

Surfaces on the Severi line

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Abstract

Let S be a minimal complex surface of general type and of maximal Albanese dimension; by the Severi inequality one has $K_S^2 \geq 4\chi(\mathcal{O}_S)$. We prove that the equality $K_S^2 = 4\chi(\mathcal{O}_S)$ holds if and only if $q(S) := h^1(\mathcal{O}_S) = 2$ and the canonical model of S is a double cover of the Albanese surface branched on an ample divisor with at most negligible singularities.

Soit S une surface complexe minimale de type général et de dimension d'Albanese maximale; par l'inégalité de Severi on a $K_S^2 \geq 4\chi(\mathcal{O}_S)$. Nous montrons que on a l'égalité $K_S^2 = 4\chi(\mathcal{O}_S)$ si et seulement si $q(S) := h^1(\mathcal{O}_S) = 2$ et le modèle canonique de S est une revêtement double de la surface d'Albanese ramifié sur une diviseur ample avec singularités au plus négligeables.

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1. Introduction

Let S be a complex surface; we write $\chi(S)$ for $\chi(\mathcal{O}_S) = \chi(\omega_S)$ and $q(S) := h^1(\mathcal{O}_S)$ for the irregularity.

Recall that S has maximal Albanese dimension if its Albanese map is generically finite; if S is minimal and of maximal Albanese dimension, then the numerical invariants of S satisfy the so-called Severi inequality:

$$K_S^2 \geq 4\chi(S). \quad (1.1)$$

In this paper we characterize the minimal surfaces of general type and maximal Albanese dimension on the Severi line, namely those for which the inequality (1.1) is actually an equality:

Theorem 1.1. *Let S be a minimal surface of general type of maximal Albanese dimension. The equality $K_S^2 = 4\chi(S)$ holds if and only if:*

- (a) $q(S) = 2$; and
- (b) *the canonical model of S is a double cover of the Albanese surface $\text{Alb}(S)$ whose branch divisor is ample and has at most negligible singularities.*

We point out that the assumption that S be of general type in Theorem 1.1 cannot be weakened: if E is a curve of genus 1 and C is a curve of genus $g > 1$, then $Y = E \times C$ has maximal Albanese dimension and lies on the Severi line but $q(Y) = g+1$ can be arbitrarily large. More generally, it is easy to see that a properly elliptic surface S of maximal Albanese dimension is isotrivial and has $\chi = 0$. (Indeed, let $f: S \rightarrow B$ be the elliptic fibration; the fibers of f are mapped by the Albanese map onto elliptic curves contained in the Albanese variety. It follows that all the fibers of f are isogenous and, more precisely f is a so-called “quasi-bundle”, namely it is isotrivial and all the fibers have smooth support. Using the results of [18] and [19], it follows that S is a free quotient of a product $E \times C$ as above by the diagonal action of an abelian group that acts on E by translation.)

In order to put our result in perspective, we give a brief account of the history of the Severi inequality and its generalizations. The inequality, claimed erroneously by Severi [20] in 1932 and then posed as a conjecture in the 1970's by Catanese and Reid [4], [17], was first proven by Manetti [12] at the end of the 1990's under the additional assumption that K_S be ample, and then a full proof was given by the second named author in 2004 [14].

The proof given in [12] yields as a byproduct the characterization of the surfaces on the Severi line with K_S ample: these are double covers of an abelian surface branched on a smooth ample divisor. The statement of Theorem 1.1 is the natural extension of this characterization to the general case, and as such has been widely believed to be true (cf. [13, §5.2]), but for a long time it could not be proved. For instance the “first” case, $K^2 = 4$ and $\chi = 1$, corresponding to a double cover of a principally polarized abelian surface branched on a divisor of $|\Theta|$, has been the object of two papers: the characterization has been proven in [5] under the assumption that the bicanonical map be non-birational and then in [6] without this assumption. Both proofs are quite technical and rely on very ad-hoc arguments, that do not extend to higher values of χ . For $\chi > 1$ up to now it was not even known whether the surfaces on the Severi line have bounded irregularity $q(S)$.

