COMPONENTS OF $V(\rho) \otimes V(\rho)$

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1. Introduction

Let $\mathfrak{g}$ be any simple Lie algebra over $\mathbb{C}$. We fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ and let ρ be the half sum of positive roots, where the roots of $\mathfrak{b}$ are called the positive roots. For any dominant integral weight $\lambda \in \mathfrak{t}^*$, let $V(\lambda)$ be the corresponding irreducible representation of $\mathfrak{g}$. B. Kostant initiated (and popularized) the study of the irreducible components of the tensor product $V(\rho) \otimes V(\rho)$. In fact, he conjectured the following.

Conjecture 1. *(Kostant) Let λ be a dominant integral weight. Then, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$ if and only if $\lambda \leq 2\rho$ under the usual Bruhat-Chevalley order on the set of weights.*

It is, of course, clear that if $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$, then $\lambda \leq 2\rho$.

One of the main motivations behind Kostant's conjecture was his result that the exterior algebra $\wedge\mathfrak{g}$, as a $\mathfrak{g}$-module under the adjoint action, is isomorphic with 2^r copies of $V(\rho) \otimes V(\rho)$, where r is the rank of $\mathfrak{g}$ (cf. [Ko]). Recall that $\wedge\mathfrak{g}$ is the underlying space of the standard chain complex computing the homology of the Lie algebra $\mathfrak{g}$, which is, of course, an object of immense interest.

Definition 2. An integer $d \geq 1$ is called a *saturation factor* for $\mathfrak{g}$, if for any $(\lambda, \mu, \nu) \in D^3$ such that $\lambda + \mu + \nu$ is in the root lattice and the space of $\mathfrak{g}$-invariants:
$$[V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^{\mathfrak{g}} \neq 0$$
for some integer $N > 0$, then
$$[V(d\lambda) \otimes V(d\mu) \otimes V(d\nu)]^{\mathfrak{g}} \neq 0,$$
where $D \subset \mathfrak{t}^*$ is the set of dominant integral weights of $\mathfrak{g}$. Such a d always exists (cf. [Ku; Corollary 44]).

Recall that 1 is a saturation factor for $\mathfrak{g} = sl_n$, as proved by Knutson-Tao [KT]. By results of Belkale-Kumar [BK_2] (also obtained by Sam [S] and Hong-Shen [HS]), d can be taken to be 2 for $\mathfrak{g}$ of types B_r, C_r and d can be taken to be 4 for $\mathfrak{g}$ of type D_r by a result of Sam [S]. As proved by Kapovich-Millson [KM_1, KM_2], the saturation factors d of $\mathfrak{g}$ of types G_2, F_4, E_6, E_7, E_8

can be taken to be 2 (in fact any $d \geq 2$), 144, 36, 144, 3600 respectively. (For a discussion of saturation factors d, see [Ku, §10].)

Now, the following result (weaker than Conjecture (1)) is our main theorem.

Theorem 3. *Let λ be a dominant integral weight such that $\lambda \leq 2\rho$. Then, $V(d\lambda) \subset V(d\rho) \otimes V(d\rho)$, where $d \geq 1$ is any saturation factor for $\mathfrak{g}$.*

In particular, for $\mathfrak{g} = sl_n$, $V(\lambda) \subset V(\rho) \otimes V(\rho)$.

The proof uses a description of the eigencone of $\mathfrak{g}$ in terms of certain inequalities due to Berenstein-Sjamaar coming from the cohomology of the flag varieties associated to $\mathfrak{g}$, a 'non-negativity' result due to Belkale-Kumar and Proposition (9).

An interesting aspect of our work is that we make an essential use of a solution of the eigenvalue problem and saturation results for any $\mathfrak{g}$.

