Luca Spataro, Luciano Fanti and Pier Mario Pacini

Savings, fertility and public policy in an OLG small open economy

Indirizzo degli Autori:

Luca Spataro, Corresponding author: Dipartimento di Economia e Management, Università di Pisa (Italy). Via Ridolfi 10, 56124 Pisa, Italy. Tel: +39 0502216217. Email: luca.spataro@unipi.it.

Luciano Fanti: Dipartimento di Economia e Management, Università di Pisa (Italy). Email: luciano.fanti@unipi.it.

PierMario Pacini, Dipartimento di Economia e Management, Università di Pisa (Italy). Email: piermario.pacini@unipi.it.

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Abstract

In an OLG model of a small open economy we analyse the characteristics of saving and fertility under two different public policies: i) constant per capita taxes (and endogenous public debt) and ii) constant per-capita debt (and endogenous stabilizing taxes). Our analysis highlights that a fertility recovery (reduction resp.) requires always a reduction (increase resp.) of taxes, although the implications for public debt management are not trivial, since they depend on the regime the economy is experiencing, i.e. on the relationship between the interest rate and the fertility rate in absence of taxes.

1 Introduction

In this paper we explore the behaviour of both saving and fertility in an OLG model in the presence of two policies entailing public debt: i) constant per capita taxes (and endogenous public debt) and ii) constant per-capita debt (and endogenous-stabilizing taxes). Although such issues are not new in economic research, in general they have been analysed separately.

In fact, several scholars have focused on the relationship between saving and public debt with fixed population growth (Diamond (1965)) or between fertility and income although disregarding the role of public policies (e.g. Jones and Schoonbroodt (2010)). Conversely, other works focused on demographic issues with public debt but abstracting from the saving-fertility-wage relationship (e.g., van Groezen et al. (2003), Fenge and von Weizsäcker (2010) and Fanti and Gori (2012)).

As regards the fertility-wage relationship[1], some theories of fertility predict an inverse relationship, both because under suitable circumstances increasing wages lead to substitute quantity of children with their quality (Becker (1960), Becker and Tomes (1976)) and because of the negative substitution effect of (female) wages on fertility, due to the potential increase of female participation (Mincer (1966)).

Also the process of economic development and in particular the so-called Demographic Transition has been the object of intense research in recent years. The economic reasons lying behind the transition from a positive to a negative relationship between fertility and income (i.e. from the Malthusian to Modern fertility behaviour)[2] have been largely investigated. For instance, a negative relation between fertility and income in the models (set up in an overlapping generations context) of Becker et al. (1990), Tamura (1998), Lucas (1998), and Galor and Weil (1998) occurs as individuals begin to trade off quantity for quality. In Galor and Weil (1996) a demographic transition is generated through a difference in the endowments of men and women and a shift in comparative advantage. Becker and Barro (1988), in the context of a model of intergenerational altruism, show that increased (Harrod-neutral) technical progress brings upon a higher growth rate of consumption and a lower rate of fertility[3]; Jones (2001), by developing an idea-based growth model, introduces a third effect of increasing wages on fertility: the subsistence consumption level effect. Galor and Weil (2000) argue that the positive effect of technical progress on the return to education and the feedback effect of higher education on technical progress bring upon a rapid decline in fertil-

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[1] See Galor and Weil (1999) and Guinnane (2011)) for an overview of several different theories of the demographic transition.

[2] See Galor and Weil (1999).

[3] See also Renström and Spataro (2015) in an endogenous growth model with human capital.

ity accompanied by accelerated output growth[4]; Fanti and Gori (2007) focus on the effects of the unionization of the economies as a cause of the emergence of modern fertility behaviour in place of the Malthusian one.

Although producing relevant results, such works have somehow overlooked the role of public policies[5]. Therefore, in this paper we propose an explanation which can add to the established explanations so far emerged in the literature, by including public policies. To do so, we abstract from general equilibrium effects and economic growth, by assuming the case of a small open economy with public intervention, where the only source of growth is given by population[6]. In particular the public intervention may assume two typical forms which are at the heart of the current economic ad political debate in most Western countries and sometimes seem to characterize the policy proposals of left and right-wing Governments: raising/reducing taxes and raising/reducing debt, respectively.

We believe that our attempt is relevant for at least two reasons. First, for theoretical purposes (to the best of our knowledge the analysis of public debt in a small open economy with endogenous saving and fertility behaviour has not been carried out so far). Although based on simplistic assumptions, our insight may help to better understand the forces at work on saving and fertility in presence of public debt, which can extend to more complex models with endogenous prices and/or growth.

Second, our work is potentially relevant for policy reasons, since the recent financial crisis has raised concern about the viability of sustained growth in presence of increasing levels of public debt. Therefore it is natural to ask whether the occurred and/or soon expected sizable changes in the public debt (sometimes in opposite directions because the latter has been increased for facing the recent financial crisis in countries with previously low debt and necessarily will be decreased in countries with an existing very large debt (such as Italy)) may affect fertility behaviour.

The main finding of the present paper is that while in the absence of intergenerational transfers the standard logarithmic OLG model would predict either a positive relation between fertility behaviour, saving and wages or even independence of fertility choices from wage (because with logarithmic utility and time-cost of childrearing - as assumed in the present paper - there

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[4]Jones and Schoonbroodt (2016) analyse the behaviour of fertility along the business cycle and show that it is procyclical.

[5]In fact, there are a few exceptions. For example, Fanti and Spataro (2013) analyse the relationship between public debt and fertility in an OLG model with fixed costs for raising children and general equilibrium factor prices, and show that the latter relation can be ambiguous. We depart from such a work by assuming that childbearing is a time cost in nature and factor prices are fixed.

[6]By doing this we also abstract from the issue of debt sustainability. For an insight into this topic, see, for example, Chalk (2000), Bräuninger (2005), Futagami et al. (2008), Yakita (2008). For a textbook treatment of the sustainability of constant debt policies under endogenous interest rates see De la Croix and Michel (2004), ch. 4.

is an exact balance between income and substitution effects), when public policies are introduced things may change dramatically.

More precisely, we show that while the increase of taxes is always detrimental for both fertility and saving, the effect of per-capita public debt depends on the conditions of the economy. More precisely, an increase of per-capita public debt is detrimental (beneficial resp.) for both fertility and saving when the interest rate is greater (smaller) than the fertility rate in the absence of the public fiscal policy and this happens when i) the interest rate is sufficiently high (low), ii) the childbearing cost is sufficiently high (low) and iii) the preference for children is sufficiently low (high). Moreover, while an increase of the wage rate is always positive for both fertility and saving under a constant per-capita tax policy, it may exert a negative effect on the latter variables under a constant per-capita public debt policy: more precisely, it is beneficial when the interest rate is greater than the fertility rate in the absence of the public fiscal policy and can be detrimental in the opposite case. The latter result may have far reaching policy implications when the political debate on the effects of too low wages (and a their possible increase) is considered: in the case of rising wages the use of a tax policy (resp. a debt policy) may exacerbate the dramatic fertility drop (resp. boost a fertility recovery) in many Western countries.

The work is organized as follows: in section 2 we lay out the model and in section 3 we characterize the steady state under the two different policies; in section 4 we carry out comparative statics exercises by investigating the effects of the main parameters of the model on the long run levels of fertility and saving. Conclusions will end the paper.

2   The model

In this work we extend a standard OLG model (Diamond (1965)) in order to entail endogenous fertility[7]. In doing this we focus on the case of a small open economy, where both the interest rate r and the wage w are fixed. The basic actors of this economy are (i) a Government which can affect the levels of a lump sum taxation τ and public debt b and (ii) individuals that live for three periods (childhood, young adulthood, and old-age). When agents become active (young adult) they take their decisions about present and future consumptions and procreation, thus giving rise to a new generation: generations overlap. The behaviour of the two types of actors will be described separately.

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[7]We adopt a standard method for endogeneizing fertility in OLG models, following, for instance, Galor and Weil (1996), Strulik (1999, 2003).

2.1 Government

We imagine that the government implements a redistributive policy between generations by lump sum taxes. More precisely, following Diamond (1965), we assume that at each date t the government issues a non-negative amount B_t of national debt and finances it by partly rolling it over and partly by levying taxes upon the young adults, so that the dynamic equation of debt is:

$$B_{t+1} = R \cdot B_t - \tau_t \cdot L_t \tag{1}$$

where $R \equiv 1 + r$ (r being the interest rate), τ_t is a lump sum tax imposed on the young adult of period t and L_t is the number of workers in period t which we assume equal to the number of agents born in the previous period, i.e. $N_{t-1} = L_t$. Denoting by $n_t \equiv \frac{N_t}{N_{t-1}} = \frac{L_{t+1}}{L_t}$, the dynamic equation of debt can be expressed in per worker terms as

$$b_{t+1} \cdot n_t = R \cdot b_t - \tau_t \tag{2}$$

2.2 Individuals

As already mentioned, individuals live for three periods. In the first period they do not take any decisions. In the second period young adults are endowed with well-behaved preferences described by a utility function U. Such a function is defined over consumption in the second and third period of life (c^1, c^2) and on the number of children per adult (n)[8], respectively, which are given birth by young adults[9]. Moreover, in the second period, individuals receive a salary w for their labour services (exogenously supplied) and decide how to split such an income over life-time consumption and on child-bearing. More precisely, we assume that each child costs a fixed fraction, q, of wage w.[10] To formalize, let an agent i of the cohort born in period $t-1$ become active in period t (being a young adult) and, in this period, she chooses her lifetime consumption plan ($c^1_{i,t}$, $c^2_{i,t+1}$) and fertility ($n_{i,t}$) according to the log-linear utility function

$$U(c^1_{i,t}, c^2_{i,t+1}, n_{i,t}) = \log(c^1_{i,t}) + d \cdot \log(c^2_{i,t+1}) + f \cdot \log(n_{i,t}) \tag{3}$$

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[8]Since we imagine that every single young adult can have children, it follows that the population will be stationary or increasing if n is equal or bigger than 1, respectively.

[9]Note that we assume individuals derive utility from the number of children they have, in line with a vast literature (e.g. Eckstein and Wolpin (1984)). However, the demand of children could be also motivated supposing individuals i) derive utility from the utility of their children (e.g. Becker and Barro (1988)) or from the consumption of their children (e.g. Kohlberg (1976)); ii) aim at receiving old-age support from their children (e.g. Willis (1980)). The investigation of the effects of these alternative motivations for having children may be the object of future research.

[10]See for example Strulik (1999, 2003) and Boldrin and Jones (2002), who make the same assumption on the cost function. This function captures the modern view of a time-cost of childrearing in terms of forgone wages.

where $d \equiv \frac{1}{1+\delta}$ is the time discount factor for the utility of future consumption (δ being the time discount rate) and f a parameter appreciating the number of children per adult in the utility function. The budget constraint is

$$c_{i,t}^1 + \frac{c_{i,t+1}^2}{R} + w \cdot q \cdot n_{i,t} = w - \tau_t \tag{4}$$

Maximization of (3) in respect of (4) gives the following behavioural relations

$$c_{i,t}^1 = \frac{w - \tau_t}{p} \tag{5}$$

$$c_{i,t+1}^2 = \frac{R \cdot d \cdot (w - \tau_t)}{p} \tag{6}$$

$$n_{i,t} = \frac{f \cdot (w - \tau_t)}{p \cdot q \cdot w} \tag{7}$$

where $p \equiv 1 + d + f$. Defining saving as $s_i \equiv \frac{c_i^2}{R}$, by (6) we get

$$s_{i,t} = \frac{d \cdot (w - \tau_t)}{p} \tag{8}$$

Inspection of equations (5) to (8) reveals, as expected from the log-linear utility function, that all variables are linear in (disposable) income.