Since the full proof of the Severi inequality given in [14] consists in applying the so-called covering trick (cf. §2.1) and then taking a limit, it is not possible to extract from it information on the case where equality holds. The breakthrough that allows us to overcome this difficulty is the work [2] by the first named author, where the Severi inequality is reinterpreted as an inequality involving the self-intersection of the nef line bundle K_S and its continuous rank (cf. §2.2), and extended to the much more general situation of an arbitrary nef line bundle on a n -dimensional variety of maximal Albanese dimension (Clifford-Severi inequality). By combining this more general form of the inequality with the covering trick, we are able to avoid the limit process and give a fairly quick proof of Theorem 1.1. Another essential ingredient of the proof is the classical Castelnuovo's bound on the geometric genus of a curve of degree d in \mathbb{P}^r .

Once this paper was finished, the authors were informed by Lu and Zuo of the existence of [11], where Theorem 1.1 is also proved in a different way. The covering trick of [14] is a key tool in both arguments, but while we avoid the limiting process, Lu and Zuo take the limit as in the original paper [14], in combination with the strong inequalities for non-double cover fibrations

obtained in [10].

Notation and conventions: We work over the complex numbers. All varieties are assumed to be projective. Given a surface S with $q(S) > 0$, we denote by $\text{Alb}(S)$ its Albanese variety and by $\text{alb}_S: S \rightarrow \text{Alb}(S)$ the Albanese map.

We will use interchangeably the notion of line bundles and (Cartier) divisors, but adopt only the additive notation. We use \equiv to denote numerical equivalence of (\mathbb{Q}) -divisors. We say that a divisor D is *pseudoeffective* if $D \cdot A \geq 0$ for any nef divisor A .

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2. Set-up and preliminary results

In this section we describe the construction and preliminary results needed for the proof of the main theorem. Most of the theory (e.g. the covering trick and the continuous rank) can be developed for varieties of arbitrary dimension, but we will stick to the surface case because this is what we need. For more general treatments the reader can consult the given references.

2.1. Covering trick

In this subsection we recall the construction introduced in [14] in order to prove the Severi inequality for surfaces of maximal Albanese dimension. We shall repeatedly use it throughout the paper.

We find it convenient to introduce some terminology for maps to abelian varieties:

Definition 2.1. Let $a: X \rightarrow A$ be a morphism from a projective variety X to an abelian variety A .

We say that the map a is *generating* if $a(X)$ generates A , and that it is *strongly generating* if the map $a^*: \text{Pic}^0(A) \rightarrow \text{Pic}^0(X)$ is injective.

We say that X is of *maximal a -dimension* if the image of X via a in A has dimension equal to $\dim X$; if a is the Albanese map then we say that X has *maximal Albanese dimension*.

Remark 2.2. Notice that, as suggested by the terminology, a strongly generating map $a: X \rightarrow A$ is generating. Indeed, if Z is the abelian subvariety of A generated by $a(X)$, then the kernel of

$$a^*: \text{Pic}^0(A) \rightarrow \text{Pic}^0(X)$$

contains the kernel of the natural surjection $\text{Pic}^0(A) \rightarrow \text{Pic}^0(Z)$; therefore $\ker a^* = \{0\}$ implies that $\text{Pic}^0(A) \rightarrow \text{Pic}^0(Z)$ is an isomorphism, that is, that $A = Z$.

Notice also that, if $a: X \rightarrow A$ is a generating map, then there is a factorization $a = \phi \circ a'$ where A' is the abelian variety dual to $\text{Pic}^0(A)/\ker a^*$, the map $a': X \rightarrow A'$ is strongly generating and $f: A' \rightarrow A$ is the dual isogeny of $\text{Pic}^0(A) \rightarrow \text{Pic}^0(A)/\ker a^*$.

Observe that the Albanese map of a smooth variety X is strongly generating and the map $a: X \rightarrow A$ coincides with the Albanese map if and only if it is strongly generating and $\dim A = q(X)$.

Fix a smooth projective surface S and a strongly generating morphism $a: S \rightarrow A$ to an abelian variety A .

