Chapter 1
Invariant distances

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In this chapter we shall define the (invariant) distance we are going to use, and collect some of its main properties we shall need later on. It will not be a comprehensive treatise on the subject; much more informations can be found in, e.g., [3, 85, 105].

Before beginning, let us introduce a couple of notations we shall consistently use.

Definition 1.0.1. Let X and Y be two (finite dimensional) complex manifolds. We shall denote by $\mathrm{Hol}(X,Y)$ the set of all holomorphic maps from X to Y, endowed with the compact-open topology (which coincides with the topology of uniform convergence on compact subsets), so that it becomes a metrizable topological space. Furthermore, we shall denote by $\mathrm{Aut}(X) \subset \mathrm{Hol}(X,X)$ the set of automorphisms, that is invertible holomorphic self-maps, of X. More generally, if X and Y are topological spaces we shall denote by $C^0(X,Y)$ the space of continuous maps from X to Y, again endowed with the compact-open topology.

Definition 1.0.2. We shall denote by $\Delta = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ the unit disc in the complex plane $\mathbb{C}$, by $\mathbb{B}^n = \{z \in \mathbb{C}^n \mid \|z\| < 1\}$ (where $\|\cdot\|$ is the Euclidean norm) the unit ball in the n-dimensional space $\mathbb{C}^n$, and by $\Delta^n \subset \mathbb{C}^n$ the unit polydisc in $\mathbb{C}^n$. Furthermore, $\langle\cdot,\cdot\rangle$ will denote the canonical Hermitian product on $\mathbb{C}^n$.

1.1 The Poincaré distance

The model for all invariant distances in complex analysis is the Poincaré distance on the unit disc of the complex plane; we shall then start recalling its definitions and main properties (see also Appendix A).

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Definition 1.1.1. The *Poincaré* (or *hyperbolic*) *metric* on Δ is the Hermitian metric whose associated norm is given by

$$\kappa_\Delta(\zeta;v) = \frac{1}{1-|\zeta|^2}|v|$$

for all $\zeta \in \Delta$ and $v \in \mathbb{C} \simeq T_\zeta\Delta$. It is a complete Hermitian metric with constant Gaussian curvature -4.

Definition 1.1.2. The *Poincaré* (or *hyperbolic*) *distance* k_Δ on Δ is the integrated form of the Poincaré metric. It is a complete distance, whose expression is

$$k_\Delta(\zeta_1,\zeta_2) = \tfrac{1}{2}\log\frac{1+\left|\frac{\zeta_1-\zeta_2}{1-\bar{\zeta}_1\zeta_2}\right|}{1-\left|\frac{\zeta_1-\zeta_2}{1-\bar{\zeta}_1\zeta_2}\right|}\,.$$

In particular,

$$k_\Delta(0,\zeta) = \tfrac{1}{2}\log\frac{1+|\zeta|}{1-|\zeta|}\,.$$

Remark 1.1.3. It is useful to keep in mind that the function

$$t \mapsto \tfrac{1}{2}\log\frac{1+t}{1-t}$$

is the inverse of the hyperbolic tangent $\tanh t = (e^t - e^{-t})/(e^t + e^{-t})$.

Besides being a metric with constant negative Gaussian curvature, the Poincaré metric strongly reflects the properties of the holomorphic self-maps of the unit disc. For instance, the isometries of the Poincaré metric coincide with the holomorphic or anti-holomorphic automorphisms of Δ (see, e.g., [3, Proposition 1.1.8]):

Proposition 1.1.4. *The group of smooth isometries of the Poincaré metric consists of all holomorphic and anti-holomorphic automorphisms of* Δ.

More importantly, the famous *Schwarz-Pick lemma* says that any holomorphic self-map of Δ is nonexpansive for the Poincaré metric and distance (see, e.g., [3, Theorem 1.1.6]):

Theorem 1.1.5 (Schwarz-Pick lemma). *Let* $f \in \mathrm{Hol}(\Delta,\Delta)$ *be a holomorphic self-map of* Δ. *Then:*

(i) *we have*

$$\kappa_\Delta\bigl(f(\zeta);f'(\zeta)v\bigr) \le \kappa_\Delta(\zeta;v) \tag{1.1}$$

for all $\zeta \in \Delta$ *and* $v \in \mathbb{C}$. *Furthermore, equality holds for some* $\zeta \in \Delta$ *and* $v \in \mathbb{C}^*$ *if and only if equality holds for all* $\zeta \in \Delta$ *and all* $v \in \mathbb{C}$ *if and only if* $f \in \mathrm{Aut}(\Delta)$*;*

(ii) *we have*

$$k_\Delta\bigl(f(\zeta_1),f(\zeta_2)\bigr) \le k_\Delta(\zeta_1,\zeta_2) \tag{1.2}$$

for all ζ_1, $\zeta_2 \in \Delta$. Furthermore, equality holds for some $\zeta_1 \neq \zeta_2$ if and only if equality holds for all ζ_1, $\zeta_2 \in \Delta$ if and only if $f \in \mathrm{Aut}(\Delta)$.

In other words, *holomorphic self-maps of the unit disc are automatically* 1-*Lipschitz, and hence equicontinuous, with respect to the Poincaré distance.*

As an immediate corollary, we can compute the group of automorphisms of Δ, and thus, by Proposition 1.1.4, the group of isometries of the Poincaré metric (see, e.g., [3, Proposition 1.1.2]):

Corollary 1.1.6. *The group* $\mathrm{Aut}(\Delta)$ *of holomorphic automorphisms of* Δ *consists in all the functions* $\gamma\colon \Delta \to \Delta$ *of the form*

$$\gamma(\zeta) = e^{i\theta} \frac{\zeta - \zeta_0}{1 - \overline{\zeta_0}\zeta} \tag{1.3}$$

with $\theta \in \mathbb{R}$ and $\zeta_0 \in \Delta$. In particular, for every pair ζ_1, $\zeta_2 \in \Delta$ there exists $\gamma \in$ $\mathrm{Aut}(\Delta)$ *such that* $\gamma(\zeta_1) = 0$ *and* $\gamma(\zeta_2) \in [0,1)$.

Remark 1.1.7. More generally, given ζ_1, $\zeta_2 \in \Delta$ and $\eta \in [0,1)$, it is not difficult to see that there is $\gamma \in \mathrm{Aut}(\Delta)$ such that $\gamma(\zeta_1) = \eta$ and $\gamma(\zeta_2) \in [0,1)$ with $\gamma(\zeta_2) \geq \eta$.

A consequence of (1.3) is that all automorphisms of Δ extends continuously to the boundary. It is customary to classify the elements of $\mathrm{Aut}(\Delta)$ according to the number of fixed points in $\overline{\Delta}$:

Definition 1.1.8. An automorphism $\gamma \in \mathrm{Aut}(\Delta) \setminus \{\mathrm{id}_\Delta\}$ is called *elliptic* if it has a unique fixed point in Δ, *parabolic* if it has a unique fixed point in $\partial\Delta$, *hyperbolic* if it has exactly two fixed points in $\partial\Delta$. It is easy to check that these cases are mutually exclusive and exhaustive.

We end this brief introduction to the Poincaré distance by recalling two facts relating its geometry to the Euclidean geometry of the plane (see, e.g., [3, Lemma 1.1.5 and (1.1.11)]):

Proposition 1.1.9. *Let $\zeta_0 \in \Delta$ and $r > 0$. Then the ball $B_\Delta(\zeta_0, r) \subset \Delta$ for the Poincaré distance of center ζ_0 and radius r is the Euclidean ball with center*

$$\frac{1 - (\tanh r)^2}{1 - (\tanh r)^2 |\zeta_0|^2}\, \zeta_0$$

and radius

$$\frac{(1 - |\zeta_0|^2)\tanh r}{1 - (\tanh r)^2 |\zeta_0|^2}\,.$$

Proposition 1.1.10. *Let $\zeta_0 = re^{i\theta} \in \Delta$. Then the geodesic for the Poincaré metric connecting* 0 *to ζ_0 is the Euclidean radius $\sigma\colon [0, k_\Delta(0,\zeta_0)] \to \Delta$ given by*

$$\sigma(t) = (\tanh t) e^{i\theta}\,.$$

In particular, $k_\Delta\big(0, (\tanh t)e^{i\theta}\big) = |t|$ for all $t \in \mathbb{R}$ and $\theta \in \mathbb{R}$.