Remark 4. As informed by Papi, Berenstein-Zelevinsky had proved Conjecture (1) (by a different method) for $\mathfrak{g} = sl_n$ (cf. [BZ, Theorem 6]). They also determine in this case when $V(\lambda)$ appears in $V(\rho) \otimes V(\rho)$ with multiplicity one. To our knowledge, Conjecture 1 appears first time in this paper.

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2. Proof of Theorem (3)

We now prove Theorem (3).

Proof. Let $\Gamma_3(\mathfrak{g})$ be the *saturated tensor semigroup* defined by

$$\Gamma_3(\mathfrak{g}) = \{(\lambda, \mu, \nu) \in D^3 : [V(N\lambda) \otimes V(N\mu) \otimes V(N\nu)]^{\mathfrak{g}} \neq 0 \text{ for some } N > 0\}.$$

To prove the theorem, it suffices to prove that $(\rho, \rho, \lambda^*) \in \Gamma_3(G)$, where λ^* is the dual weight $-w_o\lambda$, w_o being the longest element of the Weyl group of $\mathfrak{g}$. Let G be the connected, simply-connected complex algebraic group with Lie algebra $\mathfrak{g}$. Let B (resp. T) be the Borel subgroup (resp. maximal torus) of G with Lie algebra $\mathfrak{b}$ (resp. $\mathfrak{t}$). Let W be the Weyl group of G. For any standard parabolic subgroup $P \supset B$ with Levi subgroup L containing T, let W^P be the set of smallest length coset representatives in W/W_L, W_L being the Weyl group of L. Then, we have the Bruhat decomposition:

$$G/P = \sqcup_{w \in W^P} \Lambda_w^P, \quad \text{where } \Lambda_w^P := BwP/P.$$

Let $\bar{\Lambda}^P_w$ denote the closure of Λ^P_w in G/P. We denote by $[\bar{\Lambda}^P_w]$ the Poincaré dual of its fundamental class. Thus, $[\bar{\Lambda}^P_w]$ belongs to the singular cohomology:

$$[\bar{\Lambda}^P_w] \in H^{2(\dim G/P-\ell(w))}(G/P,\mathbb{Z}),$$

where $\ell(w)$ is the length of w.

Let $\{x_j\}_{1\le j\le r}\subset \mathfrak{t}$ be the dual to the simple roots $\{\alpha_i\}_{1\le i\le r}$, i.e.,

$$\alpha_i(x_j)=\delta_{i,j}.$$

In view of [BS] (or [Ku; Theorem 10]), it suffices to prove that for any standard maximal parabolic subgroup P of G and triple $(u,v,w)\in (W^P)^3$ such that the cup product of the corresponding Schubert classes in G/P :

$$[\bar{\Lambda}^P_u]\cdot[\bar{\Lambda}^P_v]\cdot[\bar{\Lambda}^P_w]=k[\bar{\Lambda}^P_e]\in H^*(G/P,\mathbb{Z}),\ \text{ for some } k\neq 0, \tag{1}$$

the following inequality is satisfied:

$$\rho(ux_P)+\rho(vx_P)+\lambda^*(wx_P)\le 0. \tag{2}$$

Here, $x_P:=x_{i_P}$, where α_{i_P} is the unique simple root not in the Levi of P.

Now, by [BK$_1$; Proposition 17(a)] (or [Ku; Corollary 22 and Identity (9)]), for any $u,v,w\in (W^P)^3$ such that the equation (1) is satisfied,

$$(\chi_{w_oww_o^P}-\chi_u-\chi_v)(x_P)\ge 0, \tag{3}$$

where w_o^P is the longest element in the Weyl group of L and

$$\chi_w := \rho-2\rho^L+w^{-1}\rho$$

(ρ^L being the half sum of positive roots in the Levi of P).

