

RESEARCH ARTICLE

Minimizing properties of networks via global and local calibrations

Alessandra Pluda¹  | Marco Pozzetta² ¹Dipartimento di Matematica, Università di Pisa, Pisa, Italy²Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, Naples, Italy
Email: marco.pozzetta@unina.it**Correspondence**Alessandra Pluda, Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy.
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Abstract

In this note, we prove that minimal networks enjoy minimizing properties for the length functional. A minimal network is, roughly speaking, a subset of \mathbb{R}^2 composed of straight segments joining at triple junctions forming angles equal to $\frac{2}{3}\pi$; in particular such objects are just critical points of the length functional a priori. We show that a minimal network Γ_* : (i) minimizes mass among currents with coefficients in an explicit group (independent of Γ_*) having the same boundary of Γ_* , (ii) identifies the interfaces of a partition of a neighborhood of Γ_* solving the minimal partition problem among partitions with same boundary traces. Consequences and sharpness of such results are discussed. The proofs reduce to rather simple and direct arguments based on the exhibition of (global or local) calibrations associated to the minimal network.

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1 | INTRODUCTION

The Steiner problem in its classical formulation (see, for instance, [9]) reads as follows: given C a collection of n points p_1, \dots, p_n in the Euclidean plane, one wants to find a connected set K that contains C whose length is minimal, namely one looks for

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$$\inf\{\mathcal{H}^1(K) : K \subset \mathbb{R}^2, \text{ connected and such that } C \subset K\}. \quad (1.1)$$

This problem has a long history (for a detailed presentation, we refer to [13]) and existence of minimizers is known even in more general ambients, like suitable metric spaces [19]. In particular, whenever the problem is set in \mathbb{R}^n , it is well-known that minimizers are finite union of segments that meet at triple junctions forming angles of 120° [13]. Due to their clear relevance in the problem we give a name to the elements of this precise class of networks: we call them *minimal networks*. As the number of points of C increases, the number of configurations that are candidate minimizers rapidly increase. Hence, not only minimizers may not be unique, but identifying them is by no means an easy task, also in terms of efficient algorithms (the Steiner problem is classified as NP-hard from a computational point of view).

A possible tool to validate the minimality of a certain candidate is the notion of *calibration*. The classical concept of calibration for a d -dimensional oriented manifold M in \mathbb{R}^n is a closed d -form ω such that $|\omega| \leq 1$ and $\langle \omega, v \rangle = 1$ for every tangent vector v to M . The existence of a calibration ω for M is a sufficient condition for M to be area minimizing in its homology class. Indeed, let N be a d -dimensional oriented manifold with the same boundary of M . Using the conditions satisfied by ω and Stokes' theorem, we get

$$\text{Vol}(M) = \int_M \omega = \int_N \omega \leq \text{Vol}(N),$$

as desired.

The definition of calibration for minimal surfaces does not work directly for the Steiner problem because neither the competitors nor the minimizers of the problem admit an orientation that is compatible with their boundary. However there have been several successful adaptations of the notion: starting from the paired calibrations by Lawlor and Morgan [14] and their suitable generalizations [7, 8], passing to calibrations for integral currents with coefficient in groups [15, 16], rank-one tensor valued measures [5, 6], and calibrations to study clusters with multiplicities [18]. We also mention an adaptation of calibrations to an evolution setting presented in [11], in this case calibrations are useful to show weak-strong uniqueness of BV solutions to the multi-phase mean curvature flow.

The aim of this note is to prove minimizing properties of minimal networks in \mathbb{R}^2 via suitable notions of calibrations. We refer to Subsection 1.1 for the basic definitions about networks. Roughly speaking, a network is identified by a compact connected graph G with 1-dimensional edges and by an immersion $\Gamma : G \rightarrow \mathbb{R}^2$, see Definitions 1.3 and 1.4. A minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ is a network with straight edges ending either at endpoints or at junctions of order 3 forming angles equal to $\frac{2}{3}\pi$, see Definition 1.7. We shall see that minimal networks enjoy minimizing properties for the *length functional*, defined as the sum of the lengths of each edge. Essentially due to the possible presence of loops, minimal networks may not be strictly stable critical points of the length functional, nevertheless it turns out that they minimize the length functional among several classes of competitors.

In [15], Marchese and Massaccesi rephrased the Steiner problem as a mass minimization problem among rectifiable currents with coefficients in a discrete subgroup \mathcal{G} of \mathbb{R}^{n-1} (with n equal to the number of points to connect) and their reformulation allows for a natural notion of calibration (see [15, section 3]). In this paper, we prove that we can canonically associate to any minimal network Γ_* a current \hat{T} with coefficients in a subgroup of \mathbb{R}^2 , also producing a calibration for \hat{T} and hence showing that \hat{T} is mass minimizing among normal currents T with coefficients in \mathbb{R}^2 such that $\partial T = \partial \hat{T}$. We refer to Section 2 for definitions and terminology about currents.

Theorem 1.1 (cf. Theorem 2.9). *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network. Then there exists a normed subgroup $(\mathcal{G}, \|\cdot\|)$ of \mathbb{R}^2 such that the following holds.*

There exists a 1-rectifiable current \hat{T} with coefficients in \mathcal{G} such that $\text{supp}(\hat{T}) = \Gamma_$, $L(\Gamma_*) = \mathbb{M}(\hat{T})$, and there exists a calibration $\omega \in C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$ for \hat{T} in the sense of Definition 2.6.*

In particular, \hat{T} is a mass minimizing current among 1-normal currents with coefficients in \mathbb{R}^2 with the same boundary of \hat{T} .

In fact, the group \mathcal{G} in the previous theorem will be explicitly exhibited in the proof and we will see that the calibration turns out to be (identified by) the identity matrix. Also, the group \mathcal{G} will be independent of the topology of the minimal network Γ_* and of the number of its endpoints. We mention that an analogous argument has been employed in [18, 20], of which Theorem 1.1 can be seen as a particular case; however the proof of Theorem 1.1 is obtained here by performing a simpler and more explicit construction.

We will also derive consequences of Theorem 1.1 regarding minimizing properties of minimal networks among several classes of networks. The length of a minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ can be proved to be less than the one of suitable networks $\Gamma : H \rightarrow \mathbb{R}^2$ having (some) endpoints in common with Γ_* , possibly immersed edges and junctions of higher order. In particular, comparison networks $\Gamma : H \rightarrow \mathbb{R}^2$ may have different topology with respect to $\Gamma_* : G \rightarrow \mathbb{R}^2$; more precisely, G can be assumed to be (homeomorphic to) a suitable quotient of H , see Corollary 2.13, or, conversely, H can be assumed to be (homeomorphic to) a suitable quotient of G , see Corollary 2.14. These results give applicable comparison results among classes of networks, yielding explicit applications of the more abstract Theorem 1.1.

The Steiner problem has also been proved to be equivalent to the so-called minimal partition problem. A Caccioppoli partition \mathbf{E} of a bounded open Lipschitz set Ω is a collection of finite perimeter sets E_1, \dots, E_n sets that are essentially disjoint and that cover Ω . Given a reference partition $\tilde{\mathbf{E}} = (\tilde{E}_1, \dots, \tilde{E}_n)$ of Ω we say that $\mathbf{E} = (E_1, \dots, E_n)$ is a minimizer of the minimal partition problem if

$$\sum_{i=1}^n P(E_i, \Omega) \leq \sum_{i=1}^n P(F_i, \Omega)$$

among any partition $\mathbf{F} = (F_1, \dots, F_n)$ with the property that $\text{tr}_\Omega \chi_{\tilde{E}_i} = \text{tr}_\Omega \chi_{E_i} = \text{tr}_\Omega \chi_{F_i}$, where the latter symbols denote traces on the boundary of Ω (see, for instance, [2, 3, 17]).

In this paper, we show that we can associate to a minimal network Γ_* an open set Ω and a partition $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ of Ω such that Γ_* coincides with the set of interfaces of $\tilde{\mathbf{E}}$ in Ω , also producing a suitable calibration for $\tilde{\mathbf{E}}$ showing that such partition is a minimizer for the minimal partition problem of Ω among partitions with the same boundary traces. We refer to Section 3 for definitions and terminology on partitions.

Theorem 1.2 (cf. Theorem 3.9). *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network such that $\Gamma_*(G) \subset D$ where D is a domain of class C^1 homeomorphic to a closed disk, and $\Gamma_*(G) \cap \partial D$ is the set of endpoints of Γ_* .*

Then there exists a bounded open set $\Omega' \subset \mathbb{R}^2$ such that $\Gamma_(G) \subset \Omega'$, $\Omega := \Omega' \cap \text{int}(D)$ has Lipschitz boundary, there exists a Caccioppoli partition $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ of Ω such that $\Omega \cap \cup_i \partial \tilde{E}_i = \Omega \cap \Gamma_*(G)$ and there exists a local paired calibration for $\tilde{\mathbf{E}}$ in Ω .*

In particular, $\widetilde{\mathbf{E}}$ is a minimizer for \mathcal{P} in \mathcal{A} , that is the class of partitions having the same trace of $\widetilde{\mathbf{E}}$ on $\partial\Omega$.

In the above theorem, Ω is essentially constructed as the intersection of a suitably small tubular neighborhood of $\Gamma_*(G)$ with D . This can be interpreted as a *local* minimality result. In fact, Ω *cannot* be taken to be an arbitrary neighborhood of the given minimal network Γ_* . In Remark 3.10, we construct a counterexample to the minimality among partitions in case such tubular neighborhood is too large.

To the best of our knowledge, the tool of calibrations has never been used before to prove *local* minimality in the framework of the Steiner problem or minimal partitions.

Addendum

After this work was completed, a result analogous to Theorem 1.2 appeared in [10]. More precisely, the authors prove that if a Caccioppoli partition of a given set D is a suitable stationary point of the interface length functional, then it is minimizing for the minimal partition problem of D among partitions sufficiently close in L^1 to the stationary one. The previous result is based on a calibration argument similar to the one employed in the present work, but the construction of the calibration is different.

Organization

We collect below terminology on networks. In Section 2, we introduce calibrations for currents and we prove Theorem 1.1. In Section 3, we define the minimal partition problem, local paired calibrations, and we prove Theorem 1.2.