The covering trick goes as follows. Fix an integer $d > 0$, and consider the morphism $\mu_d: A \rightarrow A$ given by multiplication by d . Consider the surface S_d defined by the cartesian diagram:

$$\begin{array}{ccc} S_d & \xrightarrow{p} & S \\ a_d \downarrow & & \downarrow a \\ A & \xrightarrow{\mu_d} & A \end{array}$$

Lemma 2.3. *The surface S_d is smooth and connected. In addition:*

- (i) *the morphism a_d is strongly generating;*
- (ii) *the a_d -dimension of S_d is equal to the a -dimension of S ;*
- (iii) *assume that S has maximal a -dimension; then S_d is minimal if and only if S is.*

PROOF. The surface S_d is smooth since p is étale. The connectedness of S_d follows from the property of a of being strongly generating. The surface S_d is smooth, hence its number of connected components is equal to $h^0(\mathcal{O}_{S_d})$.

By construction, the surface S_d is isomorphic to $\text{Spec}(\oplus_{\eta \in A[d]} a^* \eta)$, hence $h^0(\mathcal{O}_{S_d}) = \sum_{\eta \in A[d]} h^0(a^* \eta) = 1$, since $a^* \eta$ is a non-trivial torsion line bundle for every $0 \neq \eta \in A[d]$ by the injectivity of a^* .

(i) Since a is strongly generating, the kernel of

$$(a \circ p)^*: \text{Pic}^0(A) \longrightarrow \text{Pic}^0(S_d)$$

is equal to $\text{Pic}^0(A)[d]$. From the commutativity of the diagram that defines S_d and the fact that the kernel of $\mu_d^*: \text{Pic}^0(A) \rightarrow \text{Pic}^0(A)$ is also equal to $\text{Pic}^0(A)[d]$, it follows that $\ker a_d^* = \{0\}$.

(ii) The claim follows again from the commutativity of the diagram, since μ_d is a finite map.

(iii) Both S and S_d have non-negative Kodaira dimension since they have maximal a -dimension. In addition, one has $K_{S_d} = p^* K_S$, because p is étale, and therefore K_{S_d} is nef iff K_S is.

Let H be a fixed very ample divisor on A . Set $C := a^*(H)$ and $C_d := a_d^*(H)$. By [9, Prop 2.3.5], we have that

$$\mu_d^* H \equiv d^2 H, \quad \text{and so} \quad C_d \equiv \frac{1}{d^2} p^* C \quad (2.1)$$

and

$$C_d^2 = d^{2q-4} C^2, \quad C_d \cdot K_{S_d} = d^{2q-2} C \cdot K_S. \quad (2.2)$$

The map p is étale of degree d^{2q} , so $K_{S_d} = p^* K_S$ and the following numerical invariants are multiplied by this degree:

$$K_{S_d}^2 = d^{2q} K_S^2 \quad \chi(S_d) = d^{2q} \chi(S).$$

So, given $a: S \rightarrow A$ as above and a very ample divisor H on A , for any $d > 0$ the covering trick produces the surface S_d , the map $a_d: S_d \rightarrow A$, and the divisor C_d .

The behavior of the invariants for $d \gg 0$ (and in particular the divergence between the canonical divisor and the divisor C_d) is a key point in the covering trick.

A first application is the following nefness result.

Lemma 2.4. *With the above notation, if S is minimal of general type, then for $d \gg 0$ the divisor $K_{S_d} - C_d$ is nef.*

PROOF. Let us consider the canonical model S_{can} of S and the natural map

$$S \xrightarrow{\nu} S_{\text{can}},$$

which is the contraction of the configurations of (-2) -curves of S . The map $a: S \rightarrow A$ contracts the rational curves and therefore it induces a map $\bar{a}: S_{\text{can}} \rightarrow A$; we set $\bar{C} := \bar{a}^*C$, so that $C = \nu^*\bar{C}$.

The canonical divisor $K_{S_{\text{can}}}$ is ample, hence there is $\epsilon > 0$ such that the \mathbb{Q} -divisor $K_{S_{\text{can}}} - t\bar{C}$ is ample for every rational number t with $|t| < \epsilon$. Pulling back to the smooth surface S , we obtain that

$$\nu^*(K_{S_{\text{can}}} - t\bar{C}) = K_S - tC$$

is a nef divisor.

