Rendiconti Sem. Mat. Univ. Pol. Torino
Vol. 76, 2 (2018), 57 – 67

R. Bevilacqua - G. M. Del Corso - L. Gemignani*

A QR BASED APPROACH FOR THE NONLINEAR EIGENVALUE PROBLEM

Abstract. We describe a fast and numerically robust approach based on the structured QR eigenvalue algorithm for computing approximations of the eigenvalues of a holomorphic matrix-valued function inside the unit circle. Numerical experiments confirm the effectiveness of the proposed method.

1. Introduction

We consider the nonlinear eigenvalue problem (NEP) $T(z)v = 0$, $v \neq 0$, where $T : \Omega \to \mathbb{C}^{k\times k}$ is a holomorphic matrix-valued function. In the following we consider the set of holomorphic functions defined on a connected and open set $\Omega \subseteq \mathbb{C}$ with values in $\mathbb{C}^{k\times k}$. The pair (λ, v), $v \neq 0$, is an eigenpair of T if it satisfies $T(\lambda)v = 0$, i.e., $\det(T(\lambda)) = 0$ and $v \in \mathrm{Ker}(T(\lambda))$.

Nonlinear eigenvalue problems arise in many applications [3, 12]. The most studied case is the polynomial eigenvalue problem (PEP) that can be tackled by finding suitable linearizations [16] to convert PEP into an equivalent generalized eigenproblem. Linearization methods using block companion forms [13] allow the design of fast and stable methods [1, 4, 5] which exploit the unitary plus low rank structure [1]. Various methods have been also proposed in the literature to solve a NEP directly, for example Newton's method [15] and contour integrals [7, 20, 11] techniques.

If the matrix-valued function $T(z)$ is not a matrix polynomial, it might have infinitely many eigenvalues, and, hence, the usual scenario is to focus on computing eigenvalues located in a certain subset $\Delta \subset \Omega$. A possible approach for this task consists first of approximating $T(z)$ with a matrix polynomial $P_\ell(z)$ inside Δ, and then of computing the eigenvalues of $P_\ell(z)$ to provide numerical approximations of the eigenvalues of $T(z)$ in Δ. Indeed, given an approximating polynomial $P_\ell(z) = A_0 + A_1 z + \cdots + A_\ell z^\ell$ of degree ℓ, where $A_i \in \mathbb{C}^{k\times k}$, such that $\max_{z\in\Delta} \|T(z) - P_\ell(z)\|_2 \leqslant \varepsilon$, if (λ, v) is an eigenpair for $P_\ell(z)$ with $\|v\|_2 = 1$ and $\lambda \in \Delta$ then we have

$$\|T(\lambda)v\|_2 = \|(T(\lambda) - P_\ell(\lambda))v\|_2 \leqslant \|(T(\lambda) - P_\ell(\lambda))\|_2 \leqslant \varepsilon,$$

and hence (λ, v) is, in general, a good approximation of an eigenpair of T. Moreover, as shown in [12] for ε sufficiently small any μ which is not an eigenvalue for T is such that $\det(P_\ell(\mu)) \neq 0$.

In this paper we consider the problem of approximating the eigenvalues of a holomorphic matrix-valued function $T : \Omega \to \mathbb{C}^{k\times k}$ such that $\det(T(0)) \neq 0$ inside the

*The research of the last two authors was partially supported by GNCS project "Analisi di matrici sparse e data-sparse: metodi numerici ed applicazioni" and by the project sponsored by University of Pisa under the grant PRA-2017-05

unit circle $\Delta = \{z \in \mathbb{C} \colon |z| \leqslant 1\} \subset \Omega$. This issue is frequently encountered in the stability analysis of dynamical systems. Our approach is based on the approximation of $T(z)$ by means of a sequence of interpolating polynomials $P_\ell(z)$ of increasing degree ℓ determined in such a way to guarantee the uniform convergence of $P_\ell(z)$ to $T(z)$ over some compact set inside Δ.