In the rest of the paper, as far as individual choices are concerned, we focus on fertility (n) and saving (s); the model is closed by the dynamic equation of per capita debt (2) so that we have

$$n_t = \frac{f \cdot (w - \tau_t)}{p \cdot q \cdot w} \tag{9a}$$

$$s_t = \frac{d \cdot (w - \tau_t)}{p} \tag{9b}$$

$$b_{t+1} \cdot n_t = R \cdot b_t - \tau_t \tag{9c}$$

Since all agents are assumed identical, any reference to the subscript i in the fertility and saving functions is dropped from now on.

2.3 Solving for steady states: different policies

In a steady state all per capita variables, consumption levels (and saving), fertility, taxes and debt burden are constant. Then system (9), in steady state, becomes

$$\tau = b \cdot (R - n) \tag{10a}$$

$$n = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w} \tag{10b}$$

$$s = \frac{d \cdot (w - \tau)}{p} \tag{10c}$$

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Equation (10a) relates the values of the two public variables τ and b when the system is in a steady state. Equations (10b) and (10c) represent individual decisions and depend on the basic parameters of the model and on the level of taxation τ.

Equations (10a)-(10c) form an interdependent system of three equations with fuor variables (τ, b, n and s) and, as such, it is undetermined. To solve it, we have to fix one variable and solve for the others: our choice is to focus on public policies, thus fixing either τ or b, and then determine endogenously the remaining variables. Given this framework, we can imagine two situations:

1. We fix τ and let eqs. (10a) and (10b) determine b and n; finally s is obtained from eq. (10c) given τ. In this case the Government is committed to a given level of taxation and then b, n and s are determined endogenously in system (10): we will call this the "constant lump sum tax policy" case.

2. We fix b and let eqs (10a) and (10b) determine τ and n; s is obtained once τ is determined. In this case, the Government is committed to keep a given level of per capita debt and then τ, n and s are determined endogenously in system (10): we will call this the "constant debt policy" case.

In either case, a crucial role is played by the quantity $b \cdot (R - n)$ (see eq. (10a)) that will also be one of the key elements in the analysis to follow; to interpret it, notice that b is the present generation per capita debt "endowment" and hence $R \cdot b$ is the debt "bequest" left by each member of the current generation to their children; on the other hand $b \cdot n$ is the amount of debt that the next generation should receive were each of the children to be endowed with the same per capita debt of the parents. The imbalance between these two terms is the debt left by the current generation to the next generation *in excess* of what would be required to keep per capita debt constant (the difference $R \cdot b - n \cdot b$ will be termed "excess debt" for short in what follows); in presence of a non null excess debt and in absence of any intervention, per capita debt of the new generation would differ from that of the parents. Therefore in a steady state characterized by the invariance of per capita debt across generations, this extra debt must be absorbed and the present generation must provide, through taxation τ, the necessary resources to eliminate the imbalance, i.e. $\tau = b \cdot (R - n)$; in other words, τ is the cost that the present generation must pay to guarantee the invariance of debt across generations.

Notice that when $n = R$ the excess debt is null for whatever level of b and indeed $\tau = 0$, so that disposable income is maximized, i.e. $n = R$ is a sort of golden rule for intergenerational debt invariance and for this reason the quantity $R - n$ will be termed the "golden rule gap" in what follows.

Solving for steady state with a constant lump sum tax policy: in this case the Government fixes the desired level of taxation τ and then the interplay of public decisions (τ) and individual choices ($n(\tau)$) determines the level of public debt $b(\tau)$ thorough the equation $b(\tau) = \frac{\tau}{R - n(\tau)}$ (see eq. (10a)); the latter relation highlights that the equilibrium per capita debt is affected by τ in two ways, directly and indirectly through the golden rule gap. Given the form of the fertility and saving functions, once τ is given, n and s are determined. Formally, we substitute n from eq. (10b) into eq. (10a) and solve for b; then fertility and saving levels are determined by eqs. (10b) and (10c). Hence, under this policy, system (10) has the following explicit solution

$$b(\tau) = \frac{p \cdot q \cdot w \cdot \tau}{R \cdot p \cdot q \cdot w - f \cdot (w - \tau)} \tag{11a}$$

$$n(\tau) = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w} \tag{11b}$$

$$s(\tau) = \frac{d \cdot (w - \tau)}{p} \tag{11c}$$

Notice that here the exogenous variable is the level of taxation and public debt adjusts to guarantee equilibrium.[11]

Solving for steady state with a debt stabilizing lump sum tax policy: in this case the exogenous variable is b since Government is aimed at keeping the burden of per capita debt constant across generations (as in Diamond (1965); then taxes τ are calibrated in such a way that the interplay of τ and individual choices on fertility $n(\tau)$ makes the target debt b sustainable in steady state. Indeed, this value of τ comes from the solution for τ of the equation $\tau = b \cdot (R - n(\tau))$ which highlights the twofold role played by taxes also in this case: on the one hand, they determine fertility and hence contribute to determine the golden rule gap and the excess debt; on the other hand, they serve just the purpose to cover the so generated excess debt. Once τ is determined, decisions about saving are determined as well.

To have an explicit solution in this case, it is sufficient to substitute n from eq. (10b) into eq. (10a) and solve for τ; we then substitute the so obtained value of the tax into eq. (10b) and eq. (10c), thus obtaining the following

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[11] We are aware that, in this setting, with given taxes and debt evolving according to eq. (9c), there may be problems as to the dynamics of debt. Debt dynamics as from eq. (9c) is unstable whenever $R > n_t$, diverging from the steady states of system (11) as a consequence of whatever shock. The choice here is to focus on the characteristics of equilibrium positions compatible with given taxation levels, irrespective of the ability of the economy to recover any of them. In principle we could justify this by assuming that at any moment Government could tailor the amount of debt to its equilibrium value by suitable debt consolidation or expansion.

explicit solution for system (10):

$$\tau(b) = \frac{b \cdot w \cdot (R \cdot p \cdot q - f)}{p \cdot q \cdot w - b \cdot f} \tag{12a}$$

$$n(b) = \frac{f \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} \tag{12b}$$

$$s(b) = \frac{d \cdot q \cdot w \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} \tag{12c}$$

Notice that here the exogenous variable is the per capita level of debt and taxation adjusts to guarantee equilibrium.

3   Characterization of steady states

To characterize the steady states for this economy, we assume that all exogenous economic parameters of the model are positive, i.e.

A1) $w > 0, R > 0, q > 0$

and preferences are well defined and monotone

A2) $d > 0$ and $f > 0$

Moreover we restrict attention to the economically more compelling case in which all model variables are non-negative; therefore, in the case of a constant tax policy, we impose the restriction

R1) $\tau \geq 0$ and $n(\tau) \geq 0$, $s(\tau) \geq 0$, $b(\tau) \geq 0$

while, in the case of a debt stabilizing tax policy, we impose the restriction

R2) $b \geq 0$ and $n(b) \geq 0$, $s(b) \geq 0$, $\tau(b) \geq 0$

Clearly the respect of conditions **R1)** in the case of a constant tax policy or **R2)** in the case of a debt stabilizing tax policy imposes restrictions on the values that parameters can take in steady state and on the relation among them; this is shown in the following Proposition where a crucial role is played by the relation between R and the quantity $\frac{f}{p \cdot q}$, i.e. the value of fertility in absence of taxation[12] (the proof is in the Appendix A).

Proposition 1. *Suppose **A1)** and **A2)** hold.*

  a) Assume $R > \frac{f}{p \cdot q}$; then

   *i. $(b(\tau), n(\tau), s(\tau))$ satisfy **R1)** iff $0 \leq \tau \leq w$; in this case $0 \leq b(\tau) \leq \frac{w}{R}$.*

   *ii. $(\tau(b), n(b), s(b))$ satisfy **R2)** iff $0 \leq b \leq \frac{w}{R}$; in this case $0 \leq \tau(b) \leq w$.*

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[12]Indeed by eq. (7) or eq. (10b) we have $n(\tau) = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w}$ and hence it is $n(0) = \frac{f}{p \cdot q}$.

b) Assume $R = \frac{f}{p \cdot q}$; then

     *i. $(b(\tau), n(\tau), s(\tau))$ satisfy **R1)** iff $0 \le \tau \le w$; in this case $b(\tau) = \frac{w}{R}$ for any $\tau \in [0, w]$.*

     *ii. $(\tau(b), n(b), s(b))$ satisfy **R2)** iff $b \ge 0$; in this case $\tau(b) = 0$ for any $b \ge 0$.*

c) Assume $R < \frac{f}{p \cdot q}$; then

     *i. $(b(\tau), n(\tau), s(\tau))$ satisfy **R1)** iff $0 < \underline{\tau} \equiv \frac{(f - R \cdot p \cdot q) \cdot w}{f} \le \tau \le w$; in this case $b(\tau) \ge \frac{w}{R}$.*

     *ii. $(\tau(b), n(b), s(b))$ satisfy **R2)** iff $b \ge \frac{w}{R}$; in this case $0 < \underline{\tau} \equiv \frac{(f - R \cdot p \cdot q) \cdot w}{f} \le \tau(b) \le w$.*

By the previous Proposition, we can distinguish two regimes separated by the peculiar case in which $R = \frac{f}{p \cdot q}$: the former, identified by the condition $R > \frac{f}{p \cdot q}$ (see point a) in Proposition 1), in which population in absence of taxation would grow at a rate that is slower than the interest rate (this will be termed the "low population growth" regime) and the latter, identified by the condition $R < \frac{f}{p \cdot q}$ (see point c) in Proposition 1), in which population in absence of taxation would grow at a rate that is faster than the interest rate (this will be termed the "fast population growth" regime). In the first regime, a steady state with non-negative values of all variables occurs for any τ such that $0 \le \tau \le w$, but is compatible only with moderate level of public debt ($b \le \frac{w}{R}$). In the other regime, a steady state with non-negative values of all variables is achievable only with a huge public debt ($b > \frac{w}{R}$) and strictly positive taxes ($w \ge \tau \ge \frac{(f - R \cdot p \cdot q) \cdot w}{f} \equiv \underline{\tau} > 0$). The regime separating case ($R = \frac{f}{p \cdot q}$, see point b) in Proposition 1), is characterized alternatively by the irrelevance of a tax policy as to the management of per capita debt (only private variables n and s are affected) or the irrelevance of a debt management policy as to the determination of any other variable in the model (indeed steady state values of τ, n and s are given and independent of b in this case). By these findings, the case $R = \frac{f}{p \cdot q}$ [13] configures as an exceptional case which is rather uninteresting for the purposes of the present analysis and hence it will be neglected in what follows.

Beside these restrictions, steady states of this economy can be further characterized; a first consequence of Proposition 1 concerns the extent by which population is allowed to grow in steady state when all the assumptions and restrictions imposed so far are satisfied. In particular we can show that (the proof is in the Appendix A)

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[13]Notice that this is exactly the steady state value of fertility when $R = \frac{f}{p \cdot q}$ and the Government aims at stabilizing per capita debt since in this case $\tau(b) = 0 \; \forall b \ge 0$ (see Proposition 1b)).