Now,

$$\begin{aligned}&(\chi_{w_oww_o^P}-\chi_u-\chi_v)(x_P)\\ &=(\rho-w_o^Pw^{-1}\rho-\rho-u^{-1}\rho-\rho-v^{-1}\rho)(x_P),\ \text{since } \rho^L(x_P)=0\\ &=(-\rho-u^{-1}\rho-v^{-1}\rho-w^{-1}\rho)(x_P),\ \text{since } w_o^P(x_P)=x_P.\end{aligned} \tag{4}$$

Combining (3) and (4), we get

$$(\rho+u^{-1}\rho+v^{-1}\rho+w^{-1}\rho)(x_P)\le 0\,,\ \text{if (1) is satisfied.} \tag{5}$$

We next claim that for any dominant integral weight $\lambda\le 2\rho$ and any $u,v,w\in (W^P)^3$,

$$\rho(ux_P)+\rho(vx_P)+\lambda^*(wx_P)\le(\rho+u^{-1}\rho+v^{-1}\rho+w^{-1}\rho)(x_P), \tag{6}$$

which is equivalent to

$$\lambda^*(wx_P)\le(\rho+w^{-1}\rho)(x_P). \tag{7}$$

Of course (5) and (6) together give (2). So, to prove the theorem, it suffices to prove (7). Since the assumption on λ in the theorem is invariant under the transformation $\lambda \mapsto \lambda^*$, we can replace λ^* by λ in (7). By Proposition (9), $\lambda = \rho + \beta$, where β is a weight of $V(\rho)$ (i.e., the weight space of $V(\rho)$ corresponding to the weight β is nonzero). Thus,

$$\lambda(wx_P) = \rho(wx_P) + \beta(x_P), \text{ for some weight } \beta \text{ of } V(\rho).$$

Hence,

$$\lambda(wx_P) = \rho(wx_P) + \beta(x_P) \leq (w^{-1}\rho + \rho)(x_P), \quad \text{since } \beta \leq \rho.$$

This establishes (7) and hence the theorem is proved. □

We recall the following conjecture due to Kapovich-Millson [KM$_1$] (or [Ku; Conjecture 47]).

Conjecture 5. *Let $\mathfrak{g}$ be a simple, simply-laced Lie algebra over $\mathbb{C}$. Then, $d = 1$ is a saturation factor for $\mathfrak{g}$.*

The following theorem follows immediately by combining Theorem (3) and Conjecture (5).

Theorem 6. *For any simple, simply-laced Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, assuming the validity of Conjecture (5), Conjecture (1) is valid for $\mathfrak{g}$, i.e., for any dominant integral weight $\lambda \leq 2\rho$, $V(\lambda)$ is a component of $V(\rho) \otimes V(\rho)$.*

Remark 7. By an explicit calculation using the program LIE, it is easy to see that Conjecture (1) has an affirmative answer for simple $\mathfrak{g}$ of types G_2 and F_4. Further, Paolo Papi has informed us that he has verified the validity of Conjecture (1) (by an explicit computer calculation using LIE again) for any simple $\mathfrak{g}$ of type E_6; E_7; and E_8 as well.

3. Determination of dominant weights $\leq 2\rho$

We follow the notation and assumptions from the previous sections. In particular, $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ where we have fixed a Cartan subalgebra $\mathfrak{t}$ and a Borel subalgebra $\mathfrak{b} \supset \mathfrak{t}$. Let $\{\omega_i\}_{i\in I}$ be the fundamental weights, $\{\alpha_i\}_{i\in I}$ the simple roots, and $\{s_i\}_{i\in I}$ the simple reflections, where $I := \{1 \leq i \leq r\}$. For any $J \subset I$, let W_J be the parabolic subgroup of the Weyl group W generated by s_j with $j \in J$, w_o^J be the longest element in W_J, Φ_J be the root system generated by the simple roots α_j with $j \in J$, and $\Phi_J^+ \subset \Phi_J$ the subset of positive roots.

