leq C (\| N_{e,\xi}(\phi) \|_\epsilon + \| S_{e,\xi}(\phi) \|_\epsilon + \| R_{e,\xi} \|_\epsilon) \).

**Proof of Proposition 2.5** From Proposition 2.4 we deduce

\[
\| T_{e,\xi} (\phi) \|_\epsilon \leq C (\| N_{e,\xi}(\phi) \|_\epsilon + \| S_{e,\xi}(\phi) \|_\epsilon + \| R_{e,\xi} \|_\epsilon)
\]

and

\[
\| T_{e,\xi}(\phi_1) - T_{e,\xi}(\phi_2) \|_\epsilon \leq C \| N_{e,\xi}(\phi_1) - N_{e,\xi}(\phi_2) \|_\epsilon + C \| S_{e,\xi}(\phi_1) - S_{e,\xi}(\phi_2) \|_\epsilon.
\]

Lemmas 5.4, 5.5 and 5.6 imply that \( T_{e,\xi} \) is a contraction in the ball centered at 0 of radius \( R \epsilon \) in \( K_{e,\xi}^+ \) for a suitable constant \( R \). Hence, \( T_{e,\xi} \) has a unique fixed point.

In order to prove that the map \( \xi \mapsto \phi_{e,\xi} \) is \( C^1 \) we apply the implicit function theorem to the \( C^1 \)-function \( G : M \times H_e \to H_e \) defined by

\[
G(\xi, u) := \Pi_{e,\xi} \left\{ W_{e,\xi} + \Pi_{e,\xi}^+ u - i_{e}^* \left[ b(x) f \left( W_{e,\xi} + \Pi_{e,\xi}^+ u \right) + \omega^2 b(x) g \left( W_{e,\xi} + \Pi_{e,\xi}^+ u \right) \right] \right\} + \Pi_{e,\xi} u.
\]
Indeed, by (3.5) we have

\[
\frac{\partial G}{\partial u}(\xi, \phi, \varepsilon) (u) = \Pi_{\varepsilon, \xi} (\Pi_{\varepsilon, \xi} (u) - i_{\varepsilon}^* \left[ b(x)f'(W_{\varepsilon, \xi} + \phi, \varepsilon) \Pi_{\varepsilon, \xi} (u) + \omega^2 b(x)g'(W_{\varepsilon, \xi} + \phi, \varepsilon) \Pi_{\varepsilon, \xi} (u) \right] ) + \Pi_{\varepsilon, \xi} (u)
\]

is invertible, provided \( \varepsilon \) is small enough. For any \( \phi \) with \( \| \phi \|_{\varepsilon} \leq C\varepsilon \) we have that

\[
\left\| \frac{\partial G}{\partial u}(\xi, \phi, \varepsilon) (u) \right\|_{\varepsilon} \geq C \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon}
\]

\[+ C \| \Pi_{\varepsilon, \xi} \left\{ \Pi_{\varepsilon, \xi} (u) - i_{\varepsilon}^* \left[ f'(W_{\varepsilon, \xi} + \phi, \varepsilon) \Pi_{\varepsilon, \xi} (u) + \omega^2 g'(W_{\varepsilon, \xi} + \phi, \varepsilon) \Pi_{\varepsilon, \xi} (u) \right] \right\} \|_{\varepsilon}
\]

\[\geq C \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon} + C \| L_{\varepsilon, \xi} (\Pi_{\varepsilon, \xi} (u)) \|_{\varepsilon}
\]

\[+ C \| \Pi_{\varepsilon, \xi} \left\{ i_{\varepsilon}^* \left[ f'(W_{\varepsilon, \xi} + \phi, \varepsilon) - f'(W_{\varepsilon, \xi}) \right] \Pi_{\varepsilon, \xi} (u) \right\} \|_{\varepsilon}
\]

\[+ C \| \Pi_{\varepsilon, \xi} \left\{ i_{\varepsilon}^* \left[ \omega^2 g' \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \Pi_{\varepsilon, \xi} (u) \right] \right\} \|_{\varepsilon}
\]

\[\geq C \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon} + C \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon} - o(1) \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon}
\]

\[\geq C \| u \|_{\varepsilon}.
\]

