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RESEARCH ARTICLE

Local cone approximations in mathematical programming

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We show how to use intensively local cone approximations to obtain results in some fields of optimization theory as optimality conditions, constraint qualifications, mean value theorems and error bound.

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1. Notes

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2. Introduction

Treating smooth problems the notion of gradient are used to deduce necessary optimality conditions as stationarity conditions. The theory was extended to convex nonsmooth problems via directional derivatives and subdifferentials. In the nonconvex nonsmooth case, several trends can be distinguished as, for example, all those which generalize the concept of directional derivative.

In [10–12, 18] an axiomatic approach was given for generating generalized directional derivatives. The key tool is to observe that the epigraph of a generalized directional derivative is a conic approximation of the function's epigraph.

This concept and its properties have been used in optimization context in order to obtain general first order necessary [6, 15] and sufficient optimality conditions [3], general format of mean value theorems [5] and estimates of the error bounds [8], abstract approach to constraint qualification [4].

In another paper [7] this approach has been extended to construct second order approximations of the function's epigraph and second order generalized directional derivatives. Second order optimality conditions have been deduced.

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More recently, extensions to vector functions for treating multiobjective problems, have been studied in [1, 16, 17]. In this paper we want to describe all this path.

We start Section 3 describing local cone approximations and their main properties; in particular, those which are more relevant in optimization context. In Section 4 we describe first and second-order optimality conditions for scalar and Pareto optimization problems which can be obtained through directional K-epiderivatives. The last section is devoted to furnish an abstract scheme for deriving an approximate mean value result (Section 5.1) and an error bound for an inequality system (Section 5.2).

In the sequel the open ball with center x and radius r is denoted by $B(x,r)$. Given a set A, we indicate by A^c, $\operatorname{cl} A$, $\operatorname{int} A$ and $\operatorname{conv} A$ the complementary, the closure, the interior and the convex hull of A respectively. The domain of $f : \mathbb{R}^n \to (-\infty,+\infty]$ is $\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\} \neq \emptyset$ and its epigraph is $\operatorname{epi} f = \{(x,y) \in \mathbb{R}^{n+1} : y \geq f(x)\}$. If $A \subseteq \mathbb{R}^n$ is a closed subset, the support function associated to A is

$$\sigma(x,A) = \sup\{\langle x^*, x\rangle : x^* \in A\};$$

and its domain, denoted $\operatorname{barr} A$, is called barrier cone of A. The recession cone of A is $0^+A = \{x \in \mathbb{R}^n : A + x \subseteq A\}$ and if $p : \mathbb{R}^n \to (-\infty,+\infty]$ is a positively homogeneous function, the associated recession function is

$$p^\infty(x) = \sup\{p(x+y) - p(y) : y \in \operatorname{dom} p\}.$$

In general we assume the usual convention $\inf \emptyset = +\infty$.

3. Local approximations and generalized epiderivatives

It is well known that a close relationship exists between local cone approximations which generalize the classical tangent cone and different generalized differentiability concepts. In fact the epigraph of a generalized directional derivative of a function f can be viewed as a conical approximation of the epigraph of f. Using the approach developed by Dubovitskij and Miljutin [9], in the papers [10–12, 18] Elster and Thierfelder proposed a general definition of local cone approximation and introduced the corresponding directional K-epiderivative. Using these notions it is possible to derive general necessary optimality conditions which turn out be true generalizations of the Kuhn-Tucker theory for nonsmooth optimization problems.

Definition 1 The map $K : 2^{\mathbb{R}^n} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a *local cone approximation* if

$$tK(A,x) = K(A,x), \qquad \forall (A,x) \in 2^{\mathbb{R}^n} \times \mathbb{R}^n,\ \forall t \geq 0$$

and the following properties hold:

(i) $K(A,x) = K(A-x,0)$,
(ii) $K(A \cap B(x,r),x) = K(A,x)$ for each $r > 0$,
(iii) $K(A,x) = \emptyset$ for each $x \notin \operatorname{cl} A$,
(iv) $K(A,x) = \mathbb{R}^n$ for each $x \in \operatorname{int} A$,
(v) $K(\varphi(A),\varphi(x)) = \varphi(K(A,x))$ with φ linear homeomorphism,
(vi) $0^+A \subseteq 0^+K(A,x)$ for each $x \in \operatorname{cl} A$.