Throughout this paper, we do not identify functions coinciding almost everywhere with respect to a measure.

1.1 | Networks

For a regular curve $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ of class H^2 , define

$$\tau := \frac{\gamma'}{|\gamma'|}, \quad \nu := \mathbf{R}(\tau),$$

the tangent and the normal vector, respectively, where \mathbf{R} denotes counterclockwise rotation of $\frac{\pi}{2}$. We define $ds := |\gamma'| dx$ the arclength element and $\partial_s := |\gamma'|^{-1} \partial_x$ the arclength derivative. The curvature of γ is the vector $\kappa := \partial_s^2 \gamma$.

Definition 1.3. Fix $N \in \mathbb{N}$ and let $i \in \{1, \dots, N\}$, $E_i := [0, 1] \times \{i\}$, $E := \bigcup_{i=1}^N E_i$ and $V := \bigcup_{i=1}^N \{0, 1\} \times \{i\}$. Let \sim be an equivalence relation that identifies points of V . A *graph* G is the topological quotient space of E induced by \sim , that is

$$G := E / \sim,$$

and we assume that G is connected.

Denoting by $\pi : E \rightarrow G$ the projection onto the quotient, an *endpoint* is a point $p \in G$ such that $\pi^{-1}(p) \subset V$ and it is a singleton, a *junction* is a point $m \in G$ such that $\pi^{-1}(m) \subset V$ and it is not a singleton. The *order* of a junction is the cardinality $\#\pi^{-1}(m)$. We will always assume that the order of a junction in a graph is greater or equal to 3.

A *subgraph* $G' \subset G$ is a topological subspace of a graph G that is a graph itself with the structure induced by G . More precisely, G' is a graph and there exists a subset $E' \subset E$ such that $G' = E' / \sim$ where \sim is the same equivalence relation defining G .

Definition 1.4. An *immersed network* (or, simply, *network*) is a pair $\mathcal{N} = (G, \Gamma)$ where

$$\Gamma : G \rightarrow \mathbb{R}^2$$

is a continuous map and G is a graph and each map $\gamma^i := \Gamma|_{E_i}$ is an immersion of class C^1 (up to the boundary).

Definition 1.5. A network $\mathcal{N} = (G, \Gamma)$ is an *immersed triple junctions network* if it is an immersed network and each junction of G has order 3.

Definition 1.6. Let $\mathcal{N} = (G, \Gamma)$ be a network of class C^1 and let $e \in \{0, 1\}$. The *inner tangent vector* of a regular curve γ^i of \mathcal{N} at e is the vector

$$(-1)^e \frac{(\gamma^i)'(e)}{|(\gamma^i)'(e)|}.$$

Definition 1.7. A network $\mathcal{N} = (G, \Gamma)$ is *minimal* if each map $\gamma^i := \Gamma|_{E_i}$ is an *embedding* of class H^2 , for every $i \neq j$ the curves γ^i and γ^j do not intersect in their interior, and $\pi(0, i) \neq \pi(1, i)$ for any i . Moreover, each junction of G has order 3 and the sum of the inner tangent vectors at a junction is zero. Furthermore the curvature of the parameterization of each edge is identically zero, that is, each γ^i is the embedding of a straight segment.

We shall usually denote a network by directly writing the map $\Gamma : G \rightarrow \mathbb{R}^2$. Moreover, with a little abuse of terminology, we shall employ the words junctions and endpoints also referring to their images in \mathbb{R}^2 .

Definition 1.8. Given an immersed network (G, Γ) we denote by $\ell(\gamma^i)$ the length of the curve γ^i . The *length* of the network Γ is

$$L(\Gamma) := \sum_{i=1}^N \ell(\gamma^i).$$

By computing the first variation of the length functional, one easily gets two necessary conditions that a network has to satisfy to be a critical point of L : each curve of the network is a straight segment and the inner tangent vectors of the curves meeting at a junction sum up to zero. Hence, minimal networks are critical points of the length functional. However, also networks with junctions of order higher than three may happen to be critical points.

2 | MINIMALITY AMONG CURRENTS WITH COEFFICIENTS IN A GROUP

In this section, we give a summary of the theory of currents with coefficients in a group presented in a simplified setting that is convenient for our purposes. For further details, we refer, for instance, to [12, 15, 21, 22].

Let $k \in \{0, 1\}$. Consider \mathbb{R}^2 endowed with a norm $\|\cdot\|$ and denote by $\|\cdot\|_*$ the corresponding dual norm. We denote by $\Lambda_k(\mathbb{R}^2)$ the vector space of k -vectors in \mathbb{R}^2 . In particular, $\Lambda_0(\mathbb{R}^2)$ coincides with \mathbb{R} and $\Lambda_1(\mathbb{R}^2)$ is just \mathbb{R}^2 .

Definition 2.1 (*k-covector with values in \mathbb{R}^2*). A *k-covector with values in \mathbb{R}^2* is a linear map $\omega : \Lambda_k(\mathbb{R}^2) \rightarrow \mathbb{R}^2$. We denote by $\Lambda_2^k(\mathbb{R}^2)$ the space of k -covectors with values in \mathbb{R}^2 .

We define the *comass* of a covector $\omega \in \Lambda_2^k(\mathbb{R}^2)$

$$|\omega|_{\text{com}} := \sup \{ \|\omega(\tau)\|_* : \tau \in \Lambda_k(\mathbb{R}^2) \text{ with } |\tau| \leq 1 \},$$

where $|\tau|$ is the norm of a k -vector with respect to the Euclidean norm.

Observe that 0-covectors with values in \mathbb{R}^2 are linear maps $\omega : \mathbb{R} \rightarrow \mathbb{R}^2$, while 1-covectors with values in \mathbb{R}^2 coincide with linear maps from \mathbb{R}^2 to \mathbb{R}^2 .

We remark that, as $k \in \{0, 1\}$ in the previous definition, $\tau \in \Lambda_k(\mathbb{R}^2)$ is automatically a simple k -vector, and thus the previous definition of comass coincides with the usual one considered in the theory of currents.

Definition 2.2 (*k-form with values in \mathbb{R}^2*). A *k-form with values in \mathbb{R}^2* is a function $\omega : \mathbb{R}^2 \rightarrow \Lambda_2^k(\mathbb{R}^2)$ with compact support such that $x \mapsto \omega(x)(\tau)$ is smooth for any $\tau \in \Lambda_k(\mathbb{R}^2)$. We denote by $C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ the space of k -forms with values in \mathbb{R}^2 .

The *comass* of $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ is defined by

$$\|\omega\|_{\text{com}} := \sup_{x \in \mathbb{R}^2} |\omega(x)|_{\text{com}}.$$

The space $C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ is endowed with the following notion of convergence: we say that $\omega_n \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ converges to $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ if there exists a compact set $K \subset \mathbb{R}^2$ such that the support of ω_n is contained in K for any n and $\omega_n(\cdot)(\tau)$ converges to $\omega(\cdot)(\tau)$ in $C^m(K)$ for any m for any $\tau \in \Lambda_k(\mathbb{R}^2)$.

A form $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ is identified by a couple (ω_1, ω_2) where $\omega_i : \mathbb{R}^2 \rightarrow \Lambda^k(\mathbb{R}^2)$ is a standard k -form with compact support, in the sense that $\omega(x)(\tau) = (\omega_1(x)(\tau), \omega_2(x)(\tau))$ for any $\tau \in \Lambda_k(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. We define the differential $d\omega$ of $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ component-wise by $d\omega := (d\omega_1, d\omega_2)$.

We can now define also k -currents with coefficients in \mathbb{R}^2 .

Definition 2.3 (*k-current with coefficients in \mathbb{R}^2*). A *k-current with coefficients in \mathbb{R}^2* is a linear map

$$T : C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2)) \rightarrow \mathbb{R},$$

that is continuous with respect to convergence in $C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$, that is, if $\omega_n \rightarrow \omega$ with respect to convergence in $C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2))$ then $T(\omega_n) \rightarrow T(\omega)$.

The *boundary* of a 1-current T with coefficients in \mathbb{R}^2 is the 0-current with coefficients in \mathbb{R}^2 defined by

$$\partial T(\omega) := T(d\omega), \quad \forall \omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^0(\mathbb{R}^2)).$$

Given T a k -current with coefficients in \mathbb{R}^2 , its *mass* is

$$\mathbb{M}(T) := \sup \{ T(\omega) : \omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^k(\mathbb{R}^2)) \text{ with } \|\omega\|_{\text{com}} \leq 1 \}.$$

A 1-current T with coefficients in \mathbb{R}^2 is said to be *normal* if $\mathbb{M}(T) < \infty$ and $\mathbb{M}(\partial T) < \infty$.

We are now able to define the object of our main interest for our purposes.

Definition 2.4 (1-rectifiable current with coefficients in \mathcal{G}). Let $\Sigma \subset \mathbb{R}^2$ be a 1-rectifiable set. An *orientation* τ on Σ is a measurable map $\tau : \Sigma \rightarrow \mathbb{R}^2$ such that $\tau(x) \in T_x \Sigma$ and $|\tau(x)| = 1$ for \mathcal{H}^1 -almost every $x \in \Sigma$. Let \mathcal{G} be a discrete subgroup of $(\mathbb{R}^2, +)$. A \mathcal{G} -valued *multiplicity function* θ on Σ is a function in $L^1_{\text{loc}}(\mathcal{H}^1 \llcorner \Sigma; \mathcal{G})$.

A 1-current T is *rectifiable with coefficients in \mathcal{G}* if there exist a 1-rectifiable set $\Sigma \subset \mathbb{R}^2$, an orientation τ , and a \mathcal{G} -valued multiplicity function θ on Σ such that, recalling that $\mathbb{R}^2 \equiv \Lambda_1(\mathbb{R}^2)$, there holds

$$T(\omega) = \int_{\Sigma} \langle \omega(x)(\tau(x)), \theta(x) \rangle d\mathcal{H}^1,$$

for any $\omega \in C_c^\infty(\mathbb{R}^2, \Lambda_2^1(\mathbb{R}^2))$, where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean scalar product on \mathbb{R}^2 .