Indeed, by (3.5) we have

\[
\| \Pi_{\varepsilon, \xi} \left\{ i_{\varepsilon}^* \left[ f'(W_{\varepsilon, \xi} + \phi, \varepsilon) - f'(W_{\varepsilon, \xi}) \right] \Pi_{\varepsilon, \xi} (u) \right\} \|_{\varepsilon} \leq C \left( \| \phi \|_{\varepsilon}^{p-2} + \| \phi \|_{\varepsilon} \right) \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon}
\]

\[= o(1) \| \Pi_{\varepsilon, \xi} (u) \|_{\varepsilon}.
\]

Moreover,

\[
\left\| \Pi_{\varepsilon, \xi} \left\{ i_{\varepsilon}^* \left[ \omega^2 g' \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \Pi_{\varepsilon, \xi} (u) \right] \right\} \right\|_{\varepsilon}
\]

\[\leq C \left\| \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \left( 2q - 2q^2 \Psi \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \right) \Psi' \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \| \Pi_{\varepsilon, \xi} (u) \|_{p', \varepsilon}
\]

\[+ C \left\| 2q \Psi \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) - q^2 \Psi^2 \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \right\| \Pi_{\varepsilon, \xi} (u) \|_{p', \varepsilon}
\]

\[=: I_1 + I_2.
\]

From Lemma 5.3 we derive

\[I_1 \leq \frac{C}{\varepsilon^p} \left\| W_{\varepsilon, \xi} + \phi, \varepsilon \right\|_{g, \frac{1}{2}} \left\| \Psi' \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \Pi_{\varepsilon, \xi} (u) \right\|_{g, \frac{p}{p'}} \left\| 2q - 2q^2 \Psi \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \right\|_{g, \frac{p'}{p}}
\]

\[\leq \frac{C}{\varepsilon^p} \left( 1 + \| \phi \|_{\varepsilon}^2 \right) \left\| \Pi_{\varepsilon, \xi} (u) \right\|_{g, \frac{p}{p'}} = o(1) \| \Pi_{\varepsilon, \xi} (u) \|_{g, \frac{p}{p'}}
\]

and, since \( 0 \leq \Psi (u) \leq 1/q \), from Lemma 5.3 with \( \vartheta p > 2 \) we get

\[I_2 \leq \frac{C}{\varepsilon^p} \left\| \Pi_{\varepsilon, \xi} (u) \right\|_{g, p} \left\| \Psi \left( W_{\varepsilon, \xi} + \phi, \varepsilon \right) \right\|_{g, \frac{p'}{p}}
\]

\[\leq \frac{C \varepsilon^0}{\varepsilon^p} \left( 1 + \| \phi \|_{\varepsilon}^2 \right) \left\| \Pi_{\varepsilon, \xi} (u) \right\|_{g} = o(1) \| \Pi_{\varepsilon, \xi} (u) \|_{g}
\]

This concludes the proof.
4. The reduced energy

This section is devoted to the proof of Proposition 2.6.

**Lemma 4.1.** The following estimate

\[ \overline{I}_\varepsilon (\xi) = I_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) \]

\[ = I_\varepsilon (W_{\varepsilon,\xi}) + o(1) = J_\varepsilon (W_{\varepsilon,\xi}) + \frac{\omega^2}{2} G_\varepsilon (W_{\varepsilon,\xi}) + o(1) \]

holds true \( C^0 \)-uniformly with respect to \( \xi \) as \( \varepsilon \) goes to zero. Moreover, setting \( \xi(y) := \exp_\xi(y), y \in B(0, r) \), we have that

\[ \left( \frac{\partial}{\partial y_h} \overline{I}_\varepsilon (\xi(y)) \right)_{|y=0} = \left( \frac{\partial}{\partial y_h} I_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right)_{|y=0} \]

\[ = \left( \frac{\partial}{\partial y_h} J_\varepsilon (W_{\varepsilon,\xi(y)}) \right)_{|y=0} + o(1) \]

\[ = \left( \frac{\partial}{\partial y_h} J_\varepsilon (W_{\varepsilon,\xi(y)}) \right)_{|y=0} + \frac{\omega^2}{2} \left( \frac{\partial}{\partial y_h} G_\varepsilon (W_{\varepsilon,\xi(y)}) \right)_{|y=0} + o(1), \]

\( C^0 \)-uniformly with respect to \( \xi \) as \( \varepsilon \) goes to zero.