By (2.1) and (2.2), we have

$$K_{S_d} - C_d \equiv p^* \left(K_S - \frac{1}{d^2} C \right).$$

So for any d such that $\frac{1}{d^2} < \epsilon$ we have that $K_{S_d} - C_d$ is nef.

2.2. Continuous rank and Severi-type inequalities

Let S be a smooth surface with a generating map $a: S \rightarrow A$ to an abelian variety A . Let L be a line bundle on S . For any non-negative integer i the i -th continuous rank (with respect to a) [2] of L is the integer

$$h_a^i(S, L) := \min \{ h^i(S, L + a^*\alpha) \mid \alpha \in \text{Pic}^0(A) \}.$$

If no confusion is likely to arise we shall write $L + \alpha$ for $L + a^*\alpha$.

Remark 2.5. Observe that the i -th continuous rank of L is just the rank of the i -th Fourier-Mukai transform of L with respect to the map a . If $L = K_S + D$ with D nef, and S is of maximal a -dimension, then $h_a^0(S, L) = \chi(S, L)$ by the Generic Vanishing theorem (Theorem B in [15], cf. [8] for the case $D = 0$). In particular, $h_{\text{alb}_S}^0(S, K_S) = \chi(S)$.

In [2] the following Severi-type inequalities are proved (see [2] for a much more complete statement).

Theorem 2.6 (Barja, [2]). *Let S be a smooth surface with a generating morphism $a: S \rightarrow A$ to an abelian variety A ; suppose that S is of maximal a -dimension and L is a nef divisor on S . Then*

(i) The following inequality holds:

$$L^2 \geq 2h_a^0(S, L).$$

(ii) If, moreover, $K_S - L$ is pseudoeffective, then

$$L^2 \geq 4h_a^0(S, L).$$

(iii) If $K_S \equiv L_1 + L_2$ with L_i nef, then

$$K_S^2 \geq 4\chi(S) + 4h_{\text{alb}_S}^1(S, L_1).$$

We now state some consequences of these results for the extremal cases of the above inequalities.

Remark 2.7. The proof of Theorem 2.6(iii) in [2, Corollary D (Cor. 4.9)] shows that if we have a nef decomposition $K_S \equiv L_1 + L_2$, then

$$K_S^2 - 4\chi(S) \geq (L_1^2 - 4h_{\text{alb}_S}^0(S, L_1)) + (L_2^2 - 4h_{\text{alb}_S}^0(S, L_2)) + 4h_{\text{alb}_S}^1(S, L_1).$$

Hence it follows that if we have a minimal surface of maximal Albanese dimension S with such a decomposition $K_S \equiv L_1 + L_2$ and such that $K_S^2 = 4\chi(S)$, then the following holds:

(i) $L_i^2 = 4h_{\text{alb}_S}^0(S, L_i)$, for $i = 1, 2$;

(ii) $h_{\text{alb}_S}^1(S, L_i) = 0$, for $i = 1, 2$.

2.3. Applications of Castelnuovo's bound

We now see that the classical Castelnuovo bound combined with the above results implies a condition on the asymptotic behavior of the invariants C_d^2 and $h_a^0(S_d, C_d)$ arising from the covering trick (§2.1). Let us recall the result of Castelnuovo [1, III.2].

Theorem 2.8 (Castelnuovo's bound). *Let C be a smooth curve that admits a birational g_d^r . Then the genus of C satisfies the inequality*

$$g \leq \frac{m(m-1)}{2}(r-1) + m\epsilon,$$

where $d-1 = m(r-1) + \epsilon$, and m and ϵ are non-negative integers with $0 \leq \epsilon < r-1$ (i.e., m is the quotient and ϵ the remainder of the division of $d-1$ by $r-1$).

