A note on scalar "generalized" invexity

Riccardo Cambini and Laura Carosi*

Abstract

This paper aims to study how much "generalized" invex properties differ from invexity and to establish whether or not the use of more and more parameters and functionals in the definitions is really effective and helpful. In particular, both smooth and nonsmooth scalar functions are considered. As a conclusion, by means of some equivalence results not necessarily related to invexity, it is proved that several "generalized" invexity properties are actually equivalent to invexity, and that this happens in both the differentiable case and the nondifferentiable one. In other words, the introduction of parameters in defining scalar "generalized" invexity properties does not yield "a priori" any kind of generalization.

Key words: Generalized convexity, generalized invexity.
AMS - 2010 Math. Subj. Class. 90C26, 90C30, 90C46.

1 Introduction

As it is very well known, invexity guarantees that critical points are global minimum points, Fritz John conditions are sufficient optimality conditions and weak and strong duality results hold. In order to get larger and larger classes of functions which verify those nice properties, a huge amount of research papers provides many different generalizations of invexity. Even an hasty and superficial reading of the most recent contributions tells us that many different definitions of generalized invex functions are obtained by weakening the differentiable assumption and by introducing more and more parameters and/or functionals which are required to verify some nice conditions (see for example [9, 18] and reference therein).
Unfortunately, the proposed definitions are not often so easy to be verified and in many cases no examples are provided in order to show that these classes of functions are true generalizations of some other existing classes of generalized convex functions.

*Department of Economics and Management, University of Pisa, Via Cosimo Ridolfi 10, 56124 Pisa (Italy). E-mails: riccardo.cambini@unipi.it - laura.carosi@unipi.it

Therefore two main questions arise: are all the introduced parameters really useful? Are these classes really different?
Some relatively recent papers has taken an hard look to the generalized invexity; Craven and Glover [7], Caprary [4] and Mititelu [15] proved the equivalences among some generalized invexity properties ([1]).
Moreover, Zălinescu [19] offers a critical review of some generalized invexity concepts and shows that the use of parameters in some generalized invexity definitions is not always correct from a mathematical point of view.
Our paper aims to strenghten the results by Caprari [4] and to go further in showing that various "generalized" invexity properties are not actually true generalizations of the class of scalar invex functions. This is proved for both smooth and nonsmooth functions. More precisely, we first describe several different conditions and we prove the equivalence among all of them (see Theorem 2 and Corollary 1). At a first sight, the introduced properties do not seem related with invexity and "generalized" invexity, but as soon as we properly specify them we are able to recognize many of the definition proposed in the literature of generalized invexity. For the sake of clearness, let us consider a function $f : X \to \Re$, $X \subseteq \Re^n$, and a set valued function $K : X \to \Re^n$ such that $K(x)$ is a nonempty convex and compact set for all $x \in X$. In Theorem 2 function f is said to verify condition *B1)* if there exists $\eta : (X \times X) \to \Re^n$ such that:

$$f(x_1) - f(x_2) \geq \xi^T \eta(x_1, x_2) \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X.$$

If we assume f to be differentiable and we take $K(x) = \{\nabla f(x)\}\ \forall x \in X$, condition *B1)* is nothing but invexity. In a similar way, starting from the other conditions presented in Theorem 2 and Corollary 1 we are able, for example, to recover the definition of pseudoinvexity, F-convexity, F-pseudoconvexity, strong pseudoinvexity, strong F-pseudoconvexity, up to the notion of $(\Im, b, \phi, \rho, \theta)$-univexity used by Zalmai in [20] for n-set functions.
Therefore, by proving the equivalence of all the properties given in Thereom 2 and Corollary 1 we show that all the above mentioned generalized invexity properties coincide with invexity. Thanks to their very general formulations, Theorem 2 and Corollary 1 allow us to obtain the same equivalence results even in the nondifferentiable case. Moreover, Theorem 2 and Corollary 1 allow us to state the equivalence of many other kinds of generalized invexity definitions by means of suitable specifications of the set valued function K and of the other parameters.
Furthermore, the first condition used in Theorem 2, that is Condition *A)*, does not use any kind of parameters or functionals and it is in turn equivalent to the other listed properties. As a consequence, the use of parameters and or functionals in defining "generalized" invexity properties does not yield "a priori" any kind of generalization.

[1]See also Craven [6] for generalized invex vector function

The paper is organized as follows. In Section 2 we recall the definitions of invexity, pseudoconvexity and semi-pseudoconvexity and we provide motivation for the present study. In Section 3 several properties are introduced and their equivalence is proved. These equivalence results are used in Section 4 in order to obtain the equivalence among some "generalized" invexity properties for both smooth and non-smooth scalar functions.

2 Motivation and preliminaries

The aim of this section is to point out the need of clarifying the relationships existing between invexity and "generalized" invexity. The equivalence between invexity, pseudoinvexity and the global optimality of critical points have been already given by Craven and Glover in [7]; this result seems to be unknown or considered as not true ([2]).

For this very reason, in this preliminary section it is worth taking into account the concepts of invexity and pseudoinvexity, and the definition of semi-pseudoconvexity introduced in [3], pointing out their relationships, their differences and, in some cases, equivalences. In this light, let us first recall the following definitions.

