Decentralized Robust Model Predictive Control for Multi-input Linear Systems

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***Abstract*— In this paper, a decentralized model predictive control approach is proposed for discrete linear systems with a high number of inputs and states. The system is decomposed into several interacting subsystems. The interaction among subsystems is modeled as external disturbances. Then, using the concept of robust positively invariant ellipsoids, a robust model predictive control law is obtained for each subsystem solving several linear matrix inequalities. Maintaining the recursive feasibility while considering the attenuation of mutual coupling at each time step and the stability of the overall system are investigated. Moreover, an illustrative simulation example is provided to demonstrate the effectiveness of the method.**

Keywords—Model Predictive Control, Decentralized Control, Linear Matrix Inequalities, Large Scale Systems, Robust Positively Invariant Sets

I. INTRODUCTION

Model Predictive Control (MPC) has paved its way for industrial applications due to its ability to cope with constraints and multi-input multi-output systems. However, the high computational demand of MPC makes it inapplicable for systems with a large number of inputs and states unless sufficiently large computation time is available. Thus, several researches are done to obtain more computationally efficient MPC frameworks by introducing fast online methods, designing explicit offline control laws, or a mixture of these approaches [1-3]. Another traditional solution for reducing the computational load of MPC is to decompose the large problem into several smaller subproblems [4]. In this regard, several methods have been proposed to decompose the system into some interacting subsystems with minimum couplings (e.g. [5]). Then, many decentralized and distributed MPC schemes are designed which are different based on the chosen control structure and theoretical tools, the circumstances of exploiting mutual information, treating the interaction among subsystems, differences in applications and so on [6].

Exploiting Linear Matrix Inequalities (LMIs) in the design of MPC law has led to a computationally efficient method in which the MPC law is a state feedback matched with an invariant ellipsoid to ensure the stability properties [7]. However, the dimension of LMIs may become very large for systems with a high number of inputs and states. Consequently, this may result in a large and undesirable computational complexity and a very conservative invariant ellipsoid which may even prevent finding a feasible solution. To overcome this drawback, distributed LMI-based MPC has been studied in several works [8-11]. Reference [8] presents a distributed MPC for a set of decoupled local systems with a global cost function. In [9] and [10] distributed LMI-based MPC has been designed for linear systems with polytopic and structured uncertainty, respectively. However, in these works, only inputs are designed separately using the whole dynamic of the system and doing inner iterations. Since no decomposition is done on the states, dimensions of the underlying LMIs do no reduce significantly. Thus, these methods may not always lead to a lower computation complexity. To further reduce the complexity, it would be better to decompose both inputs and states of system [11]. The main struggle then becomes how to consider the effects of other subsystems' states and inputs in the control design for each subsystem as well as maintaining the recursive feasibility of LMIs of each subsystem. The interaction between subsystems can be considered as additive disturbances [12]. Moreover, there are LMI conditions to obtain Robust Positively Invariant (RPI) ellipsoids for linear systems with additive disturbances [13]. These RPI sets have also been used to design robust LMI-based MPC for such systems [14, 15].

Motivated by the above discussion, in this paper a decentralized LMI-based MPC is proposed for linear multi-input systems. Inputs and states of the system are assumed to be clustered into several subsystems. The effects of other subsystems are modeled as external disturbances. For each subsystem, an LMI-Based MPC is designed using local cost function and local system dynamic to compute a local state feedback. Besides, to incorporate the effects of other subsystems, an RPI ellipsoid is constructed based on the information on the states of neighboring subsystems and their previous control actions. It is known that the disturbances on each subsystem are originated from the inputs and states of other subsystems. Therefore, while system's states converge to the origin, the amount of coupling will reduce. This disturbance attenuation, in turn, reduces the conservatism of the control performance [16, 17]. In the distributed and decentralized MPC frameworks which have considered input-decoupled local subsystems [12, 18, 19], the changes in the input of each subsystem do not directly affect the amount of disturbance imposed on other subsystems. However, if the system is decomposed into subsystems which have interaction with the input of other subsystems, the coupling effects of the inputs need to be considered carefully to ensure the mutual disturbance attenuation. This fact has also addressed in some works (e.g. [20]). In LMI-based MPC, input couplings will cause some LMIs corresponding to the MPC of each subsystem to be dependent on the feedback gains of other subsystems. Thus, obtaining a large feedback gain for a subsystem at a sample time may cause such large mutual disturbances that lead to the infeasibility of LMIs of other local subsystems at the next sample time. To overcome this, some new LMIs are proposed to be added in the MPC design of each subsystem to ensure that in the next sampling time, the design of control signal for each subsystem will not alter the feasibility properties of LMIs corresponding to other subsystems.

The rest of the paper is organized as follows. Section II contains the preliminary material needed to develop the results of this paper. In Section III the main results on decentralized LMI-based MPC with RPI sets are developed. The discussion on the recursive feasibility of the LMIs and stability of the overall system is also presented in this section. Simulation example and final conclusion are drawn in Section IV and V, respectively.