The convergence properties follow from suitable choices of the interpolation nodes. We consider and compare different kind of polynomial approximations:

- The Taylor polynomial of degree ℓ obtained from the Taylor series at the origin.
- The interpolating polynomial of degree ℓ over the $\ell+1$ Chebyshev points of the first kind.
- The interpolating polynomial of degree ℓ over the $(\ell+1)$-st roots of the unity.
- The interpolating polynomial of degree ℓ over the ℓ-th roots of the unity plus the origin.

Once the interpolating polynomial $P_\ell(z)$ is available then a block companion linearization can be found. A different linearization whose construction only requires the evaluation of $P_\ell(z)$ at the interpolation nodes can also be obtained by exploiting the spectral decomposition of the block downshift matrix by means of Fourier transforms. We denote by $Z = U \otimes I_k$ the block downshift matrix, where $U \in \mathbb{R}^{\ell \times \ell}$ is the generator of the algebra of the circulant matrices, i.e., $U_{ij} = 1$ if $i - j \equiv 1(\mod \ell)$ and $U_{ij} = 0$ otherwise. Both linearizations inherit the structural property of being low rank perturbations of a unitary matrix. Recently the authors proposed a method based on implicitly shifted QR iterations to compute eigenvalues of unitary plus low rank matrices [5]. We apply this method for numerically solving the associated linearized eigenproblems. Extensive numerical experiments show that the resulting approach is fast and numerically robust.

2. Interpolating over the roots of unity

Let $\Delta = \{z \in \mathbb{C} \colon |z| \leqslant 1\}$ be the unit disk, and let $Int\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open disk. Denote by $\mathcal{H}(\Delta, \mathbb{C}^{k\times k})$, the class of functions T that are holomorphic on $Int\Delta$ and continuous on Δ.

THEOREM 1 (Fejér [10]). *Let $\mathcal{S}_\ell$ be the set of $(\ell+1)$-st roots of unity, then for any $f \in \mathcal{H}(\Delta, \mathbb{C})$, the sequence of the interpolating polynomials $\{p_\ell(z)\}$ over the set $\mathcal{S}_\ell$ converges uniformly to $f(z)$ on compact subsets of $Int\Delta$.*

When $T \in \mathcal{H}(\Delta, \mathbb{C}^{k\times k})$ the polynomial $P_\ell(z) = \sum_{s=0}^{\ell} A_s z^s$, with $A_s = [a_{ij}^{(s)}] \in \mathbb{C}^{k\times k}$, is the interpolating polynomial over the set of the $(\ell+1)$-st roots of unity, we have that each $t_{ij}(z) = [T(z)]_{ij}$ is a holomorphic function interpolated by $p_{ij}(z) = [P_\ell(z)]_{ij} = \sum_{s=0}^{\ell} a_{ij}^{(s)} z^s$, for $i, j = 1, \ldots, k$. Hence, we can extend the result of Theorem 1 to holomorphic matrix-valued functions since $\|T(z) - P_\ell(z)\|_2^2 \leqslant \|T(z) - P_\ell(z)\|_F^2 = \sum_{i,j=1}^k |t_{ij}(z) - p_{ij}(z)|^2$. Other possible extensions consider modified distributions of the interpolation nodes.

LEMMA 1. *Let $P_\ell(z) = \sum_{s=0}^{\ell} A_s z^s \in \mathbb{C}^{k\times k}[z]$ be a matrix polynomial of degree less than or equal to ℓ. Then its leading coefficient is given by $A_\ell = \frac{1}{\ell}\sum_{j=1}^{\ell} P_\ell(\omega_\ell^j) - P_\ell(0)$, where $\omega_\ell = \exp^{2\pi i/\ell}$ is the ℓ-th principal root of unity.*

Proof. The proof easily follows by considering the interpolating polynomial of the function $f(z) = P_\ell(z)$ over the ℓ-th roots of the unity plus the origin which is $P_\ell(z)$ itself. The associated Vandermonde system can be expressed in terms of the Fourier matrix and hence easily inverted by yielding the sought formula. $\square$

COROLLARY 1. *Let $T \in \mathcal{H}(\Delta, \mathbb{C}^{k\times k})$. The sequence of the interpolating matrix polynomials $\{P_\ell(z)\}$ over the set $\mathcal{S}_{\ell-1} \cup \{0\}$, i.e., the ℓ-th roots of unity plus the origin, converges uniformly to $T(z)$ on compact subsets of $Int\Delta$.*