Let $A \subset \mathfrak{t}^*$ be the dominant cone, $B \subset \mathfrak{t}^*$ the cone generated by $\{\alpha_i : i \in I\}$ and $C := 2\rho - B$. We want to describe the vertices of the polytope $A \cap C$. For $J \subset I$, define

$$A_J := \mathbb{R}_{\geq 0}[\omega_j \,:\, j \in J], \quad B_J := \mathbb{R}_{\geq 0}[\alpha_j \,:\, j \in J] \text{ and } C_J := 2\rho - B_J.$$

The sets A_J and B_J are the faces of A and B. The vertices of the polytope $A \cap C$ are given by the zero dimensional nonempty intersections of the form

$A_J \cap C_H$. To describe these intersections, we introduce some notation. For any $J \subset I$, let

$$\rho_J := \sum_{j\in J} \omega_j, \quad b_J := \sum_{\alpha\in\Phi_J^+} \alpha \text{ and } c_J := 2\rho - b_J;$$

in particular, $c_I = 0$ and $c_\emptyset = 2\rho$.

Lemma 8. *For each $J \subset I$, we have*

$$A_{I\smallsetminus J} \cap C_J = \{c_J\}.$$

Moreover, none of the other intersections $A_H \cap C_K$ give a single point.

In particular, the intersection $A \cap C$ is the convex hull of the points $\{c_J : J \subset I\}$.

Proof. The lattice generated by $A_{I\smallsetminus J}$ and the one generated by B_J are orthogonal to each other so the intersection $A_{I\smallsetminus J} \cap C_J$ contains at most one point. Observe now that

$$b_J = 2\rho_J + \sum_{\ell\notin J} a_\ell\omega_\ell, \text{ where } a_\ell \le 0.$$

Hence, $c_J \in A_{I\smallsetminus J} \cap C_J$.

Consider an intersection of the form $A_{I\smallsetminus H} \cap C_K$. Assume it is not empty and that $y = 2\rho - x \in A_{I\smallsetminus H} \cap C_K$. Since $y \in A_{I\smallsetminus H}$, we have $x = 2\rho_H + \sum_{\ell\notin H} a'_\ell\omega_\ell$. Since $x \in B_K$, if $h \notin K$, the coefficient of ω_h in x can not be positive. So, we must have $H \subset K$. If $H \subset K$ and $H \neq K$, then

$$A_{I\smallsetminus H} \cap C_K \supset (A_{I\smallsetminus H} \cap C_H) \cup (A_{I\smallsetminus K} \cap C_K) \supset \{c_H, c_K\}.$$

Hence, it is not a single point. □

We apply this Lemma to obtain the following result about the weights below 2ρ.

Proposition 9. *Let $\lambda \le 2\rho$ be a dominant integral weight. Then,*

$$\lambda = \rho + \beta,$$

for some weight β of $V(\rho)$.

Proof. Let $Q \subset \mathfrak{t}^*$ be the root lattice and let H_ρ be the convex hull of the weights $\{w(\rho) : w \in W\}$. Recall that the weights of the module $V(\rho)$ are precisely the elements of the intersection

$$(\rho + Q) \cap H_\rho.$$

If λ is as in the Proposition, then it is clear that $\lambda - \rho \in \rho + Q$. So, we need to prove that it belongs to H_ρ. To check this, it is enough to check that $(A \cap C) - \rho \subset H_\rho$ or equivalently, by the previous Lemma, that

$$c_J - \rho \in H_\rho, \text{ for all } J \subset I.$$

Notice that $w_o^J(\Phi_J^+) = -\Phi_J^+$ and that, if α is a positive root, $w_J^0(\alpha)$ is a negative root if and only if $\alpha \in \Phi_J^+$. Hence,

$$w_0^J(\rho) = \sum_{\alpha\in\Phi_I^+\smallsetminus\Phi_J^+} \alpha - \sum_{\alpha\in\Phi_J^+} \alpha = \rho - b_J,$$

and $c_J - \rho = \rho - b_J \in H_\rho$. □

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