

ON SELMER GROUPS OF ABELIAN VARIETIES OVER ℓ -ADIC LIE EXTENSIONS OF GLOBAL FUNCTION FIELDS

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ABSTRACT. Let F be a global function field of characteristic $p > 0$ and A/F an abelian variety. Let K/F be an ℓ -adic Lie extension ($\ell \neq p$) unramified outside a finite set of primes S and such that $\text{Gal}(K/F)$ has no elements of order ℓ . We shall prove that, under certain conditions, $\text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.

Keywords: Selmer groups, abelian varieties, function fields, pseudo-null modules.

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1. INTRODUCTION

Let G be a compact ℓ -adic Lie group and $\Lambda(G)$ its associated Iwasawa algebra. A crucial theme in Iwasawa theory is the study of finitely generated $\Lambda(G)$ -modules and their structure, up to “pseudo-isomorphism”. When $G \simeq \mathbb{Z}_\ell^d$ for some integer $d \geq 1$, the structure theory for finitely generated $\Lambda(G)$ -modules is well known (see, e.g., [B]). For a nonabelian G , which is the case we are interested in, studying this topic is possible thanks to an appropriate definition of the concept of “pseudo-null” for modules over $\Lambda(G)$ due to Venjakob (see [V]).

Let F be a global function field of transcendence degree one over its constant field \mathbb{F}_q , where q is a power of a fixed prime $p \in \mathbb{Z}$, and K a Galois extension of F unramified outside a finite set of primes S and such that $G = \text{Gal}(K/F)$ is an infinite ℓ -adic Lie group with $\ell \neq p$. Let A/F be an abelian variety: the structure of $\mathcal{S} := \text{Sel}_A(K)_\ell^\vee$ (the Pontrjagin dual of the Selmer group of A over K) as a $\Lambda(G)$ -module has been extensively studied, for example, in [BL], [BL2] and [T] (see also the short survey in [BBL, Section 2] and the references there) for the abelian case, and in [OT], [W] and [BV] for the noncommutative one (these results cover also the case $\ell = p$). In most cases \mathcal{S} has been proved to be a finitely generated (sometimes torsion) $\Lambda(G)$ -module and here we shall deal with the presence of nontrivial pseudo-null submodules in \mathcal{S} . For the number field setting and $K = F(A[\ell^\infty])$, this issue was studied by Ochi and Venjakob ([OV, Theorem 5.1]) when A is an elliptic curve, and by Ochi for a general abelian variety in [O] (see also [HV] and [HO] for analogous results and/or alternative proofs).

In Sections 2 and 3 we give a brief description of the objects we will work with and of the main tools we shall need, adapting some of the techniques of [OV] to our function field setting and to a general ℓ -adic Lie extension (one of the main difference being the triviality of the image of the local Kummer maps).

In Section 4 we will prove the following

Theorem 1.1 (Theorem 4.1). *Let $G = \text{Gal}(K/F)$ be an ℓ -adic Lie group without elements of order ℓ and of positive dimension $d \geq 3$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and the map ψ (induced by restriction)*

$$\text{Sel}_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \text{Coind}_G^{G_v} H^1(K_w, A)[\ell^\infty]$$

is surjective, then $\text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.

For the case $d = 2$ we need more restrictive hypotheses, in particular we have the following

Proposition 1.2 (Proposition 4.3). *Let $G = \text{Gal}(K/F)$ be an ℓ -adic Lie group without elements of order ℓ and of dimension $d \geq 2$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and $\text{cd}_\ell(G_v) = 2$ for any $v \in S$, then $\text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.*

A few considerations and particular cases for the vanishing of $H^2(F_S/K, A[\ell^\infty])$ are included at the end of Section 4.

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2. SETTING AND NOTATIONS

Here we fix notations and conventions that will be used through the paper.

2.1. Fields and extensions. Let F be a global function field of transcendence degree one over its constant field $\mathbb{F}_F = \mathbb{F}_q$, where q is a power of a fixed prime $p \in \mathbb{Z}$. We put \overline{F} for an algebraic closure of F .

For any algebraic extension L/F , let \mathfrak{M}_L be the set of places of L : for any $v \in \mathfrak{M}_L$ we let L_v be the completion of L at v . Let S be a finite nonempty subset of \mathfrak{M}_F and let F_S be the maximal Galois extension of F unramified outside S with $G_S(F) := \text{Gal}(F_S/F)$. Put $\mathcal{O}_{L,S}$ as the ring of S -integers of L and \mathcal{O}_S^\times as the units of $\mathcal{O}_S = \bigcup_{L \subset F_S} \mathcal{O}_{L,S}$. Finally, $\text{Cl}_S(L)$ denotes

the S -ideal class group of $\mathcal{O}_{L,S}$: since S is nonempty, $\text{Cl}_S(L)$ is finite.

For any place $v \in \mathfrak{M}_F$ we choose (and fix) an embedding $\overline{F} \hookrightarrow \overline{F}_v$, in order to get a restriction map $G_{F_v} := \text{Gal}(\overline{F}_v/F_v) \hookrightarrow G_F := \text{Gal}(\overline{F}/F)$.

We will deal with ℓ -adic Lie extensions K/F , i.e., Galois extensions with Galois group an ℓ -adic Lie group with $\ell \neq p$. We always assume that our extensions are unramified outside a finite set S of primes of \mathfrak{M}_F .

In what follows $\text{Gal}(K/F)$ is an ℓ -adic Lie group *without points of order ℓ* , then it has finite ℓ -cohomological dimension, which is equal to its dimension as an ℓ -adic Lie group ([Se, Corollaire (1) p. 413]).

2.2. Ext and duals. For any ℓ -adic Lie group G we denote by

$$\Lambda(G) = \mathbb{Z}_\ell[[G]] := \varprojlim_U \mathbb{Z}_\ell[G/U]$$

the associated *Iwasawa algebra* (the limit is on the open normal subgroups of G). From Lazard's work (see [L]), we know that $\Lambda(G)$ is Noetherian and, if G is pro- ℓ and has no elements of order ℓ , then $\Lambda(G)$ is an integral domain.

For a $\Lambda(G)$ -module M we consider the extension groups

$$\text{E}^i(M) := \text{Ext}_{\Lambda(G)}^i(M, \Lambda(G))$$

for any integer i and put $\text{E}^i(M) = 0$ for $i < 0$ by convention.

Since in our applications G comes from a Galois extension, we denote with G_v the decomposition group of $v \in \mathfrak{M}_F$ for some prime $w|v$, $w \in \mathfrak{M}_L$, and we use the notation

$$\text{E}_v^i(M) := \text{Ext}_{\Lambda(G_v)}^i(M, \Lambda(G_v)) .$$

Let H be a closed subgroup of G . For every $\Lambda(H)$ -module N we consider the $\Lambda(G)$ -modules

$$\text{Coind}_G^H(N) := \text{Map}_{\Lambda(H)}(\Lambda(G), N) \quad \text{and} \quad \text{Ind}_H^G(N) := N \otimes_{\Lambda(H)} \Lambda(G) \text{ }^1.$$

¹We use the notations of [OV], some texts, e.g. [NSW], switch the definitions of $\text{Ind}_G^H(N)$ and $\text{Coind}_G^H(N)$.