In analogy with the usual 1-rectifiable currents, a 1-rectifiable current with coefficients in \mathcal{G} will be denoted by the triple $T = [\Sigma, \tau, \theta]$.

Thanks to the above representation, if $T = [\Sigma, \tau, \theta]$ is a 1-rectifiable current with coefficients in \mathcal{G} one can write its mass as

$$\mathbb{M}(T) = \int_{\Sigma} \|\theta(x)\| d\mathcal{H}^1.$$

Remark 2.5 (Currents with multiplicity g induced by an immersion). If $\gamma : [0, 1] \rightarrow \mathbb{R}^2$ is a Lipschitz immersion and $g \in \mathbb{R}^2$ is a fixed vector, the immersion induces a 1-rectifiable current $T = [\gamma([0, 1]), \tau, \theta]$ by choosing τ and θ as follows. Let $\tilde{\tau}(x) := \sum_{p \in \gamma^{-1}(x)} \gamma'(p) / |\gamma'(p)|$ and $\theta(x) := |\tilde{\tau}(x)| g$ for any $x \in \gamma([0, 1])$ such that γ is differentiable at any $p \in \gamma^{-1}(x)$ with $|\gamma'(p)| > 0$, and then $\tau(x) := \tilde{\tau}(x) / |\tilde{\tau}(x)|$ for any x such that also $|\tilde{\tau}(x)| \neq 0$ (τ and θ are defined arbitrarily elsewhere).

The area formula immediately yields $T(\omega) = \int_0^1 \langle \omega(\gamma(t))(\gamma'(t)), g \rangle dt$ for any 1-form ω with values in \mathbb{R}^2 . In particular $\partial T = g\delta_{\gamma(1)} - g\delta_{\gamma(0)}$.

As noticed above, the space of 1-covectors with values in \mathbb{R}^2 is the space of linear maps from \mathbb{R}^2 to \mathbb{R}^2 , hence it is isomorphic to the space of matrices $M^{2 \times 2}(\mathbb{R})$. From now on, we shall

make use of this identification without further mention and we will denote the set of 1-forms by $C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$. Therefore, for $\omega \in C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$ we shall write

$$\omega = \begin{bmatrix} \omega_1(x) \\ \omega_2(x) \end{bmatrix},$$

where $\omega_i = (\omega_{i,1}, \omega_{i,2}) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is smooth with compact support and identifies the standard 1-form $\omega_{i,1}dx^1 + \omega_{i,2}dx^2$.

We recall here the notion of calibrations for rectifiable currents with coefficients in a group introduced by Marchese and Massaccesi in [15].

Definition 2.6 (Calibration for 1-rectifiable currents). Let $T = [\Sigma, \tau, \theta]$ be a 1-rectifiable current with coefficients in \mathcal{G} and $\omega \in C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$. Then ω is a calibration for T if

- (i) $d\omega = 0$;
- (ii) $\|\omega\|_{\text{com}} \leq 1$;
- (iii) $\langle \omega(x)(\tau(x)), \theta(x) \rangle = \|\theta(x)\|$ for \mathcal{H}^1 -almost every $x \in \Sigma$.

We recall from [15] how this notion of calibration for currents implies minimality properties for the mass of the calibrated current.

For a given set of points in the plane $p_1, \dots, p_n \in \mathbb{R}^2$, let B be a 0-current of the form

$$B := c_1 \delta_{p_1} + \dots + c_n \delta_{p_n} \quad \text{with } c_i \in \mathcal{G},$$

that is, $B(f) = \sum_{i=1}^n \langle c_i, f(p_i) \rangle$ for any 0-form f with values in \mathbb{R}^2 .

Remark 2.7. It is easily checked that $B = c_1 \delta_{p_1} + \dots + c_n \delta_{p_n}$ is the boundary of a 1-rectifiable current with bounded support with coefficients in \mathcal{G} if and only if $\sum_{i=1}^n c_i = 0$.

Indeed, if $B = \partial T$, let $\omega = v\chi(x)$ be the 0-form with values in \mathbb{R}^2 where $v \in \mathbb{R}^2$ is fixed and $\chi \in C_c^\infty(\mathbb{R}^2)$ equals 1 in a neighborhood of the support of T ; hence $0 = T(d\omega) = B(\omega) = \langle v, \sum_{i=1}^n c_i \rangle$. Arbitrariness of v implies $\sum_{i=1}^n c_i = 0$. On the other hand, if $\sum_{i=1}^n c_i = 0$, let $q \neq p_i$ for any i ; hence it suffices to take $T = \sum_{i=1}^n T_i$ where T_i is the current induced by a C^1 embedding $\gamma_i : [0, 1] \rightarrow \mathbb{R}^2$ from q to p_i endowed with multiplicity c_i .

We define the classes

$$C_1 := \{T : T \text{ is a 1-rectifiable currents with coefficients in } \mathcal{G}, \partial T = B\},$$

$$C_2 := \{T : T \text{ is a 1-normal currents with coefficients in } \mathbb{R}^2, \partial T = B\}.$$

Obviously, $C_1 \subset C_2$.

In case C_1 is nonempty and contains a current \hat{T} with a calibration ω , then \hat{T} solves a mass minimization problem as stated in the next proposition.

Proposition 2.8 [15, Proposition 3.2]. *In the notation above, suppose that $\omega \in C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$ is a calibration for some $\hat{T} \in C_1$. Then*

$$\mathbb{M}(\hat{T}) \leq \mathbb{M}(T)$$

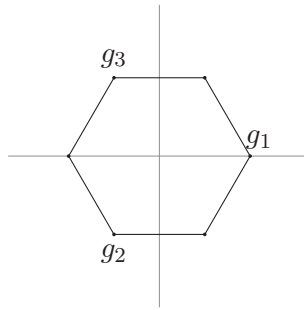


FIGURE 1 The unit ball in the norm $\| \cdot \|$.

for every $T \in C_2$.

From now on, let $\{e_1, e_2\}$ be the canonical basis of \mathbb{R}^2 . We define $g_1 := e_1$, $g_2 := (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ and $g_3 := -g_1 - g_2$. We choose a norm $\| \cdot \|$ on \mathbb{R}^2 such that $\|g_1\| = \|g_2\| = \|g_1 + g_2\| = 1$, the unit ball with respect to $\| \cdot \|$ is the regular hexagon in Figure 1, and $\| \cdot \|$ is defined on the rest of \mathbb{R}^2 by homogeneity. We define \mathcal{G} to be the discrete group generated by g_1 and g_2 with respect to addition.

Theorem 2.9. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network with endpoints $p_1, \dots, p_n \in \mathbb{R}^2$. Let $\mathcal{G}, g_1, g_2, g_3$, and $\| \cdot \|$ be as above.*

Then there exists a 1-rectifiable current \hat{T} with coefficients in \mathcal{G} with boundary $B := \partial \hat{T} = c_1 \delta_{p_1} + \dots + c_n \delta_{p_n}$ such that $c_i \in \{\pm g_1, \pm g_2, \pm g_3\}$, $\text{supp}(\hat{T}) = \Gamma_$, $L(\Gamma_*) = \mathbb{M}(\hat{T})$, and there exists a calibration $\omega \in C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$ for \hat{T} .*

In particular, \hat{T} is a mass minimizing current among 1-normal currents with coefficients in \mathbb{R}^2 with boundary B .

Proof. Up to translations we can fix one of the endpoints of Γ_* to be the origin of \mathbb{R}^2 and we can rotate Γ_* so that all its straight edges are parallel to either g_1, g_2 , or g_3 . Then the desired 1-rectifiable current $\hat{T} = [\Sigma, \tau, \theta]$ with coefficients in \mathcal{G} is defined as follows: $\Sigma = \Gamma_*(G)$ is the 1-rectifiable set, the orientation τ and the multiplicity θ are constant in the interior of each straight segment and we set $\tau(x) = \theta(x) = g_i$ if and only if x is an interior point of a straight edge of Γ_* parallel to g_i (τ and θ are defined arbitrarily at endpoints and junctions).

It is immediately checked that $B := \partial \hat{T}$ has the desired form, as no boundary is generated at triple junctions. Moreover, $\text{supp}(\hat{T}) = \Gamma_*$ and $L(\Gamma_*) = \mathbb{M}(\hat{T})$.

Finally, we claim that the identity matrix

$$\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

identifying a 1-form in $C_c^\infty(\mathbb{R}^2, M^{2 \times 2}(\mathbb{R}))$, is a calibration for \hat{T} . Indeed $d\omega = 0$ trivially. Next, to show that $\|\omega\|_{\text{com}} \leq 1$ we notice that the unit ball of the norm of the group is convex and that its extreme points are $\pm g_1, \pm g_2, \pm g_3$ and hence it is sufficient to estimate $\langle \omega(v(x)), \cdot \rangle$ against

$\pm g_1, \pm g_2, \pm g_3$, where $v(x)$ is generic unit vector $v(x) = (\cos \alpha(x), \sin \alpha(x))$. So, we have

$$|\langle \omega(v(x)), g_1 \rangle| = \left| \left\langle \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha(x) \\ \sin \alpha(x) \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \right| = |\cos \alpha(x)| \leq 1,$$

and similarly $|\langle \omega(v(x)), g_2 \rangle| = |\sin(\alpha(x) + \pi/6)| \leq 1$, and $|\langle \omega(v(x)), g_3 \rangle| = |\sin(\alpha(x) - \pi/6)| \leq 1$.

To conclude, we check that $\langle \omega(\tau(x)), \theta(x) \rangle = \|\theta(x)\|$ \mathcal{H}^1 -almost every on Σ . But this is immediate as

$$\langle \omega(\tau(x)), \theta(x) \rangle = \langle \tau(x), \theta(x) \rangle = \langle \theta(x), \theta(x) \rangle = 1 = \|\theta(x)\|$$

by definition of the orientation and of the norm $\|\cdot\|$. □

Using Theorem 2.9, we directly derive a proof of the fact that minimal networks minimize length among competitor networks having the same topology.