Corollary 1. *Under **A1)** and **A2)**, in any steady state satisfying **R1)** or **R2)** it is*

$$n(x) \leq R \qquad x \in \{\tau, b\}$$

Therefore any steady state satisfying **R1)** or **R2)**, i.e. characterized by the simultaneous presence of non negative debt and non negative taxes, is bound to have a non negative golden rule gap and a non negative excess debt.

Furthermore it is easy to see that, for any given configuration of parameters respecting **A1)** and **A2)**, the solutions $\tau(b)$ and $b(\tau)$ satisfying **R1)** or **R2)** in either regime are such that $\tau = \tau(b(\tau))$ and $b = b(\tau(b))$, i.e. the equilibrium results of the two policies are strictly interconnected, since the two functions $\tau(b)$ and $b(\tau)$ are one just the inverse of the other.

3.1   Regimes and the different effects of public policies

The results of the previous Section highlight that **R1)** and **R2)** impose restrictions on viable steady state policies and, within those bounds, the system is characterized by a persistent golden rule gap and excess debt; moreover, by Proposition 1, different population growth regimes imply different levels of debt. Indeed, we now show that the different regimes affect also the magnitude of the golden rule gap and the intensity of its reaction to changes in policy. More precisely, denoting by $\eta_\tau(R - n(\tau)) = \frac{\tau}{R - n(\tau)} \cdot \frac{\partial(R - n(\tau))}{\partial \tau}$ the elasticity of the golden rule gap to τ, we have:

Lemma 1. *Suppose **A1)** - **A2)** are satisfied; then*

$$R - n(\tau) \gtrless \frac{R \cdot \tau}{w} \quad \Leftrightarrow \quad R \gtrless \frac{f}{p \cdot q}$$

$$\eta_\tau(R - n(\tau)) \lessgtr 1 \quad \Leftrightarrow \quad R \gtrless \frac{f}{p \cdot q}$$

In words, if the economy is in a slow (fast resp.) population growth regime, the golden rule gap (i) is bounded from below (above resp.) by the quantity $\frac{R \cdot \tau}{w}$ and (ii) is inelastic (elastic resp.) with respect to changes in taxation. To exemplify: suppose that preference parameters, cost of rearing children and wages are given; then the two regimes are uniquely identified by the interest factor assuming a value $R > \frac{f}{p \cdot q}$ when the system is in a "slow population growth" regime and a value $R' < \frac{f}{p \cdot q}$ when the system is in a "fast population growth" regime. With taxes at a given level τ, fertility is the same in either regime, $n(\tau) = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w}$, but the golden rule gap is higher in the "slow" than in the "fast" population growth regime, since, by the first inequalities in Lemma 1, we have

$$R - n(\tau) > \frac{R \cdot \tau}{w} > \frac{f}{p \cdot q} \cdot \frac{\tau}{w} > \frac{R' \cdot \tau}{w} > R' - n(\tau)$$

Suppose now that, for some reason, taxes are to be reduced to a level $\tau' < \tau$. In both regimes fertility will take an identical new value $n(\tau') = \frac{f \cdot (w - \tau')}{p \cdot q \cdot w} > n(\tau)$

with an identical variation $\Delta n = n(\tau') - n(\tau)$ and hence the golden rule gap will reduce by the same amount in both regimes. However the percentage reduction of the golden rule gap is lower in the "slow" than in the "fast" population growth regime simply because its original value was greater in the first than in the second regime. Therefore, the sensitivity of the golden rule gap to identical variations in the tax level is different in the two regimes, as stated in the Lemma.

On the basis of this result, we can now show that the ruling regime affects also the way in which the two public variables τ and b are correlated; in particular we show that increasing debts must be sustained by increasing taxes in a "slow population growth" regime, while decreasing taxes lead to an increase of debt in the other regime. This is the content of the following Proposition:

Proposition 2. *Suppose **A1)** and **A2)** hold true:*

- *Under a constant tax policy, the steady state solution $b(\tau)$ satisfying **R1)** is such that $\frac{db(\tau)}{d\tau} \gtrless 0 \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$.*

- *Under a debt stabilizing policy, the steady state solution $\tau(b)$ satisfying **R2)** is such that $\frac{d\tau(b)}{db} \gtrless 0 \Leftrightarrow b \lessgtr \frac{w}{R} \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$.*

The formal proof of Proposition 2 is in the Appendix A, but we can give here a simple intuition, focusing for simplicity on just the first case (constant tax policy) in which $b(\tau) = \frac{\tau}{R - n(\tau)}$: then the overall change in per capita debt due to a change in taxation is given by

$$\frac{\partial b(\tau)}{\partial \tau} = \frac{\partial\left(\frac{\tau}{R - n(\tau)}\right)}{\partial \tau} = \frac{(R - n(\tau)) - \tau \cdot \frac{\partial(R - n(\tau))}{\partial \tau}}{(R - n(\tau))^2} = \frac{1}{R - n(\tau)} \cdot \left(1 - \eta_\tau \left(R - n(\tau)\right)\right)$$

Apart from the term $\frac{1}{R - n(\tau)}$ which is always non negative by Corollary 1, the sign of the effect of a change in taxation on the level of debt is due to the balance of the terms in $\left(1 - \eta_\tau (R - n(\tau))\right)$: the term "1" measures the direct effect on debt of a change in taxes "at constant fertility" as expressed by the numerator in $\frac{\tau}{R - n(\tau)}$; the other term $(\eta_\tau (R - n(\tau)))$ measures the indirect effect on debt of a change in taxation due to the fact that, all other things constant, taxation affects also fertility as expressed by the denominator in $\frac{\tau}{R - n(\tau)}$. Since, by Lemma 1, this second term is greater or lower than 1 depending on the population growth regime, it follows that the reaction of debt to taxation is qualitatively different in the two regimes.

The results obtained so far receive a simple graphical representation in the following diagram in which the steady state relations between τ and b in the two regimes are presented: "slow population growth" in Figure 1(A) and "fast population growth" in Figure 1(B).[14]

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[14]In this Figure the values of the parameters are: $w = 3.5$, $\tau = 1.2$, $q = 0.25$, $d = \frac{2}{3}$, $f = 1$. With these values $\frac{f}{p \cdot q} = 1.5$ and we set $R = 1.65$ in Figure 1(A) and $R' = 1.35$ in Figure 1(B).
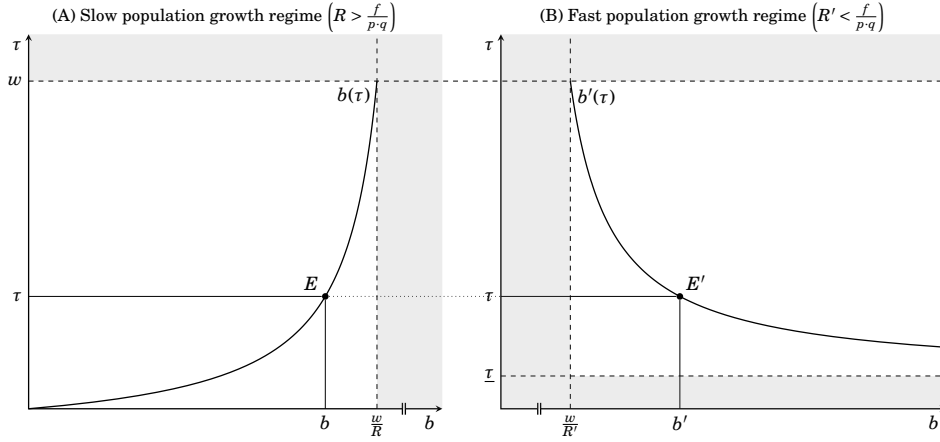
Figure 1: Characterization of steady state policies in the two different regimes.

The shadowed regions in each picture represent the bounds on public variables imposed by the restrictions **R1)** and **R2)** in the respective regime (see Proposition 1). The curves $b(\tau)$ and $b'(\tau)$ represent the equilibrium relations between the two public variables in the two regimes, i.e. the equilibrium level of debt corresponding to a taxation level τ; according to Proposition 2, the locus of $(\tau - b)$ equilibrium combinations is positively sloped in the "slow population growth regime" and negatively sloped in the "fast population growth regime", i.e. a change in either policy instrument in the first regime is accompanied by a change of the same sign in the other public variable, while the second regime entails changes of opposite sign.

3.1.1 Public policies, fertility, saving and population growth regimes

The behavioural relations in eqs. (10b) and (10c) show that there is a negative effect of taxation on both fertility and saving; given the form of the assumed utility function, n and s are normal in disposable income so that any increase (reduction resp.) in taxation, lowering (increasing resp.) disposable income, produces a reduction (increase resp.) of fertility and saving. Therefore, when public authorities are interested to a stable taxation and, by some reason, have to change the target level, the effects of this change on private decisions n and s are well determined and qualitatively identical across regimes: an increase in taxation always leads to a reduction of fertility and saving.

On the contrary, when public authorities are committed to stabilize the per capita debt and, for some reason, they have to change its level, the effects on private decisions depend on the regime the economy is in. This different effects can be appreciated in the following Figures 2(A) and 2(B), in which the picture on the left represents the equilibrium relation between public variables, the second picture represents the fertility function $n(\tau) = \frac{\hat{f} \cdot (w - \tau)}{p \cdot q \cdot w}$

drawn as a function of τ on the vertical axis and the picture on the right represents the saving function $s(\tau) = \frac{d \cdot (w - \tau)}{p}$ again drawn as a function of τ. Figure 2(A) represents the situation arising in the "slow population growth" regime following a decision by public authorities to increase the amount of per capita debt (from b to b').
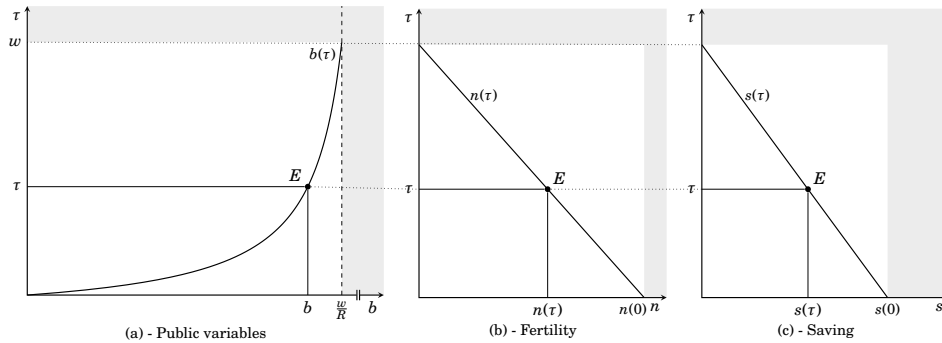

Figure 2(A): Effects of a per capita debt increase on taxes, fertility and saving in a "slow population growth regime"

In this case equilibrium values of policy variables are positively correlated so that an increase of the amount of per capita debt induces an increase of taxes which reflects in a reduction of fertility and of saving decisions.

In the other regime ("fast population growth"), the situation is portrayed in Figure 2(B). Contrary to the previous case, now equilibrium values of public variables are negatively correlated so that an increase of the amount of per capita debt from b to b' induces a reduction of taxation which reflects in an increase of fertility and of saving decisions.