- Let $f : X \to \Re$, with $X \subseteq \Re^n$ open set, be a differentiable scalar function. Function f is said to be :
  - *invex* in X (ivx) if there exists a function $\eta : (X \times X) \to \Re^n$ such that for all $x_1, x_2 \in X$ it holds:

$$f(x_1) - f(x_2) \geq \nabla f(x_2)^T \eta(x_1, x_2)$$

  - *pseudoinvex* in X (p.ivx) if there exists a function $\eta : (X \times X) \to \Re^n$ such that for all $x_1, x_2 \in X$ it holds:

$$f(x_1) < f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T \eta(x_1, x_2) < 0$$

  - *semi-pseudoconvex* in X (sm.pcx) if for all $x_1, x_2 \in X$ it holds:

$$f(x_1) < f(x_2) \quad \Rightarrow \quad \nabla f(x_2) \neq 0$$

- Let $f : X \to \Re$, with $X \subseteq \Re^n$ open set, be a differentiable scalar function and let $\eta : (X \times X) \to \Re^n$. Function f is said to be :
  - *η-invex* in X (η-ivx) if for all $x_1, x_2 \in X$ it holds:

$$f(x_1) - f(x_2) \geq \nabla f(x_2)^T \eta(x_1, x_2)$$

[2] This aspect has been already underlined by other contributions. The reader can see for example [4, 8]

– *η-pseudoinvex* in X (η-pvx) if for all $x_1, x_2 \in X$ it holds:

$$f(x_1) < f(x_2) \quad \Rightarrow \quad \nabla f(x_2)^T \eta(x_1, x_2) < 0$$

It is worth pointing out the great difference lying in the definitions of invexity and η-invexity: in the former concepts, there exists a function $\eta : (X \times X) \to \Re^n$ verifying a certain condition, in the latter ones the conditions are verified for just a specific function $\eta : (X \times X) \to \Re^n$. Hence, η-invexity concepts are far more restrictive than invexity ones. Notice also that, with a sort of grammar abuse, sometimes η-invex functions are referred to as "invex with respect to η". Finally, for the sake of completeness, notice that if $\eta(x_1, x_2) = x_1 - x_2$ then η-invexity and η-pseudoinvexity are nothing but the convexity and pseudoconvexity concepts, respectively ([3]).

The equivalence between the semi-pseudoconvexity and the global optimality of critical points have been shown in [3]; moreover, the equivalence between invexity, pseudoinvexity and the global optimality of critical points have been given by Craven and Glover in [7] (see also the simpler proof given by Ben-Israel and Mond in [14]). Just for the sake of completeness it is worth recalling these results giving also an independent proof of them.

Theorem 1 *Let $f : X \to \Re$, with $X \subseteq \Re^n$ open set, be a differentiable scalar function. The following properties are equivalent:*

i) all critical points are global minima;

ii) f is semi-pseudoconvex;

iii) f is invex;

iv) f is pseudoinvex.

Proof *i)⇒ii)* Just notice that if $f(x_1) < f(x_2)$ then x_2 is not a global minimum and hence for property *i)* it must be $\nabla f(x_2) \neq 0$.

ii)⇒iii) Let us define the following functional:

$$\eta(x_1, x_2) = \begin{cases} \frac{f(x_1) - f(x_2)}{\nabla f(x_2)^T \nabla f(x_2)} \nabla f(x_2) & \text{if } \nabla f(x_2) \neq 0,\ f(x_1) < f(x_2) \\ 0 & \text{otherwise} \end{cases}$$

In the case $\nabla f(x_2) \neq 0$, $f(x_1) < f(x_2)$, it results

$$\nabla f(x_2)^T \eta(x_1, x_2) = \nabla f(x_2)^T \frac{f(x_1) - f(x_2)}{\nabla f(x_2)^T \nabla f(x_2)} \nabla f(x_2) = f(x_1) - f(x_2).$$

[3] For a wider discussion on this topic see the book by Giorgi and Mishra [9].

In the case $\nabla f(x_2) = 0$ property *ii)* implies $f(x_1) \geq f(x_2)$ and hence $\nabla f(x_2)^T \eta(x_1, x_2) = f(x_2)^T 0 = 0 \leq f(x_1) - f(x_2)$; the same happens also in the case $\nabla f(x_2) \neq 0$, $f(x_1) \geq f(x_2)$.

iii)⇒iv) Just follows from the invexity of f by assuming $f(x_1) < f(x_2)$.

iv)⇒i) Let $x_2 \in X$ be a critical point, that is $\nabla f(x_2) = 0$. It trivially results $\nabla f(x_2)^T \eta(x_1, x_2) = 0$ so that the pseudoinvexity of f implies $f(x_2) \leq f(x_1)$ $\forall x_1 \in X$, that is to say that x_2 is a global minimum. □

Even if no kind of parameter is involved in the definition of semi-pseudoconvexity, Theorem 1 shows that this property is actually equivalent to both invexity and pseudoinvexity, thus suggesting a sort of abuse in the use of parameters in the definitions of invexity concepts.

For the sake of convenience, the inclusion relationships between the classes of functions recalled above are represented in Table 1.

ivx ≡ p.ivx ≡ sm.pcx		
∪		∪
η-ivx	⊂	η-pvx

Table 1: Inclusion relationships among the classes

It is worth providing the following examples which point out that the inclusion relationships described in Table 1 are proper (see also [2], pages 93-94).