Corollary 2.10. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network. Then $L(\Gamma_*) \leq L(\Gamma)$ for any immersed triple junctions network $\Gamma : G \rightarrow \mathbb{R}^2$ having the same endpoints of Γ_* .*

Proof. Let \hat{T} be given by Theorem 2.9, let $B := \partial \hat{T}$. The claim follows from Theorem 2.9 if we can show that given an immersed triple junctions network $\Gamma : G \rightarrow \mathbb{R}^2$ having the same endpoints of Γ_* , there exists a 1-rectifiable current with coefficient in \mathcal{G} and boundary $\partial T = B$ such that $\mathbb{M}(T) \leq L(\Gamma)$.

Recalling the construction of $\hat{T} = [\Gamma_*(G), \tau, \theta]$ in the proof of Theorem 2.9, the multiplicity θ is constant along each edge of Γ_* . Hence, for any edge E_i of the graph G we can associate $g_{E_i} \in \{g_1, g_2, g_3\}$ such that $\theta(\Gamma_*(p)) = g_{E_i}$ for \mathcal{H}^1 -almost every $p \in E_i$. Moreover, up to inverting the orientation of the edges of G , we can assume that $\tau(\Gamma_*(p)) = \tau_{\Gamma_*|_{E_i}}(p)$ for any p in the interior of E_i and any i , that is, the orientation of \hat{T} is the one induced by the immersions $\Gamma_*|_{E_i}$.

So, we can define the desired current by taking $T := [\Gamma(G), \tau_\Gamma, \theta_\Gamma]$ as follows. For any edge E_i of G let $T_{E_i} := [\Gamma|_{E_i}(E_i), \tau_i, \theta_i]$ be the current induced by $\Gamma|_{E_i}$ as in Remark 2.5 taking $g = g_{E_i}$. Hence, define $T := \sum_{E_i} T_{E_i}$.

As Γ_* and Γ have the same domain graph G , whose edge have a fixed orientation, recalling Remark 2.5 it follows that $\partial T = \partial \hat{T} = B$. Moreover, as $\|\theta_\Gamma(x)\| \leq \sum_{E_i} \#(\Gamma|_{E_i})^{-1}(x) \|g_{E_i}\| = \# \Gamma^{-1}(x)$, it follows that $\mathbb{M}(T) \leq L(\Gamma)$. □

Remark 2.11. By approximation, the result in Corollary 2.10 clearly holds also in case the maps $\Gamma|_{E_i}$ of the comparison networks are just Lipschitz immersions, that is, Lipschitz maps with almost everywhere nonvanishing derivative $(\Gamma|_{E_i})'$.

We now derive further minimizing properties of minimal networks among competitors with possibly different topologies.

Consider a graph G and another graph H that contains a copy of the graph G . The next corollary roughly states that if we consider a minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ and a network $\Gamma : H \rightarrow \mathbb{R}^2$ with the same endpoints of Γ_* , then the length of Γ is no less than the one of Γ_* .

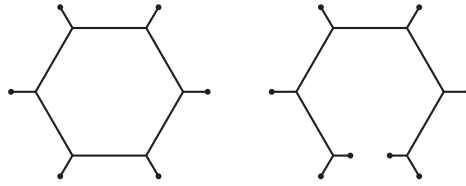


FIGURE 2 Left: A minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$. Right: A network $\Gamma : H \rightarrow \mathbb{R}^2$, the graph H does not contain a homeomorphic copy of G and clearly $L(\Gamma) < L(\Gamma_*)$.

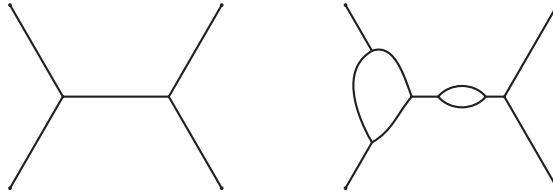


FIGURE 3 Left: A minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$. Right: A competitor $\Gamma : H \rightarrow \mathbb{R}^2$, fulfilling the hypothesis of Corollary 2.13.

We observe that some topological assumption of this kind is also necessary for the length of a given Γ_* to be minimizing. Otherwise, given a fixed minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ whose image contains a cycle, just by deleting an interior segment of an edge of the cycle one gets a strictly shorter minimal network (with two additional endpoints) as depicted in Figure 2.

Corollary 2.12. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network. Let $\Gamma : H \rightarrow \mathbb{R}^2$ be an immersed network such that there exist a subset $G' \subset H$ and a homeomorphism $f : G \rightarrow G'$ such that $\Gamma_*(q) = \Gamma(f(q))$ for any endpoint q of G . Then $L(\Gamma_*) \leq L(\Gamma)$.*

Proof. For any edge E_i of G , let J_i be the set of junctions of H contained in $f(E_i)$. Up to homeomorphism, we can assume that the restriction of f on $E_i \setminus f^{-1}(J_i)$ is a smooth diffeomorphism between intervals. Hence, for any edge E_i of G , the map $\Gamma \circ f|_{E_i}$ is Lipschitz with almost everywhere nonvanishing derivative. Therefore, the map $\Gamma' := \Gamma \circ f : G \rightarrow \mathbb{R}^2$ defines a (Lipschitz regular) immersed triple junctions network with same endpoints of Γ_* . Hence, Corollary 2.10 and Remark 2.11 apply and we get $L(\Gamma_*) \leq L(\Gamma') \leq L(\Gamma)$. □

Exploiting the previous corollary, it is possible to compare the length of a minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ with the length of a suitable immersed network $\Gamma : H \rightarrow \mathbb{R}^2$ possibly having different topology. In the next statement, we consider comparison networks $\Gamma : H \rightarrow \mathbb{R}^2$ having “richer topology,” in the sense that G is assumed to be homeomorphic to a topological quotient of H . In particular, H may have more edges and endpoints than G (see Figure 3).

Corollary 2.13. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network. Let $\Gamma : H \rightarrow \mathbb{R}^2$ be an immersed network such that*

- (1) *there exist connected pairwise disjoint subgraphs $H_i \subset H$ such that, letting $\overline{H} := H / \sim$ the quotient space that identifies each H_i with a point, there exists a homeomorphism $F : G \rightarrow \overline{H}$;*
- (2) *$\pi^{-1}(F(q)) \cap \cup_i H_i = \emptyset$ for any endpoint q of G , where $\pi : H \rightarrow \overline{H}$ is the natural projection;*

(3) for any endpoint q of G there holds $\Gamma_*(q) = \Gamma(F(q))$.[†]

Then $L(\Gamma_*) \leq L(\Gamma)$.

Proof. We want to show that there exists a subset $G' \subset H$ homeomorphic to G . In this way, employing also (3), applying Corollary 2.12 we get the result.

We are going to define G' by selecting suitable paths contained in the subgraphs H_i in order to join the connected components of $H \setminus \cup_i H_i$ so to get a subset G' homeomorphic to G .

By (1) and (2), for any subgraph H_i , using the homeomorphism F^{-1} , the point $\pi(H_i) \in \overline{H}$ corresponds either to an interior point of an edge E_j of G , or to a triple junction m of G . We distinguish the two cases.

- Suppose that, up to renaming, $\pi(H_i)$ corresponds to an interior point of the edge E_1 in G . Hence, $H_i \cap \overline{H} \setminus H_i$ consists of two points x, y that are endpoints of two (different) edges L_1, L_2 of H . As H_i is connected, there exists an embedding $\alpha : [0, 1] \rightarrow H_i$ connecting x and y .

Hence, $L_1 \cup \alpha([0, 1]) \cup L_2$ is homeomorphic to E_1 .

- Suppose that, up to renaming, $\pi(H_i)$ corresponds to a triple junction m in G where the edges E_1, E_2, E_3 concur. Hence, $H_i \cap \overline{H} \setminus H_i$ consists of three points a_1, a_2, a_3 that are endpoints of three (different) edges L_1, L_2, L_3 of H . Up to renaming, there exist embeddings $\alpha_{12} : [0, 1] \rightarrow H_i$ from a_1 to a_2 not passing through a_3 and $\alpha_{13} : [0, 1] \rightarrow H_i$ from a_1 to a_3 not passing through a_2 .

Indeed H_i is connected, thus there is an embedding $\sigma : [0, 1] \rightarrow H_i$ from a_1 to a_2 . If σ does not touch a_3 , by connectedness there is an embedding $\tilde{\sigma} : [0, 1] \rightarrow H_i$ from a_3 to a_1 . If $a_2 \notin \tilde{\sigma}([0, 1])$, we are done. If $a_2 \in \tilde{\sigma}([0, 1])$, then the claim follows by taking a_2 in place of a_1 and by splitting $\tilde{\sigma}$ into two embeddings. If otherwise $a_3 \in \sigma([0, 1])$, then the claim follows by taking a_3 in place of a_1 and by splitting σ into two embeddings.

Therefore, there is a junction w of H and times $t_2, t_3 \in (0, 1)$ such that $\alpha_{12}(t_2) = \alpha_{13}(t_3) = w$ and $\alpha_{12}((t_2, 1]) \cap \alpha_{13}((t_3, 1]) = \emptyset$.

Hence, $L_1 \cup \alpha_{12}([0, 1]) \cup \alpha_{13}((t_3, 1]) \cup L_2 \cup L_3$ is homeomorphic to $E_1 \cup E_2 \cup E_3$ in G .

Performing the selections in the previous items for any H_i , we obtain subsets $S_i \subset H_i$ such that G is homeomorphic to $G' := (H \setminus \cup_i H_i) \cup \cup_i S_i$ via a homeomorphism $f : G \rightarrow G'$ such that $\Gamma_*(q) = \Gamma(f(q))$ for any endpoint of G , by 3). Hence, Corollary 2.12 applies and the proof follows. \square

In contrast to Corollary 2.13, exploiting again the general Theorem 2.9, we can further prove that a minimal network $\Gamma_* : G \rightarrow \mathbb{R}^2$ is also length-minimizing among suitable immersed networks $\Gamma : H \rightarrow \mathbb{R}^2$ having poorer topology. In this case, “poorer topology” means that H is homeomorphic to a quotient of G . In particular, H may have less edges than G (see Figure 4).