It is possible to prove that all the six axioms are independent that means one of the axioms can't be expressed by the others. The most known tangent cones are local cone approximations. For instance the cone of feasible directions

$$Z(A,x) = \{v \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } \forall t \in (0,r)\ x + tv \in A\},$$

the radial tangent cone

$$F(A,x) = \{v \in \mathbb{R}^n : \forall r > 0\ \exists t \in (0,r)\ x + tv \in A\},$$

the cone of interior displacements

$$D(A,x) = \{v \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } \forall t \in (0,r)\ \forall v' \in B(v,r)\ x + tv' \in A\},$$

and the contingent cone

$$T(A,x) = \{v \in \mathbb{R}^n : \forall r > 0\ \exists t \in (0,r)\ \exists v' \in B(v,r) \text{ s.t. } x + tv' \in A\}$$

are local cone approximations. Other particular local cone approximations are listed in [12]. It is possible to construct further local cone approximations by common set operations assuming that cone approximations K and K_i, with $i \in I$ arbitrary index set, are given. For instance

$$\operatorname{int} K, \quad \operatorname{cl} K, \quad \operatorname{conv} K, \quad \bigcup_{i\in I} K_i, \quad \bigcap_{i\in I} K_i, \quad \sum_{i\in I} K_i$$

are local cone approximations. We pay more attention to another particular operation which is fundamental in order to derive optimality conditions for extremum problems. Starting from a local cone approximation K, the map K_c defined by

$$K_c(A,x) = (K(A^c,x))^c, \qquad \forall (A,x) \in 2^{\mathbb{R}^n} \times \mathbb{R}^n$$

is a local cone approximation. For instance $Z = F_c$ and $D = T_c$.

The notion of local cone approximation allows us to introduce a generalized directional derivatives of a proper extended-value function f.

Definition 2 Let K be a local cone approximation and $x \in \operatorname{dom} f$; the *directional K-epiderivative* of f at x is the positively homogeneous function $f^K(x,\cdot) : \mathbb{R}^n \to [-\infty,+\infty]$ defined by

$$f^K(x,v) = \inf\{y \in \mathbb{R} : (v,y) \in K(\operatorname{epi} f, (x, f(x)))\}.$$

In this way, we obtain a large family of generalized derivatives. For instance $f^Z(x,v)$ is the upper Dini derivative of f at $x \in \operatorname{dom} f$ in the direction v and

$$f^Z(x,v) = \limsup_{t\to 0^+} \frac{f(x+tv) - f(x)}{t}.$$

Analogously $f^F(x,v)$ is the lower Dini derivative and

$$f^F(x,v) = \liminf_{t\to 0^+} \frac{f(x+tv) - f(x)}{t};$$

$f^D(x,v)$ is the upper Dini-Hadamard derivative and

$$f^D(x,v) = \liminf_{(v',t)\to(v,0^+)} \frac{f(x+tv')-f(x)}{t};$$

$f^T(x,v)$ is the lower Dini-Hadamard derivative and

$$f^T(x,v) = \liminf_{(v',t)\to(v,0^+)} \frac{f(x+tv')-f(x)}{t}.$$

As a direct consequence of the definition, from property (vi), we have $(0,1) \in 0^+K(\operatorname{epi} f,(x,f(x)))$ and so the epigraph of $f^K(x,\cdot)$ is the vertical closure of the cone $K(\operatorname{epi} f,(x,f(x)))$, i.e.

$$\operatorname{epi} f^K(x,\cdot) = \{(y,\beta) \in \mathbb{R}^{n+1} : \forall\varepsilon > 0,\ (y,\beta+\varepsilon) \in K(\operatorname{epi} f,(x,f(x)))\}.$$

In particular if K is closed $\operatorname{epi} f^K(x,\cdot) = K(\operatorname{epi} f,(x,f(x)))$ and therefore $f^K(x,\cdot)$ is lower semicontinuous. This happens, for instance, for the lower Dini-Hadamard derivative.

When the cone approximation of the epigraph of f coincides with the cone approximation of the strict epigraph then

$$-f^K(x,v) = (-f)^{K_c}(x,v), \qquad \forall(x,v) \in \operatorname{dom} f \times \mathbb{R}^n.$$

In particular $-f^F(x,v) = (-f)^Z(x,v)$ and $-f^T(x,v) = (-f)^D(x,v)$.