For a $\Lambda(G)$ -module M , we denote its Pontrjagin dual by $M^\vee := \text{Hom}_{\text{cont}}(M, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In this paper, M will be a (mostly discrete) topological \mathbb{Z}_ℓ -module, so M^\vee has a natural structure of \mathbb{Z}_ℓ -module.

If M is a discrete $G_S(F)$ -module, finitely generated over \mathbb{Z} and with no p -torsion, in duality theorems we shall use also the dual $G_S(F)$ -module of M , i.e.,

$$M' := \text{Hom}(M, \mathcal{O}_S^\times) (= \text{Hom}(M, \boldsymbol{\mu}) \text{ if } M \text{ is finite}) .$$

2.3. Selmer groups. Let A be an abelian variety of dimension g defined over F : we denote by A^t its dual abelian variety. For any positive integer n we let $A[n]$ be the scheme of n -torsion points and, for any prime ℓ , we put $A[\ell^\infty] := \varinjlim A[\ell^n]$.

The *local Kummer maps* (for any $w \in \mathfrak{M}_L$)

$$\kappa_w : A(L_w) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow \varinjlim_n H^1(L_w, A[\ell^n]) := H^1(L_w, A[\ell^\infty])$$

(arising from the cohomology of the exact sequence $A[\ell^n] \hookrightarrow A \xrightarrow{\ell^n} A$) enable one to define the ℓ -part of the Selmer group of A over L as

$$\text{Sel}_A(L)_\ell = \text{Ker} \left\{ H^1(L, A[\ell^\infty]) \rightarrow \prod_{w \in \mathfrak{M}_L} H^1(L_w, A[\ell^\infty]) / \text{Im } \kappa_w \right\}$$

(where the map is the product of the natural restrictions between cohomology groups).

For infinite extensions \mathcal{L}/F the Selmer group $\text{Sel}_A(\mathcal{L})_\ell$ is defined, as usual, via direct limits.

Since $\ell \neq p$, the $\text{Im } \kappa_w$ are trivial and, assuming that S contains also all primes of bad reduction for A , we have the following equivalent

Definition 2.1. The ℓ -part of the Selmer group of A over L is

$$\text{Sel}_A(L)_\ell = \text{Ker} \left\{ H^1(F_S/L, A[\ell^\infty](F_S)) \rightarrow \bigoplus_S \text{Coind}_G^{G_v} H^1(L_w, A[\ell^\infty]) \right\} .$$

Letting L vary through subextensions of K/F , the groups $\text{Sel}_A(L)_\ell$ admit natural actions by \mathbb{Z}_ℓ (because of $A[\ell^\infty]$) and by $G = \text{Gal}(K/F)$. Hence they are modules over the Iwasawa algebra $\Lambda(G)$.

3. HOMOTOPY THEORY AND PSEUDO-NULLITY

We briefly recall the basic definitions for pseudo-null modules over a non-commutative Iwasawa algebra: a comprehensive reference is [V].

3.1. Pseudo-null $\Lambda(G)$ -modules. Let G be an ℓ -adic Lie group without ℓ -torsion, then $\Lambda(G)$ is an Auslander regular ring of finite global dimension $\mathfrak{d} = \text{cd}_\ell(G) + 1$ ([V, Theorem 3.26], cd_ℓ denotes the ℓ -cohomological dimension).

For any finitely generated $\Lambda(G)$ -module M , there is a canonical filtration

$$T_0(M) \subseteq T_1(M) \subseteq \cdots \subseteq T_{\mathfrak{d}-1}(M) \subseteq T_{\mathfrak{d}}(M) = M .$$

Definition 3.1. We say that a $\Lambda(G)$ -module M is *pseudo-null* if

$$\delta(M) := \min\{i \mid T_i(M) = M\} \leq \mathfrak{d} - 2 .$$

The quantity $\delta(M)$, called the δ -dimension of the $\Lambda(G)$ -module M , is used along with the *grade* of M , that is

$$j(M) := \min\{i \mid E^i(M) \neq 0\}.$$

As $j(M) + \delta(M) = \mathfrak{d}$ ([V, Proposition 3.5 (ii)]) we have that M is a pseudo-null module if and only if $E^0(M) = E^1(M) = 0$.

Since $\delta(T_i(M)) \leq i$ and every $T_i(M)$ is the maximal submodule of M with δ -dimension less or equal to i ([V, Proposition 3.5 (vi) (a)]), only $T_0(M), \dots, T_{\mathfrak{d}-2}(M)$ can be pseudo-null. If $T_0(M) = \dots = T_{\mathfrak{d}-2}(M) = 0$, M does not have any nonzero pseudo-null submodule. This is the case when $E^i E^i(M) = 0 \ \forall i \geq 2$ ([V, Proposition 3.5 (i) (c)]).

3.2. The powerful diagram and its consequences. In [OV, Lemma 4.5] Ochi and Venjakob generalized a result of Jannsen (see [J]) which is very powerful in applications (they call it “powerful diagram”). We provide here the statements we shall need later: for the missing details of the proofs the reader can consult [NSW, Chapter V, Section 5] and/or [OV, Section 4] (those results hold in our setting as well because we work with the $\Lambda(G)$ -module $A[\ell^\infty]$, with $\ell \neq p$).

Replacing, if necessary, F by a finite extension we can (and will) assume that K is contained in the maximal pro- ℓ extension of $F_\infty := F(A[\ell^\infty])$ unramified outside S . Then we have the following

$$\begin{array}{c} F_S \\ | \\ \Omega \\ \swarrow \mathcal{H} \quad \searrow \\ K \quad F_\infty \\ \swarrow \mathcal{G} \quad \searrow \\ F \end{array}$$

where Ω is the maximal pro- ℓ extension of F_∞ contained in F_S . We put $\mathcal{G} = \text{Gal}(\Omega/F)$, $\mathcal{H} = \text{Gal}(\Omega/K)$ and $G = \text{Gal}(K/F)$. The extension F_∞/F will be called the *trivializing extension*.