Corollary 2.14. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network. Let $\Gamma : H \rightarrow \mathbb{R}^2$ be an immersed network such that*

- (1) *there exist connected pairwise disjoint subgraphs $G_i \subset G$ such that, letting $\overline{G} := G / \sim$ the quotient space that identifies each G_i with a point, there exists a homeomorphism $F : \overline{G} \rightarrow H$;*
- (2) *for any endpoint q of G , there holds that $q \notin \cup_i G_i$ and $\Gamma_*(q) = \Gamma(F(\pi(q)))$, where $\pi : G \rightarrow \overline{G}$ is the natural projection.*

[†] $\Gamma(F(q))$ is well-defined because, by (2), $\pi^{-1}(F(q))$ is a singleton for any endpoint q of G .

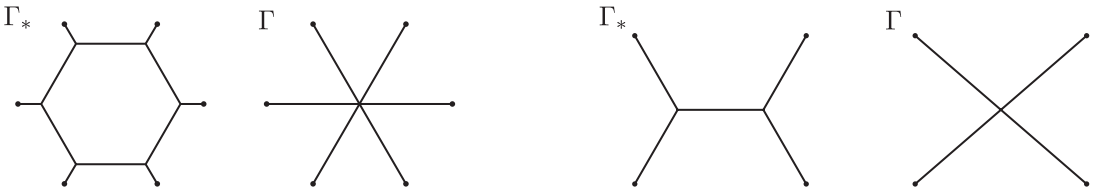


FIGURE 4 Two couples of minimal networks $\Gamma_* : G \rightarrow \mathbb{R}^2$ and competitors $\Gamma : H \rightarrow \mathbb{R}^2$ fulfilling the hypothesis of Corollary 2.14.

Then $L(\Gamma_*) \leq L(\Gamma)$.

Proof. Let $\widehat{T} = [\Gamma_*(G), \tau, \theta]$ be given by Theorem 2.9. As in the proof of Corollary 2.10, for any edge E_j of the graph G we can associate $g_{E_j} \in \{g_1, g_2, g_3\}$ such that $\theta(\Gamma_*(p)) = g_{E_j}$ for H^1 -almost every $p \in E_j$. Moreover, up to inverting the orientation of the edges of G , we can assume that $\tau(\Gamma_*(p)) = \tau_{\Gamma_*|_{E_j}}(p)$ for any p in the interior of E_j and any j . Also, for any edge E_j , we denote $\widehat{T}_{E_j} := [\Gamma_*|_{E_j}, \tau_j, \theta_j]$ where $\tau_j(x) = \tau_{\Gamma_*|_{E_j}}(\Gamma^{-1}(x))$ and $\theta_j(x) = g_{E_j}$ on $\Gamma_*|_{E_j}$ (and zero elsewhere).

For any subgraph G_i , consider the current $S_i := \sum_{E_j \subset G_i} \widehat{T}_{E_j}$. As by (2) no endpoints of G touch $\cup_i G_i$, then $\partial S_i = \sum_j g_j^i \delta_{M_j^i}$ for some junctions $M_i = \{m_j^i\}_j \subset \mathbb{R}^2$ depending on i and elements of the group $\{g_j^i\}_j$. Hence, $\sum_j g_j^i = 0$ by Remark 2.7, for any i . Moreover, for given G_i , we denote by

$$\mathcal{E}_i := \{(e_k, E_k) : e_k \in \{0, 1\}, E_k \text{ edge of } G \text{ has the endpoint } e_k \text{ lying in } G_i\}.$$

As T has no boundary at junctions, that is, ∂T is supported on endpoints, it follows that

$$\sum_{(e_k, E_k) \in \mathcal{E}_i} (-1)^{1+e_k} g_{E_k} = - \sum_j g_j^i = 0, \tag{2.1}$$

for any i .

We want to associate to $\Gamma : H \rightarrow \mathbb{R}^2$ a suitable current T with $\partial T = \partial \widehat{T}$. With little abuse of terminology, we will say that $\pi(E_k)$ is an edge of \overline{G} for any edge E_k of G not belonging to $\cup_i G_i$; in particular, for such a k , $\pi(E_k)$ is an oriented interval. By (2) we have that $F(\pi(G_i))$ is either a junction of H or an interior point of some edge of H . If the latter case happens for some G_{i_0} , then $\mathcal{E}_{i_0} = \{(e_{k_1}, E_{k_1}), (e_{k_2}, E_{k_2})\}$ has two elements. As $g_{E_j} \in \{g_1, g_2, g_3\}$ for any edge E_j of G , then (2.1) implies that $g_{E_{k_1}} = g_{E_{k_2}}$ and $e_{k_1} \neq e_{k_2}$. Hence, there is an embedding $\alpha_{k_1, k_2} : [0, 1] \rightarrow \overline{G}$ whose image is $\pi(E_{k_1}) \cup \pi(G_{i_0}) \cup \pi(E_{k_2})$ and α_{k_1, k_2} preserves the orientation of $\pi(E_{k_1}), \pi(E_{k_2})$, that is, the restriction $\alpha_{k_1, k_2}^{-1} : \alpha_{k_1, k_2}^{-1}(\pi(E_{k_\ell})) \rightarrow \pi(E_{k_\ell})$ is an orientation preserving homeomorphism for $\ell = 1, 2$.

Therefore, up to inverting orientations of edges of H , we have that the restriction $F : F^{-1}(H_s) \rightarrow H_s$ is orientation preserving for any edge H_s of H , that is, if $\pi(E_k) \subset F^{-1}(H_s)$ and $E_k \cap (\cup_i G_i) = \emptyset$ then the restriction $F : \pi(E_k) \rightarrow F(\pi(E_k)) \subset H_s$ is orientation preserving. Also, we can associate to any edge H_s of H a group element $g_s \in \{g_1, g_2, g_3\}$ where $g_s = g_{E_k}$ for any edge E_k of $G \setminus (\cup_i G_i)$ with $\pi(E_k) \subset F^{-1}(H_s)$.

Finally, we can define the desired current $T = \sum_s T_{H_s}$, where $T_{H_s} := [\Gamma|_{H_s}, \sigma_s, \vartheta_s]$ is the current induced by the immersion $\Gamma|_{H_s}$ as in Remark 2.5 taking $g = g_s$.

By the above observations, we have $\partial T_{H_s} = g_s \delta_{\Gamma|_{H_s(1)}} - g_s \delta_{\Gamma|_{H_s(0)}}$. Now junctions of H correspond: either to junctions of G not belonging to $\cup_i G_i$, or to identified graphs $\pi(G_i)$ in \bar{G} . Calling $M := \{m_i = \Gamma(F(\pi(G_i))) : F(\pi(G_i)) \text{ is a junction in } H\} \subset \mathbb{R}^2$, as F is orientation preserving, we get

$$\partial T = \sum_s \partial T_{H_s} = \partial \hat{T} + \sum_{m_i \in M} \sum_{(e_k, E_k) \in \mathcal{E}_i} (-1)^{1+e_k} g_{E_k} m_i \stackrel{(2.1)}{=} \partial \hat{T}.$$

Moreover, $\|\sum_k \vartheta_k(x)\| \leq \sum_k \#(\Gamma|_{H_k})^{-1}(x) \|g_{E_k}\| = \#\Gamma^{-1}(x)$, thus $\mathbb{M}(T) \leq L(\Gamma)$. Hence, Theorem 2.9 implies $L(\Gamma_*) \leq \mathbb{M}(T) \leq L(\Gamma)$. \square

Remark 2.15. By approximation, the results in Corollaries 2.12, 2.13, and 2.14 hold also in case the maps $\Gamma|_{E_i}$ of the comparison networks are just Lipschitz immersions.

3 | LOCAL MINIMALITY AMONG PARTITIONS

Let $\Omega \subset \mathbb{R}^2$ be open. We denote by $P(E, \Omega) := |D\chi_E|(\Omega)$ the (relative) perimeter of a measurable set $E \subset \Omega$ in Ω . The symbol $\partial^* E$ denotes the reduced boundary of E , namely the set of points $x \in \text{spt}|D\chi_E|$ where the generalized outer unit normal to E exists. Recall that $|D\chi_E|$ is concentrated on $\partial^* E$. For the theory of sets of finite perimeter and functions of bounded variation, we refer the reader to [4].

We say that $\mathbf{E} = (E_1, \dots, E_n)$, for $n \in \mathbb{N}$ with $n \geq 2$, is Caccioppoli partition of Ω if $E_i \subset \Omega$ for any i , $|E_i \cap E_j| = 0$ for $i \neq j$, $|\Omega \setminus \cup_{i=1}^n E_i| = 0$, and $\sum_{i=1}^n P(E_i, \Omega) < +\infty$. We denote by $\Sigma_{ij} := \partial^* E_i \cap \partial^* E_j$ and by $\nu_{ij} = \nu_i = -\nu_j$ the unit normal to Σ_{ij} , where ν_i is the generalized outer unit normal to the set E_i . In particular, we can think of ν_{ij} as a normal pointing from E_i into E_j .

We shall mostly focus our attention to Caccioppoli partitions $\mathbf{E} = (E_1, E_2, E_3)$ defined by three sets. We remark that, thanks to [4, Theorem 4.17], it holds $\mathcal{H}^1(\partial^* E_1 \cap \partial^* E_2 \cap \partial^* E_3) = 0$, or equivalently

$$\mathcal{H}^1(\Sigma_{ij} \cap \Sigma_{ik}) = 0 \text{ for } i, j, k \in \{1, 2, 3\} \text{ with } j \neq k.$$

Moreover, $P(E_i, \Omega) = \mathcal{H}^1(\Sigma_{ij} \cup \Sigma_{ik}) = \mathcal{H}^1(\Sigma_{ij}) + \mathcal{H}^1(\Sigma_{ik})$, for any distinct i, j, k and then

$$\frac{1}{2} \sum_{i=1}^3 P(E_i, \Omega) = \mathcal{H}^1(\Sigma_{12} \cap \Omega) + \mathcal{H}^1(\Sigma_{23} \cap \Omega) + \mathcal{H}^1(\Sigma_{31} \cap \Omega).$$

We recall from [4, Theorem 3.87] that BV functions admit (*inner*) traces on boundaries of open sets with Lipschitz boundary. More precisely, if $\Omega \subset \mathbb{R}^2$ is a bounded open set with Lipschitz boundary and $u \in [BV(\Omega)]^n$, then for \mathcal{H}^1 -almost every point $x \in \partial\Omega$ there exists $\text{tr}_\Omega u(x) \in \mathbb{R}^n$ such that

$$\lim_{r \rightarrow 0^+} \frac{1}{r^2} \int_{B_r(x) \cap \Omega} |u(y) - \text{tr}_\Omega u(x)| dy = 0.$$

Moreover, the function $\text{tr}_\Omega u$ belongs to $[L^1(\partial\Omega, \mathcal{H}^1)]^n$.