In optimization, the calculation of the generalized directional derivative of the pointwise maximum of a family of functions is quite common. Let $\{f_i\}_{i\in I}$ be a finite family of upper semicontinuous functions and $f_{\max}(x) = \max_{i\in I}\{f_i(x)\}$. If K is a local cone approximation such that $K(A\cap B,x) \subseteq K(A,x)\cap K(B,x)$ for all pairs of sets A and B (and this is, for example, verified by all above mentioned four tangent cones) then

$$f^K_{\max}(x,v) \geq \max\{f_i^K(x,v) : i \in I_{\max}(x)\}, \qquad \forall(x,v) \in \bigcap_{i\in I} \operatorname{dom} f_i \times \mathbb{R}^n$$

where $I_{\max}(x) = \{i \in I : f_{\max}(x) = f_i(x)\}$. Analogous result holds for the pointwise minimum of lower semicontinuous functions changing $\cap$ with $\cup$, $\subseteq$ with $\supseteq$, max with min and $\geq$ with $\leq$. Other interesting properties of the local cone approximations and the related directional K-epiderivatives are collected in [11, 12, 15].

In [2], it has been shown that there exists a dual characterization of a very large class of positively homogeneous functions and, by means of this dual representation, it is possible to deduce theorems of the alternative. A positively homogeneous function $p : \mathbb{R} \to (-\infty,+\infty]$ is said to be the *pointwise minimum of sublinear functions* (in short MSL function) if there exists a family $M(p)$ of nonempty closed and convex sets such that

$$p(x) = \min\{\sigma(x,C) : C \in M(p)\}, \qquad \forall x \in \mathbb{R}^n.$$

Using this class of functions, it is possible to furnish a dual representation of a directional K-epiderivative.

Definition 3 Let K be a local cone approximation and $x \in \operatorname{dom} f$; the function f is said *K-MSL-differentiable* at x if there exist an index set T and a family $\{\partial_t^K f(x)\}_{t \in T}$ of nonempty closed and convex sets such that

$$f^K(x,v) = \min\{\sigma(v, \partial_t^K f(x)) : t \in T\}, \qquad \forall v \in \mathbb{R}^n.$$

In particular, when T is a singleton the function f is said *K-subdifferentiable* at x and the unique closed and convex set $\partial^K f(x)$ is called *K-subdifferential.*

The class of K-MSL-differentiable functions is quite wide. Indeed, adapting the results in [2] to the directional K-epiderivatives, it is possible to show that given a local cone approximation K and $x \in \operatorname{dom} f$ then f is K-MSL-differentiable if and only if $f^K(x,\cdot)$ is proper and $f^K(x,0) = 0$. Moreover if $f^K(x,\cdot)$ is Lipschitzian then all the sets $\partial_t^K f(x)$ may be chosen compact.

In [7] the concept of local cone approximation has been extended in a natural way to the second-order approximation.

Definition 4 The map $K^2 : 2^{\mathbb{R}^n} \times \mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a *local second-order approximation* if

$$t^2 K^2(A,x,v) = K^2(A,x,tv), \qquad \forall (A,x,v) \in 2^{\mathbb{R}^n} \times \mathbb{R}^n \times \mathbb{R}^n,\ \forall t \geq 0$$

and the following properties hold:

(i) $K^2(A,x,v) = K^2(A-x,0,v)$;
(ii) $K^2(A \cap B(x,r),x,v) = K^2(A,x,v)$, for each $r > 0$;
(iii) $K^2(A,x,v) = \emptyset$, for each $x \notin \operatorname{cl} A$;
(iv) $K^2(A,x,v) = \mathbb{R}^n$, for each $x \in \operatorname{int} A$;
(v) $K^2(\varphi(A),\varphi(x),\varphi(v)) = \varphi(K^2(A,x,v))$, with φ linear homeomorphism;
(vi) $0^+A \subseteq 0^+K^2(A,x,v)$, for each $x \in \operatorname{cl} A$ and $v \in \operatorname{cl} K^2(A,x,0)$.