1.2 The Kobayashi distance in complex manifolds

Our next aim is to build on any complex manifold a (pseudo)distance enjoying the main properties of the Poincaré distance; in particular, we would like to preserve the 1-Lipschitz property of holomorphic maps, that is to generalize to several variables Schwarz-Pick lemma. There are several ways for doing this; historically, the first such generalization has been introduced by Carathéodory [45] in 1926, but the most well-known and most useful has been proposed in 1967 by Kobayashi [102, 103]. Here we shall concentrate on the Kobayashi (pseudo)distance; but several other similar metrics and distances have been introduced (see, e.g., [28, 74, 15, 51, 101, 140, 141, 151]; see also [77] for a general context explaining why in a very precise sense the Carathéodory distance is the smallest and the Kobayashi distance is the largest possible invariant distance, and [13] for a different differential geometric approach). Furthermore, we shall discuss only the Kobayashi *distance*; it is possible to define a Kobayashi metric, which is a complex Finsler metric whose integrated form is exactly the Kobayashi distance, see Section 4.1. It is also possible to introduce a Kobayashi pseudodistance in complex analytic spaces; again, see [3], [85] and [105] for details and much more.

To define the Kobayashi pseudodistance we first introduce an auxiliary function.

Definition 1.2.1. Let X be a connected complex manifold. The *Lempert function* $\delta_X\colon X\times X\to\mathbb{R}^+\cup\{+\infty\}$ is defined by

$$\delta_X(z,w)=\inf\bigl\{k_\Delta(\zeta_0,\zeta_1)\bigm|\exists\varphi\in\mathrm{Hol}(\Delta,X)\colon\varphi(\zeta_0)=z,\varphi(\zeta_1)=w\bigr\}$$

for every z, $w\in X$.

Remark 1.2.2. Corollary 1.1.6 yields the following equivalent definition of the Lempert function:

$$\delta_X(z,w)=\inf\bigl\{k_\Delta(0,\zeta)\bigm|\exists\varphi\in\mathrm{Hol}(\Delta,X)\colon\varphi(0)=z,\varphi(\zeta)=w\bigr\}\ .$$

The Lempert function in general (but there are exceptions; see Theorem 1.4.7 below) does not satisfy the triangular inequality (see, e.g., [113] for an example), and so it is not a distance. But this is a problem easily solved:

Definition 1.2.3. Let X be a connected complex manifold. The *Kobayashi (pseudo) distance* $k_X\colon X\times X\to\mathbb{R}^+$ is the largest (pseudo)distance bounded above by the Lempert function, that is

$$k_X(z,w)=\inf\left\{\sum_{j=1}^k\delta_X(z_{j-1},z_j)\ \middle|\ k\in\mathbb{N},z_0=z,z_k=w,z_1,\dots,z_{k-1}\in X\right\}$$

for all z, $w\in X$.

A few remarks are in order. First of all, it is easy to check that since X is connected then k_X is always finite. Furthermore, it is clearly symmetric, it satisfies the triangle inequality by definition, and $k_X(z,z) = 0$ for all $z \in X$. On the other hand, it might well happen that $k_X(z_0,z_1) = 0$ for two distinct points $z_0 \neq z_1$ of X (it might even happen that $k_X \equiv 0$; see Proposition 1.2.5 below); so k_X in general is only a pseudodistance. Anyway, the definition clearly implies the following generalization of the Schwarz-Pick lemma:

Theorem 1.2.4. *Let X, Y be two complex manifolds, and $f \in \mathrm{Hol}(X,Y)$. Then*

$$k_Y\big(f(z), f(w)\big) \leq k_X(z,w)$$

for all z, $w \in X$. In particular:

(i) *if X is a submanifold of Y then $k_Y|_{X\times X} \leq k_X$;*
(ii) *biholomorphisms are isometries with respect to the Kobayashi pseudodistances.*

A statement like this is the reason why the Kobayashi (pseudo)distance is said to be an *invariant* distance: it is invariant under biholomorphisms.

Using the definition, it is easy to compute the Kobayashi pseudodistance of a few of interesting manifolds (see, e.g., [3, Proposition 2.3.4, Corollaries 2.3.6, 2.3.7]):

Proposition 1.2.5. (i) *The Poincaré distance is the Kobayashi distance of the unit disc Δ.*
(ii) *The Kobayashi distances of $\mathbb{C}^n$ and of the complex projective space $\mathbb{P}^n(\mathbb{C})$ vanish identically.*
(iii) *For every $z = (z_1,\dots,z_n)$, $w = (w_1,\dots,w_n) \in \Delta^n$ we have*

$$k_{\Delta^n}(z,w) = \max_{j=1,\dots,n}\{k_\Delta(z_j,w_j)\} .$$

(iv) *The Kobayashi distance of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ coincides with the classical Bergman distance; in particular, if $O \in \mathbb{C}^n$ is the origin and $z \in \mathbb{B}^n$ then*

$$k_{\mathbb{B}^n}(O,z) = \tfrac{1}{2}\log\frac{1+\|z\|}{1-\|z\|} .$$

Remark 1.2.6. As often happens with objects introduced via a general definition, the Kobayashi pseudodistance can seldom be explicitly computed. Besides the cases listed in Proposition 1.2.5, as far as we know there are formulas only for some complex ellipsoids [86], bounded symmetric domains [85], the symmetrized bidisc [12] and a few other scattered examples. On the other hand, it is possible and important to estimate the Kobayashi distance; see Subsection 1.5 below.

We shall be interested in manifolds where the Kobayashi pseudodistance is a true distance, that is in complex manifolds X such that $k_X(z,w) > 0$ as soon as $z \neq w$.

Definition 1.2.7. A connected complex manifold X is *(Kobayashi) hyperbolic* if k_X is a true distance. In this case, if $z_0 \in X$ and $r > 0$ we shall denote by $B_X(z_0,r)$ the ball for k_X of center z_0 and radius r; we shall call $B_X(z_0,r)$ a *Kobayashi ball*. More generally, if $A \subseteq X$ and $r > 0$ we shall put $B_X(A,r) = \bigcup_{z\in A} B_X(z,r)$.

In hyperbolic manifolds the Kobayashi distance induces the topology of the manifold. More precisely (see, e.g., [3, Proposition 2.3.10]):

Proposition 1.2.8 (Barth, [17]). *A connected complex manifold X is hyperbolic if and only if k_X induces the manifold topology on X.*

To give a first idea of how one can work with the Kobayashi distance, we describe two large classes of examples of hyperbolic manifolds:

Proposition 1.2.9 (Kobayashi, [102, 103]).

(i) *A submanifold of a hyperbolic manifold is hyperbolic. In particular, bounded domains in $\mathbb{C}^n$ are hyperbolic.*
(ii) *Let $\pi\colon \tilde{X} \to X$ be a holomorphic covering map. Then X is hyperbolic if and only if $\tilde{X}$ is. In particular, a Riemann surface is hyperbolic if and only if it is Kobayashi hyperbolic.*

Proof. (i) The first assertion follows immediately from Theorem 1.2.4.(i). For the second one, we remark that the unit ball $\mathbb{B}^n$ is hyperbolic by Proposition 1.2.5.(iv). Then Theorem 1.2.4.(ii) implies that all balls are hyperbolic; since a bounded domain is contained in a ball, the assertion follows.

(ii) First of all we claim that

$$k_X(z_0,w_0) = \inf\left\{k_{\tilde{X}}(\tilde{z}_0,\tilde{w}) \;\middle|\; \tilde{w} \in \pi^{-1}(w_0)\right\} , \tag{1.4}$$

for any z_0, $w_0 \in X$, where $\tilde{z}_0$ is any element of $\pi^{-1}(z_0)$. Indeed, first of all Theorem 1.2.4 immediately implies that

$$k_X(z_0,w_0) \le \inf\left\{k_{\tilde{X}}(\tilde{z}_0,\tilde{w}) \;\middle|\; w \in \pi^{-1}(w_0)\right\} .$$

Assume now, by contradiction, that there is $\varepsilon > 0$ such that

$$k_X(z_0,w_0) + \varepsilon \le k_{\tilde{X}}\bigl(\tilde{z}_0,\tilde{w}\bigr)$$

for all $\tilde{w} \in \pi^{-1}(w_0)$. Choose $z_1,\dots,z_k \in X$ with $z_k = w_0$ such that

$$\sum_{j=1}^{k} \delta_X(z_{j-1},z_j) < k_X(z_0,w_0) + \varepsilon/2 .$$

By Remark 1.2.2, we can find $\varphi_1,\dots,\varphi_k \in \mathrm{Hol}(\Delta,X)$ and $\zeta_1,\dots,\zeta_k \in \Delta$ such that $\varphi_j(0) = z_{j-1}$, $\varphi_j(\zeta_j) = z_j$ for all $j = 1,\dots,k$ and

$$\sum_{j=1}^{k} k_\Delta(0,\zeta_j) < k_X(z_0,w_0) + \varepsilon .$$

Let $\tilde{\varphi}_1,\dots,\tilde{\varphi}_k \in \mathrm{Hol}(\Delta,\tilde{X})$ be the liftings of $\varphi_1,\dots,\varphi_k$ chosen so that $\tilde{\varphi}_1(0) = \tilde{z}_0$ and $\tilde{\varphi}_{j+1}(0) = \tilde{\varphi}_j(\zeta_j)$ for $j = 1,\dots,k-1$, and set $\tilde{w}_0 = \tilde{\varphi}_k(\zeta_k) \in \pi^{-1}(w_0)$. Then

$$k_{\tilde{X}}(\tilde{z}_0,\tilde{w}_0) \le \sum_{j=1}^{k} \delta_{\tilde{X}}\big(\tilde{\varphi}_j(0),\tilde{\varphi}_j(\zeta_j)\big) \le \sum_{j=1}^{k} k_\Delta(0,\zeta_j) < k_X(z_0,w_0)+\varepsilon \le k_{\tilde{X}}(\tilde{z}_0,\tilde{w}_0)\ ,$$

contradiction.