Lemma 2.9. *Let S be a surface of Kodaira dimension $\text{kod}(S) > 0$. Let $a: S \rightarrow A$ be a strongly generating map to an abelian variety such that S is of maximal a -dimension. Let C_d be the nef divisor arising from the covering trick (§2.1). Suppose that for $d \gg 0$ and for general $\alpha \in \text{Pic}^0(A)$ the linear system $|C_d + \alpha|$ on S_d is birational; then for $d \gg 0$ the ratio $M(d) := C_d^2/h_a^0(S_d, C_d)$ tends to infinity with order at least quadratical in d .*

PROOF. Let us fix d and suppose that for general $\alpha \in \text{Pic}^0(A)$ the linear system $|C_d + \alpha|$ on S_d is birational. Notice that by generic vanishing [8] we have that, for general $\alpha \in \text{Pic}^0(A)$ again,

$$h_a^0(S_d, C_d) = h^0(S_d, C_d + \alpha) = h^0(C_d, (C_d + \alpha)|_{C_d}).$$

Let us suppose that C_d is a general element in its linear system; the linear system $|C_d|$ is base point free, since it contains the linear system $a_d^*[H]$ and H is very ample on A , therefore C_d is smooth and irreducible. We want to apply Castelnuovo's bound to the linear series defined by $|(C_d + \alpha)|_{C_d}|$, which induces a birational map.

We can compute the genus $g(C_d)$ of C_d using the adjunction formula and (2.1), (2.2):

$$g(C_d) = \frac{K_{S_d} \cdot C_d}{2} + \frac{C_d^2}{2} + 1 = d^{2q-2} \frac{K_S \cdot C}{2} + d^{2q-4} \frac{C^2}{2} + 1.$$

The bound in Theorem 2.8 is

$$g(C_d) \leq \frac{m_d(m_d - 1)}{2} (h_a^0(S_d, C_d) - 2) + m_d \epsilon,$$

where $m_d = \left\lfloor \frac{C_d^2 - 1}{h_a^0(C_d) - 2} \right\rfloor$. Observe that

$$m_d \leq \frac{C_d^2 - 1}{h_a^0(C_d) - 2} \leq 3M(d).$$

The inequality of Castelnuovo and (2.2) imply that

$$d^{2q-2} \frac{K_S C}{2} + d^{2q-4} \frac{C^2}{2} + 1 \leq \frac{3}{2} M(d)^2 h_a^0(S_d, C_d) = \frac{3}{2} M(d) d^{2q-4} C^2.$$

Observe that $K_S C > 0$ as S has strictly positive Kodaira dimension, is of maximal a -dimension and C is the pullback of a very ample divisor on A . Hence we necessarily have that $M(d)$ grows at least as d^2 .

3. Proof of the main result

In this section we prove Theorem 1.1. We start with the “only if” part, assuming that S is a minimal surface of general type and of maximal Albanese dimension with $K_S^2 = 4\chi(S)$.

For $d > 0$, let C_d be the divisor given by the covering trick applied to $a = \text{alb}_S$ (cf. §2.1). Let us fix a general α in $\text{Pic}^0(A)$ and consider the following diagram:

(3.1)

$$\begin{array}{ccccc}
 S_d & \xrightarrow{a_d} & A & \hookrightarrow & \mathbb{P}(H^0(A, H + \alpha)^\vee) \\
 \downarrow f_d & \nearrow g_d & \uparrow & & \uparrow \vdots \\
 R_d & \xrightarrow{\nu} & \overline{R_d} & \hookrightarrow & \mathbb{P}(H^0(S_d, C_d + \alpha)^\vee)
 \end{array}$$

Where:

- the surface $\overline{R_d}$ is the image of the map induced by $|C_d + \alpha|$ on S_d .
- $\nu: R_d \longrightarrow \overline{R_d}$ is the normalization.

Let us define $D_d := g_d^*H$ on R_d .

Remark 3.1. Let us make the following remarks:

- (i) The linear system $|D_d + \alpha| = |g_d^*(H + \alpha)|$ on R_d is birational by construction, and $h^0(R_d, D_d + \alpha) = h^0(S_d, C_d + \alpha)$.
- (ii) If $q(S) > 2$ then the Kodaira dimension of R_d is strictly positive, for any d , because R_d dominates the image of the map a_d which generates A by construction.
- (iii) By Lemma 2.4, for $d \gg 0$ we have that both $K_{S_d} - C_d$ and C_d are nef. Since $K_S^2 = 4\chi(S)$ then, by Lemma 2.7, we must have $C_d^2 = 4h_{a_d}^0(S_d, C_d)$.
- (iv) The map f_d is generically finite, and it is not birational for $d \gg 0$ by (iii) and Lemma 2.9.