Tensoring the natural exact sequence $I(\mathcal{G}) \hookrightarrow \Lambda(\mathcal{G}) \twoheadrightarrow \mathbb{Z}_\ell$ with $A[\ell^\infty]^\vee \simeq \mathbb{Z}_\ell^{2g}$, one gets

$$I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee \hookrightarrow \Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee \twoheadrightarrow A[\ell^\infty]^\vee.$$

Since the mid term is projective ([OV, Lemma 4.2]), the previous sequence yields

$$(1) \quad H_1(\mathcal{H}, A[\ell^\infty]^\vee) \hookrightarrow (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \rightarrow (\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \twoheadrightarrow (A[\ell^\infty]^\vee)_{\mathcal{H}}.$$

In order to shorten notations we put:

- $X = H_1(\mathcal{H}, A[\ell^\infty]^\vee)$;
- $Y = (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}}$;
- $J = \text{Ker}\{(\Lambda(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}} \twoheadrightarrow (A[\ell^\infty]^\vee)_{\mathcal{H}}\}.$

So the sequence (1) becomes

$$(2) \quad X \hookrightarrow Y \twoheadrightarrow J.$$

For our purpose it is useful to think of X as $H^1(F_S/K, A[\ell^\infty])^\vee$ (note that $H_1(\mathcal{H}, A[\ell^\infty]^\vee) \simeq H^1(\Omega/K, A[\ell^\infty])^\vee \simeq H^1(F_S/K, A[\ell^\infty])^\vee$).

Let $\mathcal{F}(d)$ denote a free pro- ℓ -group of rank $d = \dim \mathcal{G}$ and denote by \mathcal{N} (resp. \mathcal{R}) the kernel of the natural map $\mathcal{F}(d) \rightarrow \mathcal{G}$ (resp. $\mathcal{F}(d) \rightarrow G$). For any profinite group H , we denote by

$H^{ab}(\ell)$ the maximal pro- ℓ -quotient of the maximal abelian quotient of H . With this notations the powerful diagram reads as

$$(3) \quad \begin{array}{ccccccc} H^2(\mathcal{H}, A[\ell^\infty])^\vee & \hookrightarrow & (H^1(\mathcal{N}^{ab}(\ell), A[\ell^\infty])^{\mathcal{H}})^\vee & \longrightarrow & H^1(\mathcal{R}, A[\ell^\infty]) & \twoheadrightarrow & X \\ \parallel & & \simeq \downarrow & & \downarrow & & \downarrow \\ H^2(\mathcal{H}, A[\ell^\infty])^\vee & \hookrightarrow & (\mathcal{N}^{ab}(\ell) \otimes A[\ell^\infty]^\vee)_{\mathcal{H}} & \longrightarrow & \Lambda(G)^{2gd} & \twoheadrightarrow & Y \\ & & & & \downarrow & & \downarrow \\ & & & & J & \xlongequal{\quad} & J. \end{array}$$

Moreover, since $cd_\ell(\mathcal{G}) \leq 2$ (just use [NSW, Theorem 8.3.17] and work as in [OV, Lemma 4.4, (iv)]), the module $\mathcal{N}^{ab}(\ell) \otimes A[\ell^\infty]^\vee$ is free over $\Lambda(\mathcal{G})$ ([OV, Lemma 4.2]), hence $(\mathcal{N}^{ab}(\ell) \otimes A[\ell^\infty]^\vee)_{\mathcal{H}}$ is projective as a $\Lambda(\mathcal{G}/\mathcal{H}) = \Lambda(G)$ -module. Therefore, if $H^2(F_S/K, A[\ell^\infty]) = 0$, the module Y has projective dimension ≤ 1 . Whenever this is true the definition of J provides the isomorphisms

$$(4) \quad E^i(X) \simeq E^{i+1}(J) \quad \text{and} \quad E^i(J) \simeq E^{i+1}((A[\ell^\infty]^\vee)_{\mathcal{H}}) \quad \forall i \geq 2,$$

which will be repeatedly used in our computations.

We shall need also a “localized” version of the sequence (2). For every $v \in S$ and a $w \in \mathfrak{M}_K$ dividing v , we define

$$X_v = H^1(K_w, A[\ell^\infty])^\vee \quad \text{and} \quad Y_v = (I(\mathcal{G}_v) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v}$$

(with \mathcal{G}_v the decomposition groups of v in \mathcal{G} and $\mathcal{H}_v = \mathcal{H} \cap \mathcal{G}_v$). The exact sequence

$$(5) \quad X_v \hookrightarrow Y_v \twoheadrightarrow J_v$$

fits into the localized version of diagram (3). If K_w is still a local field, then Tate local duality ([NSW, Theorem 7.2.6]) yields

$$H^2(K_w, A[\ell^\infty]) = H^2(K_w, \varprojlim_n A[\ell^n]) \simeq \varprojlim_n H^0(K_w, A^t[\ell^n])^\vee = 0.$$

If K_w is not local, then ℓ^∞ divides the degree of the extension K_w/F_v and $H^2(K_w, A[\ell^\infty]) = 0$ by [NSW, Theorem 7.1.8 (i)]. Therefore Y_v always has projective dimension ≤ 1 and

$$(6) \quad E^i(X_v) \simeq E^{i+1}(J_v) \simeq E^{i+2}((A[\ell^\infty]^\vee)_{\mathcal{H}_v}) \quad \forall i \geq 2.$$

We note that, since $\ell \neq p$, the image of the local Kummer maps is always 0, hence

$$X_v = H^1(K_w, A[\ell^\infty])^\vee = (H^1(K_w, A[\ell^\infty])/Im \kappa_w)^\vee \simeq H^1(K_w, A)[\ell^\infty]^\vee.$$

Then Definition 2.1 for $L = K$ can be written as

$$Sel_A(K)_\ell = Ker \left\{ \psi : X^\vee \longrightarrow \bigoplus_S Coind_G^{G_v} X_v^\vee \right\}$$

and, dualizing, we get a map

$$\psi^\vee : \bigoplus_S Ind_{G_v}^G X_v \longrightarrow X$$

whose cokernel is exactly $\mathcal{S} := Sel_A(K)_\ell^\vee$.

The following result will be fundamental for our computations.

Theorem 3.2 (U. Jannsen). *Let G be an ℓ -adic Lie group without elements of order ℓ and of dimension d . Let M be a $\Lambda(G)$ -module which is finitely generated as \mathbb{Z}_ℓ -module. Then $E^i(M)$ is a finitely generated \mathbb{Z}_ℓ -module and, in particular,*

1. *if M is \mathbb{Z}_ℓ -free, then $E^i(M) = 0$ for any $i \neq d$ and $E^d(M)$ is free;*

2. if M is finite, then $E^i(M) = 0$ for any $i \neq d+1$ and $E^{d+1}(M)$ is finite.

Proof. See [J, Corollary 2.6]. □

Corollary 3.3. *With notations as above:*

1. if $H^2(F_S/K, A[\ell^\infty]) = 0$, then, for $i \geq 2$,

$$E^i(X) \text{ is } \begin{cases} \text{finite} & \text{if } i = d-1 \\ \text{free} & \text{if } i = d-2 \\ 0 & \text{otherwise} \end{cases} ;$$

2. $E_v^i E_v^{i-1}(X_v) = 0$ for $i \geq 3$.