Example 1 *Let us consider the following functions.*

i) *Let $f(x_1, x_2) = x_1^2 x_2^2$. All the critical points, that are the points on the two lines $x = 0$ and $y = 0$, results to be global minima so that f is invex. On the other hand, it results $f(0, -4) < f(3, -1)$ while $\nabla f(3, -1)^T [(0, -4) - (3, -1)] > 0$, so that f is neither quasiconvex nor pseudoconvex, and hence is neither η-pseudoinvex nor η-invex with $\eta(x_1, x_2) = x_1 - x_2$.*

ii) *Let $f(x) = ln(x)$. Function f is pseudoconvex but not convex, hence given $\eta(x_1, x_2) = x_1 - x_2$ it results to be η-pseudoinvex but not η-invex.*

Unfortunately, in the literature pseudoinvexity is often misconstrued as a generalization of invexity. Actually, what is true is that, given a specific function $\eta : (X \times X) \to \Re^n$, η-pseudoinvexity is a generalization of η-invexity.

Starting from the concepts of invex and convex functions, many generalizations have been proposed in the last decades literature. New classes have been introduced in order to derive more general optimality conditions and to deepen on the study of duality for both scalar and vector optimization problems (see for all [9, 10, 18] and reference therein). It is impossible to take into account of all the various proposed definitions, hence for the sake of convenience just the most used will be considered in the rest of this paper. These definitions share the same approach, that is the use of parameters and functionals aimed to weaken the invexity property. As an example, just take a look at the paper by Zalmai [20] who proposes for n-set functions the notions of $(F, b, \varphi, \rho, \vartheta)$-univexity and $(F, b, \varphi, \rho, \vartheta)$-pseudounivexity ([4]), where four functions and one real value are used as parameters. The following questions arise straightforward:

- are these classes concrete generalizations of invexity?
- are they useful?
- is there a sort of abuse in the use of parameters?

The next sections are aimed to answer to these very questions.

3 Fundamental equivalences

In what follows we provide some key results which allow to answer to the questions addressed at the end of the last section. In Theorem 2 and Corollary 1 several properties are defined and their equivalences are proved. The considered properties are not necessarily related to "generalized" invexity properties and can be used also in different contexts. Nevertheless by means of suitable specifications it is possible to reduce them to some invexity concepts proposed in the recent literature (see Section 4).

Theorem 2 *Let $f : X \to \Re$, $X \subseteq \Re^n$, be a scalar function and let $K : X \to \Re^n$ be a set valued function such that $K(x)$ is a nonempty convex and compact set for all $x \in X$. The following properties A), B1)-B5), C1)-C5), are all equivalent:*

A) it holds:

$$f(x_1) < f(x_2) \Longrightarrow \xi \neq 0 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

B1) there exists $\eta : (X \times X) \to \Re^n$ such that:

$$f(x_1) - f(x_2) \geq \xi^T \eta(x_1, x_2) \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

B2) there exists $\eta : (X \times X) \to \Re^n$ such that:

$$f(x_1) < f(x_2) \Longrightarrow \xi^T \eta(x_1, x_2) < 0 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

B3) there exists $\eta : (X \times X) \to \Re^n$, there exists $\vartheta : (X \times X) \to \Re^n$, there exists $\rho \in \Re$, $\rho > 0$, such that:

$$f(x_1) - f(x_2) \geq \xi^T \eta(x_1, x_2) + \rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

[4] In the definition by Zalmai [20] the parameter ρ is not limited in sign.

B4) there exists $\eta : (X \times X) \to \Re^n$*, there exists* $\vartheta : (X \times X) \to \Re^n$*, there exists* $\rho \in \Re$*,* $\rho > 0$*, such that:*

$$f(x_1) < f(x_2) \Longrightarrow \xi^T \eta(x_1, x_2) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

B5) for every $\vartheta : (X \times X) \to \Re^n$ *there exists* $\eta : (X \times X) \to \Re^n$ *and there exists* $\rho \in \Re$*,* $\rho > 0$*, such that:*

$$f(x_1) < f(x_2) \Longrightarrow \xi^T \eta(x_1, x_2) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

C1) there exists $F : (X \times X \times \Re^n) \to \Re$*, with* $F(x_1, x_2, 0) \geq 0\ \forall x_1, x_2 \in X$*, there exists* $\psi : \Re^2 \to \Re$*, with* $\psi(y_1, y_2) < 0\ \forall y_1, y_2 \in \Re$ *such that* $y_1 < y_2$*, there exists* $b : (X \times X) \to \Re$*, with* $b(x_1, x_2) > 0\ \forall x_1, x_2 \in X$*, such that:*

$$\psi(f(x_1), f(x_2)) \geq F(x_1, x_2, b(x_1, x_2)\xi) \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

C2) there exists $F : (X \times X \times \Re^n) \to \Re$*, with* $F(x_1, x_2, 0) \geq 0\ \forall x_1, x_2 \in X$*, there exists* $\psi : \Re^2 \to \Re$*, with* $\psi(y_1, y_2) < 0\ \forall y_1, y_2 \in \Re$ *such that* $y_1 < y_2$*, there exists* $b : (X \times X) \to \Re$*, with* $b(x_1, x_2) > 0\ \forall x_1, x_2 \in X$*, such that:*

$$\psi(f(x_1), f(x_2)) < 0 \Longrightarrow F(x_1, x_2, b(x_1, x_2)\xi) < 0 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