It is immediate to observe that if K is a local cone approximation then the map $K^2(A,x,v) = K(A,x)$ is a local second-order approximation. Moreover, if K^2 is a local second-order approximation, the map $K(A,x) = K^2(A,x,0)$ is a local cone approximation which will be called local cone approximation associated to K^2. Therefore axiom (vi) might be written as

$$0^+A \subseteq 0^+K^2(A,x,v), \quad \forall x \in \operatorname{cl} A,\ \forall v \in \operatorname{cl} K(A,x).$$

The following maps are local second-order approximations which are not local cone approximations. The set of second-order feasible directions is

$$Z^2(A,x,v) = \{u \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } \forall t \in (0,r)\ x + tv + t^2u \in A\},$$

the set of the second-order radial directions is

$$F^2(A,x,v) = \{u \in \mathbb{R}^n : \forall r > 0\ \exists t \in (0,r)\ x + tv + t^2u \in A\},$$

the set of the second-order interior displacements is

$$D^2(A,x,v) = \{u \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } \forall t \in (0,r)\ \forall u' \in B(u,r)\ x + tv + t^2u' \in A\},$$

the set of the second-order contingent directions is

$$T^2(A,x,v)=\{u\in\mathbb{R}^n:\forall r>0\ \exists t\in(0,r)\ \exists u'\in B(u,r)\text{ s.t. } x+tv+t^2u'\in A\}.$$

As stated for local cone approximations, the formula

$$K_c^2(A,x,v)=(K^2(A^c,x,v))^c,\quad \forall(A,x,v)\in 2^{\mathbb{R}^n}\times\mathbb{R}^n\times\mathbb{R}^n$$

produces new local second-order approximations. For instance $Z^2=F_c^2$ and $D^2=T_c^2$.

Following the same scheme used for the first-order approximations, we define second-order directional K^2-epiderivatives.

Definition 5 Let K^2 be a local second-order approximation, $x\in\operatorname{dom}f$ and $v\in\operatorname{dom}f^K(x,\cdot)$; the *second-order directional K^2-epiderivative* of f at x in the directions v and y is

$$f^{K^2}(x,v,y)=\inf\{\beta\in\mathbb{R}:(y,\beta)\in K^2(\operatorname{epi}f,(x,f(x)),(v,f^K(x,v)))\}.$$

Notice that if $f^K(x,0)=0$ then $f^{K^2}(x,0,y)=f^K(x,y)$. Moreover, if $f^{K^2}(x,v,y)\in\mathbb{R}$, for each $t>0$, we get $f^{K^2}(x,tv,t^2y)=t^2f^{K^2}(x,v,y)$.

The following second-order directional epiderivatives arise from the definition

$$f^{Z^2}(x,v,y)=\limsup_{t\to0^+}\frac{f(x+tv+t^2y)-f(x)-tf^Z(x,v)}{t^2}$$

$$f^{F^2}(x,v,y)=\liminf_{t\to0^+}\frac{f(x+tv+t^2y)-f(x)-tf^F(x,v)}{t^2}$$

$$f^{D^2}(x,v,y)=\limsup_{(y',t)\to(y,0^+)}\frac{f(x+tv+t^2y')-f(x)-tf^D(x,v)}{t^2}$$

$$f^{T^2}(x,v,y)=\liminf_{(y',t)\to(y,0^+)}\frac{f(x+tv+t^2y')-f(x)-tf^T(x,v)}{t^2}$$

Properties of the second-order directional epiderivatives can be found in [7]. Here we want only to draw attention to a discrepancy between first and second-order directional K-epiderivatives. The chain of inclusions

$$D(A,x)\subseteq F(A,x)\subseteq Z(A,x)\subseteq T(A,x)$$

implies that the chain of inequalities

$$f^D(x,v)\geq f^F(x,v)\geq f^Z(x,v)\geq f^T(x,v).$$

The same chain of inclusions holds for the second-order local approximations but the presence of $(v,f^K(x,v))$ in the definition of second-order directional K^2-epiderivative implies that no relationship exists, in general, between f^{D^2}, f^{F^2}, f^{Z^2} and f^{T^2}.