Proof. 1. The hypothesis yields the isomorphism $E^i(X) \simeq E^{i+2}((A[\ell^\infty]^\vee)_{\mathcal{H}})$. Since

$$(A[\ell^\infty]^\vee)_{\mathcal{H}} \simeq (A[\ell^\infty]^{\mathcal{H}})^\vee = A[\ell^\infty](K)^\vee \simeq \mathbb{Z}_\ell^r \oplus \Delta$$

(with $0 \leq r \leq 2g$ and Δ a finite group) and $E^i(\mathbb{Z}_\ell^r \oplus \Delta) = E^i(\mathbb{Z}_\ell^r) \oplus E^i(\Delta)$, the claim follows from Theorem 3.2.

2. Use Theorem 3.2 and the isomorphism in (6). □

Lemma 3.4. *If $H^2(F_S/K, A[\ell^\infty]) = 0$, then there is the following commutative diagram*

$$\begin{array}{ccccc} E^1(Y) & \xrightarrow{g_1} & \bigoplus_S \text{Ind}_{G_v}^G E_v^1(Y_v) & \twoheadrightarrow & \text{Coker}(g_1) \\ \downarrow & & \downarrow & & \downarrow \\ E^1(X) & \xrightarrow{h_1} & \bigoplus_S \text{Ind}_{G_v}^G E_v^1(X_v) & \twoheadrightarrow & \text{Coker}(h_1) \\ \downarrow & & \downarrow & & \downarrow f \\ E^2(J) & \xrightarrow{\bar{g}_1} & \bigoplus_S \text{Ind}_{G_v}^G E_v^2(J_v) & \twoheadrightarrow & \text{Coker}(\bar{g}_1). \end{array}$$

Proof. The inclusions $\mathcal{G}_v \subseteq \mathcal{G}$ and $\mathcal{H}_v \subseteq \mathcal{H}$ induce the maps

$$(I(\mathcal{G}_v) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \rightarrow (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}_v} \rightarrow (I(\mathcal{G}) \otimes_{\mathbb{Z}_\ell} A[\ell^\infty]^\vee)_{\mathcal{H}}.$$

We have a homomorphism of $\Lambda(G)$ -modules $g : \bigoplus_S \text{Ind}_{G_v}^G Y_v \rightarrow Y$ which, restricted to the X_v 's, provides the map $h : \bigoplus_S \text{Ind}_{G_v}^G X_v \rightarrow X$. So we have the following situation

$$(7) \quad \begin{array}{ccc} X & \xleftarrow{h} & \bigoplus_S \text{Ind}_{G_v}^G X_v \\ \downarrow & & \downarrow \\ Y & \xleftarrow{g} & \bigoplus_S \text{Ind}_{G_v}^G Y_v \\ \downarrow & & \downarrow \\ J & \xleftarrow{\bar{g}} & \bigoplus_S \text{Ind}_{G_v}^G J_v \end{array}$$

where \bar{g} is induced by g and the diagram is obviously commutative.

Since Y and the Y_v 's have projective dimension ≤ 1 (i.e., $E^2(Y) = E^2(Y_v) = 0$), the lemma follows by taking Ext in diagram (7) and recalling that, for any $i \geq 0$, $E_v^i(\text{Ind}_{G_v}^G(X_v)) = \text{Ind}_{G_v}^G E_v^i(X_v)$ (see [OV, Lemma 5.5]). □

In the next subsection we are going to describe the structure of $\text{Coker}(g_1)$.

3.3. Homotopy theory and $\text{Coker}(g_1)$. For every finitely generated $\Lambda(G)$ -module M choose a presentation $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of M by projectives and define the *transpose* functor DM by the exactness of the sequence

$$0 \rightarrow E^0(M) \rightarrow E^0(P_0) \rightarrow E^0(P_1) \rightarrow DM \rightarrow 0.$$

Then it can be shown that the functor D is well-defined and one has $D^2 = Id$ (see [J]).

Definition 3.5. Let L be an extension of F contained in F_S . Then we define

$$Z(L) := H^0(F_S/L, \varinjlim_m D_2(A[\ell^m]))^\vee$$

where

$$D_2(A[\ell^m]) = \varinjlim_{F \subset E \subset F_S} (H^2(F_S/E, A[\ell^m]))^\vee$$

and the limit in $\varinjlim_m D_2(A[\ell^m])$ is taken with respect to the ℓ -power map $A[\ell^{m+1}] \xrightarrow{\ell} A[\ell^m]$.

In the same way we define $Z(L)$ for any Galois extension L of F_v .

An alternative description of the module Z is provided by the following

Lemma 3.6. *Let K be a fixed extension of F contained in F_S and K_w its completion for some $w|v \in S$. Then*

$$Z(K) \simeq \varinjlim_{F \subset L \subset K} H^2(F_S/L, T_\ell(A)) \quad \text{and} \quad Z(K_w) \simeq \varinjlim_{F_v \subset L \subset K_w} H^2(L, T_\ell(A)) .$$

Proof. Global case. For any global field L , let

$$\text{III}^i(F_S/L, A[\ell^\infty]) := \text{Ker} \left\{ H^i(F_S/L, A[\ell^\infty]) \rightarrow \bigoplus_S H^i(L_w, A[\ell^\infty]) \right\} .$$

We have already seen that $H^2(L_w, A[\ell^\infty]) = 0$, hence $H^2(F_S/L, A[\ell^\infty]) = \text{III}^2(F_S/L, A[\ell^\infty])$. Using the pairing of [M, Ch. I, Proposition 6.9], we get

$$\begin{aligned} Z(K) &= H^0(F_S/K, \varinjlim_m \varinjlim_{F \subset L \subset F_S} \text{III}^2(F_S/L, A[\ell^m]))^\vee \\ &= H^0(F_S/K, \varinjlim_m \varinjlim_{F \subset L \subset F_S} \text{III}^0(F_S/L, A^t[\ell^m]))^\vee \\ &= (\varinjlim_m \varinjlim_{F \subset L \subset F_S} \text{III}^0(F_S/L, A^t[\ell^m])^{\text{Gal}(F_S/K)})^\vee \\ &= (\varinjlim_m \varinjlim_{F \subset L \subset K} \text{III}^0(F_S/L, A^t[\ell^m]))^\vee \\ &= \varinjlim_m \varinjlim_{F \subset L \subset K} (H^2(F_S/L, A[\ell^m]))^\vee \\ &= \varinjlim_{F \subset L \subset K} H^2(F_S/L, T_\ell(A)) . \end{aligned}$$

Local case. The proof is similar (using Tate local duality). \square

We recall that our group G has no elements of order ℓ , hence $\Lambda(G)$ is a domain. Moreover for any open subgroup U of G we have that (see [J, Lemma 2.3])

$$E^i(U) \simeq E^i(G) \quad \forall i \in \mathbb{Z}$$

is an isomorphism of $\Lambda(U)$ -modules. An ℓ -adic Lie group G always contains an open pro- ℓ subgroup ([DdSMS, Corollary 8.34]), so, in order to use properly the usual definitions of “torsion submodule” and “rank” for a finitely generated $\Lambda(G)$ -module, with no loss of generality,

we will assume that G is pro- ℓ .