Notation. Throughout this paper, the time step k is dropped, whenever convenient, for the sake of compactness of the equations. $[A_i]_{i\in M}$ denotes the horizontal concatenation of matrices A_i where $i \in M$, i.e. $[A_1,\dots,A_i,\dots,A_M]$. $\operatorname{diag}(P_j), j \in 1,\dots,N$ is the block diagonal collection of P_1 to P_N matrices. $\|.\|$ means the 2-norm unless otherwise stated. $\operatorname{Card}(N)$ is the number of elements in set N. The sign $*$ in some matrix expressions expresses symmetric transpose structure.

II. PRELIMINARIES

Consider a discrete time multi-input linear system as

$$x^+ = Ax + Bu, \tag{1}$$

where $u \in R^m, m \geq 2$ is the input signal and $x \in R^n$ is the state vector. There are element-wise state and input constraints as $X = \{|x_i| \leq x_{i,\max}\}$ and $U = \{|u_i| \leq u_{i,\max}\}$ respectively. Let us suppose that system (1) can be decomposed into $N \in \{2,\dots,m\}$ interacting subsystems. The dynamic of each subsystem can be expressed as

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + \textstyle\sum_{j=1,j\neq i}^{N}(A_{ij}\,x_j + B_{ij}u_j), \tag{2}$$

where $x_i \in R^{n_i}, \sum_{i=1}^{N} n_i = n$ and $u_i \in R^{m_i}, \sum_{i=1}^{N} m_i = m$. The goal is to design a state feedback control law using LMI-based MPC for each subsystem. Let N_i be the index set of neighbors of subsystem i defined as $N_i = \{j \in N, j \neq i \,|\, [A_{ij}, B_{ij}] \neq 0\}$. Thus, by substituting $u_j = K_j x_j, j \in N_i$, where K_j are the state feedback gains of other neighbor subsystems, (2) can be rewritten as

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + E_i d_i, \tag{3}$$

where $E_i = A_{c,i} + B_{c,i}K_{c,i}$, $A_{c,i} = [A_{ij}]_{j\in N_i}$, $B_{c,i} = [B_{ij}]_{j\in N_i}$, $K_{c,i} = \operatorname{diag}([K_1,\dots,K_j,\dots,K_N])$, and $d_i = [x_j]_{j\in N_i}^T$.

Assumption 1. There exists a state feedback $K_i, i = 1,\dots,N$ which can stabilize the pair (A_{ii}, B_{ii}).

Assumption 2. At the initial point the mutual disturbances are bounded inside an ellipsoidal set $\Omega_{d,i} = \{d_i | d_i^T P_{d,i} d_i \leq 1\}$.

Definition 1 [13]. Consider a discrete time linear system $x^+ = Ax + Bu + Ed$, with constraints $x \in X, u \in U$, and $d \in D$. A set $\Omega \in X$ is an RPI set for this system, if and only if, $\forall x \in \Omega$ and $\forall d \in D$ there exists $u \in U$ such that $(Ax + Bu + Ed) \in \Omega$.

Lemma 1 [13]. For the linear system described in Definition 1, assume that $\Omega_d(P_d) = \{d | d^T P_d d \leq 1\}$ is the smallest outer ellipsoid that contains D. Then, an ellipsoidal set $\Omega(P) = \{x | x^T P x \leq \xi\}$ corresponding with a state feedback law $u = Kx$ is an RPI set for this system if and only if there exists positive definite matrix M such that the following LMI condition holds for some scalar $0 < \lambda < 1$ where $P = \xi M^{-1}$ and $K = YM^{-1}$.

$$\begin{bmatrix} (1-\lambda)M & 0 & * \\ 0 & \lambda P_d & * \\ AM + BY & E & M \end{bmatrix} \geq 0 \tag{4}$$

Definition 2 [14]. The linear system $x^+ = Ax + Bu + Ed$ is said to be Input to State Stable (ISS), if there exists a $\mathcal{KL}$ function $\beta(.)$ and a $\mathcal{K}$ function $\gamma(.)$ such that for all $k \geq 0$

$$\|x\| \leq \beta(\|x_0\|, k) + \gamma(\|d\|). \tag{5}$$

Lemma 2 [14]. The system $x^+ = Ax + Bu + Ed$ is ISS if it admits an ISS-Lyapunov function $V(x)$ defined as

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{6}$$

$$V(x^+) - V(x) \leq -\alpha_3(\|x\|) + \rho(\|d\|), \tag{7}$$

where $\alpha_1(.)$, $\alpha_2(.)$, and $\alpha_3(.)$ are $\mathcal{K}_\infty$ functions and $\rho(.)$ is a $\mathcal{K}$ function.

Lemma 3 [15]. In constrained finite set X, for a quadratic function $V(x) = x^T P x$, $P > 0$, there exists a finite constant $L_P > 0$ such that for all $x_1, x_2 \in X$ the following holds.

$$|V(x_1) - V(x_2)| \leq L_P \|x_1 - x_2\| \tag{8}$$

III. DECENTRALIZED MPC DESIGN

In this section, an LMI-based MPC state feedback law is designed for each subsystem. Besides, instead of using a conventional invariant set as [7], additional LMI conditions are presented to construct an RPI set for each subsystem to consider the effect of other subsystems in the design. Note that, one can calculate a priori an upper bound for the effects of other subsystems and thus solve a complete decentralized MPC for each subsystem with a local cost function and local dynamic with a fixed amount of disturbance. However, this may lead to a poor performance. Moreover, as the states of each subsystem go towards the origin, one can expect that they impose smaller disturbances on the other subsystems. Consequently, the information of shrinking RPI ellipsoids of each subsystem can be used to improve performance. Moreover, for subsystems with input couplings, the inputs have also effect on the amount of mutual disturbances. Thus, some constraints need to be imposed to ensure that the coupling disturbances decrease at each sample time.