4. Optimality conditions in mathematical programming

Now we focus our attention on the optimization problem

$$\min\{f(x) : x \in S\}, \tag{1}$$

where $f : \mathbb{R}^n \to (-\infty, +\infty]$ is a proper function and $S \subseteq \mathbb{R}^n$ is the feasible region. It is immediate to observe that $\overline{x} \in S$ is a local solution for (1) if and only if there exists $r > 0$ such that

$$\operatorname{epi} f \cap [S \times (-\infty, f(\overline{x}))] \cap B((\overline{x}, f(\overline{x})), r) = \emptyset. \tag{2}$$

Even if this expression is easy and quite elegant from the formal viewpoint, in general it is an arduous task to verify them. For this reason it is suitable to replace the sets in (2) with approximations having a simpler structure: local cone or second-order approximations. The pair of local second-order approximations (K^2, H^2) will be called *admissible* if for all $A, B \subseteq \mathbb{R}^n$ such that $A \cap B = \emptyset$ we have

$$K^2(A, x, v) \cap H^2(B, x, v) = \emptyset, \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

An easy way to obtain admissible pairs is the following. Let K^2 be an *isotone* local second-order approximation, that means, for every $A, B \subseteq \mathbb{R}^n$ with $A \subseteq B$

$$K^2(A, x, v) \subseteq K^2(B, x, v), \qquad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n;$$

then the pair (K^2, K^2_c) is admissible. For instance F^2 and T^2 are isotone and therefore the pairs (F^2, Z^2) and (T^2, D^2) are admissible. Therefore, if (K^2, H^2) is an admissible pair, a necessary condition for (2) is

$$K^2(\operatorname{epi} f, (\overline{x}, f(\overline{x})), (v, \alpha)) \cap H^2(S \times (-\infty, f(\overline{x})), (\overline{x}, f(\overline{x})), (v, \alpha)) = \emptyset \tag{3}$$

for all $(v, \alpha) \in \mathbb{R}^{n+1}$. When $S = \mathbb{R}^n$, choosing an isotone local second-order approximation K^2, from (3), we deduce the following first [6] and second-order [7] optimality condition:

$$f^K(\overline{x}, v) \geq 0, \qquad \forall v \in \mathbb{R}^n \tag{4}$$

and, for each $v, w \in \mathbb{R}^n$ such that $f^K(\overline{x}, v) = 0$ we get

$$f^{K^2}(\overline{x}, v, w) \geq 0.$$

If f is K-MSL-differentiable the first order necessary condition (4) can be equivalently written in the dual space

$$0 \in \bigcap_{t \in T} \partial^K_t f(\overline{x}).$$

A general theorem which establishes a necessary optimality condition for problems with abstract constraints is the following [7].

THEOREM 4.1 *Let $\overline{x}$ be a local solution for (1), (K^2, H_0^2) be an admissible pair of local second-order approximations and H^2 be a local second-order approximation such that for each point $(x, a) \in \mathbb{R}^{n+1}$ and for each direction $(v, \beta) \in \mathbb{R}^{n+1}$ we have*

$$H_0^2(S \times (-\infty, a), (x, a), (v, \beta)) \supseteq H^2(\Omega, x, v) \times \begin{cases} \emptyset & \text{if } \beta > 0 \\ (-\infty, 0) & \text{if } \beta = 0 \\ \mathbb{R} & \text{if } \beta < 0 \end{cases} \tag{5}$$

Then

$$f^K(\overline{x}, v) \geq 0, \qquad \forall v \in H(S, \overline{x}). \tag{6}$$

Moreover for each $v, w \in \mathbb{R}^n$ such that $f^K(\overline{x}, v) \leq 0$ and $w \in H^2(S, \overline{x}, v)$ we get

$$f^{K^2}(\overline{x}, v, w) \geq 0.$$

In particular, from the proof of Theorem 4.1, it is possible to deduce that if $f^K(\overline{x}, v) < 0$ then $f^{K^2}(\overline{x}, v, w) = +\infty$. Let us observe that condition (5) holds choosing H and H_0 as one of the four above mentioned local second-order approximations. Moreover, if f is K-MSL-differentiable, if the cone $H(S, \overline{x})$ is convex and if the following condition holds

$$-H^\circ(S, \overline{x}) \cap \left(\operatorname{barr} \partial_t^K f(\overline{x})\right)^\circ = \{0\}, \qquad \forall t \in T,$$

then first order optimality condition (6) can be equivalently written in dual form

$$0 \in \bigcap_{t \in T} \left(\partial_t^K f(\overline{x}) + H^\circ(S, \overline{x})\right).$$

Now we assume that the feasible region S is expressed by inequality constraints

$$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \ i \in I\}, \tag{7}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ and I is a finite index set. As usual we denote by $I(x) = \{i \in I : g_i(x) = 0\}$ the set of active constraints and by $S_i = \{x \in \mathbb{R}^n : g_i(x) \leq 0\}$ the region identified by a single constraint.