If $F \subset \mathbb{R}^2$ is measurable and $u : F \rightarrow \mathbb{R}^n$ is a (possibly vector-valued) measurable function, we say that u has *approximate limit* $\text{ap} - \lim_{y \rightarrow x} u = u_0 \in \mathbb{R}^n$ at $x \in F$ if

$$\lim_{r \rightarrow 0^+} \frac{1}{r^2} \int_{B_r(x) \cap F} |u(y) - u_0| \, dy = 0.$$

Adopting the above terminology, $\text{tr}_\Omega u(x) = \text{ap} - \lim_{y \rightarrow x} u$ for \mathcal{H}^1 -almost every $x \in \partial\Omega$, for $u \in BV(\Omega)$ with Ω Lipschitz and bounded.

Let us now fix a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and let $\widehat{\mathbf{E}} = (\widehat{E}_1, \widehat{E}_2, \widehat{E}_3)$ be a fixed Caccioppoli partition of Ω . We define

$$\mathcal{A} := \left\{ \mathbf{E} = (E_1, E_2, E_3) : \mathbf{E} \text{ is a Caccioppoli partition of } \Omega, \text{tr}_\Omega \chi_{E_i} = \text{tr}_\Omega \chi_{\widehat{E}_i} \text{ } \mathcal{H}^1\text{-almost every on } \partial\Omega \right\}.$$

Clearly, $\widehat{\mathbf{E}}$ itself is an element of \mathcal{A} . For every $\mathbf{E} \in \mathcal{A}$ we introduce the energy

$$\mathcal{P}(\mathbf{E}) := \frac{1}{2} \sum_{i=1}^3 P(E_i, \Omega).$$

A partition $\widetilde{\mathbf{E}} \in \mathcal{A}$ is a *minimizer* of \mathcal{P} in Ω if

$$\mathcal{P}(\widetilde{\mathbf{E}}) \leq \mathcal{P}(\mathbf{E})$$

for every $\mathbf{E} \in \mathcal{A}$.

Definition 3.1 (Approximately regular vector fields on \mathbb{R}^2). Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. A measurable vector field $\Phi : \overline{\Omega} \rightarrow \mathbb{R}^2$ is said to be *approximately regular* if it is bounded and for every Lipschitz curve[†] $\gamma \subset \overline{\Omega}$ there holds

$$\text{ap} - \lim_{y \rightarrow x} (\Phi(y) \cdot \nu_\gamma(x)) = \Phi(x) \cdot \nu_\gamma(x),$$

at \mathcal{H}^1 -almost every $x \in \gamma$, for any chosen unit normal to γ at x .

We recall that we are assuming measurable functions to be defined pointwise, hence $\Phi(x) \cdot \nu_\gamma(x)$ is a well-defined number in the above definition.

In this paper, when we talk about the divergence of a bounded vector field $\Phi : \Omega \rightarrow \mathbb{R}^2$ we refer to the distributional divergence of Φ , that is $\int_\Omega u \, \text{div} \, \Phi := - \int_\Omega \nabla u \cdot \Phi \, dx$ for any $u \in C_c^1(\Omega)$. We say that $\text{div} \, \Phi \in L^p(\Omega)$ whenever the distributional divergence is represented by an $L^p(\Omega)$ -function, still denoted by $\text{div} \, \Phi$.

If $u \in BV(\Omega)$, we denote by $Du = (\partial_1 u, \partial_2 u)$ the vector valued measure satisfying $\sum_i \int_\Omega \Phi_i \, d(\partial_i u) = - \int_\Omega u \, \text{div} \, \Phi \, dx$ for any $\Phi \in C_c^1(\Omega)$, and we denote $\int_\Omega \Phi \cdot Du := \sum_i \int_\Omega \Phi_i \, d(\partial_i u)$ for any field Φ such that the latter expression makes sense.

[†] Here by Lipschitz curve, we mean the image of a Lipschitz embedding $\sigma : [0, 1] \rightarrow \overline{\Omega}$.

Lemma 3.2 (Divergence theorem for approximately regular vector field, [1, Lemma 2.4]). *Let Ω be a bounded open set in \mathbb{R}^2 with Lipschitz boundary and let $\nu_{\partial\Omega}$ be its inner normal, which is well-defined \mathcal{H}^1 -almost every on $\partial\Omega$. Let $\Phi : \overline{\Omega} \rightarrow \mathbb{R}^2$ be an approximately regular vector field and let $u \in BV(\Omega)$. Assume that $\operatorname{div} \Phi \in L^\infty(\Omega)$ and $\operatorname{tr}_\Omega u \Phi \in L^1(\partial\Omega, \mathcal{H}^1)$. Then*

$$\int_\Omega \Phi \cdot Du = - \int_\Omega u \operatorname{div} \Phi \, dx - \int_{\partial\Omega} \operatorname{tr}_\Omega u \Phi \cdot \nu_{\partial\Omega} \, d\mathcal{H}^1, \quad (3.1)$$

where $\Phi \cdot \nu_{\partial\Omega}(x) = \operatorname{ap-lim}_{y \rightarrow x} (\Phi(y) \cdot \nu_\gamma(x))$ at \mathcal{H}^1 -almost every $x \in \partial\Omega$.

In what follows we will apply Lemma 3.2 to characteristic functions of sets of finite perimeter.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and let $\mathbf{E}, \widetilde{\mathbf{E}} \in \mathcal{A}$. Let $\Phi : \overline{\Omega} \rightarrow \mathbb{R}^2$ be an approximately regular vector field with $\operatorname{div} \Phi = 0$. Then*

$$\int_\Omega \Phi \cdot D\chi_{\widetilde{E}_i} = \int_\Omega \Phi \cdot D\chi_{E_i},$$

for any $i = 1, 2, 3$.

Proof. Using the fact that $\operatorname{div} \Phi = 0$, formula (3.1) applied with $u = \chi_{\widetilde{E}_i}$ or $u = \chi_{E_i}$ reduces to

$$\begin{aligned} \int_\Omega \Phi \cdot D\chi_{\widetilde{E}_i} &= - \int_{\partial\Omega} \operatorname{tr}_\Omega \chi_{\widetilde{E}_i} \Phi \cdot \nu_{\partial\Omega} \, d\mathcal{H}^1, \\ \int_\Omega \Phi \cdot D\chi_{E_i} &= - \int_{\partial\Omega} \operatorname{tr}_\Omega \chi_{E_i} \Phi \cdot \nu_{\partial\Omega} \, d\mathcal{H}^1, \end{aligned}$$

for any $i = 1, 2, 3$. By definition of the set \mathcal{A} , the two right-hand sides above are equal, and the claim follows. \square

Roughly speaking, the next lemma states the well-known fact that if a piecewise smooth vector field is such that its normal component along its jump set is well-defined, then the distributional divergence is simply given by the pointwise divergence of the field computed where it is smooth. We provide a proof for the convenience of the reader.

Lemma 3.4. *Let Ω be a bounded open set with Lipschitz boundary. Let $\Psi : \overline{\Omega} \rightarrow \mathbb{R}^2$ be a measurable bounded vector field. Suppose that there exists a Caccioppoli partition $\mathbf{F} = (F_1, \dots, F_n)$ of Ω , where F_i is open with Lipschitz boundary, such that $\Psi|_{F_i} \in C^1(F_i)$ with $\operatorname{div}(\Psi|_{F_i}) \in L^\infty(F_i)$ and*

$$\operatorname{tr}_{F_i}(\Psi|_{F_i}) \cdot \nu_i = -\operatorname{tr}_{F_j}(\Psi|_{F_j}) \cdot \nu_j, \quad (3.2)$$

for any $i \neq j$, at \mathcal{H}^1 -almost every point on $\partial^* F_i \cap \partial^* F_j$, where ν_i is the \mathcal{H}^1 -almost every defined outer normal to F_i .

Then the distributional divergence of Ψ is given by

$$\operatorname{div} \Psi = \sum_{i=1}^n \chi_{F_i} \operatorname{div}(\Psi|_{F_i}) \, dx.$$

If also $\operatorname{div}(\Psi|_{F_i}) = 0$ on F_i for any i , then $\operatorname{div} \Psi = 0$.

Proof. By assumptions, we see that $\Psi|_{F_i} \in [BV(F_i)]^2$. As BV functions admit inner traces on boundaries of Lipschitz domains, we see that the field $\Psi_i : \overline{F_i} \rightarrow \mathbb{R}^2$ defined by

$$\Psi_i(x) := \begin{cases} \Psi(x) & x \in F_i, \\ \operatorname{tr}_{F_i}(\Psi|_{F_i}) & \mathcal{H}^1\text{-almost every on } \partial^*F_i, \end{cases}$$

and arbitrarily defined on ∂F_i where $\operatorname{tr}_{F_i} \Psi|_{F_i}$ does not exist, is approximately regular.

Let $u \in C^1_c(\Omega)$. Applying Lemma 3.2 on each F_i , we can compute

$$\begin{aligned} \int_{\Omega} \Psi \cdot \nabla u \, dx &= \sum_i \int_{F_i} \nabla u \cdot \Psi_i \, dx = - \sum_i \int_{F_i} u \operatorname{div}(\Psi|_{F_i}) \, dx + \sum_i \int_{\partial^*F_i} u \Psi_i \cdot \nu_i \, d\mathcal{H}^1 \\ &= - \sum_i \int_{F_i} u \operatorname{div}(\Psi|_{F_i}) \, dx. \end{aligned}$$

□

Definition 3.5 (Local paired calibration). A *local paired calibration* for a Caccioppoli partition $\mathbf{E} = (E_1, E_2, E_3)$ is a collection of three approximately regular vector fields $\Phi_1, \Phi_2, \Phi_3 : \overline{\Omega} \rightarrow \mathbb{R}^2$ such that

- (1) $\operatorname{div} \Phi_i = 0$ for $i = 1, 2, 3$,
- (2) $|\Phi_i - \Phi_j| \leq 1$ \mathcal{H}^1 -almost every in Ω , for $i, j = 1, 2, 3, i \neq j$,
- (3) $(\Phi_i - \Phi_j) \cdot \nu_{ij} = 1$ \mathcal{H}^1 -almost every in Σ_{ij} , for $i, j = 1, 2, 3, i \neq j$.