Proof. From Theorem 1, given a compact subset of $Int\Delta$ containing $z = 0$, for any $\varepsilon > 0$ there exists an N, such that $\forall \ell > N$ the polynomial $Q_{\ell-1}(z)$ interpolating $T(z)$ over the ℓ-th roots of unity is such that $\|T(z) - Q_{\ell-1}(z)\|_F \leqslant \frac{\varepsilon}{3}$. Then the polynomial $P_\ell(z) = \sum_{s=0}^{\ell} A_s z^s$ interpolating $T(z)$ on $\mathcal{S}_{\ell-1} \cup \{0\}$ is such that

$$P_\ell(z) = Q_{\ell-1}(z) + (z^\ell - 1)A_\ell = Q_{\ell-1}(z) + (z^\ell - 1)(1/\ell\sum_{j=1}^{\ell} P_n(\omega_\ell^j) - P_\ell(0)).$$

We get, $\|T(z) - P_\ell(z)||_F \leqslant \|T(z) - Q_{\ell-1}(z)\|_F + |z^\ell - 1|\,\|\frac{1}{\ell}\sum_{j=1}^{\ell} P_\ell(\omega_\ell^j) - T(0)\|_F$. From Lemma 1 we have $\frac{1}{\ell}\sum_{j=1}^{\ell} P_\ell(\omega_\ell^j) = Q_{\ell-1}(0)$, and hence we get

$$\|T(z) - P_\ell(z)||_F \leqslant \|T(z) - Q_{\ell-1}(z)\|_F + 2\|Q_{\ell-1}(0) - T(0)\|_F \leqslant \varepsilon.$$

$\square$

3. Linearizations of the matrix polynomial interpolating $T(z)$

Linearization methods provide a popular way to compute the eigenvalues of matrix polynomials. Given $P_\ell(z) = \sum_{s=0}^{\ell} A_s z^s$, with $\det(A_\ell) \neq 0$, a matrix polynomial of degree ℓ, the associated block companion matrix is $C = Z + XY^H$, where $Z = U \otimes I_k$, $U \in \mathbb{R}^{\ell\times\ell}$ is the downshift matrix and $X, Y \in \mathbb{C}^{\ell k\times k}$ are such that

$$X = \left[-A_0A_\ell^{-1} - I_k; -A_1A_\ell^{-1}; \ldots; -A_{\ell-1}A_\ell^{-1}\right], \quad \text{and} \quad Y = e_\ell \otimes I_k. \tag{3.1}$$

It is shown that $\det(P_\ell(z)) = 0$ iff $\det(zI_{\ell k} - C) = 0$.

In general the leading coefficient of the interpolating polynomial is tiny in magnitude, and hence the block-vector X in (3.1) can have very large entries affecting the stability of QR-based methods which depends on $\|X\|$. It is often more convenient in the case where $\det(P_\ell(0)) \neq 0$ to approximate the eigenvalues of the companion matrix associated with the reversal polynomial $Q_\ell(z) = z^\ell P_\ell(z^{-1}) = \sum_{s=0}^{\ell} A_{\ell-s} z^s$ and then return as approximation of the original problem the value $1/\lambda$, for each λ eigenvalue of $Q_\ell(z)$. In Section 5 we report the results obtained using this technique referred to as the reverse companion approach. Further, it is worth noting that Z is unitary and therefore C can be represented as a rank$-k$ perturbation of a unitary matrix. This properties plays a fundamental role for the development of computationally efficient eigensolvers (see [5] and the references given therein).