C3) there exists $F : (X \times X \times \Re^n) \to \Re$*, with* $F(x_1, x_2, 0) \geq 0\ \forall x_1, x_2 \in X$*, there exists* $\psi : \Re^2 \to \Re$*, with* $\psi(y_1, y_2) < 0\ \forall y_1, y_2 \in \Re$ *such that* $y_1 < y_2$*, there exists* $\vartheta : (X \times X) \to \Re^n$*, there exists* $\rho \in \Re$*,* $\rho > 0$*, there exists* $b : (X \times X) \to \Re$*, with* $b(x_1, x_2) > 0\ \forall x_1, x_2 \in X$*, such that:*

$$\psi(f(x_1), f(x_2)) \geq F(x_1, x_2, b(x_1, x_2)\xi) + \rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

C4) there exists $F : (X \times X \times \Re^n) \to \Re$*, with* $F(x_1, x_2, 0) \geq 0\ \forall x_1, x_2 \in X$*, there exists* $\psi : \Re^2 \to \Re$*, with* $\psi(y_1, y_2) < 0\ \forall y_1, y_2 \in \Re$ *such that* $y_1 < y_2$*, there exists* $\vartheta : (X \times X) \to \Re^n$*, there exists* $\rho \in \Re$*,* $\rho > 0$*, there exists* $b : (X \times X) \to \Re$*, with* $b(x_1, x_2) > 0\ \forall x_1, x_2 \in X$*, such that:*

$$\psi(f(x_1), f(x_2)) < 0 \Longrightarrow F(x_1, x_2, b(x_1, x_2)\xi) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

C5) for every $\vartheta : (X \times X) \to \Re^n$ *there exists* $F : (X \times X \times \Re^n) \to \Re$*, with* $F(x_1, x_2, 0) \geq 0\ \forall x_1, x_2 \in X$*, there exists* $\psi : \Re^2 \to \Re$*, with* $\psi(y_1, y_2) < 0$ $\forall y_1, y_2 \in \Re$ *such that* $y_1 < y_2$*, and there exists* $\rho \in \Re$*,* $\rho > 0$*, there exists* $b : (X \times X) \to \Re$*, with* $b(x_1, x_2) > 0\ \forall x_1, x_2 \in X$*, such that:*

$$\psi(f(x_1), f(x_2)) < 0 \Longrightarrow F(x_1, x_2, b(x_1, x_2)\xi) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in K(x_2),\ \forall x_1, x_2 \in X$$

Proof It is first worth recalling that being $K(x)$ a nonempty convex and compact set then there exists an unique element $\bar{\xi}(x) \in K(x)$ such that $\|\bar{\xi}(x)\| \leq \|\xi\| \ \forall \xi \in K(x)$, and that it results $\bar{\xi}(x)^T\bar{\xi}(x) \leq \xi^T\bar{\xi}(x) \ \forall \xi \in K(x)$.

A)⇒B1). Let us define the following functional:

$$\eta(x_1,x_2) = \begin{cases} \frac{f(x_1)-f(x_2)}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \ \bar{\xi}(x_2) & \text{if } 0 \notin K(x_2), \ f(x_1) < f(x_2) \\ 0 & \text{otherwise} \end{cases}$$

In the case $0 \notin K(x_2)$, $f(x_1) < f(x_2)$, for all $\xi \in K(x_2)$ it is:

$$\xi^T\eta(x_1,x_2) = (f(x_1) - f(x_2)) \ \frac{\xi^T\bar{\xi}(x_2)}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \leq f(x_1) - f(x_2)$$

since $(f(x_1) - f(x_2)) < 0$ and $\frac{\xi^T\bar{\xi}(x_2)}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \geq 1$. In the case $0 \in K(x_2)$ property *A)* implies $f(x_1) \geq f(x_2)$ and hence for all $\xi \in K(x_2)$ it is $\xi^T\eta(x_1,x_2) = 0 \leq f(x_1) - f(x_2)$; the same happens also in the case $0 \notin K(x_2)$, $f(x_1) \geq f(x_2)$. The whole result is then proved.

B1)⇒B2), B3)⇒B4). Follows trivially just assuming $f(x_1) < f(x_2)$.

B1)⇒B3), B2)⇒B4). Just choose $\vartheta(x_1,x_2) = 0 \ \forall x_1,x_2 \in X$.

A)⇒B5). Let us define the following functional where ρ is fixed to any positive value while ϑ is not fixed:

$$\eta(x_1,x_2) = \begin{cases} \frac{f(x_1)-f(x_2)-\rho\|\vartheta(x_1,x_2)\|^2}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \ \bar{\xi}(x_2) & \text{if } 0 \notin K(x_2), \ f(x_1) < f(x_2) \\ 0 & \text{otherwise} \end{cases}$$

Assuming $f(x_1) < f(x_2)$ property *A)* implies $0 \notin K(x_2)$. Hence,

$$\begin{aligned} \xi^T\eta(x_1,x_2) &= \left(f(x_1) - f(x_2) - \rho\|\vartheta(x_1,x_2)\|^2\right) \frac{\xi^T\bar{\xi}(x_2)}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \\ &\leq f(x_1) - f(x_2) - \rho\|\vartheta(x_1,x_2)\|^2 < -\rho\|\vartheta(x_1,x_2)\|^2 \end{aligned}$$

since $f(x_1) - f(x_2) < 0$, $\rho\|\vartheta(x_1,x_2)\|^2 \geq 0$ and $\frac{\xi^T\bar{\xi}(x_2)}{\bar{\xi}(x_2)^T\bar{\xi}(x_2)} \geq 1$. The result is then proved.