Having proved (1.4), let us assume that $\tilde{X}$ is hyperbolic. If there are z_0, $w_0 \in X$ such that $k_X(z_0,w_0) = 0$, then for any $\tilde{z}_0 \in \pi^{-1}(z_0)$ there is a sequence $\{\tilde{w}_\nu\} \subset \pi^{-1}(w_0)$ such that $k_{\tilde{X}}(\tilde{z}_0,\tilde{w}_\nu) \to 0$ as $\nu \to +\infty$. Then $\tilde{w}_\nu \to \tilde{z}_0$ (Proposition 1.2.8) and so $\tilde{z}_0 \in \pi^{-1}(w_0)$, that is $z_0 = w_0$.

Conversely, assume X hyperbolic. Suppose $\tilde{z}_0$, $\tilde{w}_0 \in \tilde{X}$ are so that $k_{\tilde{X}}(\tilde{z}_0,\tilde{w}_0) = 0$; then $k_X\big(\pi(\tilde{z}_0),\pi(\tilde{w}_0)\big) = 0$ and so $\pi(\tilde{z}_0) = \pi(\tilde{w}_0) = z_0$. Let $\widetilde{U}$ be a connected neighborhood of $\tilde{z}_0$ such that $\pi|_{\widetilde{U}}$ is a biholomorphism between $\widetilde{U}$ and the (connected component containing z_0 of the) Kobayashi ball $B_X(z_0,\varepsilon)$ of center z_0 and radius $\varepsilon > 0$ small enough; this can be done because of Proposition 1.2.8. Since $k_{\widetilde{X}}(\tilde{z}_0,\tilde{w}_0) = 0$, we can find $\varphi_1,\dots,\varphi_k \in \mathrm{Hol}(\Delta,\tilde{X})$ and $\zeta_1,\dots,\zeta_k \in \Delta$ with $\varphi_1(0) = \tilde{z}_0$, $\varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 1,\dots,k-1$ and $\varphi_k(\zeta_k) = \tilde{w}_0$ such that

$$\sum_{j=1}^{k} k_\Delta(0,\zeta_j) < \varepsilon\ .$$

Let σ_j be the radial segment in Δ joining 0 to ζ_j; by Proposition 1.1.10 the σ_j are geodesics for the Poincaré metric. The arcs $\varphi_j \circ \sigma_j$ in $\widetilde{X}$ connect to form a continuous curve σ from $\tilde{z}_0$ to $\tilde{w}_0$. Now the maps $\pi \circ \varphi_j \in \mathrm{Hol}(\Delta,X)$ are non-expanding; therefore every point of the curve $\pi \circ \sigma$ should belong to $B_X(z_0,\varepsilon)$. But then σ is contained in $\widetilde{U}$, and this implies $\tilde{z}_0 = \tilde{w}_0$.

The final assertion on Riemann surfaces follows immediately because hyperbolic Riemann surfaces can be characterized as the only Riemann surfaces whose universal covering is the unit disc. □

It is also possible to prove the following (see, e.g., [3, Proposition 2.3.13]):

Proposition 1.2.10. *Let X_1 and X_2 be connected complex manifolds. Then $X_1 \times X_2$ is hyperbolic if and only if both X_1 and X_2 are hyperbolic.*

Remark 1.2.11. The Kobayashi pseudodistance can be useful even when it is degenerate. For instance, the classical Liouville theorem (a bounded entire function is constant) is an immediate consequence, thanks to Theorem 1.2.4, of the vanishing of the Kobayashi pseudodistance of $\mathbb{C}^n$ and the fact that bounded domains are hyperbolic.

A technical fact we shall need later on is the following:

Lemma 1.2.12. *Let X be a hyperbolic manifold, and choose $z_0 \in X$ and r_1, $r_2 > 0$. Then*

$$B_X\big(B_X(z_0,r_1),r_2\big) = B_X(z_0,r_1+r_2)\ .$$

Proof. The inclusion $B_D\big(B_D(z_0,r_1),r_2\big) \subseteq B_D(z_0,r_1+r_2)$ follows immediately from the triangular inequality. For the converse, let $z \in B_D(z_0,r_1+r_2)$, and set $3\varepsilon = r_1 +$

$r_2 - k_X(z_0, z)$. Then there are $\varphi_1, \dots, \varphi_m \in \mathrm{Hol}(\Delta, X)$ and $\zeta_1, \dots, \zeta_m \in \Delta$ so that $\varphi_1(0) = z_0$, $\varphi_j(\zeta_j) = \varphi_{j+1}(0)$ for $j = 1, \dots, m-1$, $\varphi_m(\zeta_m) = z$ and

$$\sum_{j=1}^{m} k_\Delta(0, \zeta_j) < r_1 + r_2 - 2\varepsilon \ .$$

Let $\mu \le m$ be the largest integer such that

$$\sum_{j=1}^{\mu-1} k_\Delta(0, \zeta_j) < r_1 - \varepsilon \ .$$

Let η_μ be the point on the Euclidean radius in Δ passing through $\zeta_{\mu+1}$ (which is a geodesic for the Poincaré distance) such that

$$\sum_{j=1}^{\mu-1} k_\Delta(0, \zeta_j) + k_\Delta(0, \eta_\mu) = r_1 - \varepsilon \ .$$

If we set $w = \varphi_\mu(\eta_\mu)$, then $k_X(z_0, w) < r_1$ and $k_X(w, z) < r_2$, so that

$$z \in B_D(w, r_2) \subseteq B_D\big(B_D(z_0, r_1), r_2\big) \ ,$$

and we are done. □

A condition slightly stronger than hyperbolicity is the following:

Definition 1.2.13. A hyperbolic complex manifold X is *complete hyperbolic* if the Kobayashi distance k_X is complete.

Complete hyperbolic manifolds have a topological characterization (see, e.g., [3, Proposition 2.3.17]):

Proposition 1.2.14. *Let X be a hyperbolic manifold. Then X is complete hyperbolic if and only if every closed Kobayashi ball is compact. In particular, compact hyperbolic manifolds are automatically complete hyperbolic.*

Examples of complete hyperbolic manifolds are contained in the following (see, e.g., [3, Propositions 2.3.19 and 2.3.20]):

Proposition 1.2.15. (i) *A homogeneous hyperbolic manifold is complete hyperbolic. In particular, both $\mathbb{B}^n$ and Δ^n are complete hyperbolic.*
(ii) *A closed submanifold of a complete hyperbolic manifold is complete hyperbolic.*
(iii) *The product of two hyperbolic manifolds is complete hyperbolic if and only if both factors are complete hyperbolic.*
(iv) *If $\pi\colon \tilde{X} \to X$ is a holomorphic covering map, then $\tilde{X}$ is complete hyperbolic if and only if X is complete hyperbolic.*

We shall see more examples of complete hyperbolic manifolds later on (Proposition 1.4.8 and Corollary 1.5.20). We end this subsection recalling the following important fact (see, e.g., [105, Theorem 5.4.2]):

Theorem 1.2.16. *The automorphism group* $\mathrm{Aut}(X)$ *of a hyperbolic manifold* X *has a natural structure of real Lie group.*

1.3 Taut manifolds

For our dynamical applications we shall need a class of manifolds which is intermediate between complete hyperbolic and hyperbolic manifolds. To introduce it, we first show that hyperbolicity can be characterized as a precompactness assumption on the space $\mathrm{Hol}(\Delta, X)$.

If X is a topological space, we shall denote by $X^* = X \cup \{\infty\}$ its one-point (or Alexandroff) compactification; see, e.g., [95, p. 150] for details.