THEOREM 4.2 *Let $\overline{x}$ be a local solution for (1) with S expressed by (7), (K, H_0) be an admissible pair of local cone approximation, and g_i be upper semicontinuous for each $i \in I \setminus I(\overline{x})$. Suppose there exist a family of local cone approximation $\{H_i\}_{i \in I(\overline{x})}$ and a local cone approximation H satisfying assumption (5),*

$$\bigcap_{i \in I(\overline{x})} H(S_i, \overline{x}) \subseteq H(S, \overline{x}). \tag{8}$$

and $\{v \in \mathbb{R}^n : g_i^{H_i}(\overline{x}, v) < 0\} \subseteq H(S_i, \overline{x})$ for every $i \in I(\overline{x})$. Then the system

$$\begin{cases} f^K(\overline{x}, v) < 0, \\ g_i^{H_i}(\overline{x}, v) < 0, \quad i \in I(\overline{x}). \end{cases} \tag{9}$$

is impossible.

If all the directional K-epiderivatives are MSL functions, by means of a theorem of alternative [4], the impossibility of the system (9) coincides with the following generalized John necessary optimality condition:

$$0 \in \operatorname{cl}\operatorname{conv}\left(\partial_t^K f(\overline{x}) \cup \bigcup_{i \in I(\overline{x})} \partial_{t_i}^{H_i} g_i(\overline{x})\right) \tag{10}$$

for each $t \in T$ and $t_i \in T_i$ with $i \in I(\overline{x})$.

Moreover, if all the sets $\partial_t^K f(\overline{x})$ and $\partial_{t_i}^{K_i} g_i(\overline{x})$ are compact, then (10) assumes the following simpler form

$$0 \in \theta \partial_t^K f(\overline{x}) + \sum_{i \in I(\overline{x})} \lambda_i \partial_{t_i}^{K_i} g_i(\overline{x}). \tag{11}$$

A crucial point in optimization theory is to establish conditions which guarantee the multiplier $\theta \neq 0$. Thanks to a generalized Farkas lemma for MSL systems proved in [13], the KKT optimality condition

$$0 \in \operatorname{cl}\left(\partial_t^K f(\overline{x}) + \operatorname{cone}\operatorname{conv} \bigcup_{i \in I(\overline{x})} \partial_{t_i}^{H_i} g_i(\overline{x})\right)$$

for each $t \in T$ and $t_i \in T_i$ with $i \in I(\overline{x})$, coincides with the impossibility of the system

$$\begin{cases} f^K(\overline{x}, v) < 0, \\ g_i^{H_i}(\overline{x}, v) \leq 0, \quad i \in I(\overline{x}). \end{cases} \tag{12}$$

For this reason, a regularity condition can be viewed as a condition that guarantees the impossibility of the system (12) starting from the impossibility of the system (9). This point of view allows us to furnish various regularity conditions without requiring any assumption of convexity or its generalizations [4].

Theorem 4.3 *Assume that all the assumptions of Theorem 4.2 are satisfied and $f^K(\overline{x}, \cdot)$ and $g_i^{H_i}(\overline{x}, \cdot)$ are proper. Define*

$$G(\overline{x}, v) = \max\{g_i^{H_i}(\overline{x}, v) : i \in I(\overline{x})\}$$

and suppose that one of the following regularity condions holds:

(i) $\operatorname{dom} f^{0^+K}(\overline{x}, \cdot) \cap \{v \in \mathbb{R}^n : G^\infty(\overline{x}, v) < 0\} \neq \emptyset$;

(ii) $f^K(\overline{x}, \cdot)$ *upper semicontinuous and*

$$\forall v \in \mathbb{R}^n : G(\overline{x}, v) = 0, \ \exists w \in \mathbb{R}^n \ : \ \liminf_{t \downarrow 0} \frac{G(\overline{x}, v + tw) - G(\overline{x}, v)}{t} < 0;$$

(iii) $f^K(\overline{x}, \cdot)$ *upper semicontinuous and*

$$\operatorname{cl}\{v \in \mathbb{R}^n : G(\overline{x}, v) < 0\} = \{v \in \mathbb{R}^n : G(\overline{x}, v) \leq 0\}.$$

Then the system (12) is impossible.