Bi)⇒Ci), $i \in \{1,\ldots,5\}$. Just choose $F(x_1,x_2,\xi) = \xi^T\eta(x_1,x_2)$, $\psi(y_1,y_2) = y_1 - y_2$, $b(x_1,x_2) = 1$.

C1)⇒C2), C3)⇒C4). Follows trivially just assuming $\psi(f(x_1),f(x_2)) < 0$.

C1)⇒C3), C2)⇒C4). Just choose $\vartheta(x_1,x_2) = 0 \ \forall x_1,x_2 \in X$.

C4)⇒A), C5)⇒A). Assuming $f(x_1) < f(x_2)$ it follows that $\psi(f(x_1),f(x_2)) < 0$ and hence for all $\xi \in K(x_2)$ it is:

$$F(x_1,x_2,b(x_1,x_2)\xi) < -\rho\|\vartheta(x_1,x_2)\|^2 \leq 0$$

Being $F(x_1,x_2,b(x_1,x_2)\xi) < 0$ it follows $b(x_1,x_2)\xi \neq 0$ so that $\xi \neq 0$ and the result is proved.

□

The previous theorem shows that the use of parameter functionals in properties *B1)-B5)* and *C1)-C5)* is useless since they are equivalent to property *A)* which has no parameter functionals at all.

It is worth noticing that many other properties can be obtained from *C1)-C5)* by properly fixing some of the parameters. It is important to point out that the properties obtained in this way are not more restrictive than the original ones, they are actually equivalent to *A)*, *B1)-B5)* and *C1)-C5)*, as it is stated in the following corollary.

Corollary 1 *Let $f : X \to \Re$, $X \subseteq \Re^n$, be a scalar function and let $K : X \to \Re^n$ be a set valued function such that $K(x)$ is a nonempty convex and compact set for all $x \in X$. Let P be any property obtained from C1)-C5) in Theorem 2 by assuming one or more of the following conditions:*

i) $\psi(y_1, y_2) = y_1 - y_2$,

ii) $\psi(y_1, y_2) = \varphi(y_1 - y_2)$ *for a suitable* $\varphi : \Re \to \Re$, *with* $\varphi(z) < 0 \ \forall z < 0$,

iii) $F(x_1, x_2, \xi) = \xi^T \eta(x_1, x_2)$ *for a suitable* $\eta : (X \times X) \to \Re^n$,

iv) $b(x_1, x_2) = 1$.

Then, P is equivalent to properties A), B1)-B5), C1)-C5), listed in Theorem 2.

Proof First note that conditions *i)-iv)* verify the assumptions of properties *C1)-C5)* in Theorem 2. Let P be obtained from property *Ci)*, with $i \in \{1, \ldots, 5\}$, by assuming one or more of conditions *i)-iv)*. The result then follows from Theorem 2 by noticing that property *Bi)* implies P and that property P itself implies *Ci)*. □

Remark 1 *The results provided in this section could be generalized by replacing $\Re^n$ with a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, by taking the set X as arbitrary and by defining K as a multifunction $K : X \rightrightarrows H$ whose values are closed and convex (possibly empty). This could be useful in the forthcoming Subsection 4.2 in order to extend the provided results by dealing with the Clarke-Rockafellar subdifferential of a general function $f : H \to \bar{\Re}$ (which may be empty at certain $x \in H$ with $f(x) \in \Re$). For the sake of simplicity, we leave such generalizations to the interested reader.*

4 "Generalized" invexity for scalar functions

The equivalences discussed in Section 3 seem not necessarily related to invexity and optimality conditions. Neverthless, some "generalized" invexity properties for scalar functions can be recognized in Theorem 2 and Corollary 1 by properly specifying the set valued function K and the other parameters.

In this section both the differentiable case and the nondifferentiable one will be handled.

4.1 The differentiable case

Let f be differentiable and let $K(x) = \{\nabla f(x)\}$, so that $\xi \in K(x_2)$ is replaced by $\nabla f(x_2)$. Under such assumptions, it is very easy to recognize in property *B1)* the very well known concept of invexity, in property *B2)* the concept of pseudoinvexity and in property *A)* the semi-pseudoconvexity concept considered in Section 2 (see also [3]). Thanks to Theorem 2 we can go further in analyzing other classes of "generalized" invex function. More precisely we are able to prove that several notions of "generalized" invexity actually coincide with the invexity. With this aim, let us recall the following definitions.