Theorem 1.3.1 ([5]). *Let* X *be a connected complex manifold. Then* X *is hyperbolic if and only if* $\mathrm{Hol}(\Delta, X)$ *is relatively compact in the space* $C^0(\Delta, X^*)$ *of continuous functions from* Δ *into the one-point compactification of* X*. In particular, if* X *is compact then it is hyperbolic if and only if* $\mathrm{Hol}(\Delta, X)$ *is compact. Finally, if* X *is hyperbolic then* $\mathrm{Hol}(Y, X)$ *is relatively compact in* $C^0(Y, X^*)$ *for any complex manifold* Y.

If X is hyperbolic and not compact, the closure of $\mathrm{Hol}(\Delta, X)$ in $C^0(\Delta, X^*)$ might contain continuous maps whose image might both contain ∞ and intersect X, exiting thus from the realm of holomorphic maps. Taut manifolds, introduced by Wu [150], are a class of (not necessarily compact) hyperbolic manifolds where this problem does not appear, and (as we shall see) this will be very useful when studying the dynamics of holomorphic self-maps.

Definition 1.3.2. A complex manifold X is *taut* if it is hyperbolic and every map in the closure of $\mathrm{Hol}(\Delta, X)$ in $C^0(\Delta, X^*)$ either is in $\mathrm{Hol}(\Delta, X)$ or is the constant map ∞.

This definition can be rephrased in another way not requiring the one-point compactification.

Definition 1.3.3. Let X and Y be topological spaces. A sequence $\{f_\nu\} \subset C^0(Y, X)$ is *compactly divergent* if for every pair of compacts $H \subseteq Y$ and $K \subseteq X$ there exists $\nu_0 \in \mathbb{N}$ such that $f_\nu(H) \cap K = \emptyset$ for every $\nu \geq \nu_0$. A family $\mathscr{F} \subseteq C^0(Y, X)$ is *normal* if every sequence in $\mathscr{F}$ admits a subsequence which is either uniformly converging on compact subsets or compactly divergent.

By the definition of one-point compactification, a sequence in $C^0(Y, X)$ converges in $C^0(Y, X^*)$ to the constant map ∞ if and only if it is compactly divergent. When X and Y are manifolds (more precisely, when they are Hausdorff, locally compact, connected and second countable topological spaces), a subset in $C^0(Y, X^*)$ is compact if and only if it is sequentially compact; therefore we have obtained the following alternative characterization of taut manifolds:

Corollary 1.3.4. *A connected complex manifold X is taut if and only if the family* $\mathrm{Hol}(\Delta, X)$ *is normal.*

Actually, it is not difficult to prove (see, e.g., [3, Theorem 2.1.2]) that the role of Δ in the definition of taut manifolds is not essential:

Proposition 1.3.5. *Let X be a taut manifold. Then* $\mathrm{Hol}(Y, X)$ *is a normal family for every complex manifold Y.*

It is easy to find examples of hyperbolic manifolds which are not taut:

Example 1.3.6. Let $D = \Delta^2 \setminus \{(0,0)\}$. Since D is a bounded domain in $\mathbb{C}^2$, it is hyperbolic. For $\nu \geq 1$ let $\varphi_\nu \in \mathrm{Hol}(\Delta, D)$ given by $\varphi_\nu(\zeta) = (\zeta, 1/\nu)$. Clearly $\{\varphi_\nu\}$ converges as $\nu \to +\infty$ to the map $\varphi(\zeta) = (\zeta, 0)$, whose image is not contained either in D or in ∂D. In particular, the sequence $\{\varphi_\nu\}$ does not admit a subsequence which is compactly divergent or converging to a map with image in D—and thus D is not taut.

On the other hand, complete hyperbolic manifolds are taut. This is a consequence of the famous Ascoli-Arzelà theorem (see, e.g., [95, p. 233]):

Theorem 1.3.7 (Ascoli-Arzelà theorem). *Let X be a metric space, and Y a locally compact metric space. Then a family* $\mathscr{F} \subseteq C^0(Y, X)$ *is relatively compact in* $C^0(Y, X)$ *if and only if the following two conditions are satisfied:*

(i) $\mathscr{F}$ *is equicontinuous;*
(ii) *the set* $\mathscr{F}(y) = \{f(y) \mid f \in \mathscr{F}\}$ *is relatively compact in X for every* $y \in Y$.

Then:

Proposition 1.3.8. *Every complete hyperbolic manifold is taut.*

Proof. Let X be a complete hyperbolic manifold, and $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, X)$ a sequence which is not compactly divergent; we must prove that it admits a subsequence converging in $\mathrm{Hol}(\Delta, X)$.

Up to passing to a subsequence, we can find a pair of compacts $H \subset \Delta$ and $K \subseteq X$ such that $\varphi_\nu(H) \cap K \neq \emptyset$ for all $\nu \in \mathbb{N}$. Fix $\zeta_0 \in H$ and $z_0 \in K$, and set $r = \max\{k_X(z, z_0) \mid z \in K\}$. Then for every $\zeta \in \Delta$ and $\nu \in \mathbb{N}$ we have

$$k_X\big(\varphi_\nu(\zeta), z_0\big) \leq k_X\big(\varphi_\nu(\zeta), \varphi_\nu(\zeta_0)\big) + k_X\big(\varphi_\nu(\zeta_0), z_0\big) \leq k_\Delta(\zeta, \zeta_0) + r\,.$$

So $\{\varphi_\nu(\zeta)\}$ is contained in the closed Kobayashi ball of center z_0 and radius $k_\Delta(\zeta, \zeta_0) + r$, which is compact since X is complete hyperbolic (Proposition 1.2.14); as a consequence, $\{\varphi_\nu(\zeta)\}$ is relatively compact in X. Furthernore, since X is hyperbolic, the whole family $\mathrm{Hol}(\Delta, X)$ is equicontinuous (it is 1-Lipschitz with respect to the Kobayashi distances); therefore, by the Ascoli-Arzelà theorem, the sequence $\{\varphi_\nu\}$ is relatively compact in $C^0(\Delta, X)$. In particular, it admits a subsequence converging in $C^0(\Delta, X)$; but since, by Weierstrass theorem, $\mathrm{Hol}(\Delta, X)$ is closed in $C^0(\Delta, X)$, the limit belongs to $\mathrm{Hol}(\Delta, X)$, and we are done. □

Thus complete hyperbolic manifolds provide examples of taut manifolds. However, there are taut manifolds which are not complete hyperbolic; an example has been given by Rosay (see [135]). Finally, we have the following equivalent of Proposition 1.2.15 (see, e.g., [3, Lemma 2.1.15]):

Proposition 1.3.9. (i) *A closed submanifold of a taut manifold is taut.*
(ii) *The product of two complex manifolds is taut if and only if both factors are taut.*

Just to give an idea of the usefulness of the taut condition in studying holomorphic self-maps we end this subsection by quoting Wu's generalization of the classical Cartan-Carathéodory and Cartan uniqueness theorems (see, e.g., [3, Theorem 2.1.21 and Corollary 2.1.22]):

Theorem 1.3.10 (Wu, [150]). *Let X be a taut manifold, and let $f \in \mathrm{Hol}(X,X)$ with a fixed point $z_0 \in X$. Then:*

(i) *the spectrum of $\mathrm{d}f_{z_0}$ is contained in $\overline{\Delta}$;*
(ii) $|\det \mathrm{d}f_{z_0}| \leq 1$*;*
(iii) $|\det \mathrm{d}f_{z_0}| = 1$ *if and only if $f \in \mathrm{Aut}(X)$;*
(iv) $\mathrm{d}f_{z_0} = \mathrm{id}$ *if and only if f is the identity map;*
(v) *$T_{z_0}X$ admits a $\mathrm{d}f_{z_0}$-invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $\mathrm{d}f_{z_0}|_{L_N}$ is contained in Δ, the spectrum of $\mathrm{d}f_{z_0}|_{L_U}$ is contained in $\partial\Delta$, and $\mathrm{d}f_{z_0}|_{L_U}$ is diagonalizable.*

Corollary 1.3.11 (Wu, [150]). *Let X be a taut manifold, and $z_0 \in X$. Then if f, $g \in \mathrm{Aut}(X)$ are such that $f(z_0) = g(z_0)$ and $\mathrm{d}f_{z_0} = \mathrm{d}g_{z_0}$ then $f \equiv g$.*

Proof. Apply Theorem 1.3.10.(iv) to $g^{-1} \circ f$. □

1.4 Convex domains

In the following we shall be particularly interested in two classes of bounded domains in $\mathbb{C}^n$: convex domains and strongly pseudoconvex domains. Consequently, in this and the next section we shall collect some of the main properties of the Kobayashi distance respectively in convex and strongly pseudoconvex domains.