A. Main Results

Consider the following local dynamic for each subsystem without the effects of the other subsystems

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}u_i \tag{9}$$

Let us assume a local objective function for each subsystem corresponding with its local dynamic (9) as

$$J_i = \sum_{l=0}^{\infty} \bar{x}_{i,(k+l|k)}^T Q_i \bar{x}_{i,(k+l|k)} + u_{i,(k+l|k)}^T R_i u_{i,(k+l|k)}, \quad (10)$$

where Q_i and R_i are positive definite weighting matrices. $\bar{x}_{i,(k+l|k)}$ are the predicted states of the nominal i^{th} subsystem at time $k+l$ from measurements of time k. Consider the quadratic function $V_i(\bar{x}_i) = \bar{x}_i{}^T P_i \bar{x}_i$ with $P_i > 0$ and $V_i(0) = 0$ which satisfies the following inequality at time k:

$$V_i(\bar{x}_{i,(k+l+1|k)}) - V_i(\bar{x}_{i,(k+l|k)}) \leq -\bar{x}_{i,(k+l|k)}^T Q_i \bar{x}_{i,(k+l|k)} - u_{i,(k+l|k)}^T R_i u_{i,(k+l|k)}. \quad (11)$$

Calculating the summation of both side of (11) from $l = 0$ to $l = \infty$ gives

$$J_i \leq \bar{x}_{i,(k|k)}^T P_i \bar{x}_{i,(k|k)} - \bar{x}_{i,(\infty|k)}^T P_i \bar{x}_{i,(\infty|k)}. \quad (12).$$

For J_i defined in (10) to be finite, $\bar{x}_{i,(\infty|k)}$ should go toward zero. Therefore, from (12) the upper bound of objective function (10) can be obtained as $J_i \leq \bar{x}_i^T P_i \bar{x}_i \leq \xi_i$ where ξ_i is a positive scalar. Then, the MPC law can be designed using Theorem 1.

Remark 1. Note that the predicted states of the nominal i^{th} subsystem at time k based on measurements of time k, i.e. $\bar{x}_{i,(k|k)} = \bar{x}_i$, equals to x_i which is the measured state at time k.

Theorem 1. Consider subsystem (3) with constraints on the inputs and states as $|u_{i_r}| \leq u_{i_{r,\max}}, r = 1, \dots, m_i$ and $|x_{i_r}| \leq x_{i_{r,\max}}, r = 1, \dots, n_i$. The gain of the state feedback controller $u_i = K_i x_i$ which minimizes the upper bound of objective function (10) at time k is obtained as $K_i = Y_i G_i^{-1}$. Matrices $M_i \in R^{n_i \times n_i}$, $G_i \in R^{n_i \times n_i}$, and $Y_i \in R^{m_i \times n_i}$ are obtained from the following optimization problem with $0 < \lambda_i < 1$.

$$\min_{M_i, G_i, Y_i, \xi_i} \xi_i \quad (13a)$$

$$\begin{bmatrix} I & \bar{x}_i^T \\ \bar{x}_i & M_i \end{bmatrix} \geq 0 \quad (13b)$$

$$\begin{bmatrix} G_i + G_i^T - M_i & * & * & * \\ A_{ii} G_i + B_{ii} Y_i & M_i & * & * \\ Q_i^{1/2} G_i & 0 & \xi_i I & * \\ R_i^{1/2} Y_i & 0 & 0 & \xi_i I \end{bmatrix} \geq 0 \quad (13c)$$

$$\begin{bmatrix} (1-\lambda_i)(G_i + G_i^T - M_i) & * & * \\ 0 & \lambda_i P_{d,i} & * \\ A_{ii} G_i + B_{ii} Y_i & E_i & M_i \end{bmatrix} \geq 0 \quad (13d)$$

$$\begin{bmatrix} G_i + G_i^T - \kappa_i^2 I & * \\ A_{ji} G_i + B_{ji} Y_i & I \end{bmatrix} \geq 0, \kappa_i = \left\| (A_{ji} + B_{ji} \bar{K}_i) \right\| \quad (13e)$$

$$X_i - M_i \geq 0 \text{ with } X_{i_{rr}} \leq x_{i_{r,\max}}^2, \quad r = 1, \dots, n_i \quad (13f)$$

$$\begin{bmatrix} W_i & Y_i \\ Y_i^T & G_i + G_i^T - M_i \end{bmatrix} \geq 0, \ W_{i_{rr}} \leq u_{i_{r,\max}}^2, r = 1, \dots, m_i \quad (13g)$$

Proof. Inequality (13b) is obtained by substituting $P_i = \xi_i M_i^{-1}$ into $V_i(\bar{x}_i) = \bar{x}_i^T P_i \bar{x}_i \leq \xi_i$ and employing the Schur complement. To derive inequality (13c), substitute the i^{th} subsystem's local dynamic equation (9) into (11) and apply Schur complement to it. Then, multiply $\text{diag}(G_i, I, I, I)$ to its right hand and $\text{diag}(G_i^T, I, I, I)$ to its left hand side. Note that $G_i + G_i^T - M_i \leq G_i^T M_i^{-1} G_i$.