Another linearization can be obtained using the eigendecomposition of $U = F^H \Omega F$, where F is the Fourier matrix* such that $F_{ij} = \omega_\ell^{ij}/\sqrt{\ell}$, $\omega_\ell = \exp^{2\pi i/\ell}$, $i, j = 1, \dots, \ell$ and $\Omega = diag(\omega_\ell, \omega_\ell^2, \dots, \omega_\ell^\ell)$. Then $C = (F^H \otimes I_k)(\Omega \otimes I_k + \tilde{X}\tilde{Y}^H)(F \otimes I_k)$, and denoting by $\tilde{X} = (F \otimes I_k)X$ and by $\tilde{Y} = (F \otimes I_k)Y$, from $(F^H \otimes I_k)(F \otimes I_k) = I_{\ell k}$ one obtains that C and $\tilde{C} = \Omega \otimes I_k + \tilde{X}\tilde{Y}^H$ are similar and therefore this latter matrix can also be used to convert the PEP into a linear eigenproblem. In addition, $\tilde{C}$ and C inherit the structural property of being low rank corrections of unitary matrices and thus they are both eligible for the design of fast eigensolvers. Nevertheless, using the unitary diagonal plus low rank structure of $\tilde{C}$ instead of the structure of C is particularly suited if $P_\ell(z)$ is the interpolating polynomial over the $(\ell+1)$-st roots of unity or over $S_{\ell-1} \cup \{0\}$. In fact, in this cases we can directly find a closed expression for $\tilde{X}$ and $\tilde{Y}$ from the value of $T(z)$ on the interpolating nodes by avoiding the explicit computation of the coefficients of the polynomial in the monomial basis.

Consider for instance the case we are interpolating on the ℓ-th roots of unity plus the origin. The coefficients of the polynomial in the monomial basis can be obtained solving the Vandermonde linear systems $VP = T$ where $P = [A_\ell; \dots; A_0] \in \mathbb{C}^{(\ell+1)k \times k}$, $T = [T(\omega); T(\omega^2); \dots T(\omega^\ell); T(0)] \in \mathbb{C}^{(\ell+1)k \times k}$, $\omega = \omega_\ell = \exp^{2\pi i/\ell}$,

$$V = \left[\begin{array}{c|c} \sqrt{\ell} F J & u \\ \hline 0 & 1 \end{array}\right] \otimes I_k, \quad F_{ij} = \omega^{ij}/\sqrt{\ell}, \quad u_j = 1, j = 1 \dots, \ell$$

and J is the reverse permutation matrix. After some manipulations we get

$$\tilde{P} = [A_0; A_1; \dots; A_{\ell-1}] = \frac{1}{\sqrt{\ell}}(UF^H \otimes I_k)T - (e_1 \otimes I_k)A_\ell.$$

Hence,

$$X = \left[-A_0 A_\ell^{-1} - I_k; -A_1 A_\ell^{-1}; \dots; -A_{\ell-1} A_\ell^{-1}\right] = -\frac{1}{\sqrt{\ell}}(UF^H \otimes I_k)\tilde{T} A_\ell^{-1},$$

where $\tilde{T} = [T(\omega); T(\omega^2); \dots T(\omega^\ell)]$. Since $U = F^H \Omega F$, we have

$$\tilde{X} = (F \otimes I_k)X = -\frac{1}{\sqrt{\ell}}(\Omega \otimes I_k)\tilde{T} A_\ell^{-1}, \qquad \tilde{Y} = \frac{1}{\sqrt{\ell}} u \otimes I_k,$$

*Note that in the literature the Fourier matrix is customarily defined in a slightly different way: $F_{ij} = \omega_\ell^{(i-1)(j-1)}/\sqrt{\ell}$, $i, j = 1, \dots, \ell$.

where from Lemma 1 $A_\ell = \frac{1}{\ell}\sum_{j=1}^{\ell} T(\omega^j) - T(0)$.

As we have already observed the leading block coefficient A_ℓ may have a very small norm and consequently $\|A_\ell^{-1}\|$ may be huge. When $T(0)$ is invertible and numerically well conditioned this issue can be circumvented by considering reversed polynomials. The resulting scheme proceeds as follows.

1. Compute $T(0)$ and consider $G(z) = T(z)T(0)^{-1}$. $G(z)$ has the same eigenvalues of $T(z)$ and $G(0) = I_k$.
2. Let $P(z)$ be the interpolating polynomial of $G(z)$ on the ℓ-th roots of unity plus the origin .
3. Consider the reversed polynomial $Q(z) = z^\ell P(z^{-1}) = \sum_{s=0}^{\ell} A_{\ell-s} z^s$. Note that $Q(z)$ is a monic polynomial since $A_0 = I_k$.
4. Compute the eigenvalues λ of $Q(z)$, and return $1/\lambda$ as estimates of the eigenvalues of $T(z)$.