Therefore a same debt expansion has opposite effects on private decisions, depending on the ruling regime and this is finally due to the different ways in which taxes must be calibrated to keep equilibrium in the two cases. Clearly,
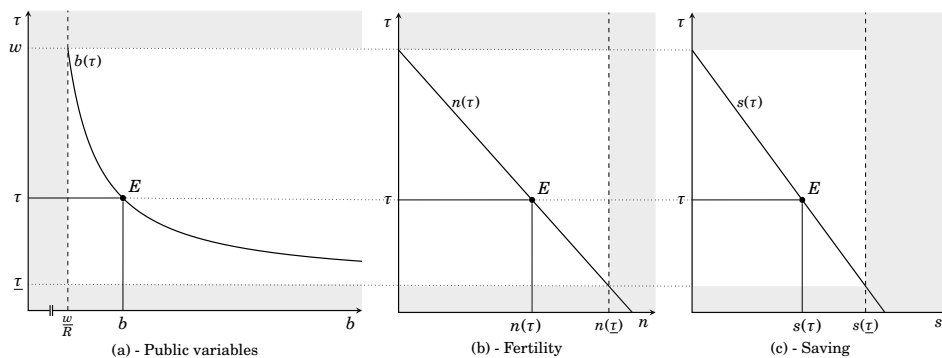

Figure 2(B): Effects of a per capita debt increase on taxes, fertility and saving in a "fast population growth regime"

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these effects could also be interpreted the other way round; suppose that public authorities are interested in stimulating fertility and saving. Given behavioural relations (10b) and (10c), this stimulation can be obtained by a reduction of taxes, i.e. increasing disposable income. Given the form of the equilibrium relation as in Figure 1, the required reduction of taxation must be accomplished through a reduction of per capita debt in the "slow population growth" regime, while in the other regime this can be obtained at the cost of an increase in the per capita debt level.

4 Comparative statics on the steady state solutions

In this Section we extend the previous analysis and, fixed a policy in terms of either taxes or per capita debt, we analyze the effects on the other public variable and individual decisions of a change in underlying parameters of the model (w, R, q) through comparative static exercises. As it can be expected, results are not identical under the two regimes. Throughout this Section we assume **A1)** and **A2)** and that the system satisfies **R1)** or **R2)**.

4.1 Changes of wage w

The following table summarizes the results for changes in wage w; they are obtained by simple differentiation of the steady state relations in systems (11) and (12) with respect to w (the proofs are in the Appendix C):

	Constant tax policy		**Debt stabilizing policy**	
	Public debt	Individual choices	Taxation	Individual choices
$R > \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial w} > 0$	$\frac{\partial n(\tau)}{\partial w} > 0 \quad \frac{\partial s(\tau)}{\partial w} > 0$	$\frac{\partial \tau(b)}{\partial w} < 0$	$\frac{\partial n(b)}{\partial w} > 0 \quad \frac{\partial s(b)}{\partial w} > 0$
$R < \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial w} > 0$	$\frac{\partial n(\tau)}{\partial w} > 0 \quad \frac{\partial s(\tau)}{\partial w} > 0$	$\frac{\partial \tau(b)}{\partial w} > 0$	$\frac{\partial n(b)}{\partial w} < 0 \quad \frac{\partial s(b)}{\partial w} \lessgtr 0$ [15]

As we can see, within a same regime, a change of w produces different effects depending on the policy target public authorities are committed to. To analyze these diverging implications, we focus on each regime separately, but preliminarily notice that there is a common effect at work in both regimes: all other things constant, an increase of wage increases disposable income and hence, for a same level of taxation, all curves as in Figures 2(A) and 2(B) move to the right. More precisely, given τ, an increase of w (a) stimulates fertility[16] (b) increases saving due to the increase of disposable income (and

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[15]Indeed $\frac{\partial s(b)}{\partial w} \leq 0$ for $b \in \left[\frac{w}{R}, \frac{w}{R} \cdot \left(1 + \sqrt{\frac{R \cdot f}{p \cdot q} - 1} \right) \right]$ and $\frac{\partial s(b)}{\partial w} > 0$ for $b > \frac{w}{R} \cdot \left(1 + \sqrt{\frac{R \cdot f}{p \cdot q} - 1} \right)$ (see Appendix C).

[16]The fact that higher wages promote higher fertility may appear surprising since w is both the individual wage and a determinant of the cost of having children ($q \cdot w$); hence an increase in w entails both a positive income effect and a negative substitution effect. As shown in the Appendix B, the two effects cancel out in absence of taxation, but the income effect prevails if

to the fact that, given the assumed form of the utility function, future consumption is normal in disposable income) and finally (c) produces an increase of the equilibrium per capita debt corresponding to that level of taxation; this latter effect is due to the fact that an increase of wages increases fertility and this reduces the golden rule gap, i.e. the excess debt, thus making an higher value of debt sustainable in equilibrium with unchanged taxes.

4.1.1 Changes of w in the "slow population growth" regime

The effects on public variables, fertility and saving of an increase of w can be summarized in the following Figure in which we assume that model parameters are such that $R > \frac{f}{p \cdot q}$ i.e. the economy is within the "slow population growth" regime. As outlined previously, all curves shift to the right due to the increase of wage:
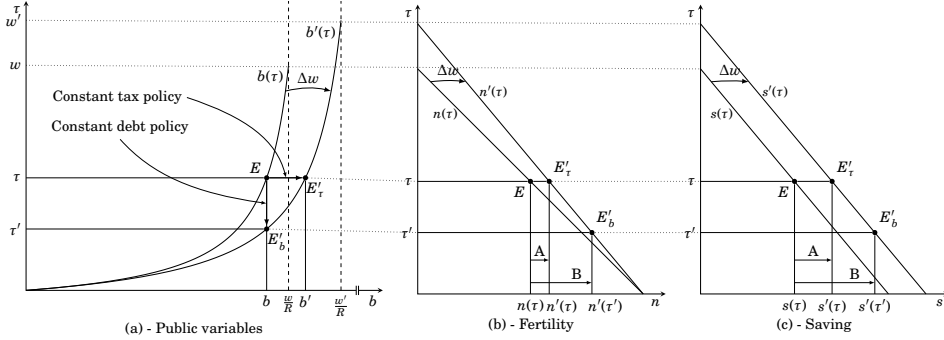

Figure 3(A): The effects on public variables, fertility and saving of an increase of wages in the "slow population growth" regime

Constant tax policy: if public authorities are committed to keep a constant level of τ, we can see that the system passes from the original equilibrium configuration as represented by the points labeled as E to the new equilibrium configuration as represented by the points labeled as E'_τ, i.e. we observe an increase of equilibrium per capita debt from b to b', an increase in the value of n passing from $n(\tau)$ to $n'(\tau)$ and finally an increase of the amount of saving passing from $s(\tau)$ to $s'(\tau)$.

Debt stabilizing policy: suppose now that public authorities were committed to keep per capita debt stable at the original level b. To do that they have to suitably tailor taxation and, looking at the first picture in Figure 3(A), this is accomplished by a reduction of taxes from the original level τ to the new level τ'; this is possible since the excess debt is now decreased (due to the augmented fertility) and can be matched by a parallel reduction of τ since in

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$\tau > 0$, as in the cases we examine.

this regime the golden rule gap is relatively inelastic to τ itself. This reduction of taxation reinforces the increase in disposable income already produced by the increase of wages and consequently we have further stimulus to private fertility and saving decisions, passing from $n(\tau)$ to $n'(\tau')$ and from $s(\tau)$ to $s'(\tau')$ respectively.

4.1.2 Changes of w in the "fast population growth" regime

In this case, determined by a configuration of parameters such that $R < \frac{f}{p \cdot q}$, the effects of an increase of wage can be summarized as in the following Figure 3(B), where for the same reasons previously outlined, all the curves shift to the right:
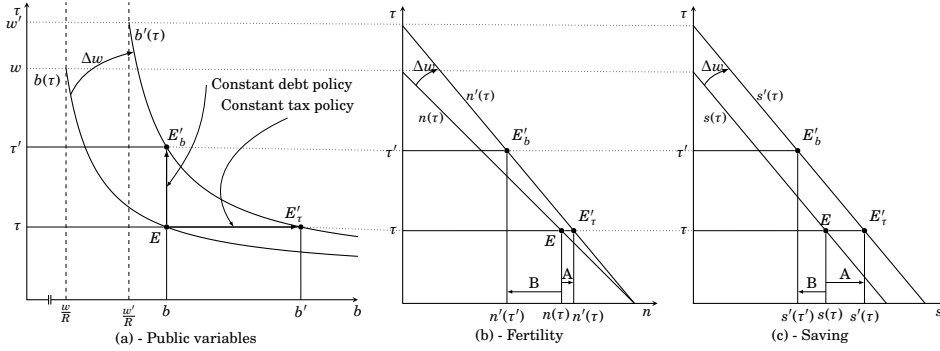

Figure 3(B): The effects on public variables, fertility and saving of an increase of wages in the "fast population growth" regime.

Constant tax policy: if public authorities are committed to a constant level of τ, the situation is the same as the one described in Section 4.1.1. Keeping taxes constant, the increase of wages translates in an increase of disposable income that stimulates fertility from $n(\tau)$ to $n'(\tau)$ and saving from $s(\tau)$ to $s'(\tau)$; the reduced golden rule gap ($R - n'(\tau) < R - n(\tau)$ since $n'(\tau) > n(\tau)$) allows to sustain a larger per capita debt, that indeed passes from b to b'.

Debt stabilizing policy: under this policy, public authorities want to keep debt at level b in face of a reduction of the golden rule gap (due to the augmented fertility). However in this regime, as seen in Proposition 2, the golden rule gap is elastic to τ and a decrease of taxation would reduce so much the value of the golden rule gap that the overall effect on debt ($b = \frac{\tau}{R - n(\tau)}$) would be actually an increase of b. The impossibility of matching the reduced excess debt with a reduction of taxes, makes a tax increase the only instrument to restore equilibrium as can be seen in Figure 3(B), where the new equilibrium with constant debt, following a wage increase, occurs in position E'_b with an increase in taxation from τ to τ'. As a consequence, we observe an increase

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of both wages and taxation and the effect on private variables n and s will depend on the balance between these two variations, resulting either in an increase or in a reduction of disposable income. Actually Figure 3(B) represents the case in which the two variations balance in a reduction of disposable income and hence both fertility and saving contract to the new values $n'(\tau')$ and $s'(\tau')$.

However, in this regime, a particular situation may occur when debt is very high (and taxes are very low): in this case, the increase of w produces an increase in τ that is so small that increments in wages and taxes balance in an increase of net disposable income, making saving to increase (this effect is not appreciated in fertility decisions since there is also an increase in the cost of bearing children ($q \cdot w$) outweighing the increase in disposable income). This special situation is portrayed in the following Figure:
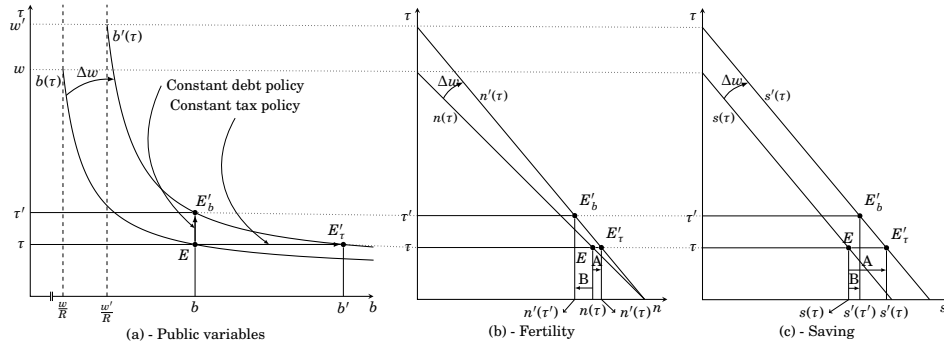

Figure 3(C): The effects on public variables, fertility and saving of an increase of wages in the "fast population growth" regime with very high debt

4.2 Changes of the interest factor R

The interest rate is one of the determinants of the regime the economy is in; therefore we have to distinguish two cases: changes of R keeping the economy within the same regime and changes of R producing a regime switch. They will be analyzed separately.