Inequality (12d) comes from Lemma 1 and constructs an RPI set for the i^{th} subsystem and guaranties that the states of the subsystem i will stay inside this RPI set regardless of disturbances imposed by other subsystems. It is known from the proof of Lemma 1 [13, 14] that the set $\Omega_i(P_i) = \{x_i \in R^{n_i} | x_i^T P_i x_i - \xi_i \leq 0\}$ is an RPI set for system (3) if the following two conditions hold simultaneously.

$$\frac{1}{\xi_i} x_i^{+T} P_i x_i^+ - \frac{1}{\xi_i} x_i^T P_i x_i \leq 0, \quad (14)$$

$$\frac{1}{\xi_i} x_i^T P_i x_i \geq d_i^T P_{d,i} d_i, \quad (15)$$

where $P_{d,i} = \text{diag}(M_j^{-1})/n_c, j \in N_i, n_c = \text{Card}(N_i)$. Applying S-procedure, an equivalent inequality can be obtained for (14) and (15) as

$$\left(\frac{1}{\xi_i} x_i^{+T} P_i x_i^+ - \frac{1}{\xi_i} x_i^T P_i x_i\right) - \lambda_i \left(d_i^T P_{d,i} d_i - \frac{1}{\xi_i} x_i^T P_i x_i\right) \leq 0, (16)$$

Substituting the subsystem's dynamic in (3) and considering the state feedback controller $u_i = K_i x_i$ and $P_i = \xi_i M_i^{-1}$ yields

$$\left((A_{ii} + B_{ii} K_i) x_i + E_i d_i\right)^T M_i^{-1} \left((A_{ii} + B_{ii} K_i) x_i + E_i d_i\right) - x_i^T M_i^{-1} x_i - \lambda_i \left(d_i^T P_{d,i} d_i - x_i^T M_i^{-1} x_i\right) \leq 0, \quad (17)$$

which can be rewritten as

$$\begin{bmatrix} x_i \\ d_i \end{bmatrix}^T \begin{bmatrix} \Gamma_i & * \\ E_i^T M_i^{-1} (A_{ii} + B_{ii} K_i) & E_i^T M_i^{-1} E_i - \lambda_i P_{d,i} \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} \leq 0, (18)$$

where $\Gamma_i = (A_{ii} + B_{ii} K_i)^T M_i^{-1} (A_{ii} + B_{ii} K_i) + (-1 + \lambda_i) M_i^{-1}$.

For (18) to be true, it is sufficient that the following inequality holds

$$\begin{bmatrix} (-1 + \lambda_i) M_i^{-1} & 0 \\ 0 & -\lambda_i P_{d,i} \end{bmatrix} + \begin{bmatrix} (A_{ii} + B_{ii} K_i)^T \\ E_i^T \end{bmatrix} M_i^{-1} [(A_{ii} + B_{ii} K_i) \quad E_i] \leq 0. \quad (19)$$

By applying the Schur complement to (19) one can obtain

$$\begin{bmatrix} (1 - \lambda_i) M_i^{-1} & 0 & (A_{ii} + B_{ii} K_i)^T \\ 0 & \lambda_i P_{d,i} & E_i^T \\ A_{ii} + B_{ii} K_i & E_i & M_i \end{bmatrix} \geq 0. \quad (20)$$

Multiplying $\text{diag}(G_i, I, I)$ and its transpose to the right and left hand sides of (20) respectively, and having $G_i + G_i^T - M_i \leq G_i^T M_i^{-1} G_i$ in mind, will lead to (13d). Moreover, (13f) and (13g) ensure the constraint satisfaction on states and inputs, respectively [7].

Dealing with systems with input couplings, for each j the effect of subsystem i on subsystem j at the next sample time is proportional to $E_j = (A_{ji} + B_{ji} K_i)$. Hence, while designing K_i for the i^{th} subsystem, some constraints should be considered to prevent subsystem i from altering the feasibility of LMIs of subsystem j in the next sample time. Therefore, the following condition must hold.

$$(A_{ji} + B_{ji} K_i)^T (A_{ji} + B_{ji} K_i) \leq \kappa_i^2 I, \quad (21)$$

where $\kappa_i = \|(A_{ji} + B_{ji}\bar{K}_i)\|$ and $\bar{K}_i$ is the feedback gain calculated in the previous sample time. Using the Schur complement, (21) turns into condition (13e). ■

Recursive feasibility is very important in MPC design. In most LMI-based MPC formulations [7, 9, 10], from the assumption of initial feasibility, recursive feasibility is verified by the fact that the only LMI containing the state information, or in other word, the only LMI which may change in each sample time and thus need to be checked for feasibility, is the one ensuring the states being in the invariant set, which is always satisfied by construction. However, in the framework of this paper, this fact is not valid because the feedback gain of other subsystems will change the E_i in (13d).

Theorem. 2. If the LMIs of Theorem 1 are feasible at the initial point, they will be feasible for the rest of the time.