**Proof.** In Lemma 5.1 of \[3\] we have proved the following two estimates:

\[ J_\varepsilon (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon (W_{\varepsilon,\xi(y)}) = o(1), \]

\[ (J_\varepsilon' (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon' (W_{\varepsilon,\xi(y)})) \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right] = o(1). \]

To complete the proof we shall prove the three following estimates:

\[ G_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon (W_{\varepsilon,\xi}) = o(1), \]

\[ [G_\varepsilon' (W_{\varepsilon,\xi_0} + \phi_{\varepsilon,\xi_0}) - G_\varepsilon' (W_{\varepsilon,\xi_0})] \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right)_{|y=0} \right] = o(1), \]

\[ \left( J_\varepsilon' (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) + \frac{\omega^2}{2} G_\varepsilon' (W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) \right) \left[ \frac{\partial}{\partial y_h} \phi_{\varepsilon,\xi(y)} \right] = o(1). \]

We start with (4.2). For some \( \theta \in [0, 1] \) we have

\[ G_\varepsilon (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - G_\varepsilon (W_{\varepsilon,\xi}) \]

\[ = \frac{1}{\varepsilon^2} \int_M b(x) \left[ \Psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})^2 - \Psi (W_{\varepsilon,\xi}) (W_{\varepsilon,\xi})^2 \right] \]

\[ = \frac{1}{\varepsilon^2} \int_M b(x) \Psi' (W_{\varepsilon,\xi} + \theta \phi_{\varepsilon,\xi}) [\phi_{\varepsilon,\xi}] (W_{\varepsilon,\xi})^2 \]

\[ + \frac{1}{\varepsilon^2} \int_M b(x) \Psi (W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) [2\phi_{\varepsilon,\xi} W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}^2]. \]

Since \( \|\phi_{\varepsilon,\xi}\|_e \leq C_\varepsilon \), from Lemma 5.4 we obtain (4.2).
Next, we prove \[4.3\]. For some \(\theta \in [0, 1]\) we have

\[
\begin{align*}
[G'_e (W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - G'_e (W_{\varepsilon, \xi_0})] 
& \leq \frac{q}{2\varepsilon^2} \int_M b(x) \left\{ [2\Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - \Psi(W_{\varepsilon, \xi_0})] - [q\Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_{\varepsilon, \xi_0})] \right\} \\
& \quad \left. \cdot W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& \quad + \frac{q}{2\varepsilon^2} \int_M 2b(x) \left[ \Psi(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0}) \right] \phi_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& \quad + \frac{q}{\varepsilon^2} \int_M b(x) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& \quad + \frac{q}{\varepsilon^2} \int_M b(x) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& \quad + \frac{q}{\varepsilon^2} \int_M b(x) \Psi'(W_{\varepsilon, \xi_0} + \theta\phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) W_{\varepsilon, \xi_0} \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& \quad + \frac{q}{2\varepsilon^2} \int_M b(x) \Psi^2(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0}) \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right|_{y=0} \\
& := I_1 + I_2 + I_3 + I_4 + I_5
\end{align*}
\]

From Lemma 5.4, Remark 5.2 and equations (2.8), (2.9), (2.6), (2.7), recalling that \(\|\phi_{\varepsilon, \xi(y)}\|_{\varepsilon} \leq C\varepsilon\), we get

\[
I_1 \leq C \frac{\varepsilon^4}{\varepsilon^2} \left( \int_M [\Psi'(W_{\varepsilon, \xi_0} + \phi_{\varepsilon, \xi_0})(\phi_{\varepsilon, \xi_0})]^3 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi_0}^3 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon^2} \int_M \left[ \left( \frac{\partial}{\partial y_h} W_{\varepsilon, \xi(y)} \right) \right]_{y=0}^3 \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^4 \int_{\mathbb{S}^2} \sum_{k=1}^2 \left| \frac{1}{\varepsilon} \frac{\partial U}{\partial z_k}(\varepsilon z) \right| \left( \varepsilon z \right) + \left( \frac{\partial Y}{\partial z_k}(\varepsilon z) \right) U(z) \right|_{d\varepsilon}^3 \right)^{\frac{1}{2}}
\]

\[
\leq C \frac{1}{\varepsilon} = O(\varepsilon^4)
\]