For the linearizations associated with $Q(z)$ we have that $A_0 = I_k$ and therefore $X = [-A_\ell - I_k, -A_{\ell-1}, \cdots - A_1]^T$ and

$$\tilde{X} = -\frac{1}{\sqrt{\ell}}(\Omega \otimes I_k)\tilde{G}, \quad \tilde{G} = [G(\omega^{\ell-1}); G(\omega^{\ell-2}); \ldots; G(\omega); G(\omega^\ell)] = \tilde{T}[T(0)^{-1}]$$

where $\tilde{T} = [T(\omega^{\ell-1}); T(\omega^{\ell-2}); \ldots; T(\omega); T(\omega^\ell)]$ while $\tilde{Y} = \frac{1}{\sqrt{\ell}} u \otimes I_k$. It follows that the linearization of $Q(z)$ using the unitary diagonal plus a rank$-k$ form can be directly generated from the values attained by $T(z)$ at the interpolation nodes without requiring the explicit computation of the coefficients A_i of the interpolating polynomial. This can yield benefits in terms of accuracy and computational costs.

4. The structured QR method

In order to solve the linearized problems described in Section 3 we can use the algorithm presented in [5] which is specifically designed to take advantage from the unitary plus low rank structure of the problem. Other algorithms in the literature, such as the one proposed in [1] can be applied in the case of companion linearization but not for the unitary diagonal plus low rank linearization. The method proposed in [5] can be applied to unreduced, i.e., with non-zero subdiagonal entries, Hessenberg matrices that are the sum of a unitary part and a low rank part. It is shown that $A \in \mathbb{C}^{m\times m}$ can be embedded in a larger matrix $\hat{A} \in \mathbb{C}^{N\times N}$ with $N = m+k$, which is still unitary plus rank k. The matrix $\hat{A} = \hat{U} + \hat{X}\hat{Y}^H$, with $\hat{U}$ unitary, and $\hat{X}, \hat{Y} \in \mathbb{C}^{N\times k}$ can then be factorized as $\hat{A} = L \cdot F \cdot R$, where L is an unreduced unitary k-lower Hessenberg matrix, R is a unitary k-upper Hessenberg matrix. Moreover, the leading $m-1$ entries in the outermost diagonal of R, $r_{i+k,i}, i = 1, \ldots, m-1$, are nonzero and $F = Q + TZ^H$, where Q is a block diagonal unitary Hessenberg matrix, $Q = \left[\begin{array}{c|c} I_k & \\ \hline & \hat{Q} \end{array}\right]$, with $\hat{Q}$ unreduced, $T = [T_k, O_{k,m}]^T$ with T_k upper triangular, and $Z \in \mathbb{C}^{N\times k}$, with full rank.

Time-delay

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$	$err(\lambda_2)$	$err(\lambda_3)$
Taylor	comp.	32	6.90e+11	5.28e-02	5.28e-02	4.67e-02
Cheb.	comp.	64	2.92e-01	2.97e-09	2.74e-08	2.79e-08
$\omega_{\ell+1}^j$	comp.	32	1.15e+11	3.76e-05	6.43e-05	2.24e-04
$\omega_{\ell+1}^j$	rev comp	64	1.00e+00	1.19e-14	8.19e-15	1.07e-14
$\omega_\ell^j, 0$	comp.	32	1.15e+11	2.88e-05	4.12e-05	2.54e-04
$\omega_\ell^j, 0$	rev comp	64	1.00e+00	5.74e-15	1.40e-14	1.12e-14
$\omega_\ell^j, 0$	unit diag	32	1.15e+11	3.19e-05	9.84e-05	3.02e-04
$\omega_\ell^j, 0$	rev unit d.	64	1.00e+00	1.50e-15	1.89e-15	3.24e-15

Based on this representation in [5] an implicitly shifted QR method is proposed, aimed at the efficient computation of the eigenvalues of $\hat{A}$ and *a fortiori* of A. The backward stability of the algorithm is theoretically proven. The cost of the algorithm is $O(m^2k)$ and then, since $m = \ell k$ in our case turns out to be $O(\ell^2 k^3)$ where ℓ is the degree of the polynomial approximating the non linear function.