The convexity and its generalizations play a fundamental role in order to derive sufficient optimality conditions. A quite weak concept of generalized convexity for differentiable functions was introduced in [14] with the name of invexity. By exploiting the concept of directional K-epiderivative it is possible to give [3] a unifying definition of invexity, quasiinvexity and pseudoinvexity for nonsmooth functions for obtaining sufficient optimality conditions.

Definition 6 Let K be a local cone approximation; the function f is said

- K-*invex* if there exists a function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$f(x_1) - f(x_2) \geq f^K(x_2, \eta(x_1, x_2)), \qquad \forall (x_1, x_2) \in \mathbb{R}^n \times \operatorname{dom} f;$$

- K-*quasiinvex* if there exists a function η such that $f(x_1) \leq f(x_2)$ implies $f^K(x_2, \eta(x_1, x_2)) \leq 0$ for all $x_1 \in \mathbb{R}^n$ and $x_2 \in \operatorname{dom} f$;
- *strictly* K-*pseudoinvex* if there exists a function η such that $f(x_1) \leq f(x_2)$ implies $f^K(x_2, \eta(x_1, x_2)) < 0$ for all $x_1 \in \mathbb{R}^n \setminus \{x_2\}$ and $x_2 \in \operatorname{dom} f$.

Clearly

$$K\text{-invexity} \Rightarrow K\text{-quasiinvexity} \Rightarrow \text{strict } K\text{-pseudoinvexity}.$$

Moreover it is clear that f is K-invex if and only if every point x satisfying (4) is a global minimum point. The concept of K-invexity and its generalizations allows us to deduce sufficient optimality conditions directly from the impossibility either of the system (9) or of the system (12) as reported by the following theorem [3].

THEOREM 4.4 *Let* $\overline{x} \in S$.

- *If the system (9) is impossible,* f *is* K*-invex and* g_i *are strictly* H_i*-pseudoinvex with respect to the same kernel* η*, then* $\overline{x}$ *is a global solution.*
- *If the system (12) is impossible,* f *is* K*-invex and* g_i *are* H_i*-quasiinvex with respect to the same kernel* η*, then* $\overline{x}$ *is a global solution.*

We have noted that the impossibility of (12) descends from the impossibility of (9) in presence of a regularity condition. Nevertheless, even if we have not regularity but we strengthen the hypothesis of invexity of the constraint functions, the impossibility of the system (9) implies the optimality of $\overline{x}$.

The described approach via local approximations can be extended in a natural way to the multiobjective optimization with inequality constraints. Let us consider the following problem

$$\operatorname{Min}\{(f_1(x), \ldots, f_m(x)) : x \in S\} \tag{13}$$

where S is described as in (7) and $f_j : \mathbb{R}^n \to \mathbb{R}$ for all $j \in J = \{1, \ldots, m\}$. Here Min means the minimum with respect to the Pareto cone. Let us recall that a feasible point $\overline{x}$ is an efficient solution of (13) if there exists no feasible solution x such that $f_{j_0}(x) < f_{j_0}(\overline{x})$ for some $j_0 \in J$ and $f_j(x) \leq f_j(\overline{x})$ for all $j \in J \setminus \{j_0\}$. Clearly every efficient solution of (13) solves the scalar optimization problem

$$\min\{f_{j_0}(x) : x \in S \cap S_{j_0}\} \tag{14}$$

where $S_{j_0} = \{x \in \mathbb{R}^n : f_j(x) \leq f_j(\overline{x}),\ \forall j \in J \setminus \{j_0\}\}$. Hence, choosing two suitable families of local cone approximations $\{K_j : j \in J\}$ and $\{H_i : i \in I(\overline{x})\}$, it is possible to apply Theorem 4.2 achieving the impossibility of the system

$$\begin{cases} f_j^{K_j}(\overline{x}, y) < 0, & j \in J \\ g_i^{H_i}(\overline{x}, y) < 0, & i \in I(\overline{x}). \end{cases} \tag{15}$$

In [16] a KKT condition has been achieved via a theorem of alternative (there is a further extension in [17] in presence of equality constraints).

Again, using suitable assumptions of invexity, from the impossibility of system (15) some sufficient conditions similar to the ones expressed in Theorem 4.4 are deduced for the multiobjective problem. Other optimality conditions with local cone approximations are presented also in [1].