Proof. Denote the optimal solution obtained from solving the LMIs of Theorem 1 at time $k = 0$ as $\{M_i^*, G_i^*, Y_i^*, \xi_i^*\}$ and $P_i^* = \xi_i^* M_i^{*-1}$. When $x_i(1)$ is obtained, (13d) ensures that $x_i^T(1)P_i^*x_i(1) \le \xi_i^*$ which also implies that $\bar{x}_i(1)^T P_i^* \bar{x}_i(1) \le \xi_i^*$. Thus, (13b) also holds for time $k = 1$. On the other hand, (17) is true if the following inequality holds true where ε_i is a positive scalar and σ_i is the largest eigenvalue of M_i^{-1}.

$$\begin{aligned}(1+\varepsilon_i)(x_i^T(A_{ii}+B_{ii}K_i)^T M_i^{-1}(A_{ii}+B_{ii}K_i)x_i) \\ +(1+{\varepsilon_i}^{-1})(\sigma_i d_i^T E_i^T E_i d_i) - x_i^T M_i^{-1} x_i \\ -\lambda_i\left(d_i^T P_{d,i} d_i - x_i^T M_i^{-1} x_i\right) \le 0\end{aligned} \quad (22)$$

Furthermore, satisfaction of (13e) ensures that $E_i^T E_i$ decreases at each sample time. In fact, by implementing condition (13e), the local state feedback gain of each subsystem will be obtained in a way that it will not ruin the feasibility of LMIs of other subsystem at the next sample time. Besides, by definition, $P_{d,i}$ will be larger at each sample time as the RPI ellipsoid for each subsystem shrinks. Thus, it can be concluded from (22) that (13d) also remains feasible for each subsystem. ■

Theorem 3. All subsystems (3) will be input to state stable. Moreover, the overall system is stable.

Proof. Based on Lemma 2, it needs to be shown that $V_i^*(x) = x_i^T P_i^* x_i$ is an ISS Lyapunov function where P_i^* is the optimal value of P_i at time k. At first, it is obvious that

$$\sigma_{i_{\min}}\|x_i\|^2 \le V_i(x_i) \le \sigma_{i_{\max}}\|x_i\|^2, \quad (23)$$

where $\sigma_{i_{\min}} = \min\{\underline{\sigma}(P_i^*)|k \ge 0\}$ and $\sigma_{i_{\max}} = \max\{\bar{\sigma}(P_i^*)|k \ge 0\}$. $\underline{\sigma}(.)$ and $\bar{\sigma}(.)$ are the minimum and maximum singular values. Since (11) is guaranteed when (13c) holds, one can write

$$V^*(\bar{x}_i^+) - V^*(\bar{x}_i) < -\bar{x}_i^T Q_i \bar{x}_i - u_i^T R_i u_i \le -\|\bar{x}_i\|_Q^2. \quad (24)$$

On the other hand, it is known from (9) and (3) that $x_i^+ = \bar{x}_i^+ + E_i d_i$. Therefore, there exists a $\mathcal{K}_\infty$ function $\rho_d(.)$ which $\|x_i^+ - \bar{x}_i^+\| \le \rho_d(\|d_i\|)$. Besides, Lemma 3 implies that $\|V_i^*(x_i^+) - V_i^*(\bar{x}_i^+)\| \le L_P(k)\|x_i^+ - \bar{x}_i^+\|$. Therefore, we have

$$|V_i^*(x_i^+) - V_i^*(\bar{x}_i^+)| < L_P(k)\rho_d(\|d_i\|). \quad (25)$$

By considering $\bar{x}_{i,(k|k)} = x_i$, the following inequality can be derived from (24) and (25).

$$V_i^*(x_i^+) - V_i^*(x_i) \le -\|x_i\|_Q^2 + \bar{L}_P\rho_d(\|d_i\|), \quad (26)$$

where $\bar{L}_P = \max\{L_P(k)|k \ge 0\}$. Eventually, (23) and (26) demonstrate that $V_i^*(x_i)$ is an ISS Lyapunov function. Thus, subsystem (3) will be ISS in the presence of d_i.

However, d_i is imposed by the other subsystems' states and it is known that the states of each subsystem lie in its corresponding RPI set $\Omega_{x_i} = \{x_i|x_i^T P_i x_i \le \xi_i\}$ which shrinks at each time step by solving (13). Therefore, the disturbance set imposed on each subsystem $\Omega_{d,i} = \{d_i|d_i^T P_{d,i} d_i \le 1\}$ also shrinks at the next time step. Repeating this, finally d_i will vanish and asymptotic stability can be obtained. Consequently, the state of the overall system will also go toward origin. ■

Remark 2. Note that, it is possible that the norm of x_j, $j \in N_i$ and consequently the norm of the mutual disturbances d_i does not decrease at each time step. However, as stated in the proof of Theorem 3, x_j, $j \in N_i$ will always remain in their RPI set which is shrinking at each time step and Theorem 1 only uses this information.

Remark 3. Inequality (13d) is an LMI when λ_i is pre-specified. Otherwise, it is a Bilinear Matrix Inequality (BMI). Hence, to avoid computational complexity, one can choose a value for λ_i offline, which leads to larger RPI set and use it at each time step.

The results of this section are summarized in Algorithm 1.

Algorithm 1

Offline part: Step 1. Decompose the system to some subsystems as (3).