A differentiable scalar function $f : X \to \Re$, with $X \subseteq \Re^n$ open set, is said to be :

- *F-convex* if there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) = 0 \ \forall x_1, x_2 \in X$, such that
$$f(x_1) \geq f(x_2) + F(x_1, x_2, \nabla f(x_2)) \quad \forall x_1, x_2 \in X$$
- *F-pseudoconvex* if there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) = 0 \ \forall x_1, x_2 \in X$, such that
$$f(x_1) < f(x_2) \implies F(x_1, x_2, \nabla f(x_2)) < 0 \quad \forall x_1, x_2 \in X$$
- *strong pseudoinvex* if for every $\vartheta : (X \times X) \to \Re^n$ there exists $\eta : (X \times X) \to \Re^n$ and there exists $\rho \in \Re$, $\rho > 0$, such that
$$f(x_1) < f(x_2) \implies \nabla f(x_2)^T \eta(x_1, x_2) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall x_1, x_2 \in X$$
- *strong F-pseudoconvex* if for every $\vartheta : (X \times X) \to \Re^n$ there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, and there exists $\rho \in \Re$, $\rho > 0$, such that
$$f(x_1) < f(x_2) \implies F(x_1, x_2, \nabla f(x_2)) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall x_1, x_2 \in X$$
- *$(F, b, \varphi, \rho, \vartheta)$-univex* if there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0$ $\forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:
$$\varphi(f(x_1) - f(x_2)) \geq F(x_1, x_2, b(x_1, x_2) \nabla f(x_2)) \quad \forall x_1, x_2 \in X$$
- *$(F, b, \varphi, \rho, \vartheta)$-pseudounivex* if there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:
$$\varphi(f(x_1) - f(x_2)) < 0 \implies F(x_1, x_2, b(x_1, x_2) \nabla f(x_2)) < 0 \quad \forall x_1, x_2 \in X$$

- *strong* $(F,b,\varphi,\rho,\vartheta)$*-univex* if there exists $F:(X\times X\times\Re^n)\to\Re$, with $F(x_1,x_2,0)\geq 0\ \forall x_1,x_2\in X$, there exists $\varphi:\Re\to\Re$, with $\varphi(z)<0\ \forall z\in\Re$ such that $z<0$, there exists $\vartheta:(X\times X)\to\Re^n$, there exists $\rho\in\Re$, $\rho>0$, there exists $b:(X\times X)\to\Re$, with $b(x_1,x_2)>0\ \forall x_1,x_2\in X$, such that:

$$\varphi(f(x_1)-f(x_2))\geq F(x_1,x_2,b(x_1,x_2)\nabla f(x_2))+\rho\|\vartheta(x_1,x_2)\|^2\quad\forall x_1,x_2\in X$$

- *strong* $(F,b,\varphi,\rho,\vartheta)$*-pseudounivex* if there exists $F:(X\times X\times\Re^n)\to\Re$, with $F(x_1,x_2,0)\geq 0\ \forall x_1,x_2\in X$, there exists $\varphi:\Re\to\Re$, with $\varphi(z)<0\ \forall z\in\Re$ such that $z<0$, there exists $\vartheta:(X\times X)\to\Re^n$, there exists $\rho\in\Re$, $\rho>0$, there exists $b:(X\times X)\to\Re$, with $b(x_1,x_2)>0\ \forall x_1,x_2\in X$, such that:

$$\varphi(f(x_1)-f(x_2))<0\Longrightarrow F(x_1,x_2,b(x_1,x_2)\nabla f(x_2))<-\rho\|\vartheta(x_1,x_2)\|^2\quad\forall x_1,x_2\in X$$

The F-convex functions have been introduced by Hanson and Mond in [12] and for this reason they are also called Hanson and Mond functions; Craven and Glover in [7] prove that this class of function is nothing but the class of invex function. Furthermore, Caprari [4] proves the equivalence between the strong F-convexity and the strong pseudoinvexity.

Referring again to Theorem 2 and Corollary 1, strong pseudoinvex is nothing but a particular case of property *B5)* (let us recall to see $\xi\in K(x_2)$ as $\nabla f(x_2)$), while F-convex comes from *C1)* by replacing $\xi\in K(x_2)$ with $\nabla f(x_2)$ and by setting $\psi(f(x_1),f(x_2))=f(x_1)-f(x_2)$ and $b(x_1,x_2)=1$. Analogously, F-pseudoconvex and strong F-pseudoconvex are particular cases of properties *C2)* and *C5)*, respectively (let $\xi\in K(x_2)$ be $\nabla f(x_2)$, $\psi(f(x_1),f(x_2))=f(x_1)-f(x_2)$ and $b(x_1,x_2)=1$). Finally, $(F,b,\varphi,\rho,\vartheta)$-univexity, $(F,b,\varphi,\rho,\vartheta)$-pseudounivexity, strong $(F,b,\varphi,\rho,\vartheta)$-univexity and strong $(F,b,\varphi,\rho,\vartheta)$-pseudounivexity can be obtained from *C1)*, *C2)*, *C3)* and *C4)*, respectively, by setting $\psi(f(x_1),f(x_2))=\varphi(f(x_1)-f(x_2))$.

Therefore, we are able to extend the analysis started first by Craven and Glover and later by Caprari. As a trivial consequence of Theorem 2 and Corollary 1 we show that F-convexity coincides with both F-pseudoconvexity, strong pseudoinvexity and $(F,b,\varphi,\rho,\vartheta)$-pseudounivexity. Hence, all the previously recalled classes coincide with the invex one.