In a similar way, using Lemma 5.3 and embedding the first and the second term in \(L^6\) and the third one in \(L^{3/2}\), we get

\[
I_4 \leq C \frac{1}{\varepsilon^2} \|\phi_{\varepsilon, \xi}\|_{\varepsilon} + \|\phi_{\varepsilon, \xi}\|_{\varepsilon}^2 \|\phi_{\varepsilon, \xi}\|_{\varepsilon}^{\frac{4}{3}} = O(\varepsilon^4).
\]
For $I_3$ by Lemma 5.3 we have
\[
I_3 \leq C \frac{\varepsilon^4}{\varepsilon^2} \left( \int_M |\Psi(W_\varepsilon, \xi_0)|^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \phi_{\varepsilon, \xi_0}^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M \left[ \frac{\partial}{\partial y_h} W_\varepsilon(x) \right]_{y=0}^3 \right)^{\frac{1}{3}}
\]
\[
\leq C \frac{\varepsilon^4}{\varepsilon^2} \|\Psi(W_\varepsilon, \xi_0)\|_{\|g\|} \|\phi_{\varepsilon, \xi_0}\|_{\varepsilon} \left( \int_{\mathbb{R}^2} \left[ \sum_{k=1}^2 \frac{1}{\varepsilon} \frac{\partial U}{\partial z^k}(z) \chi(\varepsilon z) + \left( \chi(\varepsilon z) + \frac{\partial}{\partial z^k}(\varepsilon z) \right) U(z) \right] \right)^\frac{3}{4} dz
\]
\[
\leq C \frac{\varepsilon^4}{\varepsilon^2} \varepsilon^2 \varepsilon = O(\varepsilon)
\]
and, from the estimate for $I_3$, since $0 < \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) < 1/q$, we obtain
\[
I_5 \leq CI_3 = O(\varepsilon).
\]

Finally, we prove (4.3). Following the proof of Lemma 5.1 in [3], we need only to prove that
\[
|G'_\varepsilon (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) [Z^I_{\varepsilon, \xi_0}]| = o(1),
\]
that is
\[
\left| \frac{1}{\varepsilon^2} \int_M \left[ \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) \right] (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) Z^I_{\varepsilon, \xi_0} \right| = o(1).
\]
We have
\[
\left| \frac{1}{\varepsilon^2} \int_M \left[ \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) - q\Psi^2(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) \right] (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) Z^I_{\varepsilon, \xi_0} \right|
\]
\[
\leq C \frac{\varepsilon^2}{\varepsilon^2} \int_M \left| \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) Z^I_{\varepsilon, \xi_0} \right|
\]
\[
+ C \int_M \left| \Psi^2(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) Z^I_{\varepsilon, \xi_0} \right| := I_1 + I_2.
\]
By Proposition 2.3 we have that $\|Z^I_{\varepsilon, \xi_0}\| = O(1)$. So, by Lemma 5.3 and Remark 5.2 we have
\[
I_1 \leq C \frac{\varepsilon^4}{\varepsilon^2} \left( \int_M \left| \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) \right|^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M (W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0})^3 \right)^{\frac{1}{3}} \left( \frac{1}{\varepsilon^2} \int_M |Z^I_{\varepsilon, \xi_0}(y)|^3 \right)^{\frac{1}{3}}
\]
\[
\leq C \frac{\varepsilon^4}{\varepsilon^2} \left\| \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) \right\|_{\|g\|} \|W_\varepsilon, \xi_0\|_{3, e} + \|\phi_{\varepsilon, \xi_0}\| e \|Z^I_{\varepsilon, \xi_0}\| e = O(\varepsilon).
\]
Again, as $0 < \Psi(W_\varepsilon, \xi_0, \phi_{\varepsilon, \xi_0}) < 1/q$, we obtain
\[
I_2 \leq CI_1 = O(\varepsilon).
\]
This concludes the proof. 