We start with convex domains recalling a few definitions.

Definition 1.4.1. Given x, $y \in \mathbb{C}^n$ let

$$[x,y] = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in [0,1]\} \text{ and } (x,y) = \{sx + (1-s)y \in \mathbb{C}^n \mid s \in (0,1)\}$$

denote the *closed,* respectively *open, segment* connecting x and y. A set $D \subseteq \mathbb{C}^n$ is *convex* if $[x,y] \subseteq D$ for all x, $y \in D$; and *strictly convex* if $(x,y) \subseteq D$ for all x, $y \in \overline{D}$. A convex domain not strictly convex will sometimes be called *weakly convex.*

An easy but useful observation (whose proof is left to the reader) is:

Lemma 1.4.2. *Let $D \subset \mathbb{C}^n$ be a convex domain. Then:*

(i) $(z,w) \subset D$ *for all* $z \in D$ *and* $w \in \partial D$;
(ii) *if* x, $y \in \partial D$ *then either* $(x,y) \subset \partial D$ *or* $(x,y) \subset D$.

This suggests the following

Definition 1.4.3. Let $D \subset \mathbb{C}^n$ be a convex domain. Given $x \in \partial D$, we put

$$\mathrm{ch}(x) = \{y \in \partial D \mid [x,y] \subset \partial D\} ;$$

we shall say that x is a *strictly convex point* if $\mathrm{ch}(x) = \{x\}$. More generally, given $F \subseteq \partial D$ we put

$$\mathrm{ch}(F) = \bigcup_{x \in F} \mathrm{ch}(x) .$$

A similar construction having a more holomorphic character is the following:

Definition 1.4.4. Let $D \subset \mathbb{C}^n$ be a convex domain. A *complex supporting functional* at $x \in \partial D$ is a $\mathbb{C}$-linear map $L\colon \mathbb{C}^n \to \mathbb{C}$ such that $\mathrm{Re}\, L(z) < \mathrm{Re}\, L(x)$ for all $z \in D$. A *complex supporting hyperplane* at $x \in \partial D$ is an affine complex hyperplane $H \subset \mathbb{C}^n$ of the form $H = x + \ker L$, where L is a complex supporting functional at x (the existence of complex supporting functionals and hyperplanes is guaranteed by the Hahn-Banach theorem). Given $x \in \partial D$, we shall denote by $\mathrm{Ch}(x)$ the intersection of $\overline{D}$ with of all complex supporting hyperplanes at x. Clearly, $\mathrm{Ch}(x)$ is a closed convex set containing x; in particular, $\mathrm{Ch}(x) \subseteq \mathrm{ch}(x)$. If $\mathrm{Ch}(x) = \{x\}$ we say that x is a *strictly $\mathbb{C}$-linearly convex point;* and we say that D is *strictly $\mathbb{C}$-linearly convex* if all points of ∂D are strictly $\mathbb{C}$-linearly convex. Finally, if $F \subset \partial D$ we set

$$\mathrm{Ch}(F) = \bigcup_{x \in F} \mathrm{Ch}(x) ;$$

clearly, $\mathrm{Ch}(F) \subseteq \mathrm{ch}(F)$.

Definition 1.4.5. Let $D \subset \mathbb{C}^n$ be a convex domain, $x \in \partial D$ and $L\colon \mathbb{C}^n \to \mathbb{C}$ a complex supporting functional at x. The *weak peak function* associated to L is the function $\psi \in \mathrm{Hol}(D, \Delta)$ given by

$$\psi(z) = \frac{1}{1 - \big(L(z) - L(x)\big)} .$$

Then ψ extends continuously to $\overline{D}$ with $\psi(\overline{D}) \subseteq \overline{\Delta}$, $\psi(x) = 1$, and $|\psi(z)| < 1$ for all $z \in D$; moreover $y \in \partial D$ is such that $|\psi(y)| = 1$ if and only if $\psi(y) = \psi(x) = 1$, and hence if and only if $L(y) = L(x)$.

Remark 1.4.6. If $x \in \partial D$ is a strictly convex point of a convex domain $D \subset \mathbb{C}^n$ then it is possible to find a complex supporting functional L at x so that $\mathrm{Re}\, L(z) < \mathrm{Re}\, L(x)$ for all $z \in \overline{D} \setminus \{x\}$. In particular, the associated weak peak function $\psi\colon \mathbb{C}^n \to \mathbb{C}$ is a true peak function (see Definition 1.5.17 below) in the sense that $|\psi(z)| < 1$ for all $z \in \overline{D} \setminus \{x\}$.

We shall now present three propositions showing how the Kobayashi distance is particularly well-behaved in convex domains. The first result, due to Lempert, shows that in convex domains the definition of Kobayashi distance can be simplified:

Proposition 1.4.7 (Lempert, [113]). *Let $D \subset \mathbb{C}^n$ be a convex domain. Then $\delta_D = k_D$.*

Proof. First of all, note that $\delta_D(z,w) < +\infty$ for all z, $w \in D$. Indeed, let

$$\Omega = \{\lambda \in \mathbb{C} \mid (1-\lambda)z + \lambda w \in D\} .$$

Since D is convex, Ω is a convex domain in $\mathbb{C}$ containing 0 and 1. Let $\phi\colon \Delta \to \Omega$ be a biholomorphism such that $\phi(0) = 0$; then the map $\varphi\colon \Delta \to D$ given by

$$\varphi(\zeta) = \bigl(1 - \phi(\zeta)\bigr)z + \phi(\zeta)w$$

is such that z, $w \in \varphi(\Delta)$.

Now, by definition we have $\delta_D(z,w) \geq k_D(z,w)$; to get the reverse inequality it suffices to show that δ_D satisfies the triangular inequality. Take z_1, z_2, $z_3 \in D$ and fix $\varepsilon > 0$. Then there are φ_1, $\varphi_2 \in \mathrm{Hol}(\Delta, D)$ and ζ_1, $\zeta_2 \in \Delta$ such that $\varphi_1(0) = z_1$, $\varphi_1(\zeta_1) = \varphi_2(\zeta_1) = z_2$, $\varphi_2(\zeta_2) = z_3$ and

$$k_\Delta(0,\zeta_1) < \delta_D(z_1,z_2) + \varepsilon ,$$
$$k_\Delta(\zeta_1,\zeta_2) < \delta_D(z_2,z_3) + \varepsilon .$$

Moreover, by Remark 1.1.7 we can assume that ζ_1 and ζ_2 are real, and that $\zeta_2 > \zeta_1 > 0$. Furthermore, up to replacing φ_j by a map φ_j^r defined by $\varphi_j^r(\zeta) = \varphi_j(r\zeta)$ for r close enough to 1, we can also assume that φ_j is defined and continuous on $\overline{\Delta}$ (and this for $j = 1, 2$).

Let $\lambda\colon \mathbb{C} \setminus \{\zeta_1, \zeta_1^{-1}\} \to \mathbb{C}$ be given by

$$\lambda(\zeta) = \frac{(\zeta - \zeta_2)(\zeta - \zeta_2^{-1})}{(\zeta - \zeta_1)(\zeta - \zeta_1^{-1})} .$$

Then λ is meromorphic in $\mathbb{C}$, and in a neighborhood of $\overline{\Delta}$ the only pole is the simple pole at ζ_1. Moreover, $\lambda(0) = 1$, $\lambda(\zeta_2) = 0$ and $\lambda(\partial\Delta) \subset [0,1]$. Then define $\phi\colon \overline{\Delta} \to \mathbb{C}^n$ by

$$\phi(\zeta) = \lambda(\zeta)\varphi_1(\zeta) + \bigl(1 - \lambda(\zeta)\bigr)\varphi_2(\zeta) .$$

Since $\varphi_1(\zeta_1) = \varphi_2(\zeta_1)$, it turns out that ϕ is holomorphic on Δ; moreover, $\phi(0) = z_1$, $\phi(\zeta_2) = z_3$ and $\phi(\partial\Delta) \subset \overline{D}$. We claim that this implies that $\phi(\Delta) \subset D$. Indeed, otherwise there would be $\zeta_0 \in \Delta$ such that $\phi(\zeta_0) = x_0 \in \partial D$. Let L be a complex supporting functional at x_0, and ψ the associated weak peak function. Then we would have $|\psi \circ \phi| \leq 1$ on $\partial\Delta$ and $|\psi \circ \phi(\zeta_0)| = 1$; thus, by the maximum principle, $|\psi \circ \phi| \equiv 1$, i.e., $\phi(\Delta) \subset \partial D$, whereas $\phi(0) \in D$, contradiction.