4.2.1 Intra-regime changes of the interest factor R

In this case we assume that the change of the interest factor R does not alter the ruling regime determined by the quantity $\frac{f}{p \cdot q}$. The following table summarizes the results obtained by differentiation of the steady state relations in systems (11) and (12) with respect to R (the proofs are in the Appendix C).

As usual, the effects on model variables due to a change of R vary depending on the regime and the policy adopted and will be analyzed separately. However, as in the previous case, the effect of an increase of R on the shape

	Constant tax policy		Debt stabilizing policy	
	Public debt	Individual choices	Taxation	Individual choices
$R > \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial R} < 0$	$\frac{\partial n(\tau)}{\partial R} = 0 \quad \frac{\partial s(\tau)}{\partial R} = 0$	$\frac{\partial \tau(b)}{\partial R} > 0$	$\frac{\partial n(b)}{\partial R} < 0 \quad \frac{\partial s(b)}{\partial R} < 0$
$R < \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial R} < 0$	$\frac{\partial n(\tau)}{\partial R} = 0 \quad \frac{\partial s(\tau)}{\partial R} = 0$	$\frac{\partial \tau(b)}{\partial R} < 0$	$\frac{\partial n(b)}{\partial R} > 0 \quad \frac{\partial s(b)}{\partial R} > 0$

of the curves $b(\tau)$, $n(\tau)$ and $s(\tau)$ is qualitatively identical in the two regimes: indeed simply looking at equations (10b) and (10c) we see that fertility and saving decisions are independent of R so the respective curves will not be affected by any change of the interest rate. As far as the equilibrium relation between τ and b is concerned, we observe that, all other things constant, an increase of R increases excess debt. For any given taxation level, this higher cost can be sustained only by lowering the amount of per capita debt (remember the equilibrium relation $\tau = b \cdot (R - n)$); therefore for any given level of τ, as a consequence of an increase of R we have a reduction of b, i.e. the equilibrium loci for public variables must move to the left.

4.2.1.1 Changes of R in the "slow population growth" regime

The effects on public variables, fertility and saving of an increase of the interest factor from R to R' ($R > \frac{f}{p \cdot q}$) can be summarized in the following Figure.
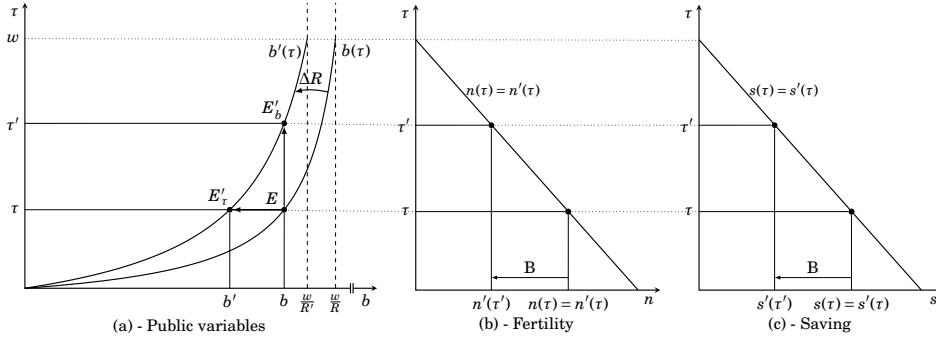

Figure 4(A): The effects on public variables, fertility and saving of an increase of R in the "slow population growth" regime.

Constant tax policy: in this case, the consequences of keeping taxes constant at level τ when R increases as in Figure 4(A) are simple: disposable income does not vary if τ is kept constant and R does not affect private decisions, so that fertility and saving remain at their original level ($n(\tau) = n'(\tau)$ and $s(\tau) = s'(\tau)$). The only effect is on the equilibrium value of debt which, for the reasons seen in Section 4.2.1, must decrease to level b'.

Debt stabilizing policy: in the "slow population growth" regime the increased excess debt can be covered by an increase in taxation, because this latter will not affect substantially the golden rule gap that is now inelastic to τ. Therefore the new equilibrium (E'_b in Figure 4(A)) will be characterized by a higher taxation level (τ') than the original one. The consequent reduction of disposable income will produce a contraction of both fertility (from $n(\tau)$ to $n'(\tau') < n(\tau)$) and saving (from $s(\tau)$ to $s'(\tau') < s(\tau)$).

4.2.1.2 Changes of R in the "fast population growth" regime

The effects of and increase of the interest factor from R to R' in a "fast population growth" regime ($R' < \frac{f}{p \cdot q}$) are summarized in the following Figure
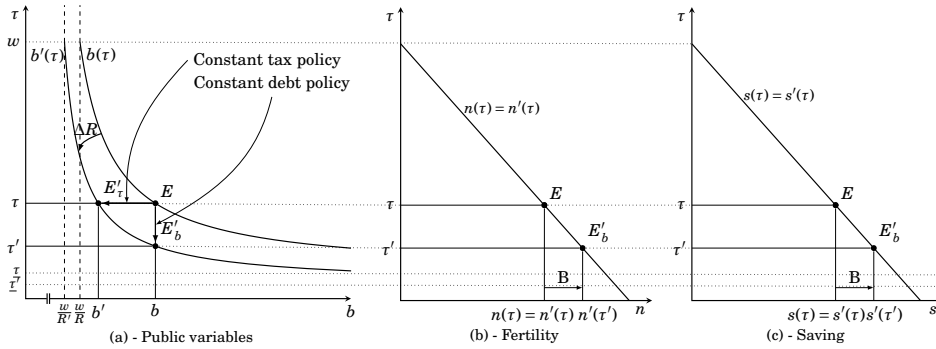

Figure 4(B): The effects on public variables, fertility and saving of an increase of R in the "fast population growth" regime.

Constant tax policy: under this policy disposable income does not vary and since R does not affect private decisions, we have that fertility and saving remain at their original level ($n(\tau) = n'(\tau)$ and $s(\tau) = s'(\tau)$). The only effect is on the equilibrium value of debt which decreases to level b', since the equilibrium loci of public variables move to the left as explained in Section 4.2.1.

Debt stabilizing policy: keeping per capita debt constant in face of the extra debt induced by a higher value of R requires an adjustment in taxation. In the "fast population growth" regime, this extra debt cannot be covered by an increase of taxation, because now the golden rule gap is elastic to τ and an increment of the latter would produce a so huge increase of the former that they would balance in a reduction of debt rather than its constancy. Therefore invariance of debt can be obtained just by a reduction of taxation stimulating fertility and hence the reduction of the golden rule gap, so that the new equilibrium (E'_b in Figure 4(B)) will be characterized by a lower taxation level (τ'). The consequent increase of disposable income will produce an expansion of fertility (from $n(\tau)$ to $n'(\tau') > n(\tau)$) and saving (from $s(\tau)$ to $s'(\tau') > s(\tau)$).

4.2.2 Regime switching changes of the interest rate R

When R varies crossing the value $\frac{f}{p \cdot q}$, we have a switch of regime passing from "slow" to "fast" population growth or vice versa. For the sake of exposition, suppose that R decreases substantially and passes from taking the value R ("slow population growth") to a value R' characterizing a "fast population growth" regime (all the other parameters being as in Figure 1). The situation is represented in the following picture (where $R = 1.65$ and $R' = 1.325$):
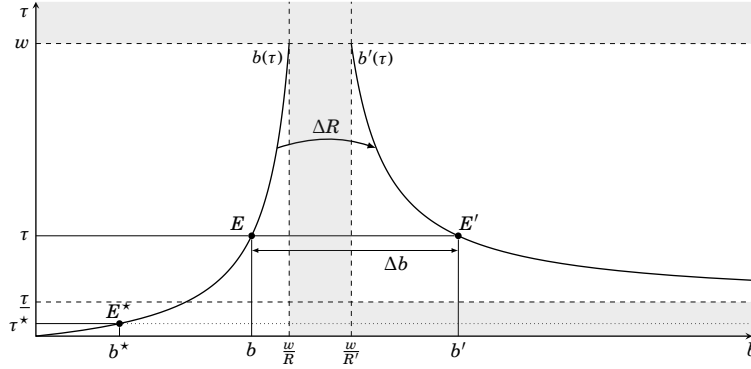

Figure 5: Public policies and regime switching due to a change of R.

As we can see, such a huge change in the interest rate brings the economy from a regime in which the relation between public variables is increasing and the system is bounded from above and right as described in Proposition 1a) (this is the region of the plane to the left of the dashed line $\frac{w}{R}$) to a regime in which the public variables are correlated in a completely different way and the system is bounded from below, above and left (see Proposition 1c)). The main conclusions that can be drawn in presence of a change of R producing a regime switch are the following and can be easily verified by appelling to Proposition 1:

- No constant debt policy is tenable across regimes (the viable supports for b in the two regimes are disjoint: see Proposition 1a)i. and 1c)i.);

- Only a constant tax policy can be held across regimes provided τ is sufficiently high $\tau > \underline{\tau} = \frac{(f - R' \cdot p \cdot q) \cdot w}{f}$ (only these values of τ are in the intersection of the viable supports for this policy in the two regimes, see Proposition 1a)ii. and 1c)ii.). If this type of policy is adopted by the Government, the effects on endogenous variables of the regime switching change in R are the same examined in Section 4.2.1 for the constant tax policy case: disposable income is not affected and hence fertility and saving choices do not vary; on the other hand per capita equilibrium level of debt moves contrariwise to the change of R, due to the fact that, other things constant, a reduction (increase resp.) of R re-

duces (increases resp.) the excess debt and this is compatible with a greater (lower resp.) amount of debt if taxes remain constant.

4.3 Changes of the child rearing cost q

Changes in q affect the other model variables in a twofold manner: on one hand it affects directly individual decisions (specifically fertility) being one of the determinants of the child bearing cost ($q \cdot w$), on the other it contributes to the determination of the regime the economy is in (regimes depend on whether $R \gtrless \frac{f}{p \cdot q}$). Both influences will be examined in the following, separating the analysis of the intra-regime changes of q from the analysis of variations in q producing a regime switch. Given similarities with other cases previously encountered, description will be shortened.