Step 2. Select the largest disturbance set on each subsystem as $\Omega_{d,i,\max} = \{d_i|d_i^T P_{d,i,\max} d_i \le 1\}$ where $P_{d,i,\max} = \text{diag}((x_{jr,\max}^{-2}))/n_c, j \in N_i, r = 1, \dots, n_j$ $n_c = \text{Card}(N_i)$. Solve an optimization problem with objective function $\max_{M_i,G_i,Y_i} \log\det(M_i)$ offline with conditions (13c), (13d), (13f), and (13g). Save the obtained solutions as $\{M_i^0, K_i^0, \lambda_i^0\}$.

Online part: Step 3. Replace λ_i^0 in (13d) to make it an LMI condition. Use K_i^0 at time $k = 0$ wherever the previous state gains is needed. Solve (13) for each subsystem.

Step 4. Apply $u_i = K_i x_i$ to the system. Update the mutual disturbance set $P_{d,i} = \text{diag}(M_j^{-1})/n_c, j \in N_i, n_c = \text{Card}(N_i)$ and E_i with obtained state feedback gain for each subsystem. Repeat solving (13).

B. Extension to uncertain systems and other properties

The results presented in Section III-A can also be easily extended to decentralized control of linear uncertain systems. The results also satisfy Plug-and-Play (PnP) decentralized control properties.

1) Linear systems with polytopic uncertainty

Suppose that system (1) is composed of N subsystems as (3) where each subsystem's dynamic can be formulated inside a polytope as

$$[A_{ii}\ B_{ii}\ E_i] = \sum_{l=1}^{L}\beta_i^l[A_{ii}^l\ B_{ii}^l\ E_i^l];\ \sum_{l=1}^{L}\beta_i^l = 1;\ \beta_i^l \ge 0. \quad (27)$$

Then, Theorem 1 can be used to compute control signal for each subsystem by solving (13) for every local vertices $l = 1, \dots, L$.

2) Linear systems with additive disturbances

Let system (1) have also additive disturbances as

$$x^+ = Ax + Bu + Hw, \qquad (28)$$

where $w \in W$, $W = \{w \in R^{n_w} | w^T w \le \gamma^2\}$. Suppose that system (28) can be decomposed into N subsystem (3) represented as

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + E_i d_i + H_i w_i, \qquad (29)$$

where $H_i = [H_{ij}]_{j\in\{i,N_i\}}$, $w_i = [w_j]^T_{j\in\{i,N_i\}}$, and $w_i^T w_i \le \gamma_i^2 \le \gamma^2$. Introducing the extended disturbance notation as $E_{i,e} = [E_i, H_i]$ and $d_{i,e} = [d_i^T, w_i^T]^T$, a similar subsystem's as (3) can be obtained $(x_i^+ = A_{ii}x_i + B_{ii}u_i + E_{i,e}d_{i,e})$. The extended disturbance $d_{i,e}$ is also bounded at each sample time as $\Omega_{d,i,e} = \{d_{i,e} | d_{i,e}^T P_{d,i,e} d_{i,e} \le 1\}$, where $P_{d,i,e} = \text{diag}([M_j^{-1}, 1/\gamma_i^2])/(n_c, n_w), j \in N_i$, $n_c = \text{Card}(N_i)$. Thus, Theorem 1 can be applied to obtain the state feedback MPC law for each subsystem and the ISS property is preserved for each subsystem and overall system. Note that in this case, the states of the system cannot reach the origin unless w be a vanishing type of disturbance.

3) Plug and Play properties

The proposed method also enjoys the so called PnP properties [12]. It means that it can be used to decentralized control of coupled subsystems where subsystem are allowed to join or leave the system offline. When a subsystem leaves, the disturbance on other subsystems will remain inside the previous disturbance set $\Omega_{d,i}$. Thus, the design will not alter. On the other hand, if an additional subsystem is added, it has effect only on its neighbor subsystems and the stability can be achieved by retuning $\Omega_{d,i}$ in the control design of its neighbors.

IV. Simulation Results

Consider a system consisted of $N = 10$ mass-spring-dampers which are coupled by inputs and states. The dynamic of the system is as follows

$$\begin{aligned} m\ddot{x}_1 &= u_1 - k_0 x_1 - h_0 \dot{x}_1 - k_c(x_1 - x_2) \\ m\ddot{x}_i &= u_i - i_c(u_{i-1} + u_{i+1}) - (k_0 + 2k_c)x_i \\ &\quad -h_0\dot{x}_i + k_c(x_{i-1} + x_{i+1}), i = 2, \dots, N-1 \\ m\ddot{x}_N &= u_N - k_0 x_N - h_0 \dot{x}_N - k_c(x_{N-1} - x_N) \end{aligned}$$

where $m = 1.5$, $k_0 = 1.05$, $h_0 = 0.3$, $k_c = 0.1$ and $i_c = 0.1$. By choosing the position and velocity of each mass as state variables, the system has 20 states and 10 inputs which can be clustered in 10 subsystems as $\{x_{i,1}, x_{i,2}, u_i\}, i = 1, \dots, 10$. The discrete time system's model can be obtained as (1) with Euler forward method with sample time $T_s = 0.1$ s.

The input and state constraints are $-2 \le u_i \le 2$ and $-2 \le x_i \le 2$, respectively. To demonstrate the applicability of the results of this paper, the state feedback MPC law is obtained for each subsystem by solving the LMIs of Theorem 1. The control parameters for each subsystem are as $Q_i = \text{diag}([1.5, 1.5])$; $R_i = 0.1; \lambda_i = 0.01; i = 1, \dots, 10$. The initial state for each mass is $x_{0,i} = [0.6, -0.15]$.