So $\phi \in \mathrm{Hol}(\Delta, D)$. In particular, then,

$$\delta_D(z_1,z_3) \le k_\Delta(0,\zeta_2) = k_\Delta(0,\zeta_1) + k_\Delta(\zeta_1,\zeta_2) \le \delta_D(z_1,z_2) + \delta_D(z_2,z_3) + 2\varepsilon \,,$$

and the assertion follows, since ε is arbitrary. □

Bounded convex domains, being bounded, are hyperbolic. But actually more is true:

Proposition 1.4.8 (Harris, [77]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded convex domain. Then D is complete hyperbolic.*

Proof. We can assume $O \in D$. By Proposition 1.2.14, it suffices to show that all the closed Kobayashi balls $\overline{B_D(O,r)}$ of center O are compact. Let $\{z_\nu\} \subset \overline{B_D(O,r)}$; we must find a subsequence converging to a point of D. Clearly, we may suppose that $z_\nu \to w_0 \in \overline{D}$ as $\nu \to +\infty$, for D is bounded.

Assume, by contradiction, that $w_0 \in \partial D$, and let $L\colon \mathbb{C}^n \to \mathbb{C}$ be a complex supporting functional at w_0; in particular, $L(w_0) \ne 0$ (because $O \in D$). Set $H = \{\zeta \in \mathbb{C} \mid \operatorname{Re} L(\zeta w_0) < \operatorname{Re} L(w_0)\}$; clearly H is a half-plane of $\mathbb{C}$, and the linear map $\pi\colon \mathbb{C}^n \to \mathbb{C}$ given by $\pi(z) = L(z)/L(w_0)$ sends D into H. In particular

$$r \ge k_D(0,z_\nu) \ge k_H\big(0,\pi(z_\nu)\big) \,.$$

Since H is complete hyperbolic, by Proposition 1.2.14 the closed Kobayashi balls in H are compact; therefore, up to a subsequence $\{\pi(z_\nu)\}$ tends to a point of H. On the other hand, $\pi(z_\nu) \to \pi(w_0) = 1 \in \partial H$, and this is a contradiction. □

Remark 1.4.9. There are unbounded convex domains which are not hyperbolic; for instance, $\mathbb{C}^n$ itself. However, unbounded hyperbolic convex domains are automatically complete hyperbolic, because Harris (see [77]) proved that a convex domain is hyperbolic if and only if it is biholomorphic to a bounded convex domain. Furthermore, Barth (see [18]) has shown that an unbounded convex domain is hyperbolic if and only if it contains no complex lines.

Finally, the convexity is reflected by the shape of Kobayashi balls. To prove this (and also because they will be useful later) we shall need a couple of estimates:

Proposition 1.4.10 ([113], [109], [88]). *Let $D \subset \mathbb{C}^n$ be a convex domain. Then:*

(i) *if z_1, z_2, w_1, $w_2 \in D$ and $s \in [0,1]$ then*

$$k_D\big(sz_1 + (1-s)z_2, sw_1 + (1-s)w_2\big) \le \max\{k_D(z_1,w_1), k_D(z_2,w_2)\} \,;$$

(ii) *if z, $w \in D$ and s, $t \in [0,1]$ then*

$$k_D\big(sz + (1-s)w, tz + (1-t)w\big) \le k_D(z,w) \,.$$

Proof. Let us start by proving (i). Without loss of generality we can assume that $k_D(z_2,w_2) \le k_D(z_1,w_1)$. Fix $\varepsilon > 0$; by Proposition 1.4.7, there are φ_1, $\varphi_2 \in \operatorname{Hol}(\Delta,D)$ and ζ_1, $\zeta_2 \in \Delta$ such that $\varphi_j(0) = z_j$, $\varphi_j(\zeta_j) = w_j$ and $k_\Delta(0,\zeta_j) <$

$k_D(z_j,w_j)+\varepsilon$, for $j=1$, 2; moreover, we may assume $0\le\zeta_2\le\zeta_1<1$ and $\zeta_1>0$. Define $\psi\colon\Delta\to D$ by

$$\psi(\zeta)=\varphi_2\left(\frac{\zeta_2}{\zeta_1}\,\zeta\right),$$

so that $\psi(0)=z_2$ and $\psi(\zeta_1)=w_2$, and $\phi_s\colon\Delta\to\mathbb{C}^n$ by

$$\phi_s(\zeta)=s\varphi_1(\zeta)+(1-s)\psi(\zeta)\,.$$

Since D is convex, ϕ_s maps Δ into D; furthermore, $\phi_s(0)=sz_1+(1-s)z_2$ and $\phi_s(\zeta_1)=sw_1+(1-s)w_2$. Hence

$$\begin{aligned}k_D\big(sz_1+(1-s)z_2,sw_1+(1-s)w_2\big)&=k_D\big(\phi_s(0),\phi_s(\zeta_1)\big)\\&\le k_\Delta(0,\zeta_1)<k_D(z_1,w_1)+\varepsilon\,,\end{aligned}$$

and (i) follows because ε is arbitrary.

Given $z_0\in D$, we obtain a particular case of (i) by setting $z_1=z_2=z_0$:

$$k_D\big(z_0,sw_1+(1-s)w_2\big)\le\max\{k_D(z_0,w_1),k_D(z_0,w_2)\}\tag{1.5}$$

for all z_0, w_1, $w_2\in D$ and $s\in[0,1]$.

To prove (ii), put $z_0=sz+(1-s)w$; then two applications of (1.5) yield

$$\begin{aligned}k_D\big(sz+(1-s)w,tz+(1-t)w\big)&\le\max\big\{k_D\big(sz+(1-s)w,z\big),k_D\big(sz+(1-s)w,w\big)\big\}\\&\le k_D(z,w)\,,\end{aligned}$$

and we are done. □

Corollary 1.4.11. *Closed Kobayashi balls in a hyperbolic convex domain are compact and convex.*

Proof. The compactness follows from Propositions 1.2.14 and 1.4.8 (and Remark 1.4.9 for unbounded hyperbolic convex domains); the convexity follows from (1.5). □

1.5 Strongly pseudoconvex domains

Another important class of domains where the Kobayashi distance has been studied in detail is given by strongly pseudoconvex domains. In particular, in strongly pseudoconvex domains it is possible to estimate the Kobayashi distance by means of the Euclidean distance from the boundary.

To recall the definition of strongly pseudoconvex domains, and to fix notations useful later, let us first introduce smoothly bounded domains. For simplicity we shall state the following definitions in $\mathbb{R}^N$, but they can be easily adapted to $\mathbb{C}^n$ by using the standard identification $\mathbb{C}^n\simeq\mathbb{R}^{2n}$.

Definition 1.5.1. A domain $D \subset \mathbb{R}^N$ has C^r *boundary* (or is a C^r *domain*), where $r \in \mathbb{N} \cup \{\infty, \omega\}$ (and C^ω means real analytic), if there is a C^r function $\rho\colon \mathbb{R}^N \to \mathbb{R}$ such that:

(a) $D = \{x \in \mathbb{R}^N \mid \rho(x) < 0\}$;
(b) $\partial D = \{x \in \mathbb{R}^N \mid \rho(x) = 0\}$; and
(c) $\operatorname{grad}\rho$ is never vanishing on ∂D.

The function ρ is a *defining function* for D. The *outer unit normal vector* $\mathbf{n}_x$ at x is the unit vector parallel to $-\operatorname{grad}\rho(x)$.

Remark 1.5.2. it is not difficult to check that if ρ_1 is another defining function for a domain D then there is a never vanishing C^r function $\psi\colon \mathbb{R}^N \to \mathbb{R}^+$ such that

$$\rho_1 = \psi\rho \,. \tag{1.6}$$

If $D \subset \mathbb{R}^N$ is a C^r domain with defining function ρ, then ∂D is a C^r manifold embedded in $\mathbb{R}^N$. In particular, for every $x \in \partial D$ the tangent space of ∂D at x can be identified with the kernel of $d\rho_x$ (which by (1.6) is independent of the chosen defining function). In particular, $T_x(\partial D)$ is just the hyperplane orthogonal to $\mathbf{n}_x$.

Using a defining function it is possible to check when a C^2-domain is convex.

Definition 1.5.3. If $\rho\colon \mathbb{R}^N \to \mathbb{R}$ is a C^2 function, the *Hessian* $H_{\rho,x}$ of ρ at $x \in \mathbb{R}^N$ is the symmetric bilinear form given by

$$H_{\rho,x}(v,w) = \sum_{h,k=1}^{N} \frac{\partial^2\rho}{\partial x_h \partial x_k}(x)\, v_h w_k$$

for every v, $w \in \mathbb{R}^N$.