4.3.1 Intra-regime changes of the child rearing cost q

In this case, the change of the child rearing cost from an original value q to a new value q' is such that it does not cause any regime shift, i.e. $R > \max\left(\frac{f}{p \cdot q}, \frac{f}{p \cdot q'}\right)$ or $R < \min\left(\frac{f}{p \cdot q}, \frac{f}{p \cdot q'}\right)$, given R, f and p. With this restriction, the effects of the change of q differ across regimes and policies and the following table summarizes the results, again obtained by differentiation of the steady state relations in systems (11) and (12) with respect to q (the proofs are in the Appendix C):

	Constant tax policy		**Debt stabilizing policy**	
	Public debt	Individual choices	Taxation	Individual choices
$R > \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial q} < 0$	$\frac{\partial n(\tau)}{\partial q} < 0 \quad \frac{\partial s(\tau)}{\partial q} = 0$	$\frac{\partial \tau(b)}{\partial q} > 0$	$\frac{\partial n(b)}{\partial q} < 0 \quad \frac{\partial s(b)}{\partial q} < 0$
$R < \frac{f}{p \cdot q}$	$\frac{\partial b(\tau)}{\partial q} < 0$	$\frac{\partial n(\tau)}{\partial q} < 0 \quad \frac{\partial s(\tau)}{\partial q} = 0$	$\frac{\partial \tau(b)}{\partial q} < 0$	$\frac{\partial n(b)}{\partial q} > 0 \quad \frac{\partial s(b)}{\partial q} > 0$

As in the previous cases, we start noticing that the effect of an increase of q on the shape of the curves $b(\tau)$, $n(\tau)$ and $s(\tau)$ is qualitatively identical across regimes: a change of q affects fertility since it is a component of the child bearing cost ($q \cdot w$): given the assumed form of the utility function, all other parameters remaining constant, an increase of q reduces n, i.e. the fertility function as represented in the middle pictures of Figures 2(A) and 2(B) moves to the left. This reduction in fertility, in turn, increases the golden rule gap so that, for any given taxation level, there will be a higher excess debt that can be eliminated just by lowering the amount of per capita debt, i.e. the equilibrium locus for public variables must move to the left, as it can be seen in the following Figures 6(A) and 6(B). Finally, observe that saving is not affected by q (see eq. (10c)) so that there will not be any shift in the respective function.

4.3.1.1 Changes of q in the "slow population growth" regime

The effects on public variables, fertility and saving of an increase of the child bearing cost from q to q', when $R > \max\left(\frac{f}{p \cdot q}, \frac{f}{p \cdot q'}\right)$, are summarized in the following Figure.
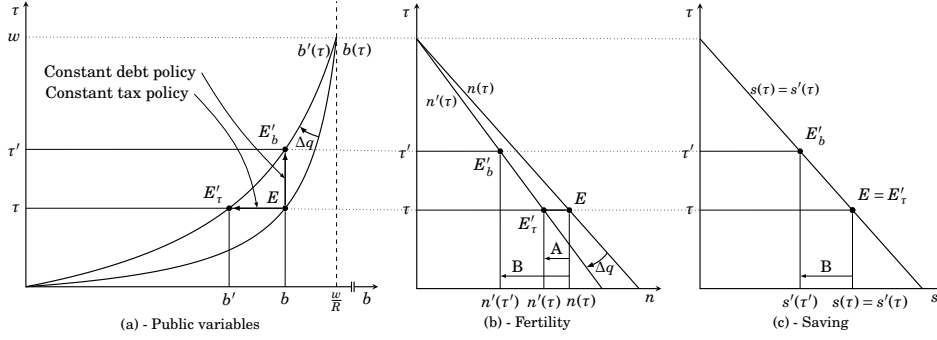

Figure 6(A): The effects on public variables, fertility and saving of an increase of q in the "slow population growth" regime.

Constant tax policy: if public authorities keep taxes constant, disposable income does not vary and hence we do not observe any modifications in saving decisions ($s(\tau) = s'(\tau)$ in the third picture), but fertility reduces since we have an increase in the cost of child bearing ($n(\tau) < n'(\tau)$ in the second picture). As explained before, this reduced fertility increase the golden rule gap and the excess debt of the present generation: this in turn, in the lack of an adjustment on taxation, is bound to reduce the per capita debt that can be sustained in equilibrium and indeed the equilibrium value of debt which shrinks to b' in the first picture of Figure 6(A).

Debt stabilizing policy: public authorities are now committed to keep debt constant in face of an increase of the golden rule gap due to the decreased fertility. As seen in Section 4.2.1.1, this calls for an increase in taxation, further depressing fertility (and saving as well) due to the consequent reduction of disposable income (see variations from $n(\tau)$ to $n'(\tau') < n(\tau)$ and from $s(\tau)$ to $s'(\tau') < s(\tau)$ in Figure 6(A)).

4.3.1.2 Changes of q in the "fast population growth" regime

The effects of an increase of the child bearing cost from q to q' such that $R < \min\left(\frac{f}{p \cdot q}, \frac{f}{p \cdot q'}\right)$ are summarized in the following Figure.

Constant tax policy: in so far as this policy is concerned, the effects and the mechanisms at work following an increase of q are identical to those
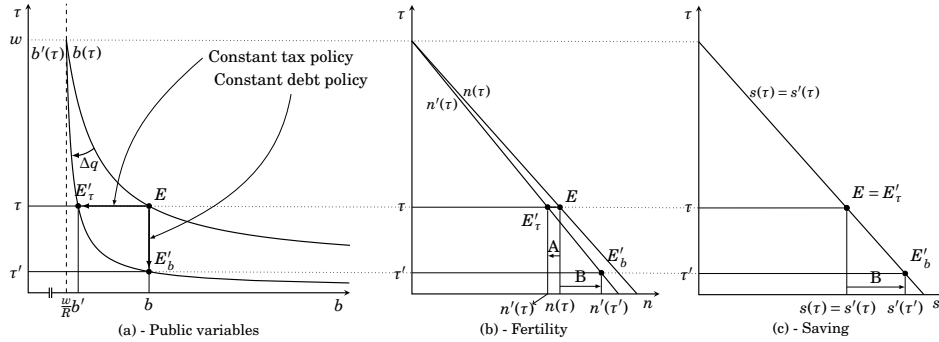
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Figure 6(B): The effects on public variables, fertility and saving of an increase of q in the "fast population growth" regime.

analyzed in the previous section under the same policy and we observe a reduction of fertility and equilibrium per capita debt (from $n(\tau)$ to $n'(\tau') < n(\tau)$ and from b to b' respectively) essentially due to the shift to the left of both the equilibrium locus of public variables and of the fertility function, while saving decisions remain unaffected.

Debt stabilizing policy: stabilizing debt at the original level b after the increase in q that reduced fertility, requires now a reduction of taxation to eliminate the generated extra debt since, in this way, fertility is stimulated and the golden rule gap reduces. Therefore, due to the decreased taxation (from τ to τ'), we have an increase in disposable income that increases saving (from $s(\tau)$ to $s'(\tau') > s(\tau)$ in Figure 6(B)) and that more than compensates the increased cost of child bearing due to the increase in q, so that finally we have the desired increase in fertility (from $n(\tau)$ to $n'(\tau') > n(\tau)$).

4.3.2 Regime switching changes of child rearing cost q

When the change of q produces a change of regime we are in a situation similar to that examined in Section 4.2.2 and is represented in the following Figure 7 where the increase of the child bearing cost from q to q' makes the system pass from the region to the right of the dashed line corresponding to $\frac{w}{R}$ (the "fast population growth" regime where $R < \frac{f}{p \cdot q}$) to the region to the left (the "slow population growth" regime where $R > \frac{f}{p \cdot q'}$):

Apart the shrinking of the separating region between the two regimes, the situation is at all similar to that presented in Figure 5 in Section 4.2.2 and also the conclusions that can be drawn, so that they will not be repeated here.
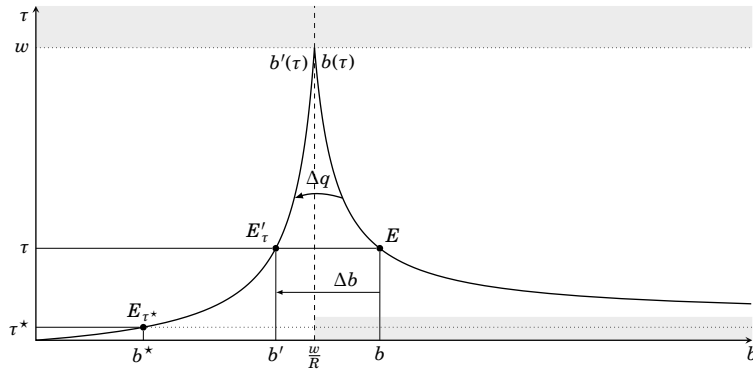
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Figure 7: Public policies and regime switching due to a change of q.

5 Conclusions

In this paper we have analyzed the behaviour of fertility and saving in an OLG model of a small open economy in presence of intergenerational distribution policies. We found that, in steady state positions, the public variables (i.e. taxation and public debt levels) have a different correlation depending on the parameters of the model defining two different possible regimes: slow and fast population growth. Public policies affect other endogenous variables such as consumption levels, saving and fertility and the calibration of policy instruments to (dis)incentivate them must be contingent on the ruling regime. In particular, our analysis highlights that a fertility recovery (reduction resp.) requires always a reduction (increase resp.) of taxes, although the implications for public debt management are not trivial, since they depend on the regime the economy is experiencing, i.e. on the relationship between the interest rate and the fertility rate in absence of taxes.

When the underlying parameters change, also the steady state values of the endogenous variables change and the two policies have different effects: these effects can be summarized in the following table where we report by row regimes and the variables whose value changes to allow for comparative statics and by column the endogenous variables under the different policies.

		Constant tax policy				Constant debt policy		
		b	n	s		τ	n	s
Slow population growth	w	+	+	+		-	+	$\pm$
	R	-	0	0		+	-	-
	q	-	-	0		+	-	-
Fast population growth	w	+	+	+		+	-	-
	R	-	0	0		-	+	+
	q	-	-	0		-	+	+

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To the extent that most Western countries, especially Italy, are currently facing decreasing fertility and saving and increasing debt, on the one hand, and, on the other hand, extremely low interest rates, some policy implications may be derived. Since a possible increase of such low interest rates could be expected, an interesting prediction of the model is that public debt would be reduced regardless of the regime types, under constant tax policy. By contrast, the effects on fertility and saving would depend crucially on the regime types. Of course, it is an empirical matter whether it is possible in practice to design a sufficiently well-targeted constant tax or debt policy. Moreover, we note that the paper's findings are only preliminary because we have chosen indeed a very simple neoclassical model. For example, we have not considered i) the endogeneity of factor prices, ii) alternative approaches to the demand for children and/or the cost of children, iii) the relationship between growth and mortality. Extensions in these directions, however, are left for further research.

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Appendices

A   Proofs of propositions in Section 3

Proof of Proposition 1. The parts are proved separately. The general idea is the following: to show that $(x, f_1(x), f_2(x), f_3(x)) \geq 0$ iff $x \in [a, b]$ we have to show:

- *sufficiency*: if $x \in [a, b]$ then $(x, f_1(x), f_2(x), f_3(x)) \geq 0$ and

- *necessity*: if $x < a$ or $x > b$ then $(x, f_1(x), f_2(x), f_3(x)) \geq 0$ is not satisfied.

a) Assume **A1)**, **A2)** and take the case $R > \frac{f}{p \cdot q}$ i.e. $R \cdot p \cdot q > f$

   i. *Sufficiency*: by eq. (11b) $n(\tau) = \frac{f \cdot (w - \tau)}{q \cdot w \cdot p}$ and by eq. (11c) $s(\tau) = \frac{d \cdot (w - \tau)}{p}$ and they are both non-negative for $0 \leq \tau \leq w$. Furthermore $R \cdot p \cdot q > f \Rightarrow R \cdot p \cdot q \cdot w - f \cdot (w - \tau) > 0$ for any $\tau \in [0, w]$; hence by eq. (11a) we have $b(0) = 0$ and $\frac{db(\tau)}{d\tau} = \frac{p \cdot q \cdot (R \cdot p \cdot q - f) \cdot w^2}{(R \cdot p \cdot q \cdot w - f \cdot (w - \tau))^2} > 0$ so that also $b(\tau) \geq 0$ for $0 \leq \tau \leq w$ i.e. **R1)** is satisfied.