Remark 3.6. The concept of local paired calibration is nothing but the notion of paired calibration introduced by Lawlor and Morgan in [14] restricted to the case of a partition of Ω composed of three sets.

Remark 3.7. Let Ω be bounded, let $\mathbf{E} \in \mathcal{A}$ and assume that there exist approximately regular fields $\Psi_{12}, \Psi_{23}, \Psi_{31} : \overline{\Omega} \rightarrow \mathbb{R}^2$ such that

- $\operatorname{div} \Psi_{12} = \operatorname{div} \Psi_{23} = \operatorname{div} \Psi_{31} = 0$,
- $|\Psi_{12}|, |\Psi_{23}|, |\Psi_{31}| \leq 1$ \mathcal{H}^1 -almost every in Ω ,
- $\Psi_{ij} \cdot \nu_{ij} = 1$ \mathcal{H}^1 -almost every in Σ_{ij} , for $i, j = 1, 2, 3$ such that Ψ_{ij} is defined,
- $\Psi_{12} + \Psi_{23} + \Psi_{31} = 0$ \mathcal{H}^1 -almost every in Ω .

Then there exists a local paired calibration for \mathbf{E} .

Indeed, fix an arbitrary approximately regular field Φ_1 with $\operatorname{div} \Phi_1 = 0$, for example, $\Phi_1(x, y) := (0, 0)|_{\overline{\Omega}}$. Hence, set $\Phi_2 := \Phi_1 - \Psi_{12}$ and $\Phi_3 := \Psi_{31} + \Phi_1$. Then $\Phi_i - \Phi_j = \Psi_{ij}$ \mathcal{H}^1 -almost every in Ω . Hence, the properties of $\Psi_{12}, \Psi_{23}, \Psi_{31}$ imply that Φ_1, Φ_2, Φ_3 give a local paired calibration for \mathbf{E} .

The next lemma states that existence of a local paired calibration for a partition implies that it is minimizing. We provide a detailed proof that formalizes the general principle introduced in [14, sections 1.1 and 1.2].

Lemma 3.8. *If Φ is a local paired calibration for $\tilde{\mathbf{E}} \in \mathcal{A}$, then $\tilde{\mathbf{E}} \in \mathcal{A}$ is a minimizer of \mathcal{P} in Ω .*

Proof. Using the third condition in the definition of local paired calibration, denoting by $\tilde{\nu}_{ij}$ the usual normal at the interfaces of $\tilde{\mathbf{E}}$, and the structure of Caccioppoli partitions [4, Theorem 4.17] we have

$$\begin{aligned} \mathcal{P}(\tilde{\mathbf{E}}) &= \frac{1}{2} \sum_{i=1}^3 P(\tilde{E}_i, \Omega) = \mathcal{H}^1(\tilde{\Sigma}_{12} \cap \Omega) + \mathcal{H}^1(\tilde{\Sigma}_{23} \cap \Omega) + \mathcal{H}^1(\tilde{\Sigma}_{31} \cap \Omega) \\ &\stackrel{(3)}{=} \int_{\tilde{\Sigma}_{12} \cap \Omega} (\Phi_1 - \Phi_2) \cdot \tilde{\nu}_{12} \, d\mathcal{H}^1 + \int_{\tilde{\Sigma}_{23} \cap \Omega} (\Phi_2 - \Phi_3) \cdot \tilde{\nu}_{23} \, d\mathcal{H}^1 + \int_{\tilde{\Sigma}_{31} \cap \Omega} (\Phi_3 - \Phi_1) \cdot \tilde{\nu}_{31} \, d\mathcal{H}^1 \\ &= \sum_{i=1}^3 \int_{\Omega} \Phi_i \cdot D\chi_{\tilde{E}_i}. \end{aligned}$$

Thanks to the divergence free condition on the vector fields, we can apply Corollary 3.3 getting

$$\sum_{i=1}^3 \int_{\Omega} \Phi_i \cdot D\chi_{\tilde{E}_i} \stackrel{(1)}{=} \sum_{i=1}^3 \int_{\Omega} \Phi_i \cdot D\chi_{E_i}.$$

To conclude, we use the second condition in the definition of local paired calibration to get

$$\begin{aligned} &\sum_{i=1}^3 \int_{\Omega} \Phi_i \cdot D\chi_{E_i} \\ &= \int_{\Sigma_{12} \cap \Omega} (\Phi_1 - \Phi_2) \cdot \nu_{12} \, d\mathcal{H}^1 + \int_{\Sigma_{23} \cap \Omega} (\Phi_2 - \Phi_3) \cdot \nu_{23} \, d\mathcal{H}^1 + \int_{\Sigma_{31} \cap \Omega} (\Phi_3 - \Phi_1) \cdot \nu_{31} \, d\mathcal{H}^1 \\ &\leq \int_{\Sigma_{12} \cap \Omega} |\Phi_1 - \Phi_2| \, d\mathcal{H}^1 + \int_{\Sigma_{23} \cap \Omega} |\Phi_2 - \Phi_3| \, d\mathcal{H}^1 + \int_{\Sigma_{31} \cap \Omega} |\Phi_3 - \Phi_1| \, d\mathcal{H}^1 \\ &\stackrel{(2)}{\leq} \mathcal{H}^1(\Sigma_{12} \cap \Omega) + \mathcal{H}^1(\Sigma_{23} \cap \Omega) + \mathcal{H}^1(\Sigma_{31} \cap \Omega) = \frac{1}{2} \sum_{i=1}^3 P(E_i, \Omega) = \mathcal{P}(\mathbf{E}). \end{aligned}$$

Following the chain of inequalities, we have $\mathcal{P}(\tilde{\mathbf{E}}) \leq \mathcal{P}(\mathbf{E})$ as desired. □

We conclude by proving the *local* minimality of a partition induced by a minimal network.

Theorem 3.9. *Let $\Gamma_* : G \rightarrow \mathbb{R}^2$ be a minimal network such that $\Gamma_*(G) \subset D$ where D is a domain of class C^1 homeomorphic to a closed disk, and $\Gamma_*(G) \cap \partial D$ is the set of endpoints of Γ_* .*

Then there exists a bounded open set $\Omega' \subset \mathbb{R}^2$ such that $\Gamma_(G) \subset \Omega'$, $\Omega := \Omega' \cap \text{int}(D)$ has Lipschitz boundary, there exists a Caccioppoli partition $\tilde{\mathbf{E}} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ of Ω such that $\Omega \cap \cup_i \partial \tilde{E}_i = \Omega \cap \Gamma_*(G)$ and there exists a local paired calibration for $\tilde{\mathbf{E}}$ in Ω .*

In particular, $\tilde{\mathbf{E}}$ is a minimizer for \mathcal{P} in \mathcal{A} , that is the class of partitions having the same trace of $\tilde{\mathbf{E}}$ on $\partial\Omega$.

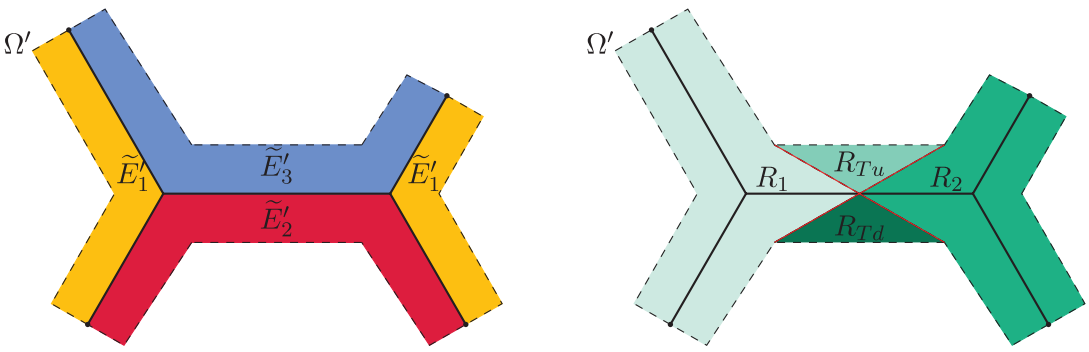


FIGURE 5 Left: Partition $\tilde{E}' = (\tilde{E}'_1, \tilde{E}'_2, \tilde{E}'_3)$ of Ω' . Right: Split of Ω' in four regions R_1, R_2, R_{Tu}, R_{Td} .

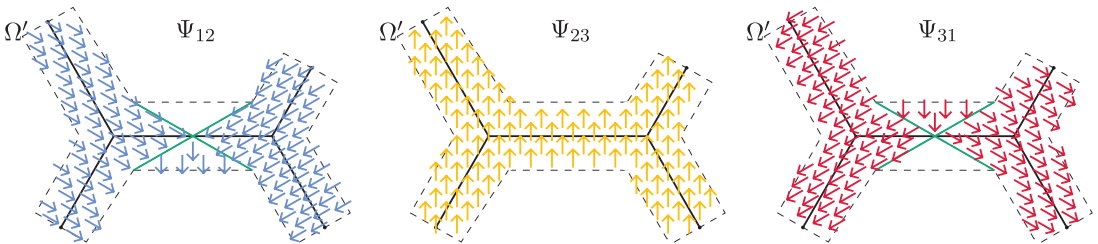


FIGURE 6 Picture of the three fields $\Psi_{12}, \Psi_{23}, \Psi_{31}$.

Proof. Let us assume that G has at least two junctions. The cases with no junctions or with one junction are easier to treat with the construction outlined in the rest of the proof.