\textbf{Lemma 4.2.} The expansion
\[
I_\varepsilon(W_\varepsilon, \xi) = \left( \frac{1}{2} - \frac{1}{p} \right) \frac{c(\xi) \varphi(\xi) \varphi^2 - \varphi}{b(\xi) \varphi^2} \int_{\mathbb{R}^n} U^p dz + o(1)
\]
holds true $C^1$-uniformly with respect to $\xi \in M$. 
Proof. In Lemma 5.2 of [3] we proved that

$$J_\varepsilon(W_\varepsilon,\xi) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{c(\xi)^{\frac{2}{\tau}} a(\xi)^{\frac{p-\tau}{\tau}}}{b(\xi)^{\frac{p-\tau}{\tau}}} \int_{\mathbb{R}^n} U^p dz + O(\varepsilon).$$

Hence, it suffices to show now that $|G_\varepsilon(W_\varepsilon,\xi)| = o(1)$, $C^1$-uniformly with respect to $\xi \in M$.

Regarding the $C^0$-convergence, by Remark 5.2 and Lemma 5.3, we have that

$$|G_\varepsilon(W_\varepsilon,\xi)| \leq \frac{C}{\varepsilon^2} \int_M \Psi(W_\varepsilon,\xi)W_{\varepsilon,\xi}^2 d\mu_g$$

$$\leq \frac{C}{\varepsilon^2} \left(\int_M \Psi(W_\varepsilon,\xi)^2\right)^{\frac{1}{2}} \left(\int_M W_{\varepsilon,\xi}^4\right)^{\frac{1}{2}}$$

$$\leq C \frac{1}{\varepsilon} \|[\Psi(W_\varepsilon,\xi)]_{\|g} \leq \frac{\varepsilon^{\frac{3}{2}}}{\varepsilon} = O(\varepsilon^\frac{3}{2}).$$

Regarding the $C^1$-convergence observe that

$$\left|\frac{\partial}{\partial y_h} G_\varepsilon(W_\varepsilon,\xi)\right|_{y=0} \leq \frac{C}{\varepsilon^2} \left|\frac{\partial}{\partial y_h} \int_M \Psi(W_\varepsilon,\xi(\xi))W_{\varepsilon,\xi(\xi)}^2 d\mu_g\right|$$

$$\leq \frac{C}{\varepsilon^2} \int_M \Psi(W_\varepsilon,\xi(\xi))2W_{\varepsilon,\xi(\xi)} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(\xi)}\right)_{y=0} d\mu_g$$

$$+ \frac{C}{\varepsilon^2} \int_M W_{\varepsilon,\xi(\xi)}^2 \Psi'(W_\varepsilon,\xi(\xi)) \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(\xi)}\right)_{y=0} d\mu_g$$

$$:= I_1 + I_2.$$ 

Now, from Remark 5.2, Lemma 5.3, and the estimates (2.8) and (2.9), we derive

$$I_1 \leq C \frac{\varepsilon^\frac{3}{2}}{\varepsilon^2} \left(\int_M \Psi(W_\varepsilon,\xi(\xi))^\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi(\xi)}^\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2} \int_M \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(\xi)}\right)_{y=0}\right)^{\frac{1}{2}}$$

$$\leq C \frac{\varepsilon^\frac{3}{2}}{\varepsilon^2} \frac{1}{\varepsilon} = o(1).$$

On the other hand, from Remark 5.2, the proof of Lemma 5.4, and the estimates (2.8) and (2.9), for some \( t \in (1, 3/2) \) we obtain

\[
I_2 \leq C \varepsilon^{-\frac{2}{3}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi(h)}^2 \right)^{\frac{1}{6}} \left( \int_M \left( \nabla_g (W_{\varepsilon, \xi(y)}) \left( \frac{\partial}{\partial y} W_{\varepsilon, \xi(h)} \right) \right) \right)^{\frac{1}{6}}
\]

\[
\leq C \varepsilon^{-\frac{2}{3}} \left( \frac{1}{\varepsilon^2} \int_M W_{\varepsilon, \xi(h)}^2 \right)^{\frac{1}{6}} \left( \frac{\partial}{\partial y} W_{\varepsilon, \xi(h)} \right) \bigg|_{y=0}^6
\]