5. Further applications of the local cone approximations

In this section we furnish two applications of the abstract concept of local cone approximation. The first part is devoted to establish a generalization of the Zagrodny mean value theorem, the last part concerns a sufficient condition which guarantees the existence of the error bound for a system of inequalities.

5.1. *An approximate mean value theorem*

Many authors have introduced different axiomatic approaches in order to derive generalizations of the Zagrodny approximate mean value theorem [19]. Such an effort has been devoted to avoid redoubling of different results which proofs follow the same principles. The core of these approaches is based on the construction of an axiomatic class of abstract subdifferentials containing as special case many well-known subdifferentials. Nevertheless an abstract form of the approximate mean value theorem can be achieved also by means of the concept of directional K-epiderivative [5].

We recall that a local cone approximation K is said *convex-regular* if for each lower semicontinuous function f and for each continuous convex function g we have

$$(f+g)^K(x,v) \leq f^K(x,v) + g'(x,v), \qquad \forall (x,v) \in \operatorname{dom} f \times \mathbb{R}^n.$$

where $g'(x,v)$ denotes the classical directional derivative. For instance Z, F, D and T are convex-regular local cone approximations.

THEOREM 5.1 *Let K be an isotone and convex-regular local cone approximation and f be lower semicontinuous; then, for each $a, b \in \mathbb{R}^n$ with $a \in \operatorname{dom} f$, and for each $r \leq f(b)$ there exist $\overline{x} \in [a,b)$ and a sequence $\{x_k\} \subseteq \operatorname{dom} f$ with $x_k \to \overline{x}$ and $f(x_k) \to f(\overline{x})$ such that*

$$\liminf_{k \to +\infty} f^K(x_k, b-a) \geq r - f(a)$$

and

$$\liminf_{k \to +\infty} f^K(x_k, b-x_k) \geq \frac{\|b-\overline{x}\|}{\|b-a\|}(r - f(a)).$$

Some consequences can be deduced from Theorem 5.1. For instance if all the assumptions of the theorem are satisfied and, in addiction, there exists $L > 0$ such that

$$f^K(x, v) \leq L\|v\|, \qquad \forall (x, v) \in \operatorname{dom} f \times \mathbb{R}^n$$

then f is Lipschitzian with constant L.

Another result is related to the monotonicity of a lower semicontinuous function. Let C be a convex and pointed cone and K be an isotone and convex-regular local cone approximation. If $f^K(x, v) \leq 0$ for all $v \in C$ then f is C-decreasing in the sense that $f(x_1) \geq f(x_2)$ for every $x_1, x_2 \in \mathbb{R}^n$ with $x_2 - x_1 \in C$.

5.2. *An error bound result for inequality systems*

Roughly speaking, the solution set of an inequality system is said to have an error bound if the involved functions provide an upper estimate for the distance from any point to the solution set. More precisely, given a function f and denoted the solution set of the inequality by

$$S = \{x \in \mathbb{R}^n : f(x) \leq 0\},$$

we say that S has a *local error bound* if it is nonempty and there exist two constants $\mu > 0$ and $a > 0$ such that

$$d(x, S) \leq \mu f_+(x), \qquad \forall x \in f^{-1}(-\infty, a)$$

where $d(x, S) = \inf_{x' \in S} \|x - x'\|$ and $f_+(x) = \max\{0, f(x)\}$. In [8] a sufficient condition for the error bound of a parametric system with lower semicontinuous functions defined on a Banach space has been established using the class of the directional K-epiderivatives. A nonparametric version of that result is the following.

Theorem 5.2 *Let K be an isotone and convex-regular local cone approximation and f be lower semicontinuous. Suppose that*

(i) there exists $a > 0$ such that $f^{-1}(-\infty, a) \neq \emptyset$,
(ii) there exists $m > 0$ such that, for each $x \in f^{-1}(0, a)$ there is $v = v(x) \in \mathbb{R}^n$ such that

$$f^K(x, v) < -m\|v\|;$$

then

$$d(x, S) \leq m^{-1} f_+(x), \qquad \forall x \in f^{-1}(-\infty, a).$$

Since $m\|v\| = \sigma(v, mB)$, if the function f is K-MSL-differentiable, we have

$$f^K(x, v) + m\|v\| = \min_{t \in T} \sigma(v, \partial_t^K f(x) + mB);$$

hence, we may write assumption (ii) in the following form

$$0 \notin \bigcap_{t \in T} \left(\partial_t^K f(x) + mB\right).$$

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