Theorem 3 *Let $f:X\to\Re$ be a differentiable scalar function on the open set $X\subseteq\Re^n$. The following conditions are equivalent:*

i) f is semi-pseudoconvex;

ii) f is invex;

iii) f is pseudoinvex;

iv) f is F-convex;

v) f *is F-pseudoconvex;*

vi) f *is strong pseudoinvex;*

vii) f *is strong F-pseudoconvex;*

viii) f *is $(F, b, \varphi, \rho, \vartheta)$-univex;*

ix) f *is $(F, b, \varphi, \rho, \vartheta)$-pseudounivex;*

x) f *is strong $(F, b, \varphi, \rho, \vartheta)$-univex;*

xi) f *is strong $(F, b, \varphi, \rho, \vartheta)$-pseudounivex.*

Deeply speaking, some of the definitions presented in the recent literature with the aim to generalize the concept of invexity do not provide actually any kind of generalization (along this line see also [4, 9, 19]). Moreover, the use of parameter functions aimed to look for invexity generalizations is useless, since those class of functions coincides with the one of semi-pseudoconvex functions which is related only to the behavior of the gradient of function f and does not involve any kind of parameter functions.

4.2 The nondifferentiable case

Invexity properties have been extended in the non differentiable case following various approaches, such as using Dini derivatives [10, 16] or Clarke's subdifferential [4, 7, 9]. The latter one has been the most used and will be analyzed in this subsection. Specifically speaking, invexity properties can be extended to the nondifferentiable case by assuming function f to be locally Lipschitz and by using the Clarke's subdifferential of f at x, denoted with $\partial^c f(x)$, which results to be nonempty, convex and compact due to the local Lipschitzianity of f. In this light, some of the "generalized" invexity properties discussed so far can be rewritten as follows:

- *invexity* : there exists a function $\eta : (X \times X) \to \Re^n$ such that

$$f(x_1) - f(x_2) \geq \xi^T \eta(x_1, x_2) \quad \forall \xi \in \partial^c f(x_2),\ \forall x_1, x_2 \in X$$

- *pseudoinvexity* : there exists a function $\eta : (X \times X) \to \Re^n$ such that

$$f(x_1) < f(x_2) \quad \Rightarrow \quad \xi^T \eta(x_1, x_2) < 0 \quad \forall \xi \in \partial^c f(x_2),\ \forall x_1, x_2 \in X$$

- *semi-pseudoconvexity* : it holds:

$$f(x_1) < f(x_2) \quad \Rightarrow \quad \xi \neq 0 \quad \forall \xi \in \partial^c f(x_2),\ \forall x_1, x_2 \in X$$

- *F-convexity* : there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) = 0 \ \forall x_1, x_2 \in X$, such that

$$f(x_1) \geq f(x_2) + F(x_1, x_2, \xi) \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *F-pseudoconvexity* : there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) = 0 \ \forall x_1, x_2 \in X$, such that

$$f(x_1) < f(x_2) \implies F(x_1, x_2, \xi) < 0 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *strong pseudoinvexity* : for every $\vartheta : (X \times X) \to \Re^n$ there exists $\eta : (X \times X) \to \Re^n$ and there exists $\rho \in \Re$, $\rho > 0$, such that

$$f(x_1) < f(x_2) \implies \xi^T \eta(x_1, x_2) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *strong F-pseudoconvexity* : for every $\vartheta : (X \times X) \to \Re^n$ there exists a functional $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, and there exists $\rho \in \Re$, $\rho > 0$, such that

$$f(x_1) < f(x_2) \implies F(x_1, x_2, \xi) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *$(F, b, \varphi, \rho, \vartheta)$-univexity* : there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:

$$\varphi(f(x_1) - f(x_2)) \geq F(x_1, x_2, b(x_1, x_2)\xi) \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *$(F, b, \varphi, \rho, \vartheta)$-pseudounivexity* : there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:

$$\varphi(f(x_1) - f(x_2)) < 0 \implies F(x_1, x_2, b(x_1, x_2)\xi) < 0 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *strong $(F, b, \varphi, \rho, \vartheta)$-univexity* : there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $\vartheta : (X \times X) \to \Re^n$, there exists $\rho \in \Re$, $\rho > 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:

$$\varphi(f(x_1) - f(x_2)) \geq F(x_1, x_2, b(x_1, x_2)\xi) + \rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

- *strong $(F, b, \varphi, \rho, \vartheta)$-pseudounivexity* : there exists $F : (X \times X \times \Re^n) \to \Re$, with $F(x_1, x_2, 0) \geq 0 \ \forall x_1, x_2 \in X$, there exists $\varphi : \Re \to \Re$, with $\varphi(z) < 0 \ \forall z \in \Re$ such that $z < 0$, there exists $\vartheta : (X \times X) \to \Re^n$, there exists $\rho \in \Re$, $\rho > 0$, there exists $b : (X \times X) \to \Re$, with $b(x_1, x_2) > 0 \ \forall x_1, x_2 \in X$, such that:

$$\varphi(f(x_1) - f(x_2)) < 0 \implies F(x_1, x_2, b(x_1, x_2)\xi) < -\rho \|\vartheta(x_1, x_2)\|^2 \quad \forall \xi \in \partial^c f(x_2), \ \forall x_1, x_2 \in X$$

By assuming $K(x) = \partial^c f(x)$ in Theorem 2 and in Corollary 1 it is then trivial to prove that all the equivalences given in Theorem 3 hold in the nondifferentiable case too, which means that the use of parameter functions aimed to look for invexity generalizations is useless even in the Clarke nondifferentiable case.