\[
\leq C \varepsilon^{-\frac{2}{3}} \varepsilon^{-\frac{2}{3}} \left( \frac{1}{\varepsilon^2} \int_M \left( \frac{\partial}{\partial y} W_{\varepsilon, \xi(h)} \right) \bigg|_{y=0}^6 \right)^{\frac{1}{6}}
\]

\[
\leq C \varepsilon^{-\frac{2}{3}} \varepsilon^{-\frac{2}{3}} \frac{1}{\varepsilon} = C \varepsilon^{-\frac{5}{3}} = o(1).
\]

This concludes the proof. \( \square \)

5. Some estimates involving \( \Psi \)

We start by pointing out the following facts.

**Remark 5.1.** There exists a constant \( C > 0 \) such that, for every \( \varphi \in H^1_g(M) \) and every \( 0 < \varepsilon < 1 \), we have

\[
C \| \varphi \|_g^2 = C \int_M (|\nabla_g \varphi|^2 + \varphi^2) \, d\mu_g
\]

\[
\leq \int_M \left( c(x) |\nabla_g \varphi|^2 + \frac{d(x)}{\varepsilon^2} \varphi^2 \right) \, d\mu_g = \| \varphi \|_\varepsilon^2.
\]

**Remark 5.2.** The following estimates

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} |W_{\varepsilon, \xi}|_p = C |U|_p^p, \quad p \geq 2,
\]

\[
\lim_{\varepsilon \to 0} \nabla_g W_{\varepsilon, \xi}^2 \leq C |\nabla U|^2
\]

hold true uniformly with respect to \( \xi \in M \).

Abusing notation we write

\[
\| u \|_g^2 = \int_M \left( c(x) |\nabla_g \varphi|^2 + b(x) u^2 \right) \, d\mu_g.
\]

This norm is equivalent to the standard norm (3.1) of \( H^1_g(M) \). From equations (2.1), (2.2) and (2.3) we obtain

\[
\| \Psi(u) \|_g^2 = \int_M b(x) qu^2 \Psi(u) \, d\mu_g - \int_M b(x) q^2 u^2 (\Psi(u))^2 \, d\mu_g
\]

\[
\leq C \int_M u^2 \Psi(u) \, d\mu_g,
\]
Proof. Since $k, h$ holds true for every $w$, we conclude that $\| \Psi(\omega, \theta) \|_g \leq C(\varepsilon^g + \| \theta \|^2_g)$ holds true for every $\varphi$. (5.2)

Thus, by Remark 5.2,

$$
\| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C \left( \epsilon^{2/\gamma} \left( \int_M W_{\varepsilon, \xi}^{2\gamma} \right)^{1/\gamma} + \left( \int_M \varphi^{2\gamma} \right)^{1/\gamma} \right) 
$$

as claimed.

Lemma 5.3. Given $\vartheta \in (1, 2)$ there is a constant $C > 0$ such that the inequality

$$
\| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C(\varepsilon^g + \| \varphi \|^2_g)
$$

holds true for every $\varphi \in H^1_g(M), \xi \in M$ and small enough $\varepsilon > 0$.

Proof. Let $t \in (2, \infty)$ be such that $t = \vartheta$ where $\vartheta$ is the exponent conjugate to $t$. From inequality (5.1) we obtain

$$
\| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C \left( \int_M \left( W_{\varepsilon, \xi} + \varphi \right)^t \right)^{1/t} \leq C \| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \| W_{\varepsilon, \xi} + \varphi \|_{g, 2\gamma}.
$$

Thus, by Remark 5.2,

$$
\| \Psi(W_{\varepsilon, \xi} + \varphi) \|_g \leq C \left( \epsilon^{2/\gamma} \left( \int_M W_{\varepsilon, \xi}^{2\gamma} \right)^{1/\gamma} + \left( \int_M \varphi^{2\gamma} \right)^{1/\gamma} \right) \leq C(\varepsilon^g + \| \varphi \|^2_g),
$$

as claimed.