The state trajectory of subsystems 1, 5 and 10 of the controlled system from the initial state and the corresponding control inputs have been depicted in Figs. 1 and 2, respectively. To compare the results, a centralized MPC with the same parameters is also applied to the system. It can be seen that the proposed decentralized MPC is able to steer the states of the system to the origin. Note that by shrinking the mutual disturbance set on each subsystem at each time step, eventually all the states can be steered to the origin.

Moreover, the average CPU time over $N_k = 80$ sample times for centralized MPC is around $t_c = 4.5$ s per step while with the proposed decentralized method, each control input can be obtained in less than around $t_{d_i,\max} = 0.02$ s per step.

This demonstrates that the proposed method can be useful to apply MPC to multi-input large scale systems with fast dynamics. All simulations has been done with Matlab 2014 LMI toolbox on a Windows 64-bit OS, with 3 GHz Core i5 CPU and 8 GB of RAM.

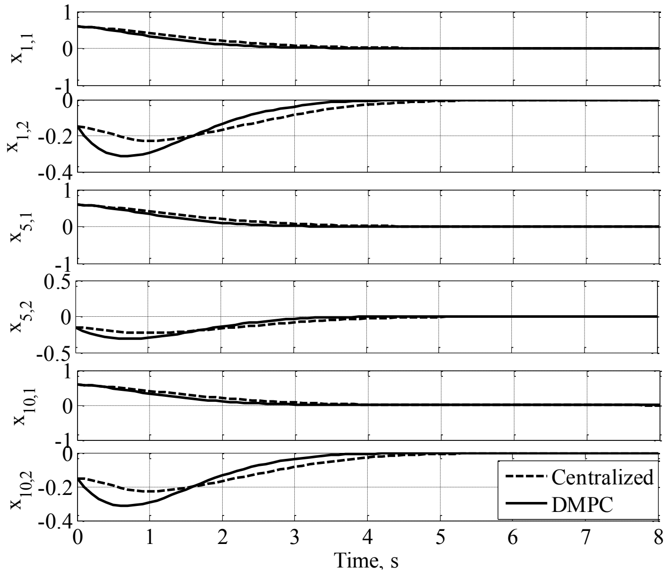

Fig. 1. State trajectory of closed loop system. Dashed line: centralized MPC and bold line: the proposed decentralized MPC.

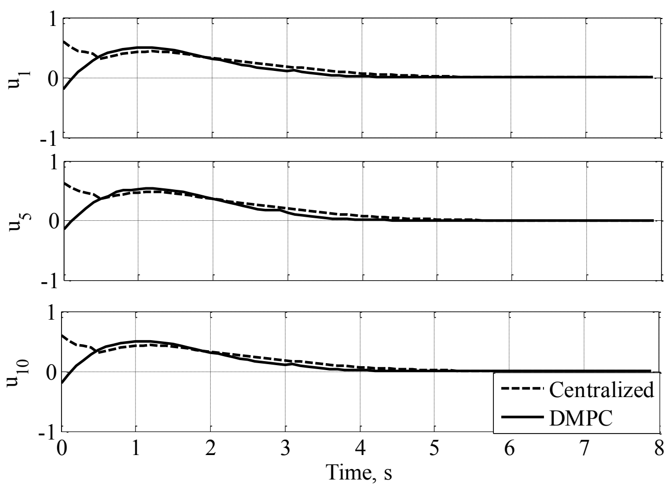

Fig. 2. Inputs of closed loop system. Dashed line:centralized MPC and bold line: the proposed decentralized MPC.

Besides, solving LMI based centralized MPC for systems with a high number of states, demand solving a set of LMIs with high dimensions and many variables. Therefore, this may result into a very conservative invariant set which is much smaller than the actual region of attraction of the system. For example, if the initial state of the system is $x_{0,1} = x_{0,10} = [1.6,0]$ and $x_{0,i} = [-1.5,0], i = 2,\dots,9$, centralized MPC cannot be used to steer this initial state to the origin, because x_0 is outside of the maximum invariant set computed by solving the centralized MPC's set of LMIs. However, the proposed method can easily drive the system's state from this initial point to the origin as presented in Fig. 3 for subsystems 1, 5 and 10, since the initial state is inside the RPI set of each subsystem.

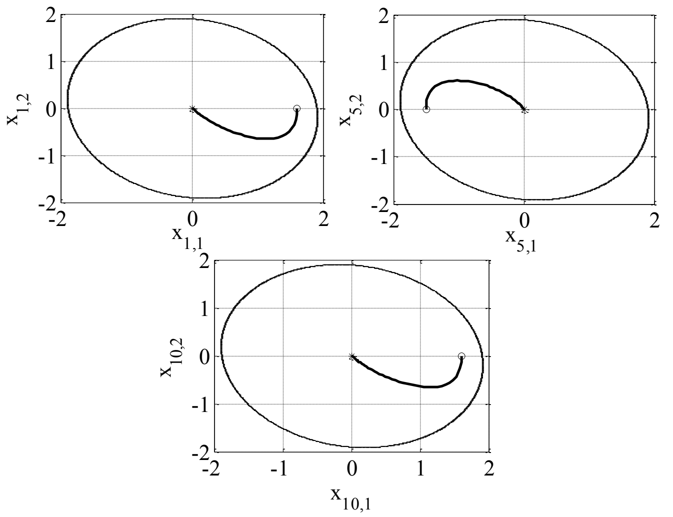

Fig. 3. State trajectory of closed loop system starting from initail point x_0 using proposed decentralized MPC.