   Necessity: $\tau < 0$ must be ruled out by the first condition in **R1)** and $\tau > w$ implies negative values for $n(\tau)$ and $s(\tau)$ (see eqs. (11b) and (11c)) so that **R1)** would not be satisfied.

   Finally simple substitution in eq. (11a) shows that the value of $b(\tau)$ at the upper bound of the relevant interval is $b(w) = \frac{w}{R}$.

ii. *Sufficiency*: $w \geq R \cdot b$ and $w \geq R \cdot b$ entails $w > \frac{f}{p \cdot q} \cdot b \Rightarrow p \cdot q \cdot w - b \cdot f > 0$. It follows that ($\alpha$) in eq. (12a) $\tau(b) = \frac{b \cdot w \cdot (R \cdot p \cdot q - f)}{p \cdot q \cdot w - b \cdot f} \geq 0$, ($\beta$) in eq. (12b) $n(b) = \frac{f \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} \geq 0$ and (γ) in eq. (12c) $s(b) = \frac{d \cdot q \cdot w \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} \geq 0$ since in all three cases the denominator is positive and the numerator is non negative. Hence **R2)** is satisfied.

Necessity: $b < 0$ is incompatible with the first condition in **R2)**; $b > \frac{w}{R}$, i.e. $b \cdot R > w$, is incompatible with the non negativity of $n(b)$ and $s(b)$ since the numerators in both eqs. (12b) and (12c) would be negative in face of a positive denominator.

Finally simple substitution in eq. (12a) shows that the value of $\tau(b)$ at the upper bound of the relevant interval is $\tau\left(\frac{w}{R}\right) = w$.

b) Now assume **A1)**, **A2)** and take the case $R = \frac{f}{p \cdot q}$, i.e. $R \cdot p \cdot q = f$.

i. *Sufficiency*: by eq. (11b) $n(\tau) = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w}$ and by eq. (11c) $s(\tau) = \frac{d \cdot (w - \tau)}{p}$ and they are both non-negative for $0 \leq \tau \leq w$. Furthermore, by eq. (11a) and the fact that $R = \frac{f}{p \cdot q}$, we have $b(\tau) = \frac{p \cdot q \cdot w \cdot \tau}{R \cdot p \cdot q \cdot w - f \cdot (w - \tau)} = \frac{p \cdot q \cdot w \cdot \tau}{p \cdot q \cdot \left(R \cdot w - \frac{f}{p \cdot q} \cdot (w - \tau)\right)} = \frac{w \cdot \tau}{R \cdot w - R \cdot (w - \tau)} = \frac{w \cdot \tau}{R \cdot \tau} = \frac{w}{R}$ for any τ so that $b(\tau) \geq 0$ as well for $0 \leq \tau \leq w$ and **R1)** is satisfied.

Necessity: $\tau < 0$ must be ruled out by the first condition in **R1)** and $\tau > w$ implies negative values for $n(\tau)$ and $s(\tau)$ (see eqs. (11b) and (11c)) so that **R1)** would not be satisfied.

By point (i) above $b(\tau) = \frac{w}{R} > 0$ for any $\tau \in [0, w]$.

ii. *Sufficiency*: since in this case $R \cdot p \cdot q - f = 0$, we have ($\alpha$) $\tau(b) = \frac{b \cdot w \cdot (R \cdot p \cdot q - f)}{p \cdot q \cdot w - b \cdot f} = \frac{b \cdot w \cdot (R \cdot p \cdot q - f)}{p \cdot q \cdot \left(w - b \cdot \frac{f}{p \cdot q}\right)} = \frac{b \cdot w \cdot (R \cdot p \cdot q - f)}{p \cdot q \cdot (w - b \cdot R)} = 0$ for any $b \geq 0$; (β) $n(b) = \frac{f \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} = \frac{f \cdot (w - R \cdot b)}{p \cdot q \cdot (w - b \cdot R)} = \frac{f}{p \cdot q} > 0$ for any $b \geq 0$; (γ) $s(b) = \frac{d \cdot q \cdot w \cdot (w - R \cdot b)}{p \cdot q \cdot w - b \cdot f} = \frac{d \cdot q \cdot w \cdot (w - R \cdot b)}{p \cdot q \cdot (w - b \cdot R)} = \frac{d \cdot w}{p} > 0$ for any $b \geq 0$. Hence **R2)** is satisfied.

Necessity: $b < 0$ is incompatible with the first condition in **R2)**; $\tau(b)$, $n(b)$ and $s(b)$ are independent of b when $R = \frac{f}{p \cdot q}$ and non negative, so that **R2)** holds for any non-negative value of b.

c) Finally assume **A1)**, **A2)** and $R < \frac{f}{p \cdot q}$.

i. *Sufficiency*: $R < \frac{f}{p \cdot q}$, i.e. $f > R \cdot p \cdot q$, and **A1)**-**A2)** entail $w > \frac{(f - R \cdot p \cdot q)}{f} \cdot w > 0$. Then by eq. (11b) $n(\tau) = \frac{f \cdot (w - \tau)}{p \cdot q \cdot w}$ and by eq. (11c) $s(\tau) = \frac{d \cdot (w - \tau)}{p}$ and they are both non-negative for $\frac{(f - R \cdot p \cdot q) \cdot w}{f} \leq \tau \leq w$. As to $b(\tau)$ observe that $b(w) = \frac{w}{R}$ and $\frac{db(\tau)}{d\tau} = \frac{p \cdot q \cdot (R \cdot p \cdot q - f) \cdot w^2}{(R \cdot p \cdot q \cdot w - f \cdot (w - \tau))^2} < 0$ so that $b(\tau) \geq \frac{w}{R} > 0$ for $\frac{(f - R \cdot p \cdot q) \cdot w}{f} \leq \tau \leq w$. Hence **R1)** is satisfied.

Necessity: $\tau < 0$ must be ruled out by the first condition in **R1)** and $\tau > w$ implies negative values for $n(\tau)$ and $s(\tau)$ (see eqs. (11b) and (11c)) so that **R1)** would not be satisfied.

ii. *Sufficiency*: simple substitutions into eq. (12a) show that $\tau\left(\frac{w}{R}\right) = w$, $\lim_{b\to\infty}\tau(b) = \frac{(f - R \cdot p \cdot q)\cdot w}{f} > 0$ and $\frac{d\tau(b)}{db} = \frac{p \cdot q \cdot (R \cdot p \cdot q - f)\cdot w^2}{(p \cdot q \cdot w - b \cdot f)^2} < 0$ so that $0 < \frac{(f - R \cdot p \cdot q)\cdot w}{f} \leq \tau(b) \leq w$ for any $b \geq \frac{w}{R}$. Consequently $\forall b \geq 0$ $n(b) \geq 0$ (by (12b)) and $s(b) \geq 0$ (by (12c)) as in **R2)**, so that $b \geq \frac{w}{R}$ is indeed sufficient for all relations to be non-negative.

Necessity: to show necessity we have to show that for $b < \frac{w}{R}$ some of the relations become negative. Indeed, as we have shown above, $\tau(b)$ is monotonically decreasing in b and $\tau\left(\frac{w}{R}\right) = w$ so that $\tau(b) > w$ for $b < \frac{w}{R}$ implying $n(b) < 0$; hence $b < \frac{w}{R}$ is necessary for **R2)**.

This completes the proof. $\qquad\square$

Proof of Corollary 1. In the case of a constant lump sum tax policy, in equilibrium we have $R - n(\tau) = \frac{\tau}{b(\tau)} = \frac{R \cdot p \cdot q \cdot w - f \cdot (w - \tau)}{p \cdot q \cdot w}$. Suppose $R > \frac{f}{p \cdot q}$; then **R1)** requires $0 \leq \tau \leq w$ (see the proof of Proposition 1a)i.) and the numerator is positive so that $R - n(\tau) > 0$. Suppose now $R < \frac{f}{p \cdot q}$; then **R1)** requires $w \geq \tau \geq \frac{(f - R \cdot p \cdot q)\cdot w}{f} > 0$ (see the proof of Proposition 1c)i.) and it is easy to verify that the numerator is positive at the upper bound of variation ($\tau = w$) and vanishes at the lower bound $\tau = \frac{(f - R \cdot p \cdot q)\cdot w}{f}$, so that $R - n(\tau) \geq 0$.

In the case of a debt stabilizing lump sum tax policy the proof is at all similar and is not repeated here for the sake of brevity. $\qquad\square$

Proof of Lemma 1. First we show that $R - n(\tau) \gtrless \frac{R \cdot \tau}{w} \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$. Substituting $n(\tau)$ from (11b) into the definition of the golden rule gap we have

$$R - n(\tau) = R - \frac{f \cdot (w - \tau)}{p \cdot q \cdot w} = R \cdot \left[1 - \frac{f}{\frac{p \cdot q}{R}} \cdot \frac{w - \tau}{w}\right]$$

Notice that $\frac{f}{\frac{p \cdot q}{R}} \lessgtr 1 \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$ and hence

$$R - n(\tau) = R \cdot \left[1 - \frac{f}{\frac{p \cdot q}{R}} \cdot \frac{w - \tau}{w}\right] \gtrless R \cdot \left[1 - \frac{w - \tau}{w}\right] = \frac{R \cdot \tau}{w} \Leftrightarrow R \gtrless \frac{f}{p \cdot q} \qquad (13)$$

Now we show that $\eta_\tau(R - n(\tau)) \lessgtr 1 \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$. By definition $\eta_\tau(R - n(\tau)) = \frac{\partial(R - n(\tau))}{\partial \tau} \cdot \frac{\tau}{R - n(\tau)}$; by (11b) we have (a) $\frac{\partial(R - n(\tau))}{\partial \tau} = \frac{f}{p \cdot q \cdot w}$ and by (13) we have (b) $\frac{\tau}{R - n(\tau)} \lessgtr \frac{\tau}{\frac{R \cdot \tau}{w}} = \frac{w}{R} \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$. Combining (a) and (b) we get

$$\eta_\tau(R - n(\tau)) \lessgtr \frac{f}{p \cdot q \cdot w} \cdot \frac{w}{R} = \frac{f}{p \cdot q} \cdot \frac{1}{R} \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$$

Notice that if $R > \frac{f}{p \cdot q}$ then $\frac{f}{p \cdot q} \cdot \frac{1}{R} < 1$ and hence $\eta_\tau (R - n(\tau)) < \frac{f}{p \cdot q} \cdot \frac{1}{R} < 1$. If instead $R < \frac{f}{p \cdot q}$ then $\frac{f}{p \cdot q} \cdot \frac{1}{R} > 1$ and hence $\eta_\tau (R - n(\tau)) > \frac{f}{p \cdot q} \cdot \frac{1}{R} > 1$ thus proving the Lemma. $\qquad \square$

Proof of Proposition 2. The two cases are examined separately. The first deals with system (11) in which $b(\tau) = \frac{\tau}{R - n(\tau)}$. Simple differentiation gets

$$
\begin{aligned}
\frac{\partial b(\tau)}{\partial \tau} = \frac{\partial \frac{\tau}{R - n(\tau)}}{\partial \tau} \quad &= \quad \frac{(R - n(\tau)) + \tau \cdot \frac{\partial n(\tau)}{\partial \tau}}{(R - n(\tau))^2} \\
&= \quad \frac{1}{R - n(\tau)} \cdot \left(1 + \frac{\tau}{R - n(\tau)} \cdot \frac{\partial n(\tau)}{\partial \tau}\right) \\
&= \quad \frac{1}{R - n(\tau)} \cdot \left(1 - \eta_\tau (R - n(\tau))\right)
\end{aligned}
$$

Since $R - n(\tau) \geq 0$ by Corollary 1, the sign of $\frac{\partial b(\tau)}{\partial \tau}$ is the sign of $\left(1 - \eta_\tau (R - n(\tau))\right)$ and by Lemma 1 we have $\frac{\partial b(\tau)}{\partial \tau} \gtrless 0 \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$.