Lemma 5.4. Given $s \in (1, 2)$ there is a constant $C > 0$ such that the inequality

$$
\| \Psi'(W_{\varepsilon, \xi} + k) [h] \|_g \leq C \| h \|_g \left( \varepsilon^{\frac{2}{\gamma}} + \| k \|_g \right)
$$

holds true for every $k, h \in H^1_g(M), \xi \in M$ and small enough $\varepsilon > 0$.

Proof. From inequality (5.2) we obtain,

$$
\| \Psi'(W_{\varepsilon, \xi} + k) [h] \|_g \leq C \int_M \left| W_{\varepsilon, \xi} + k \right| [h] \| \Psi'(W_{\varepsilon, \xi} + k) [h] \|_g \ d\mu_g 
$$

$$
\leq C \left( \int_M \left| W_{\varepsilon, \xi} \right| [h] \| \Psi'(W_{\varepsilon, \xi} + k) [h] \|_g \ d\mu_g + \int_M \left| k \right| [h] \| \Psi'(W_{\varepsilon, \xi} + k) [h] \|_g \ d\mu_g \right) =: I_1 + I_2.
$$

Set $t := 2s' \in (4, \infty)$, where $s'$ is the conjugate exponent to $s$. Using Remark 5.2 we conclude that

$$
I_1 \leq C \left| \Psi'(W_{\varepsilon, \xi} + k) [h] \right|_{g, s} \| h \|_{g, t} \left| W_{\varepsilon, \xi} \right|_{g, s} 
$$

$$
= C \left| \Psi'(W_{\varepsilon, \xi} + k) [h] \right|_g \| h \|_g \varepsilon^{\frac{2}{\gamma}} \left( \int_M W_{\varepsilon, \xi}^{s} \right)^{1/s} 
$$

$$
= C \left| \Psi'(W_{\varepsilon, \xi} + k) [h] \right|_g \| h \|_g \varepsilon^{\frac{2}{\gamma}}.
$$

Since

$$
I_2 \leq C \left| \Psi'(W_{\varepsilon, \xi} + k) [h] \right|_{g, 3} \| h \|_{g, 3} \left| k \right|_{g, 3} \leq C \left| \Psi'(W_{\varepsilon, \xi} + k) [h] \right|_g \| h \|_g \| k \|_g,
$$

we have

$$
I \leq C \left( \varepsilon^{\frac{2}{\gamma}} + \| k \|_g \right) \| h \|_g.
$$
the claim follows. □

Lemma 5.5. Consider the functions
\[ v_{\varepsilon, \xi}(z) := \begin{cases} \Psi(W_{\varepsilon, \xi})(\exp(\epsilon z)) & \text{for } z \in B(0, r/\varepsilon), \\ 0 & \text{for } z \in \mathbb{R}^2 \setminus B(0, r/\varepsilon). \end{cases} \]

Then, for any \( \vartheta \in (1, 2) \), there exists a constant \( C > 0 \), independent of \( \varepsilon, \xi \), such that
\[ |v_{\varepsilon, \xi}(z)|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\vartheta-1}, \]
\[ |\nabla v_{\varepsilon, \xi}(z)|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\vartheta}. \]

Proof. After a change of variables we have that
\[ \int_{B(\xi, r)} |\nabla \Psi(W_{\varepsilon, \xi})|^2 + |\Psi(W_{\varepsilon, \xi})|^2 d\mu_g \]
\[ = \varepsilon^2 \int_{B(0, r/\varepsilon)} |g_{\xi}(\varepsilon z)|^{1/2} \left( \sum_{ij} g_{\xi}^{ij}(\varepsilon z) \frac{1}{\epsilon^2} \frac{\partial v_{\varepsilon, \xi}(z)}{\partial z_i} \frac{\partial v_{\varepsilon, \xi}(z)}{\partial z_i} + v_{\varepsilon, \xi}(z) \right) d\mu_g. \]

Thus
\[ \|\Psi(W_{\varepsilon, \xi})\|_g^2 \geq C\left( |\nabla v_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^2 |v_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)}^2 \right). \]
This, combined with Lemma 5.3 gives
\[ |\nabla v_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)} + \varepsilon |v_{\varepsilon, \xi}|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{\vartheta}, \]
as claimed. □

REFERENCES


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