Let us first observe that up to passing to a desingularization, we can suppose R_d smooth. In what follows we assume d large enough. Let us consider the divisor D_d on R_d , which is big and nef by construction.

By Remark 3.1(iii) we have for α general the following chain of equalities

$$(\deg f_d)D_d^2 = C_d^2 = 4h^0(S_d, C_d + \alpha) = 4h^0(R_d, D_d + \alpha). \quad (3.2)$$

Theorem 2.6 applies to the pair (R_d, D_d) giving the Severi-type inequality:

$$D_d^2 \geq 2h_{g_d}^0(R_d, D_d) = 2h^0(R_d, D_d + \alpha). \quad (3.3)$$

Combining (3.2), (3.3) and Remark 3.1 (iv), we obtain that $\deg f_d = 2$ and $D_d^2 = 2h_{g_d}^0(R_d, D_d)$.

We now prove that the Kodaira dimension of R_d is 0. Let us denote $r := h_{g_d}^0(R_d, D_d) - 1$. Observe that we have $K_{R_d}D_d \geq 0$ and $K_{R_d}D_d = 0$ only if $\text{kod}(R_d) = 0$. We can suppose that D_d is a smooth curve. The adjunction formula gives

$$2r + 2 = D_d^2 \leq 2g(D_d) - 2,$$

i.e. $r \leq g(D_d) - 2$, with equality holding only if $\text{kod}(R_d) = 0$.

By the previous computation and generic vanishing on R_d , we have that for a general $\alpha \in \text{Pic}^0(A)$ the system $|(D_d + \alpha)_{|D_d}|$ is a g_{2r+2}^r on D_d .

Thus we have a map $\text{Pic}^0(A) \rightarrow W_{2r+2}^r(D_d)$ that is injective because the map g_d is strongly generating, and $D_d^2 > 0$.

If $r \leq g(D_d) - 3$ we can apply an inequality of Debarre and Fahlaoui [7, Proposition 3.3] on the dimension of abelian varieties contained in the Brill-Noether locus, that in our case gives $\dim A \leq (D_d^2 - 2r)/2 = 1$, a contradiction.

So we have that $r = g(D_d) - 2$ and $D_d^2 = 2g(D_d) - 2$, and therefore $\text{kod}(R_d) = 0$ by the previous remarks.

Hence for $d \gg 0$ the surface R_d is birational to an abelian surface by classification. Observe that the map g_d is birational. Indeed, assume otherwise; then g_d induces a non trivial étale map between the minimal model T_d of R_d and the abelian surface A . Since a map from a smooth surface to an abelian variety is a morphism, the morphism $a_d: S_d \rightarrow A$ factorizes as $S_d \rightarrow T_d \rightarrow A$ and therefore a_d is not strongly generating, contradicting Lemma 2.3. It follows that for $d \gg 0$ the map g_d is birational and $\deg a_d = \deg f_d = 2$.

Remark 3.2. It is worth remarking that a key point of our proof is that we are not taking the limit in the covering trick as in [14]. But it is crucial

that we are allowed to take $d \gg 0$, because the properties proved above for the factorization (3.1) can fail to hold for small d , as we now observe. For d small, and H ample enough, it can well happen that C_d is birational on S_d . Consider for instance for the case $d = 1$ the following situation: $a = \text{alb}_S$ is a double covering, and B is its ample branch locus. Then choosing $H = kB$, we have that $(\text{alb}_S)^*(H) = 2kK_S$, which is very ample on S for $k \geq 3$. In case C_d is very ample, then the map f_d is an isomorphism between S_d and R_d (in particular $\text{kod}(R_d) = 2$), and g_d coincides with a_d .

Indeed, the process of taking d big enough corresponds to choosing on A a divisor which is very ample but also “comparatively small”.