The following result is well-known (see, e.g, [107, p.102]):

Proposition 1.5.4. *A C^2 domain $D \subset \mathbb{R}^N$ is convex if and only if for every $x \in \partial D$ the Hessian $H_{\rho,x}$ is positive semidefinite on $T_x(\partial D)$, where ρ is any defining function for D.*

This suggests the following

Definition 1.5.5. A C^2 domain $D \subset \mathbb{R}^N$ is *strongly convex* at $x \in \partial D$ if for some (and hence any) C^2 defining function ρ for D the Hessian $H_{\rho,x}$ is positive definite on $T_x(\partial D)$. We say that D is *strongly convex* if it is so at each point of ∂D.

Remark 1.5.6. It is easy to check that strongly convex C^2 domains are strictly convex. Furthermore, it is also possible to prove that every strongly convex domain D has a C^2 defining function ρ such that $H_{\rho,x}$ is positive definite on the whole of $\mathbb{R}^N$ for every $x \in \partial D$ (see, e.g., [107, p. 101]).

Remark 1.5.7. If $D \subset \mathbb{C}^n$ is a convex C^1 domain and $x \in \partial D$ then the unique (up to a positive multiple) complex supporting functional at x is given by $L(z) = \langle z, \mathbf{n}_x \rangle$. In particular, $\mathrm{Ch}(x)$ coincides with the intersection of the associated complex supporting hyperplane with ∂D. But non-smooth points can have more than one complex supporting hyperplanes; this happens for instance in the polydisc.

Let us now move to a more complex setting.

Definition 1.5.8. Let $D \subset \mathbb{C}^n$ be a domain with C^2 boundary and defining function $\rho \colon \mathbb{C}^n \to \mathbb{R}$. The *complex tangent space* $T_x^{\mathbb{C}}(\partial D)$ of ∂D at $x \in \partial D$ is the kernel of $\partial \rho_x$, that is

$$T_x^{\mathbb{C}}(\partial D) = \left\{ v \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(x)\, v_j = 0 \right\} .$$

As usual, $T_x^{\mathbb{C}}(\partial D)$ does not depend on the particular defining function. The *Levi form* $L_{\rho,x}$ of ρ at $x \in \mathbb{C}^n$ is the Hermitian form given by

$$L_{\rho,x}(v,w) = \sum_{h,k=1}^n \frac{\partial^2 \rho}{\partial z_h \partial \bar{z}_k}(x)\, v_h \overline{w_k}$$

for every v, $w \in \mathbb{C}^n$.

Definition 1.5.9. A C^2 domain $D \subset \mathbb{C}^n$ is called *strongly pseudoconvex* (respectively, *weakly pseudoconvex*) at a point $x \in \partial D$ if for some (and hence all) C^2 defining function ρ for D the Levi form $L_{\rho,x}$ is positive definite (respectively, weakly positive definite) on $T_x^{\mathbb{C}}(\partial D)$. The domain D is *strongly pseudoconvex* (respectively, *weakly pseudoconvex*) if it is so at each point of ∂D.

Remark 1.5.10. If D is strongly pseudoconvex then there is a defining function ρ for D such that the Levi form $L_{\rho,x}$ is positive definite on $\mathbb{C}^n$ for every $x \in \partial D$ (see, e.g., [107, p. 109]).

Roughly speaking, strongly pseudoconvex domains are locally strongly convex. More precisely, one can prove (see, e.g., [3, Proposition 2.1.13]) the following:

Proposition 1.5.11. *A bounded C^2 domain $D \subset\subset \mathbb{C}^n$ is strongly pseudoconvex if and only if for every $x \in \partial D$ there is a neighborhood $U_x \subset \mathbb{C}^n$ and a biholomorphism $\Phi_x \colon U_x \to \Phi_x(U_x)$ such that $\Phi_x(U_x \cap D)$ is strongly convex.*

From this one can prove that strongly pseudoconvex domains are taut; but we shall directly prove that they are complete hyperbolic, as a consequence of the boundary estimates we are now going to state.

Definition 1.5.12. If $M \subset \mathbb{C}^n$ is any subset of $\mathbb{C}^n$, we shall denote by $d(\cdot, M) \colon \mathbb{C}^n \to \mathbb{R}^+$ the Euclidean distance from M, defined by

$$d(z,M) = \inf\{\|z - x\| \mid x \in M\} .$$

To give an idea of the kind of estimates we are looking for, we shall prove an easy lemma:

Lemma 1.5.13. *Let $\mathbb{B}_r \subset \mathbb{C}^n$ be the euclidean ball of radius $r > 0$ in $\mathbb{C}^n$ centered at the origin. Then*

$$\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r) \le k_{\mathbb{B}_r}(O, z) \le \tfrac{1}{2}\log(2r) - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r)$$

for every $z \in \mathbb{B}_r$.

Proof. We have

$$k_{B_r}(O, z) = \tfrac{1}{2}\log\frac{1 + \|z\|/r}{1 - \|z\|/r} ,$$

and $d(z, \partial\mathbb{B}_r) = r - \|z\|$. Then, setting $t = \|z\|/r$, we get

$$\begin{aligned}\tfrac{1}{2}\log r - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r) &= \tfrac{1}{2}\log\frac{1}{1-t} \le \tfrac{1}{2}\log\frac{1+t}{1-t} = k_{B_r}(O, z) \\ &\le \tfrac{1}{2}\log\frac{2}{1-t} = \tfrac{1}{2}\log(2r) - \tfrac{1}{2}\log d(z, \partial\mathbb{B}_r) ,\end{aligned}$$

as claimed. □

Thus in the ball the Kobayashi distance from a reference point is comparable with one-half of the logarithm of the Euclidean distance from the boundary. We would like to prove similar estimates in strongly pseudoconvex domains. To do so we need one more definition.

Definition 1.5.14. Let M be a compact C^2-hypersurface of $\mathbb{R}^N$, and fix an unit normal vector field $\mathbf{n}$ on M. We shall say that M has a *tubular neighborhood* of *radius* $\varepsilon > 0$ if the segments $\{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\}$ are pairwise disjoint, and we set

$$U_\varepsilon = \bigcup_{x \in M} \{x + t\mathbf{n}_x \mid t \in (-\varepsilon, \varepsilon)\} .$$

Note that if M has a tubular neighborhood of radius ε, then $d(x + t\mathbf{n}_x, M) = |t|$ for every $t \in (-\varepsilon, \varepsilon)$ and $x \in M$; in particular, U_ε is the union of the Euclidean balls $\mathbb{B}(x, \varepsilon)$ of center $x \in M$ and radius ε.

Remark 1.5.15. A proof of the existence of a tubular neighborhood of radius sufficiently small for any compact C^2-hypersurface of $\mathbb{R}^N$ can be found, e.g., in [111, Theorem 10.19].

And now, we begin proving the estimates. The upper estimate does not even depend on the strong pseudoconvexity:

Theorem 1.5.16 ([147, 1]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded C^2 domain, and $z_0 \in D$. Then there is a constant $c_1 \in \mathbb{R}$ depending only on D and z_0 such that*

$$k_D(z_0, z) \le c_1 - \tfrac{1}{2}\log d(z, \partial D) \tag{1.7}$$

for all $z \in D$.

Proof. Since D is a bounded C^2 domain, ∂D admits tubular neighborhoods U_ε of radius $\varepsilon < 1$ small enough. Put

$$c_1 = \sup\{k_D(z_0, w) \mid w \in D \setminus U_{\varepsilon/4}\} + \max\{0, \tfrac{1}{2}\log \operatorname{diam}(D)\} ,$$

where $\operatorname{diam}(D)$ is the Euclidean diameter of D.

There are two cases:

(i) $z \in U_{\varepsilon/4} \cap D$. Let $x \in \partial D$ be such that $\|x - z\| = d(z, \partial D)$. Since $U_{\varepsilon/2}$ is a tubular neighborhood of ∂D, there exists $\lambda \in \mathbb{R}$ such that $w = \lambda(x - z) \in \partial U_{\varepsilon/2} \cap D$ and the Euclidean ball $\mathbb{B}$ of center w and radius $\varepsilon/2$ is contained in $U_\varepsilon \cap D$ and tangent to ∂D in x. Therefore Lemma 1.5.13 yields

$$\begin{aligned} k_D(z_0, z) &\le k_D(z_0, w) + k_D(w, z) \le k_D(z_0, w) + k_B(w, z) \\ &\le k_D(z_0, w) + \tfrac{1}{2}\log \varepsilon - \tfrac{1}{2}\log d(z, \partial B) \\ &\le c_1 - \tfrac{1}{2}\log d(z, \partial D) , \end{aligned}$$

because $w \notin U_{\varepsilon/4}$ (and $\varepsilon < 1$).