Proposition 3.7. *Let M be a finitely generated $\Lambda(G)$ -module. Then $E^i(M)$ is a finitely generated torsion $\Lambda(G)$ -module for any $i \geq 1$.*

Proof. Take a finite presentation $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ with finitely generated and projective $\Lambda(G)$ -modules P_1 and P_0 , and the consequent exact sequence

$$(8) \quad 0 \rightarrow R_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

for a suitable submodule R_1 of P_1 . Since M and $\text{Hom}_{\Lambda(G)}(M, \Lambda(G))$ have the same $\Lambda(G)$ -rank, computing ranks in the sequence coming from (8)

$$\begin{aligned} \text{Hom}_{\Lambda(G)}(M, \Lambda(G)) \hookrightarrow \text{Hom}_{\Lambda(G)}(P_0, \Lambda(G)) \rightarrow \text{Hom}_{\Lambda(G)}(R_1, \Lambda(G)) \rightarrow E^1(M) \rightarrow \\ \rightarrow 0 \rightarrow E^1(R_1) \rightarrow E^2(M) \rightarrow 0 \rightarrow \cdots \rightarrow 0 \rightarrow E^{i-1}(R_1) \rightarrow E^i(M) \rightarrow 0 \rightarrow \cdots \end{aligned}$$

one finds $\text{rank}_{\Lambda(G)}(E^1(M)) = 0$ for any finitely generated $\Lambda(G)$ -module M . Therefore $E^1(R_1)$ is torsion, which yields $E^2(M) \simeq E^1(R_1)$ is torsion. Iterating $E^i(M) \simeq E^{i-1}(R_1)$ is $\Lambda(G)$ -torsion $\forall i \geq 2$. \square

Lemma 3.8. *Let F_n be subfields of K such that $\text{Gal}(K/F) = \varprojlim_n \text{Gal}(F_n/F)$. Then*

$$H_{Iw}^2(K_w, T_\ell(A)) := \varprojlim_{n,m} H^2(F_{v_n}, A[\ell^m])$$

is a torsion $\Lambda(G_v)$ -module. If $H^2(F_S/K, A[\ell^\infty]) = 0$, then

$$H_{Iw}^2(K, T_\ell(A)) := \varprojlim_{n,m} H^2(F_S/F_n, A[\ell^m])$$

is a $\Lambda(G)$ -torsion as well.

Proof. The proofs are identical so we only show the second statement. From the spectral sequence

$$E_2^{p,q} = E^p(H^q(F_S/K, A[\ell^\infty])^\vee) \implies H_{Iw}^{p+q}(K, T_\ell(A))$$

due to Jannsen (see [J1]), we have a filtration for $H_{Iw}^2(K, T_\ell(A))$

$$(9) \quad 0 = H_3^2 \subseteq H_2^2 \subseteq H_1^2 \subseteq H_0^2 = H_{Iw}^2(K, T_\ell(A)) ,$$

which provides the following sequences:

$$\begin{aligned} E^0(H^1(F_S/K, A[\ell^\infty])^\vee) \rightarrow E^2(H^0(F_S/K, A[\ell^\infty])^\vee) \rightarrow H_1^2 \\ \rightarrow E^1(H^1(F_S/K, A[\ell^\infty])^\vee) \rightarrow E^3(H^0(F_S/K, A[\ell^\infty])^\vee) \end{aligned}$$

and

$$H_1^2 \hookrightarrow H_{Iw}^2(K, T_\ell(A)) \rightarrow E_\infty^{0,2} .$$

By hypothesis $E_\infty^{0,2} \simeq E_2^{0,2} = 0$, so $H_1^2 \simeq H_{Iw}^2(K, T_\ell(A))$.

Since $H^i(F_S/K, A[\ell^\infty])^\vee$ is a finitely generated $\Lambda(G)$ -module for $i \in \{0, 1\}$ (for $i = 1$ just look at X in diagram (3)), Proposition 3.7 yields that the groups $E^2(H^0(F_S/K, A[\ell^\infty])^\vee)$ and $E^1(H^1(F_S/K, A[\ell^\infty])^\vee)$ are $\Lambda(G)$ -torsion. Hence H_1^2 is torsion as well. \square

Lemma 3.9. *With notations and hypotheses as in Lemma 3.4, $\text{Coker}(g_1)$ is finitely generated as \mathbb{Z}_ℓ -module.*

Proof. Lemma 3.6 yields $Z(K) = H_{Iw}^2(K, T_\ell(A))$ so, using [OV, Proposition 4.10], one has $DH_{Iw}^2(K, T_\ell(A)) \simeq Y$. Therefore $E^1(DH_{Iw}^2(K, T_\ell(A))) \simeq E^1(Y)$. Since $H_{Iw}^2(K, T_\ell(A))$ is a $\Lambda(G)$ -torsion module, [OV, Lemma 3.1] implies $E^1(DH_{Iw}^2(K, T_\ell(A))) \simeq H_{Iw}^2(K, T_\ell(A))$, i.e.,

$$H_{Iw}^2(K, T_\ell(A)) \simeq E^1(Y)$$

(the same holds for the “local” modules). The map g_1 of Lemma 3.4 then reads as

$$g_1 : \varprojlim_n H^2(F_S/F_n, T_\ell(A)) \rightarrow \bigoplus_S \text{Ind}_{G_v}^G \varprojlim_n H^2(F_{v_n}, T_\ell(A)) .$$

The claim follows from the Poitou-Tate sequence (see [NSW, 8.6.10 p. 488]), since

$$\text{Coker}(g_1) \simeq \varprojlim_{n,m} H^0(F_S/F_n, (A[\ell^m])') .$$

□

4. MAIN THEOREM

We are now ready to prove the following

Theorem 4.1. *Let $G = \text{Gal}(K/F)$ be an ℓ -adic Lie group without elements of order ℓ and of positive dimension $d \geq 3$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and the map ψ in the sequence*

$$(10) \quad \text{Sel}_A(K)_\ell \hookrightarrow H^1(F_S/K, A[\ell^\infty]) \xrightarrow{\psi} \bigoplus_S \text{Coind}_G^{G_v} H^1(K_w, A)[\ell^\infty]$$

is surjective, then $\mathcal{S} := \text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.

Proof. We need to prove that

$$E^i E^i(\mathcal{S}) = 0 \quad \forall i \geq 2 ,$$

and we consider two cases.