The second statement follows immediately by the fact that the functions $\tau(b)$ and $b(\tau)$ are one the inverse of the other so that $\frac{d\tau(b)}{db} \gtrless 0 \Leftrightarrow R \gtrless \frac{f}{p \cdot q}$. $\qquad \square$

B   Fertility and changes in income

From eq. (10b) fertility choices are governed by the relation $n = \frac{f \cdot (w - \tau)}{q \cdot w \cdot p}$ which shows the twofold dependency of n on w: directly (at the numerator) reflecting individual positive attitude toward spending out of income for child rearing and indirectly (at the denominator) where the term $q \cdot w$ stands for the cost of child rearing, so that an increase in w ends up in a higher cost. The effect of a change in w on the choice of n can be analysed through the Slutsky equation

$$
\begin{aligned}
\frac{\partial n}{\partial w} = \left(\frac{\partial n}{\partial w}\right)^{SE} - \left(\frac{\partial n}{\partial w} \cdot n\right)^{IE} \quad &= \quad -\frac{f^2 \cdot (w - \tau) - f \cdot \tau \cdot p}{q \cdot p^2 \cdot w^2} - \left(-\frac{f^2 \cdot (w - \tau)}{q \cdot p^2 \cdot w^2}\right) \\
&= \quad 0 + \frac{f \cdot \tau \cdot p}{q \cdot p^2 \cdot w^2}
\end{aligned}
$$

As it can be expected given the Cobb-Douglas form of the utility function, in absence of taxation ($\tau = 0$) the substitution and the income effect cancel out and $\frac{\partial n}{\partial w} = 0$. If $\tau \neq 0$, then the income effect prevails by that part directly linked to taxation and we have

$$
\frac{\partial n}{\partial w} = \frac{f \cdot \tau}{q \cdot p \cdot w^2} \gtrless 0 \Leftrightarrow \tau \gtrless 0.
$$

C Derivation of formulas for comparative statics

$\dfrac{\partial b(\tau)}{\partial w} = \dfrac{f \cdot p \cdot q \cdot \tau^2}{\left(w \cdot p \cdot q \cdot \left(\frac{f}{p \cdot q} - R\right) - f \cdot \tau\right)^2} \geq 0$	By **A1)** and **A2)**
$\dfrac{\partial n(\tau)}{\partial w} = \dfrac{f \cdot \tau}{p \cdot q \cdot w^2} \geq 0$	By **A1)**, **A2)** and **R1)**
$\dfrac{\partial s(\tau)}{\partial w} = \dfrac{d}{p} > 0$	By **A2)**
$\dfrac{\partial \tau(b)}{\partial w} = -\dfrac{b^2 \cdot f \cdot p \cdot q \cdot \left(R - \frac{f}{p \cdot q}\right)}{(b \cdot f - p \cdot q \cdot w)^2} \begin{cases} \leq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \geq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	By **A1)** and **A2)**
$\dfrac{\partial n(b)}{\partial w} = \dfrac{b \cdot f \cdot p \cdot q \cdot \left(R - \frac{f}{p \cdot q}\right)}{(b \cdot f - p \cdot q \cdot w)^2} \begin{cases} \geq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \leq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	By **A1)**, **A2)** and **R2)**
$\dfrac{\partial s(b)}{\partial w} \begin{cases} > 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \leq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \quad and \quad b \in \left[\frac{w}{R}, \frac{w}{R} \cdot \left(1 + \sqrt{\frac{R \cdot f}{p \cdot q} - 1}\right)\right] \\ > 0 & \text{if} \quad R < \frac{f}{p \cdot q} \quad and \quad b > \frac{w}{R} \cdot \left(1 + \sqrt{\frac{R \cdot f}{p \cdot q} - 1}\right) \end{cases}$	See Proof 1 below
$\dfrac{\partial b(\tau)}{\partial R} = \dfrac{p^2 \cdot q^2 \cdot w^2 \cdot \tau}{(R \cdot p \cdot q \cdot w - f \cdot (w - \tau))^2} \geq 0$	By **R1)**
$\dfrac{\partial n(\tau)}{\partial R} = 0$	By eq. (11b)
$\dfrac{\partial s(\tau)}{\partial R} = 0$	By eq. (11c)
$\dfrac{\partial \tau(b)}{\mathrm{d}R} = \dfrac{b \cdot p \cdot q \cdot w}{p \cdot q \cdot w - b \cdot f} \begin{cases} \geq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \leq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	See Proof 2 below
$\dfrac{\partial n(b)}{\partial R} = \dfrac{b \cdot f}{b \cdot f - p \cdot q \cdot w} \begin{cases} \leq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \geq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	See Proof 2 below
$\dfrac{\partial s(b)}{\mathrm{d}R} = \dfrac{b \cdot p \cdot q \cdot w}{b \cdot f - p \cdot q \cdot w} \begin{cases} \leq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \geq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	See Proof 2 below
$\dfrac{\partial b(\tau)}{\partial q} = -\dfrac{f \cdot p \cdot \tau \cdot w \cdot (w - \tau)}{(R \cdot p \cdot q \cdot w - f \cdot (w - \tau))^2} \geq 0$	By **A1)**, **A2)** and Proposition 1
$\dfrac{\partial n(\tau)}{\partial q} = -\dfrac{f \cdot (w - \tau)}{p \cdot q^2 \cdot w} \leq 0$	By **A1)**, **A2)** and Proposition 1
$\dfrac{\partial s(\tau)}{\partial q} = 0$	By eq. (11c)
$\dfrac{\partial \tau(b)}{\partial q} = \dfrac{b \cdot f \cdot p \cdot w \cdot (w - R \cdot b)}{(b \cdot f - p \cdot q \cdot w)^2} \begin{cases} \geq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \leq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	By **A1)**, **A2)** and Proposition 1
$\dfrac{\partial n(b)}{\partial q} = -\dfrac{f \cdot p \cdot w \cdot (w - R \cdot b)}{(b \cdot f - p \cdot q \cdot w)^2} \begin{cases} \leq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \geq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	By **A1)**, **A2)** and Proposition 1
$\dfrac{\partial s(b)}{\partial q} = -\dfrac{b \cdot d \cdot f \cdot w \cdot (w - R \cdot b)}{(b \cdot f - p \cdot q \cdot w)^2} \begin{cases} \leq 0 & \text{if} \quad R > \frac{f}{p \cdot q} \\ \geq 0 & \text{if} \quad R < \frac{f}{p \cdot q} \end{cases}$	By **A1)**, **A2)** and Proposition 1

Proof 1. Differentiating $s(b)$ in eq. (12c) with respect to w gets

$$\frac{\partial s(b)}{\partial w} = d \cdot p \cdot q^2 \cdot \frac{w^2 + \frac{f \cdot b}{p \cdot q} \cdot (R \cdot b - 2 \cdot w)}{(b \cdot f - p \cdot q \cdot w)^2} \tag{14}$$

Assume **A1)** and **A2)** and examine (14) separately in the two regimes.

- $R > \frac{f}{p \cdot q}$ which, by Proposition 1, entails $0 \le b \le \frac{w}{R}$. Then

$$
\begin{aligned}
\left.\frac{\partial s(b)}{\partial w}\right|_{b=0} &= \frac{d}{p} > 0 \\
\left.\frac{\partial s(b)}{\partial w}\right|_{b=\frac{w}{R}} &= \frac{R \cdot p \cdot q}{R \cdot p \cdot q - f} > 0 \\
\frac{\partial \frac{\partial s(b)}{\partial w}}{\partial b} &= \frac{2 \cdot b \cdot d \cdot f \cdot q \cdot w \cdot (f - R \cdot p \cdot q)}{(b \cdot f - p \cdot q \cdot w)^3} > 0
\end{aligned}
$$

where the last inequality follows from the fact that in this regime the numerator is negative and the denominator is negative as well since $w \ge R \cdot b$ and $R > \frac{f}{p \cdot q}$ entail $w > \frac{f}{p \cdot q} \cdot b$ i.e. $p \cdot q \cdot w > b \cdot f$. Therefore $\frac{\partial s(b)}{\partial w}$ is monotonically increasing from $0 < \frac{d}{p} < 1$ to $\frac{R \cdot p \cdot q}{R \cdot p \cdot q - f} > 1$ for b in the interval $\left[0, \frac{w}{R}\right]$.

- $R < \frac{f}{p \cdot q}$ which, by Proposition 1, entails $b > \frac{w}{R}$. Then in this case

$$
\begin{aligned}
\lim_{b \to \frac{w}{R}} \frac{\partial s(b)}{\partial w} &= \frac{R \cdot p \cdot q}{R \cdot p \cdot q - f} < 0 \\
\lim_{b \to \infty} \frac{\partial s(b)}{\partial w} &= \frac{R \cdot d \cdot q}{f} > 0 \\
\frac{\partial \frac{\partial s(b)}{\partial w}}{\partial b} &= \frac{2 \cdot b \cdot d \cdot f \cdot q \cdot w \cdot (f - R \cdot p \cdot q)}{(b \cdot f - p \cdot q \cdot w)^3} > 0
\end{aligned}
$$

where the last inequality follows from the fact that in this regime both the numerator and the denominator are positive. Therefore $\frac{\partial s(b)}{\partial w}$ is monotonically increasing from $\frac{R \cdot p \cdot q}{R \cdot p \cdot q - f} < 0$, when b tends to its minimum value in this regime, to $\frac{R \cdot d \cdot q}{f} > 0$ when b becomes arbitrarily high and vanishes when $b = \frac{w}{R} \cdot \left(1 + \sqrt{\frac{R \cdot f}{p \cdot q} - 1}\right)$ as simple calculations can show.

$\square$

Proof 2. Assuming **A1)**, **A1)** and **R2)**, the sign of $\frac{b \cdot p \cdot q \cdot w}{p \cdot q \cdot w - b \cdot f}$ is the sign of the denominator and in the two regimes we have:

- $R > \frac{f}{p \cdot q}$ entails $b \ge \frac{w}{R}$ or $w \ge R \cdot b$. Hence $w \ge R \cdot b$ and $R > \frac{f}{p \cdot q}$ together entail $w > \frac{f}{p \cdot q} \cdot b$ i.e. $p \cdot q \cdot w > b \cdot f$.

- $R < \frac{f}{p \cdot q}$ entails $b > \frac{w}{R}$ or $w < R \cdot b$. Hence $w < R \cdot b$ and $R < \frac{f}{p \cdot q}$ together entail $w < \frac{f}{p \cdot q} \cdot b$ i.e. $p \cdot q \cdot w < b \cdot f$.

$\square$