(ii) $z \in D \setminus U_{\varepsilon/4}$. Then

$$k_D(z_0, z) \le c_1 - \tfrac{1}{2}\log \operatorname{diam}(D) \le c_1 - \tfrac{1}{2}\log d(z, \partial D) ,$$

because $d(z, \partial D) \le \operatorname{diam}(D)$, and we are done. □

To prove the more interesting lower estimate, we need to introduce the last definition of this subsection.

Definition 1.5.17. Let $D \subset \mathbb{C}^n$ be a domain in $\mathbb{C}^n$, and $x \in \partial D$. A *peak function* for D at x is a holomorphic function $\psi \in \operatorname{Hol}(D, \Delta)$ continuous up to the boundary of D such that $\psi(x) = 1$ and $|\psi(z)| < 1$ for all $z \in \overline{D} \setminus \{x\}$.

If $D \subset \mathbb{C}^n$ is strongly convex and $x \in \partial D$ then by Remark 1.4.6 there exists a peak function for D at x. Since a strongly pseudoconvex domain D is locally strongly convex, using Proposition 1.5.11 one can easily build peak functions defined in a neighborhood of a point of the boundary of D. To prove the more interesting lower estimate on the Kobayashi distance we shall need the non-trivial fact that in a strongly pseudoconvex domain it is possible to build a family of *global* peak functions continuously dependent on the point in the boundary:

Theorem 1.5.18 (Graham, [71]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^2 domain. Then there exist a neighborhood D' of $\overline{D}$ and a continuous function $\Psi \colon \partial D \times D' \to \mathbb{C}$ such that $\Psi_{x_0} = \Psi(x_0, \cdot)$ is holomorphic in D' and a peak function for D at x_0 for each $x_0 \in \partial D$.*

With this result we can prove

Theorem 1.5.19 ([147, 1]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex C^2 domain, and $z_0 \in D$. Then there is a constant $c_2 \in \mathbb{R}$ depending only on D and z_0 such that*

$$c_2 - \tfrac{1}{2}\log d(z,\partial D) \le k_D(z_0,z) \tag{1.8}$$

for all $z \in D$.

Proof. Let $D' \supset\supset D$ and $\Psi\colon \partial D \times D' \to \mathbb{C}$ be given by Theorem 1.5.18, and define $\phi\colon \partial D \times \Delta \to \mathbb{C}$ by

$$\phi(x,\zeta) = \frac{1-\overline{\Psi(x,z_0)}}{1-\Psi(x,z_0)} \cdot \frac{\zeta - \Psi(x,z_0)}{1-\overline{\Psi(x,z_0)}\zeta}\ . \tag{1.9}$$

Then the map $\Phi(x,z) = \Phi_x(z) = \phi\big(x,\Psi(x,z)\big)$ is defined on a neighborhood $\partial D \times D_0$ of $\partial D \times \overline{D}$ (with $D_0 \subset\subset D'$) and satisfies

(a) Φ is continuous, and Φ_x is a holomorphic peak function for D at x for any $x \in \partial D$;
(b) for every $x \in \partial D$ we have $\Phi_x(z_0) = 0$.

Now set $U_\varepsilon = \bigcup_{x\in\partial D} P(x,\varepsilon)$, where $P(x,\varepsilon)$ is the polydisc of center x and polyradius $(\varepsilon,\dots,\varepsilon)$. The family $\{U_\varepsilon\}$ is a basis for the neighborhoods of ∂D; hence there exists $\varepsilon > 0$ such that $U_\varepsilon \subset\subset D_0$ and U_ε is contained in a tubular neighborhood of ∂D. Then for any $x \in \partial D$ and $z \in P(x,\varepsilon/2)$ the Cauchy estimates yield

$$\begin{aligned}|1-\Phi_x(z)| = |\Phi_x(x) - \Phi_x(z)| &\le \left\| \frac{\partial \Phi_x}{\partial z} \right\|_{P(x,\varepsilon/2)} \|z-x\| \\ &\le \frac{2\sqrt{n}}{\varepsilon} \|\Phi\|_{\partial D \times U_\varepsilon} \|z-x\| = M\|z-x\|\ ,\end{aligned}$$

where M is independent of z and x; in these formulas $\|F\|_S$ denotes the supremum of the Euclidean norm of the map F on the set S.

Put $c_2 = -\frac{1}{2}\log M$; note that $c_2 \le \frac{1}{2}\log(\varepsilon/2)$, for $\|\Phi\|_{\partial D\times U_\varepsilon} \ge 1$. Then we again have two cases:

(i) $z \in D \cap U_{\varepsilon/2}$. Choose $x \in \partial D$ so that $d(z,\partial D) = \|z-x\|$. Since $\Phi_x(D) \subset \Delta$ and $\Phi_x(z_0) = 0$, we have

$$k_D(z_0,z) \ge k_\Delta\big(\Phi_x(z_0),\Phi_x(z)\big) \ge \tfrac{1}{2}\log\frac{1}{1-|\Phi_x(z)|}\ .$$

Now,

$$1-|\Phi_x(z)| \le |1-\Phi_x(z)| \le M\|z-x\| = M\,d(z,\partial D)\ ;$$

therefore

$$k_D(z_0,z) \ge -\tfrac{1}{2}\log M - \tfrac{1}{2}\log d(z,\partial D) = c_2 - \tfrac{1}{2}\log d(z,\partial D)$$

as desired.

(ii) $z \in D \setminus U_{\varepsilon/2}$. Then $d(z,\partial D) \geq \varepsilon/2$; hence

$$k_D(z_0,z) \geq 0 \geq \tfrac{1}{2}\log(\varepsilon/2) - \tfrac{1}{2}\log d(z,\partial D) \geq c_2 - \tfrac{1}{2}\log d(z,\partial D) ,$$

and we are done. □

A first consequence is the promised:

Corollary 1.5.20 (Graham, [71]). *Every bounded strongly pseudoconvex C^2 domain D is complete hyperbolic.*

Proof. Take $z_0 \in D$, $r > 0$ and let $z \in B_D(z_0,r)$. Then (1.8) yields

$$d(z,\partial D) \geq \exp\bigl(2(c_2 - r)\bigr) ,$$

where c_2 depends only on z_0. Then $B_D(z_0,r)$ is relatively compact in D, and the assertion follows from Proposition 1.2.14. □

For dynamical applications we shall also need estimates on the Kobayashi distance $k_D(z_1,z_2)$ when both z_1 and z_2 are close to the boundary. The needed estimates were proved by Forstnerič and Rosay (see [58], and [3, Corollary 2.3.55, Theorem 2.3.56]):

Theorem 1.5.21 ([58]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded strongly pseudoconvex C^2 domain, and choose two points x_1, $x_2 \in \partial D$ with $x_1 \neq x_2$. Then there exist $\varepsilon_0 > 0$ and $K \in \mathbb{R}$ such that for any z_1, $z_2 \in D$ with $\|z_j - x_j\| < \varepsilon_0$ for $j = 1, 2$ we have*

$$k_D(z_1,z_2) \geq -\tfrac{1}{2}\log d(z_1,\partial D) - \tfrac{1}{2}\log d(z_2,\partial D) + K . \tag{1.10}$$

Theorem 1.5.22 ([58]). *Let $D \subset\subset \mathbb{C}^n$ be a bounded C^2 domain and $x_0 \in \partial D$. Then there exist $\varepsilon > 0$ and $C \in \mathbb{R}$ such that for all z_1, $z_2 \in D$ with $\|z_j - x_0\| < \varepsilon$ for $j = 1, 2$ we have*

$$k_D(z_1,z_2) \leq \tfrac{1}{2}\log\left(1 + \frac{\|z_1 - z_2\|}{d(z_1,\partial D)}\right) + \tfrac{1}{2}\log\left(1 + \frac{\|z_1 - z_2\|}{d(z_2,\partial D)}\right) + C . \tag{1.11}$$

We end this section by quoting a theorem, that we shall need in Chapter 6, giving a different way of comparing the Kobayashi geometry and the Euclidean geometry of strongly pseudoconvex domains:

Theorem 1.5.23 ([9]). *Let $D \subset\subset \mathbb{C}^n$ be a strongly pseudoconvex C^∞ domain, and $R > 0$. Then there exist $C_R > 0$ depending only on R and D such that*

$$\frac{1}{C_R} d(z_0,\partial D)^{n+1} \leq \nu\bigl(B_D(z_0,R)\bigr) \leq C_R d(z_0,\partial D)^{n+1}$$

for all $z_0 \in D$, where $\nu\bigl(B_D(z_0,R)\bigr)$ denotes the Lebesgue volume of the Kobayashi ball $B_D(z_0,R)$.

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