Case $i = 2$. Let $\mathcal{D} := \bar{g}_1(E^2(J))$. Then

$$\text{Coker}(\bar{g}_1) = \bigoplus_S \text{Ind}_{G_v}^G E^2(J_v) / \mathcal{D} .$$

Observe that $\mathcal{D} \simeq \bar{g}_1(E^3(A[\ell^\infty]_{\mathcal{H}}^\vee))$ is a finitely generated \mathbb{Z}_ℓ -module (it is zero if $d \neq 3$ and free as \mathbb{Z}_ℓ -module if $d = 3$), so $E^1(\mathcal{D}) = 0$. Even if the theorem is limited to $d \geq 3$ we remark here that, for $d = 2$, \mathcal{D} is finite and, for $d = 1$, $\mathcal{D} = 0$: hence $E^1(\mathcal{D}) = 0$ in any case.

Moreover

$$\begin{aligned} E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^2(J_v)\right) &= E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^3(A[\ell^\infty]_{\mathcal{H}_v}^\vee)\right) \\ &= \bigoplus_S \text{Ind}_{G_v}^G E^2 E^3(A[\ell^\infty]_{\mathcal{H}_v}^\vee) = 0 , \end{aligned}$$

so, taking Ext in the sequence,

$$(11) \quad \mathcal{D} \hookrightarrow \bigoplus_S \text{Ind}_{G_v}^G E^2(J_v) \twoheadrightarrow \bigoplus_S \text{Ind}_{G_v}^G E^2(J_v) / \mathcal{D} ,$$

one finds

$$E^1(\mathcal{D}) \rightarrow E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^2(J_v) / \mathcal{D}\right) \rightarrow E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^2(J_v)\right) .$$

Therefore

$$(12) \quad E^2\left(\bigoplus_S \text{Ind}_{G_v}^G E^2(J_v) / \mathcal{D}\right) = 0 .$$

Recall the sequences

$$(13) \quad \bigoplus_S \operatorname{Ind}_{G_v}^G X_v \hookrightarrow X \twoheadrightarrow \mathcal{S}$$

$$(14) \quad \operatorname{Ker}(f) \hookrightarrow \operatorname{Coker}(h_1) \twoheadrightarrow \operatorname{Coker}(\bar{g}_1)$$

provided (respectively) by the hypothesis on ψ and by Lemma 3.4. Take Ext on (13) to get

$$\operatorname{E}^1(X) \xrightarrow{h_1} \operatorname{E}^1\left(\bigoplus_S \operatorname{Ind}_{G_v}^G X_v\right) \rightarrow \operatorname{E}^2(\mathcal{S}) \rightarrow \operatorname{E}^2(X) .$$

If $d \geq 5$, then $\operatorname{E}^2(X) \simeq \operatorname{E}^3(J) \simeq \operatorname{E}^4(A[\ell^\infty]_{\mathcal{H}}^\vee) = 0$. When this is the case $\operatorname{Coker}(h_1) \simeq \operatorname{E}^2(\mathcal{S})$ and sequence (14) becomes

$$\operatorname{Ker}(f) \hookrightarrow \operatorname{E}^2(\mathcal{S}) \twoheadrightarrow \bigoplus_S \operatorname{Ind}_{G_v}^G \operatorname{E}^2(J_v)/\mathcal{D} .$$

By Lemma 3.9, $\operatorname{Ker}(f)$ is a finitely generated \mathbb{Z}_ℓ -module. Taking Ext , one has

$$\operatorname{E}^2\left(\bigoplus_S \operatorname{Ind}_{G_v}^G \operatorname{E}^2(J_v)/\mathcal{D}\right) \rightarrow \operatorname{E}^2 \operatorname{E}^2(\mathcal{S}) \rightarrow \operatorname{E}^2(\operatorname{Ker}(f)) ,$$

where the first and third term are trivial, so $\operatorname{E}^2 \operatorname{E}^2(\mathcal{S}) = 0$ as well.

We are left with $d = 3, 4$. We know that $\operatorname{E}^4(A[\ell^\infty]_{\mathcal{H}}^\vee) = \operatorname{E}^2(X)$ is free over \mathbb{Z}_ℓ if $d = 4$ or finite if $d = 3$ (again we remark it is 0 if $d = 1, 2$). Anyway $\operatorname{E}^2 \operatorname{E}^2(X) = 0$ in all cases. From the sequence

$$\operatorname{Coker}(h_1) \hookrightarrow \operatorname{E}^2(\mathcal{S}) \xrightarrow{\eta} \operatorname{E}^2(X)$$

one writes

$$(15) \quad \operatorname{Coker}(h_1) \hookrightarrow \operatorname{E}^2(\mathcal{S}) \twoheadrightarrow \operatorname{Im}(\eta)$$

where $\operatorname{Im}(\eta)$ is free over \mathbb{Z}_ℓ if $d = 4$ or finite if $d = 3$.

Taking Ext in (14) one has

$$\operatorname{E}^2(\operatorname{Coker}(\bar{g}_1)) \rightarrow \operatorname{E}^2(\operatorname{Coker}(h_1)) \rightarrow \operatorname{E}^2(\operatorname{Ker}(f))$$

with the first (see equation (12)) and third term equal to zero, so $\operatorname{E}^2(\operatorname{Coker}(h_1)) = 0$. This fact in sequence (15) implies

$$0 = \operatorname{E}^2(\operatorname{Im}(\eta)) \rightarrow \operatorname{E}^2 \operatorname{E}^2(\mathcal{S}) \rightarrow \operatorname{E}^2(\operatorname{Coker}(h_1)) = 0 ,$$

so $\operatorname{E}^2 \operatorname{E}^2(\mathcal{S}) = 0$.

Case $i \geq 3$. From sequence (13) we get the following

$$(16) \quad \operatorname{E}^{i+1}(A[\ell^\infty]_{\mathcal{H}}^\vee) \simeq \operatorname{E}^{i-1}(X) \rightarrow \bigoplus_S \operatorname{Ind}_{G_v}^G \operatorname{E}_v^{i-1}(X_v) \rightarrow \operatorname{E}^i(\mathcal{S}) \rightarrow \operatorname{E}^i(X) \simeq \operatorname{E}^{i+2}(A[\ell^\infty]_{\mathcal{H}}^\vee) .$$

We have four cases, depending on whether $\operatorname{E}^{i-1}(X)$ and $\operatorname{E}^i(X)$ are trivial or not.

Case 1. Assume $\operatorname{E}^{i-1}(X) = \operatorname{E}^i(X) = 0$.

From (16) we obtain the isomorphism

$$\bigoplus_S \operatorname{Ind}_{G_v}^G \operatorname{E}_v^{i-1}(X_v) \simeq \operatorname{E}^i(\mathcal{S}) ,$$

so

$$\bigoplus_S \operatorname{Ind}_{G_v}^G \operatorname{E}_v^i \operatorname{E}_v^{i-1}(X_v) \simeq \operatorname{E}^i \operatorname{E}^i(\mathcal{S}) = 0$$

thanks to Corollary 3.3 part **2**. We remark that this is the only case to consider when $d = 1, 2$.