5. Numerical Experiments

Based on the results of the previous sections some methods for approximating the eigenvalues inside the unit circle Δ of a holomorphic matrix-valued function $T : \Omega \to \mathbb{C}^{k\times k}$ such that $\Delta \subset \Omega$ and $\det(T(0)) \neq 0$ are proposed. The methods proceed as: (1) Compute a polynomial approximation $P_\ell(z)$ by interpolation; (2) Solve the corresponding PEP by linearization using the algorithm presented in [5]. Several strategies can be considered for carrying out both stages. By combining different options the following methods are taken into consideration:

1. Taylor/companion (Taylor approximation at the origin + companion linearization);
2. Chebyshev/companion (polynomial approximation at Chebyshev points as described in [8] + companion linearization);
3. roots-of-unity/companion (polynomial interpolation at the roots of unity + companion linearization);
4. roots-of-unity-plus-zero/companion (polynomial interpolation at the roots of unity complemented with zero + companion linearization);
5. roots-of-unity-plus-zero/ reversed-companion (polynomial interpolation at the roots of unity complemented with zero + companion linearization of the reversed polynomial);
6. roots-of-unity-plus-zero/diagonal (polynomial interpolation at the roots of unity complemented with zero + unitary diagonal plus low rank linearization);

Model of Cancer-growth

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$	$err(\lambda_2)$	$err(\lambda_3)$
Cheb.	compan.	64	4.35e+01	2.46e-12	3.42e-12	1.92e-09
$\omega_{\ell+1}^j$	compan	32	2.26e+15	2.50e-03	9.54e-03	5.61e-02
$\omega_{\ell+1}^j$	rev. comp.	64	1.00e+00	2.18e-16	7.58e-16	1.33e-15
$\omega_\ell^j, 0$	compan	16	1.41e+15	6.00e-03	7.88e-03	1.28e-01
$\omega_\ell^j, 0$	rev. comp.	64	1.00e+00	1.79e-16	7.93e-16	7.95e-16
$\omega_\ell^j, 0$	unit. diag.	64	1.45e+15	2.94e-02	4.80e-02	4.84e-02
$\omega_\ell^j, 0$	rev unit. d.	32	1.00e+00	6.32e-16	3.57e-16	2.23e-16

Neutral functional Differential equation

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$	$err(\lambda_2)$	$err(\lambda_3)$
Chebyshev	compan.	32	1.62e+04	1.14e-09	1.14e-09	5.86e-03
$\omega_{\ell+1}^j$	compan	64	4.45e+12	1.70e-13	1.57e-11	3.06e-10
$\omega_{\ell+1}^j$	rev. comp.	64	1.00e+00	3.27e-13	3.50e-13	1.16e-11
$\omega_\ell^j, 0$	compan	64	5.99e+12	1.45e-11	5.02e-11	5.04e-10
$\omega_\ell^j, 0$	rev. comp.	64	1.00e+00	7.61e-14	1.11e-13	1.22e-11
$\omega_\ell^j, 0$	unit. diag.	32	4.35e+07	5.64e-06	6.60e-06	9.68e-06
$\omega_\ell^j, 0$	rev unit. d.	64	1.00e+00	1.48e-13	2.18e-13	2.18e-13

7. roots-of-unity-plus-zero/reversed-diagonal (polynomial interpolation at the roots of unity complemented with zero + unitary diagonal plus low rank linearization of the reversed polynomial).

All these methods have been tested numerically using MatLab. Our test suite consists of the following problems:

Time-delay equation [7]. The matrix function is $T(z) = z + T_0 + T_1 \exp(6z - 1)$ with

$$T_0 = \begin{bmatrix} 4 & -1 \\ -2 & 5 \end{bmatrix}; \quad T_1 = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}.$$

This function has three eigenvalues inside the unit circle.