Up to now we have proven that for a surface S satisfying the assumptions of Theorem 1.1, the Albanese morphism

$$a: S \longrightarrow \text{Alb}(S)$$

is a generically finite map of degree 2 onto the Albanese surface $\text{Alb}(S)$. We now see that the canonical model of S is a double covering of $\text{Alb}(S)$, and describe its possible singularities.

Let us now consider S as in Theorem 1.1, and consider the Stein factorization of a :

$$S \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\alpha} S' \xrightarrow{\pi} \text{Alb}(S) \end{array}$$

where S' is normal and π is a double covering. Let $B \subset A$ be the branch locus of π , which is reduced. Note that the map α factors through the canonical model S_{can} of S . We are going to prove that indeed $S' = S_{\text{can}}$.

Note that, as S' is of general type, B is ample; we can compute the canonical invariants of S' as follows:

$$K_{S'}^2 = \frac{B^2}{2}; \quad \chi(S') = \frac{B^2}{8}, \quad (3.4)$$

so that S' satisfies the Severi equality.

Let us perform the canonical resolution of the double covering π [3, III.7]:

$$\begin{array}{ccccccc} \tilde{S} := S_k & \xrightarrow{\sigma_k} & S_{k-1} & \longrightarrow & \cdots & \longrightarrow & S_1 \xrightarrow{\sigma_1} S_0 = S' \\ \downarrow & & \downarrow & & & & \downarrow \pi \\ \tilde{A} := A_k & \xrightarrow{\tau_k} & A_{k-1} & \longrightarrow & \cdots & \longrightarrow & A_1 \xrightarrow{\tau_1} A_0 = A \end{array}$$

Recall that the maps τ_j are successive blow-ups that resolve the singularities of B ; the morphism $S_j \rightarrow A_j$ is the double covering with branch locus defined by the inductive formula:

$$B_j := \tau_j^* B_{j-1} - 2 \left\lfloor \frac{m_{j-1}}{2} \right\rfloor E_j,$$

where E_j is the exceptional divisor of τ_j , m_{j-1} is the multiplicity for B_{j-1} of the blown-up point. The surface \tilde{S} is smooth, not necessarily minimal, birational to S , and therefore $K_{\tilde{S}}^2 = K_S^2 - \delta$, with $\delta \geq 0$. One has the following relations [16, Obs.1.16]:

$$K_S^2 = K_{\tilde{S}}^2 + \delta = K_{S'}^2 - 2 \sum_{i=1}^k \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right)^2 + \delta,$$

and

$$\chi(S) = \chi(\tilde{S}) = \chi(S') - \frac{1}{2} \sum_{i=1}^k \left\lfloor \frac{m_i}{2} \right\rfloor \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right).$$

Recall now that a singularity P of the branch locus B on A is called *negligible* (or simple, inessential, etc...) if $K_S^2 = K_{S'}^2$ and $\chi(S) = \chi(S')$. By the above formulae [16, Prop. 1.8], this is equivalent to P having multiplicity ≤ 3 and all the points infinitely near to P having multiplicity ≤ 2 .

So we have by assumption and by (3.4)

$$\begin{aligned} 0 &= K_S^2 - 4\chi(S) = K_{\tilde{S}}^2 - 4\chi(\tilde{S}) + \delta = \\ &= K_{S'}^2 - 4\chi(S') + 2 \sum_{i=1}^k \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right) + \delta = 2 \sum_{i=1}^k \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right) + \delta. \end{aligned}$$

In particular we deduce that $\delta = 0$ and $2 \sum_{i=1}^k \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right) = 0$, and the singularities of B are necessarily negligible.

Now we need only to observe that, since the branch locus B has negligible singularities, then the double cover S' has Du Val singularities [16, Table 1.18]; since there is a birational morphism $S_{\text{can}} \rightarrow S'$, we conclude that $S' = S_{\text{can}}$.

On the other hand, it is clear from the above discussion that a double cover of an abelian surface with ample branch locus B with negligible singularities is the canonical model of a general type surface of maximal Albanese dimension lying on the Severi line.

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