Theorem 4 *Let $f : X \to \Re$, with $X \subseteq \Re^n$ open set, be locally Lipschitz and let $\partial^c f(x)$ be the Clarke's subdifferential of f at x. Then, conditions i)-xi) given in Theorem 3 are equivalent.*

5 Conclusions

In this paper we prove the equivalence of various classes of "generalized" invex functions pointing out that the use of parameters and functionals does not yield "a priori" any kind of generalization. This has been shown in both the differentiable case and the nondifferentiable one. In this latter case locally Lipschitz functions are considered as well as their Clarke's subdifferential.

Looking at further classes of generalized invex functions, Mititelu [15] shows preinvexity and prepseudoinvexity are equivalent to invexity and pseudoinvexity respectively. Therefore, taking into account our equivalence results we get that even prepseudoinvexity coincides with invexity. It is worth noticing that we just consider differentiable functions on an open set, while Mititelu [15] and Mititelu and Postoloache [16] deal with non-smooth functions on arbitrary set by using the upper Dini derivatives. A possible extension of our analysis is to consider invexity properties based on the use of upper Dini derivatives.

It is worth noticing that the general formulation of Theorem 2 and Corollary 1 could suggest further equivalence results among other classes of "generalized" invex functions the interested reader can find in the huge literature on this topic.

Regarding vector valued functions, in [6] Craven presents various inclusions among different classes of generalized invexity. Following his lines and taking into account our equivalence results it could be interesting analyzing how they can be extended to the vector case.

References

[1] M. Avriel, W. E.. Diewert, S. Schaible and I. Zang (2010), *Generalized Concavity*, SIAM Classics in Applied Mathematics vol.63 (original book *Generalized Concavity*, Mathematical Concepts and Methods in Science and Engineering, vol.36, Plenum Press, 1988, New York).

[2] A. Cambini, L. Martein (2009), *Generalized Convexity and Optimization: Theory and Applications*, Lecture Notes in Economics and Mathematical Systems, vol.616, Springer, Berlin.

[3] R. Cambini, L. Carosi (2010), *Mixed Type Duality for Multiobjective Optimization Problems with Set Constraints*, in *Optimality Conditions in Vector Optimization*, M. A. Jimenéz, G. Ruiz-Garzón and A. Rufián-Lizana eds., Bentham Science Publishers, pp. 119-142.

[4] E. Caprari (2003), *ρ-invex functions and (F, ρ)-convex functions: properties and equivalencies*, Optimization, vol.65, pp. 65-74.

[5] B.D. Craven (1981), *Invex functions and constrained local minima*, Bulletin of the Australian Mathematical Society, Series B, vol. 24, pp. 357-366.

[6] B.D. Craven, *Kinds of Vector Invex* (2011), Taiwanese Journal of Mathematics, vol. 14, no. 5, pp. 1925-1933.

[7] B.D. Craven, B.M. Glover (1985), *Invex functions and duality*, Bulletin of the Australian Mathematical Society, Series A, vol.39, pp. 1-20.

[8] G. Giorgi, A. Guerraggio, J. Thierfelder (2004), *Mathematics of Optimization: Smooth and Nonsmooth Case*, Elsevier, Amsterdam.

[9] G. Giorgi, S.K. Mishra (2008), *Invexity and Optimization*, Nonconvex Optimization and its Applications, Springer, Berlin.

[10] G. Giorgi, B. Mititelu (1993), *Convexités généralisées et propriétés*, Revue Roumain de mathèmatiques pures et appliquèe, vol.38, pp. 125-142.

[11] M.A. Hanson (1981), *On sufficiency of the Kuhn-Tucker conditions*, Journal of Mathematical Analysis and Applications, vol.80, pp. 545-550.

[12] M.A. Hanson, B. Mond (1982), *Further Generalizations of Convexity in Mathematical Programming*, Journal of Information and Optimization Sciences, vol.3, pp. 25-32.

[13] V. Jeyakumar (1985), *Strong and weak invexity in mathematical programming*, Mathematical Methods of Operation Research, vol.55, pp. 109-125.

[14] B. Israel, B. Mond (1986), *What is invexity?* Bull. Bulletin of the Australian Mathematical Society, Series B, vol.26, pp. 1-9.

[15] B. Mititelu (2009), Invex sets and preinvex functions. Journal of Advanced Mathematical Studies. vol.2, no. 2, pp. 41-52.

[16] B. Mititelu (2011), Nonsmooth invex functions via upper directional derivative of Dini. Journal of Advanced Mathematical Studies, vol.4, no. 1, pp. 57-76.

[17] B. Mond, T. Weir (1981), *Generalized concavity and duality*, in *Generalized Concavity and Duality in Optimization and Economics*, S. Schaible and W.T. Ziemba eds., Academic Press, New York, pp. 263-279.

[18] R. Pini, C. Singh (1997), *A survey of recent[1985-1995]advances in generalized convexity with applications to duality theory and optimality conditions.* Optimization, vol.39, pp.311-360,

[19] C. Zălinescu (2014), *A critical view on invexity*, Journal of Optimization Theory and Applications, vol.162, no.3, pp. 695-704.

[20] G.J. Zalmai (2005), *Generalized* $(\Im, b, \phi, \rho, \theta)$*-univex n-set functions and global parametric sufficient optimality conditions in minimax fractional subset programming*, International Journal of Mathematics and Mathematical Sciences, vol.8, pp. 1253-1276.