Model of cancer growth [2]. The matrix function is $T(z) = z - A_0 - A_1 \exp(-rz)$, where

$$A_0 = \begin{bmatrix} -\mu_1 & 0 & 0 \\ 2b1 & -(\mu_0+\mu_Q) & b_Q \\ 0 & \mu_Q & -(b_Q+\mu_G) \end{bmatrix}, A_1 = \exp(-(\mu_0+\mu_Q)r) \begin{bmatrix} 2b_1 & 0 & b_Q \\ -2b_1 & 0 & -b_Q \\ 0 & 0 & 0 \end{bmatrix}.$$

The parameters are chosen as suggested in [2] by setting $r = 5; b_1 = 0.13; b_Q = 0.2; \mu_1 = 0.28; \mu_0 = 0.11; \mu_Q = 0.02; \mu_G = 0.0001$. We refer to [2] for the physical meaning of the constants and for the description of the model. This function has three eigenvalues inside the unit circle.

Neutral functional differential equation [9]. The function is scalar $t(z) = -1 + 0.5z + z^2 + hz^2 \exp(\tau z)$. The case $h = -0.82465048736655, \tau = 6.74469732735569$

Spectral Abscissa Optimization

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$	$err(\lambda_2)$	$err(\lambda_3)$	$err(\lambda_4)$
Taylor	comp.	16	4.44e+17	5.32e-02	2.35e-01	7.39e-01	7.60e-01
Cheb.	compan.	32	2.66e+06	6.50e-12	9.90e-12	3.92e-08	3.93e-08
$\omega_{\ell+1}^j$	compan	32	1.64e+15	9.35e-04	1.35e-02	1.32e-03	3.72e-03
$\omega_{\ell+1}^j$	rev. comp.	32	1.00e+00	1.64e-16	3.02e-16	7.25e-14	8.18e-14
$\omega_\ell^j,0$	compan	32	1.91e+15	1.53e-02	6.96e-02	1.02e-01	8.67e-02
$\omega_\ell^j,0$	rev. comp.	64	1.00e+00	1.03e-15	1.07e-15	7.76e-14	8.95e-14
$\omega_\ell^j,0$	unit. diag.	32	1.91e+15	9.51e-03	1.69e-02	1.11e-03	5.25e-03
$\omega_\ell^j,0$	rev unit. d.	64	1.00e+00	3.76e-16	8.52e-16	1.73e-14	5.53e-14

is analyzed in [14] corresponding to a Hopf bifurcation point. This function has three eigenvalues inside the unit circle.

Spectral abscissa optimization [17]. The function is $T(z) = zI_3 - A - B\exp(-z\tau)$ with $\tau = 5$, $B = bq^T$ and

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix},\ b = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix},\ q = \begin{bmatrix} 0.47121273 \\ 0.50372106 \\ 0.60231834 \end{bmatrix}.$$

Abscissa optimization techniques favor multiple roots and clustered eigenvalues with potential numerical difficulties. This function has 4 eigenvalues inside the unit circle.

Hadeler problem [3]. The matrix function is $T(z) = (\exp(z)-1)A_2 + z^2A_1 - \alpha A_0$ where $A_0, A_1, A_2 \in \mathbb{R}^{k\times k}$, and $V = \text{ones}(k,1) * [1:k]$ $A_0 = \alpha I_k, A_1 = k * I_k + 1./(V + V'), A_2 = (k+1-\max(V,V')).*([1:k]*[1:k]')$. In our experiments we set $k = 8$ and $\alpha = 100$. Ruhe [18] proved that the problem has k real and positive eigenvalues, in particular two of them are $0 < \lambda < 1$, and hence lay inside the unit circle.

Vibrating string [3, 19]. The model refers to a string of unit length clamped at one end, while the other one is free but is loaded with a mass m attached by an elastic spring of stiffness k_p. Assuming $m = 1$, and discretizing the differential equation one gets the non linear eigenvalue problem $F(z)v = 0$, where $F(z) = A - Bz + k_pC\frac{z}{z-k_p}$ is rational, $A, B, C \in \mathbb{R}^{k\times k}$, $k_p = 0.01, h = 1/k$,

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 1 \end{bmatrix},\ B = \frac{h}{6}\begin{bmatrix} 4 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & 4 & 1 \\ & & 1 & 2 \end{bmatrix},\ C = e_k e_k^T.$$

The dominant eigenvalues is $\lambda \approx 2.4874$. To apply our method to this problem we need to shift the spectrum to have the interesting root inside the unit circle. We considered instead the matrix function $T(z) = A - B(z+1.5) + k_pC\frac{z+1.5}{z+1.5-k_p}$ and we use $k = 50$ for our experiments. Note that the pole $z = k_p - 1.5$ is outside the region of interest. The shifted function has only an eigenvalue inside the unit circle.