V. Conclusion

In this paper, a decentralized MPC algorithm based on LMI has been presented to reduce the computational time of MPC for linear discrete time system with a high number of inputs and states. Each subsystem just uses its local objective function and local dynamics to compute the state feedback control gain. However, in order to account for interactions with other subsystems, some information from neighbor subsystems has been used to compute an RPI set for each subsystem. The ISS stability for each subsystem is proved. Besides, the attenuation of state dependent disturbances for each subsystem is also considered in the design and recursive feasibility of LMIs of each subsystem is ensured despite the coupling between the inputs and states of subsystems. Thus, the overall system's states can be steered to the origin with lower required computational time. Simulation results show the effectiveness of proposed method. Moreover, it has been shown that with some minor modifications, the results are applicable to design control action for polytopic uncertain systems and also systems with external additive disturbances.

References

[1] Y. Wang and S. Boyd, "Fast model predictive control using online optimization," *IEEE Transactions on Control Systems Technology*, vol. 18, no. 2, pp. 267-278, 2010.

[2] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos, "The explicit linear quadratic regulator for constrained systems," *Automatica*, vol. 38, no. 1, pp. 3-20, 2002.

[3] G. Pannocchia, J.B. Rawlings, and S.J. Wright, "Fast, large-scale model predictive control by partial enumeration," *Automatica*, vol. 43, no. 5, pp. 852-860, 2007.

[4] B. Hernandez, P. Baldivieso, and P. Trodden, "Distributed MPC: Guaranteeing Global Stability from Locally Designed Tubes," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 11829-11834, 2017.

[5] R. Guicherd, P.A. Trodden, A.R. Mills, and V. Kadirkamanathan, " Weak Interactions Based System Partitioning Using Integer Linear Programming," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 3635-3641, 2017.

[6] P.D. Christofides, R. Scattolini, D.M. de la Pena, and J. Liu, "Distributed model predictive control: A tutorial review and future research directions," Computers & Chemical Engineering, vol. 51, pp. 21-41, 2013.

[7] M.V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361-1379, 1996.

[8] B. Ding, "Distributed robust MPC for constrained systems with polytopic description," *Asian Journal of Control*, vol. 13, no. 1, pp. 198-212, 2011.

[9] W. Al-Gherwi, H. Budman, and A. Elkamel, "A robust distributed model predictive control algorithm," Journal of Process Control, vol. 21, no. 8, pp. 1127-1137, 2011.

[10] L. Zhang, W. Xie, and J. Wang, "Robust distributed model predictive control of linear systems with structured time-varying uncertainties," *International Journal of Control*, vol. 90, no. 11, pp. 2449-2460, 2017.

[11] S. Adelipour, M. Rastgar, and M. Haeri, "Computational load reduction in model predictive control of nonlinear systems via decomposition," *In IEEE 2017 5th International Conference on Control, Instrumentation, and Automation (ICCIA),* pp. 216-221, Shiraz, Iran, Nov 2017.

[12] S. Riverso, M. Farina, and G. Ferrari-Trecate, "Plug-and-play decentralized model predictive control for linear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 10, pp. 2608-2614, 2013.

[13] H.N. Nguyen, S. Olaru, P.O. Gutman, and M. Hovd, "Constrained control of uncertain, time-varying linear discrete-time systems subject to bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 831-836, 2015.

[14] W. Yang, G. Feng, and T. Zhang, "Robust model predictive control for discrete-time Takagi–Sugeno fuzzy systems with structured uncertainties and persistent disturbances," *IEEE Transactions on Fuzzy Systems*, vol. 22, no. 5, pp. 1213-1228. 2014.

[15] W. Yang, G. Feng, and T. Zhang, "Robust model predictive control of uncertain linear systems with persistent disturbances and input constraints," *In IEEE European Control Conference (ECC)*, pp. 542-547, Zurich, Switzerland, July 2013.

[16] P.A. Trodden and J.M. Maestre, "Distributed predictive control with minimization of mutual disturbances," *Automatica*, vol. 77, pp. 31-43, 2017.

[17] S, Riverso and G. Ferrari-Trecate, "Plug-and-play distributed model predictive control with coupling attenuation," *Optimal Control Applications and Methods*, vol. 36, no. 3, pp. 292-305, 2015.

[18] C. Conte, C.N. Jones, M. Morari, and M.N. Zeilinger, "Distributed synthesis and stability of cooperative distributed model predictive control for linear systems," *Automatica*, vol. 69, pp. 117-125, 2016.

[19] Z. Wang and C.J. Ong, "Distributed MPC of constrained linear systems with time-varying terminal sets," *Systems & Control Letters*, vol. 88, pp. 14-23, 2016.

[20] S. Riverso, "Distributed and plug-and-play control for constrained systems". Doctoral dissertation, University of Pavia, 2014.