Case 2. Assume $E^{i-1}(X) = 0$ and $E^i(X) \neq 0$.

This happens when $i = d - 2$ or $i = d - 1$ and $A[\ell^\infty]_{\mathcal{H}}^\vee$ is finite. From (16) we have

$$\bigoplus_S \text{Ind}_{G_v}^G E_v^{d-3} \hookrightarrow E^{d-2}(\mathcal{S}) \twoheadrightarrow N$$

$$(\text{ resp. } \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-2} \hookrightarrow E^{d-1}(\mathcal{S}) \twoheadrightarrow N)$$

where N is a submodule of the free module $E^{d-2}(X)$ (resp. of the finite module $E^{d-1}(X)$). Therefore $E^{d-2}(N) = 0$ (resp. $E^{d-1}(N) = 0$) and, moreover, $E_v^{d-2}E_v^{d-3}(X_v) = 0$ (resp. $E_v^{d-1}E_v^{d-2}(X_v) = 0$) by Corollary 3.3 part 2. Hence $E^{d-2}E^{d-2}(\mathcal{S}) = 0$ (resp. $E^{d-1}E^{d-1}(\mathcal{S}) = 0$).

Case 3. Assume $E^{i-1}(X) \neq 0$ and $E^i(X) = 0$.

This happens when $i = d$ or $i = d - 1$ and $A[\ell^\infty]_{\mathcal{H}}^\vee$ is free. The sequence (16) gives

$$N \hookrightarrow \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-1}(X_v) \twoheadrightarrow E^d(\mathcal{S})$$

$$(\text{ resp. } N \hookrightarrow \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-2}(X_v) \twoheadrightarrow E^{d-1}(\mathcal{S}))$$

where now N is a quotient of the finite module $E^{d-1}(X)$ (resp. of the free module $E^{d-2}(X)$). Then $E^d(N) = 0$ (resp. $E^{d-1}(N) = 0$) and

$$\bigoplus_S \text{Ind}_{G_v}^G E_v^d E_v^{d-1}(X_v) \simeq E^d E^d(\mathcal{S}) = 0$$

$$(\text{ resp. } \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-1} E_v^{d-2}(X_v) \simeq E^{d-1} E^{d-1}(\mathcal{S}) = 0) .$$

Case 4. Assume $E^{i-1}(X) \neq 0$ and $E^i(X) \neq 0$.

This happens when $i = d - 1$ and $A[\ell^\infty]_{\mathcal{H}}^\vee$ has nontrivial rank and torsion. From sequence (16) we have

$$E^{d-2}(X) \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-2}(X_v) \rightarrow E^{d-1}(\mathcal{S}) \rightarrow E^{d-1}(X) .$$

Let N_1, N_2 and N_3 be modules such that:

- N_1 is a quotient of $E^{d-2}(X)$ (which is torsion free so that $E^{d-2}(N_1) = 0$);
- N_2 is a submodule of $E^{d-1}(X)$ (which is finite so that $E^{d-1}(N_2) = 0$);
- N_3 is a module such that the sequences

$$N_1 \hookrightarrow \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-2}(X_v) \twoheadrightarrow N_3 \quad \text{and} \quad N_3 \hookrightarrow E^{d-1}(\mathcal{S}) \twoheadrightarrow N_2$$

are exact.

Applying the functor Ext we find

$$E^{d-2}(N_1) \rightarrow E^{d-1}(N_3) \rightarrow \bigoplus_S \text{Ind}_{G_v}^G E_v^{d-1} E_v^{d-2}(X_v)$$

(which yields $E^{d-1}(N_3) = 0$), and

$$E^{d-1}(N_2) \rightarrow E^{d-1} E^{d-1}(\mathcal{S}) \rightarrow E^{d-1}(N_3)$$

which proves $E^{d-1} E^{d-1}(\mathcal{S}) = 0$. □

Remark 4.2. As pointed out in various steps of the previous proof, most of the statements still hold for $d = 1, 2$. The only missing part is $E^2(\text{Ker}(f)) = 0$ for $i = 2$, in that case only our calculations to get $E^2 E^2(\mathcal{S}) = 0$ fail. In particular the same proof shows that $E^2 E^2(\mathcal{S}) = 0$ when $\text{Ker}(f)$ is free and $d = 1$ or when $\text{Ker}(f)$ is finite and $d = 2$ or, obviously, for any d if f is injective.

We can extend the previous result to the $d \geq 2$ case with some extra assumptions.

Proposition 4.3. *Let $G = \text{Gal}(K/F)$ be an ℓ -adic Lie group without elements of order ℓ and of dimension $d \geq 2$. If $H^2(F_S/K, A[\ell^\infty]) = 0$ and $\text{cd}_\ell(G_v) = 2$ for any $v \in S$, then $\text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule.*

Proof. Since $\text{cd}_\ell(F_v) = 2$ (by [NSW, Theorem 7.1.8]), our hypothesis implies that $\text{Gal}(\overline{F}_v/K_w)$ has no elements of order ℓ (see also [NSW, Theorem 7.5.3]). Hence $H^1(K_w, A[\ell^\infty])^\vee = 0$ and $\text{Sel}_A(K)_\ell^\vee \simeq X$ embeds in Y . Now $H^2(F_S/K, A[\ell^\infty]) = 0$ yields Y has projective dimension ≤ 1 , so Y has no nontrivial pseudo-null submodule (by [OV, Proposition 2.5]). \square

4.1. The hypotheses on $H^2(F_S/K, A[\ell^\infty])$ and ψ . Let F_m be extensions of F such that $\text{Gal}(K/F) \simeq \varprojlim_m \text{Gal}(F_m/F)$. To provide some cases in which the main hypotheses hold we consider the Poitou-Tate sequence for the module $A[\ell^n]$, from which one can extract the sequence

$$(17) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker}(\psi_{m,n}) & \longrightarrow & H^1(F_S/F_m, A[\ell^n]) & \xrightarrow{\psi_{m,n}} & \prod_{v \in S} H^1(F_{v_m}, A[\ell^n]) \\ & & & & & & \downarrow \phi_{m,n} \\ & & \prod_{v \in S} H^2(F_{v_m}, A[\ell^n]) & \longleftarrow & H^2(F_S/F_m, A[\ell^n]) & \longleftarrow & \text{Ker}(\psi_{m,n}^t)^\vee \\ & & \downarrow & & & & \\ & & H^0(F_S/F_m, A^t[\ell^n])^\vee & \longrightarrow & 0 & & \end{array}$$