Hadeler function

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$	$err(\lambda_2)$
Taylor	companion	16	2.75e+13	2.84e-05	2.13e-04
Cheb	companion	32	5.09e+05	6.55e-12	4.53e-08
$\omega_{\ell+1}^j$	companion	16	2.58e+13	1.30e-03	6.50e-04
$\omega_{\ell+1}^j$	rev comp	64	1.00e+00	5.32e-15	2.69e-15
$\omega_\ell^j, 0$	companion	16	3.41e+13	1.90e-04	2.10e-03
$\omega_\ell^j, 0$	rev comp	64	1.00e+00	1.08e-15	5.61e-12
$\omega_\ell^j, 0$	unit diag	32	2.58e+14	1.52e-02	2.34e-02
$\omega_\ell^j, 0$	rev unit diag	16	1.00e+00	1.67e-15	9.84e-16

Vibrating string

nodes	Structure	ℓ	$\|A_\ell^{-1}\|_2$	$err(\lambda_1)$
Cheb	compan.	6	8.92e+15	3.37e-08
$\omega_{\ell+1}^j$	rev. comp	6	1.00e+00	3.78e-11
$\omega_\ell^j, 0$	rev comp	6	1.00e+00	3.80e-11
$\omega_\ell^j, 0$	rev unit diag	6	1.00e+00	3.78e-11

To measure the accuracy in retrieving the eigenvalues inside the unit circle of the test functions, we consider the following forward error estimate

$$err(\lambda) = \frac{\|T(\lambda)v\|_2}{\|T(\lambda)\|_2\|v\|_2} = \frac{\sigma_k}{\sigma_1}, \text{for } k>1, \quad err(\lambda) = \sigma_1, \text{for } k=1,$$

where λ is the approximated eigenvalue inside the unit circle, $\sigma_i = \sigma_i(T(\lambda)), 1 = 1,\ldots,k$ are the singular values of $T(\lambda)$ and v is its $k-$th right singular vector.

We performed a wide range of tests, in particular for each of the above problems[†] we test the methods with values of $\ell = \{16, 32, 64\}$. Sometime the linearization based on the Taylor polynomial cannot be applied due the singularity of the leading term. In the tables, for each combination of nodes and structures, we report the best results obtained on our test functions, highlighting the methods which give the best results. We report also the norm of the inverse of the leading coefficients of the approximating polynomial. We see that the when the norm of $\|A_\ell^{-1}\|_2$ becomes very large also the norm of the linearization grows, affecting the stability of the algorithm. For the Vibrating string problem, we report only the results obtained using the methods based on the linearizations of the reverse polynomial which in all the other cases show better converging properties.

Numerical results show that polynomial interpolation techniques at the roots-of-unity-plus-zero set combined with unitary-diagonal plus low rank linearization method applied to the reversed polynomial provides an effective tool to compute numerical approximations of the zeros of a holomorphic matrix-valued function inside the unit

[†]With the exception of the Vibrating string problem where the value of $k = 50$ prevents the investigation of the behavior of the methods with large ℓ.

circle. This method is also the less expensive since we do not need the explicit computation of the coefficients of the polynomial as the solution of the Vandermonde linear system but only the evaluation of the non-linear function at the roots of unity plus the origin. The resulting approach can also benefit of fast QR-based eigensolvers exploiting additional structural properties of the considered linearized eigenproblems.

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AMS Subject Classification: 65F15

Roberto BEVILACQUA, Gianna M. DEL CORSO, Luca GEMIGNANI
Department of Computer Science, University of Pisa
Largo Pontecorvo 3, 56127 Pisa, ITALY
e-mail: {roberto.bevilacqua,gianna.delcorso,luca.gemignani}@unipi.it

Lavoro pervenuto in redazione il 16-5-19.