(where $\psi_{m,n}^t$ is the analogue of $\psi_{m,n}$ for the dual abelian variety A^t , i.e., their kernels represent the Selmer groups over F_m for the modules $A^t[\ell^n]$ and $A[\ell^n]$ respectively). Taking direct limits on n and recalling that $H^2(F_{v_m}, A[\ell^\infty]) = 0$, the sequence (17) becomes

$$(18) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Sel}_A(F_m)_\ell & \longrightarrow & H^1(F_S/F_m, A[\ell^\infty]) & \xrightarrow{\psi_m} & \prod_{v \in S} H^1(F_{v_m}, A[\ell^\infty]) \\ & & & & & & \downarrow \phi_m \\ & & 0 & \longleftarrow & H^2(F_S/F_m, A[\ell^\infty]) & \longleftarrow & (\varprojlim_n \text{Ker}(\psi_{m,n}^t))^\vee \end{array}$$

(for more details one can consult [CS, Chapter 1]). One way to prove that $H^2(F_S/K, A[\ell^\infty]) = 0$ and ψ is surjective is to show that $(\varprojlim_n \text{Ker}(\psi_{m,n}^t))^\vee = 0$ for any m . We mention here two cases in which the hypothesis on the vanishing of $H^2(F_S/K, A[\ell^\infty])$ is verified. The following is basically [CS, Proposition 1.9].

Proposition 4.4. *Let F_m be as above and assume that $|\text{Sel}_{A^t}(F_m)_\ell| < \infty$ for any m , then*

$$H^2(F_S/K, A[\ell^\infty]) = 0.$$

Proof. From [M, Chapter I Remark 3.6] we have the isomorphism

$$A^t(F_{v_m})^* \simeq H^1(F_{v_m}, A[\ell^\infty])^\vee,$$

where $A^t(F_{v_m})^* \simeq \varprojlim_n A^t(F_{v_m})/\ell^n A^t(F_{v_m})$.

Taking inverse limits on n in the exact sequence

$$A^t(F_m)/\ell^n A^t(F_m) \hookrightarrow \text{Ker}(\psi_{m,n}^t) \twoheadrightarrow \text{III}(A^t/F_m)[\ell^n],$$

and noting that $|\text{III}(A^t/F_m)[\ell^\infty]| < \infty$ yields $T_\ell(\text{III}(A^t/F_m)) = 0$, we find

$$A^t(F_m)^* \simeq \varprojlim_n \text{Ker}(\psi_{m,n}^t).$$

Therefore (18) becomes

$$(19) \quad 0 \longrightarrow \text{Sel}_A(F_m)_\ell \longrightarrow H^1(F_S/F_m, A[\ell^\infty]) \xrightarrow{\psi} \prod_{v \in S} (A^t(F_{v_m})^*)^\vee$$

$$\downarrow \tilde{\phi}$$

$$0 \longleftarrow H^2(F_S/F_m, A[\ell^\infty]) \longleftarrow (A^t(F_m)^*)^\vee$$

By hypothesis $A^t(F_m)^*$ is finite, therefore $H^2(F_S/F_m, A[\ell^\infty])$ is finite as well. From the cohomology of the sequence

$$A[\ell] \hookrightarrow A[\ell^\infty] \xrightarrow{\ell} A[\ell^\infty]$$

(and the fact that $H^3(F_S/F_m, A[\ell]) = 0$, because $cd_\ell(\text{Gal}(F_S/F_m)) = 2$), one finds

$$H^2(F_S/F_m, A[\ell^\infty]) \xrightarrow{\ell} H^2(F_S/F_m, A[\ell^\infty]),$$

i.e., $H^2(F_S/F_m, A[\ell^\infty])$ is divisible. Being divisible and finite $H^2(F_S/F_m, A[\ell^\infty])$ must be 0 for any m and the claim follows. \square

We can also prove the vanishing of $H^2(F_S/K, A[\ell^\infty])$ for the extension $K = F(A[\ell^\infty])$.

Proposition 4.5. *If $K = F(A[\ell^\infty])$, then $H^2(F_S/K, A[\ell^\infty]) = 0$.*

Proof. $\text{Gal}(F_S/K)$ has trivial action on $A[\ell^\infty]$ and (by the Weil pairing) on μ_{ℓ^∞} , so

$$H^2(F_S/K, A[\ell^\infty]) \simeq H^2(F_S/K, (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{2g}) \simeq H^2(F_S/K, (\mu_{\ell^\infty})^{2g}).$$

Let $F_n = F(A[\ell^n])$, using the notations of Lemma 3.6, Poitou-Tate duality ([NSW, Theorem 8.6.7]) and the isomorphism $\text{III}^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z}) \simeq \text{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})$ ([NSW, Lemma 8.6.3]), one has

$$\begin{aligned} H^2(F_S/K, \mu_{\ell^\infty}) &\simeq \text{III}^2(F_S/K, \mu_{\ell^\infty}) \simeq \varinjlim_{n,m} \text{III}^2(F_S/F_n, \mu_{\ell^m}) \\ &\simeq \varinjlim_{n,m} \text{III}^1(F_S/F_n, \mu_{\ell^m}')^\vee \simeq \varinjlim_{n,m} \text{III}^1(F_S/F_n, \mathbb{Z}/\ell^m\mathbb{Z})^\vee \\ &\simeq \varinjlim_{n,m} \text{Hom}(C\ell_S(F_n), \mathbb{Z}/\ell^m\mathbb{Z})^\vee \simeq \varinjlim_{n,m} C\ell_S(F_n)/\ell^m \\ &\simeq \varinjlim_n C\ell_S(F_n) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell = 0 \end{aligned}$$

since $C\ell_S(F_n)$ is finite. \square

Remark 4.6. The above proposition works in the same way for a general ℓ -adic Lie extensions, unramified outside S , which contains the trivializing extension.

Example 4.7. Let A be an abelian variety without complex multiplication: by Proposition 4.5, the extension $K = F(A[\ell^\infty])$ realizes the hypothesis of Proposition 4.3 when every bad reduction prime is of split multiplicative reduction (in order to have $cd_\ell(G_v) = 2$) and $\ell > 2g + 1$ (by [ST] and the embedding $\text{Gal}(K/F) \hookrightarrow \text{GL}_{2g}(\mathbb{Z}_\ell)$). Therefore $\text{Sel}_A(K)_\ell^\vee$ has no nontrivial pseudo-null submodule. When $A = \mathcal{E}$ is an elliptic curve (using Igusa's theorem, see, e.g., [BLV]) one can prove that $\dim \text{Gal}(K/F) = 4$ and also the surjectivity of the map ψ (which, in this case, is not needed to prove the absence of pseudo-null submodules): more details can be found in [S].

The same problem over number fields cannot (in general) be addressed in the same way and one needs the surjectivity of the map ψ . The topic is treated (for example) in [C, Section 4.2] and [HV, Section 7.1].

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