We first consider a simple configuration in which $\Gamma_*(G)$ is composed of five curves that meet at two triple junctions. Let $\Gamma'_*(G)$ be the network having all edges equal to those of $\Gamma_*(G)$ except for those ending at an endpoint, which are lengthened by an additive length equal to $\delta' \in (0, 1)$ (that will be suitably chosen later). We first construct a calibration for the Caccioppoli partition identified by the new network $\Gamma'_*(G)$ containing $\Gamma_*(G)$, over an open set Ω' containing $\Gamma'_*(G)$ (see Figure 5). Eventually, in the general case, a restriction of both the partition and the calibration to D will give the desired calibration of a partition of the final set Ω .

We call O^1, O^2 the two junctions and we let d be the distance between O^1 and O^2 . We define the set Ω' to be the δ -tubular neighborhood of $\Gamma'_*(G)$ with $\delta = \frac{d\sqrt{3}}{8}$ truncated at each of the four new endpoints with the line passing through the endpoint that is orthogonal to the corresponding edge. We thus define the partition $\tilde{E}'_1, \tilde{E}'_2, \tilde{E}'_3$ as in Figure 5 on the left.

Let M be the midpoint between O^1 and O^2 . We consider two lines ℓ_1, ℓ_2 through M forming an angle of 30° with the segment connecting O^1 and O^2 . By the choice of δ , the two lines meet $\partial\Omega'$ at points whose projections on the edge $O^1 O^2$ lie in the interior of the edge; hence the lines split Ω' in four open regions that we denote R_1, R_2, R_{Tu}, R_{Td} , as in Figure 5. Now we construct a calibration for \tilde{E}' . Thanks to Remark 3.7 we can exhibit directly the “difference fields” $\Psi_{12}, \Psi_{23}, \Psi_{31}$ (see Figure 6). We define the fields pointwise on each region by

$$\begin{aligned}
\text{Region } R_1 : & \quad \Psi_{12} = \left(\sqrt{3}/2, -1/2 \right), & \Psi_{23} = (0, 1), & \Psi_{31} = \left(-\sqrt{3}/2, -1/2 \right), \\
\text{Region } R_2 : & \quad \Psi_{12} = \left(-\sqrt{3}/2, -1/2 \right), & \Psi_{23} = (0, 1), & \Psi_{31} = \left(\sqrt{3}/2, -1/2 \right), \\
\text{Region } R_{T_u} : & \quad \Psi_{12} = (0, 0), & \Psi_{23} = (0, 1), & \Psi_{31} = (0, -1), \\
\text{Region } R_{T_d} : & \quad \Psi_{12} = (0, -1), & \Psi_{23} = (0, 1), & \Psi_{31} = (0, 0).
\end{aligned}$$

Moreover, the fields Ψ_{ij} are pointwise defined on $\partial\Omega' \cup \ell_1 \cup \ell_2$ by taking traces from suitable corresponding regions. More precisely, we set $\Psi_{ij}|_{\partial R \setminus \{M\}} = \text{tr}_R(\Psi_{ij}|_R)$ for regions $R \in \{R_1, R_2\}$, and $\Psi_{ij}|_{\partial\Omega' \cap \overline{S} \setminus (\overline{R_1} \cup \overline{R_2})} = \text{tr}_S(\Psi_{ij}|_S)$ for regions $S \in \{R_{T_u}, R_{T_d}\}$, and $\Psi_{ij}(M) = 0$.

To prove that $\Psi_{12}, \Psi_{23}, \Psi_{31}$ induce a local paired calibration, we check the four conditions of Remark 3.7.

The divergence free condition immediately follows from Lemma 3.4. Indeed, the fields are constant in the regions R_1, R_2, R_{T_u} and R_{T_d} , so within each regions they have zero divergence. Moreover, one can easily prove that the condition (3.2) is satisfied. As an example, we check it for Ψ_{12} in the transition between R_1 and R_{T_u} .

As $\nu_{R_1} = (1/2, \sqrt{3}/2)$ and $\nu_{R_{T_u}} = -\nu_{R_1} = (-1/2, -\sqrt{3}/2)$, by direct computation we have:

$$\begin{aligned}
\text{tr}_{R_1}(\Psi_{12}|_{R_1}) \cdot \nu_{R_1} &= (\sqrt{3}/2, -1/2) \cdot (1/2, \sqrt{3}/2) = 0, \\
\text{tr}_{R_{T_u}}(\Psi_{12}|_{R_{T_u}}) \cdot \nu_{R_{T_u}} &= (0, 0) \cdot (-1/2, -\sqrt{3}/2) = 0
\end{aligned}$$

hence $\text{tr}_{R_1}(\Psi_{12}|_{R_1}) \cdot \nu_{R_1} = -\text{tr}_{R_{T_u}}(\Psi_{12}|_{R_{T_u}}) \cdot \nu_{R_{T_u}}$. The computations are completely analogous for all the other transitions and fields.

Moreover, we have $|\Psi_1| = |\Psi_2| = |\Psi_3| = 1$. Finally, the third and fourth condition of Remark 3.7 are trivial to check.

We now pass to the general case. Let Γ_*, D be as in the statement. Denote by $\{D_j\}$ the collection of the connected components of $\text{int}(D) \setminus \Gamma_*(G)$. For any j , at most six edges of $\Gamma_*(G)$ are contained in the boundary ∂D_j , and in particular ∂D_j is a hexagon whenever $\partial D_j \cap \partial D = \emptyset$. Indeed, if at least seven edges of $\Gamma_*(G)$ are contained in the boundary ∂D_j , as D_j and ∂D_j are connected, we can split ∂D_j as a union $\partial D_j = \partial(\mathbb{R}^2 \setminus D_j) \cup \bigsqcup_k S_k^i$, where each S_k^i is (the image of) a minimal network with only one endpoint, such endpoint being the intersection of S_k^i with $\partial(\mathbb{R}^2 \setminus D_j)$. But minimal networks S with only one endpoint do not exist, otherwise the distance function from the endpoint would not achieve a maximum on S . Nonexistence of minimal networks with one endpoint also implies that if a path in $\text{int}(D)$ crosses an edge of $\Gamma_*(G)$, then it passes from a component D_j to a different component $D_{j'}$. Similarly, one notices that minimal networks with only two endpoints (and at least a triple junction) do not exist, and thus either $\partial D_i \cap \partial D_k = \emptyset$ or $\partial D_i \cap \partial D_k$ is an edge of $\Gamma_*(G)$, for $i \neq k$.

We define a planar graph Q whose vertices are the components $\{D_j\}$ and we say that D_j and D_k , for $j \neq k$, are connected by an edge if and only if $\partial D_i \cap \partial D_k$ is an edge of $\Gamma_*(G)$. By the above observations it readily follows that Q is homeomorphic to a subset of the planar triangular graph Q' identified by the lattice $\{n(1, 0) + m(1/2, \sqrt{3}/2) : n, m \in \mathbb{Z}\}$. In fact, a homeomorphism can be constructed by iteration as follows. Starting from a first component D_{j_1} , if $x \in \partial D_{j_1}$ is a triple junction of $\Gamma_*(G)$, parameterizing a circle of small radius centered at x starting from a point in D_{j_1} identifies a triangle in Q . Repeating the construction for the other triple junctions lying on ∂D_{j_1} ,

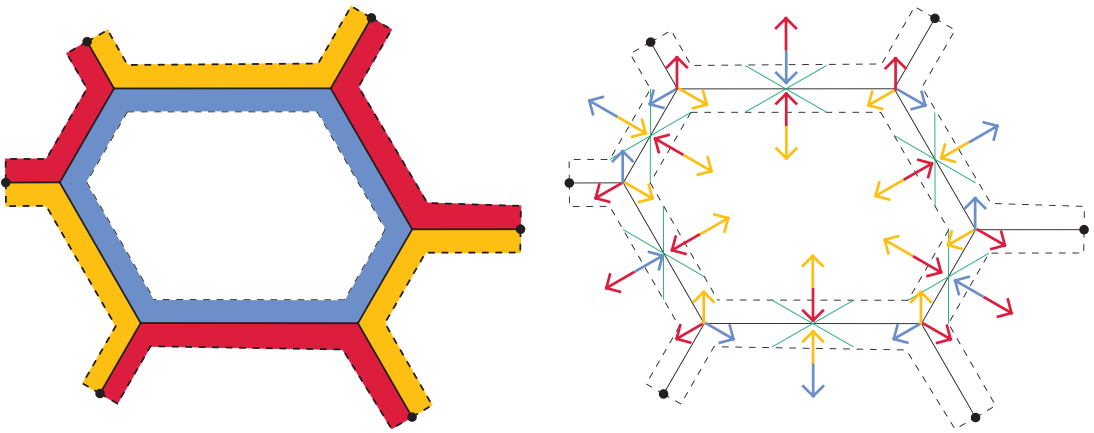


FIGURE 7 Left: Partition $\tilde{E}' = (\tilde{E}'_1, \tilde{E}'_2, \tilde{E}'_3)$, in yellow the set \tilde{E}'_1 , in red \tilde{E}'_2 and in blue \tilde{E}'_3 . Right: The three fields: Ψ_{12} in blue, Ψ_{23} in yellow and Ψ_{31} in red.

which are six at most, and then iterating the construction on adjacent components eventually leads to the desired homeomorphism.

As the triangular graph Q' can be colored with three colors, there exists a Caccioppoli partition $F = (F_1, F_2, F_3)$, consisting of just three sets, of D such that $\text{int}(D) \cap \Gamma_*(G) = \text{int}(D) \cap \cup_i \partial F_i$.

Now let d be the minimal length among the edges of Γ_* . For $\delta \in (0, \frac{d\sqrt{3}}{8})$ small enough and $\delta' \in (0, 1)$ that will be suitably chosen we consider the new network $\Gamma'_*(G)$ as before having the same edges of $\Gamma_*(G)$ except for those connected to endpoints that are extended by a length equal to δ' , and we take Ω' equal to the δ -tubular neighborhood of $\Gamma'_*(G)$, orthogonally truncated at endpoints. By assumptions, as D has boundary of class C^1 , for almost every δ', δ suitably small, $\partial\Omega'$ intersects ∂D transversely. Therefore, choosing any such δ', δ , the set $\Omega := \Omega' \cap \text{int}(D)$ is an open set with Lipschitz boundary. We define the desired Caccioppoli partition $\tilde{E} = (\tilde{E}_1, \tilde{E}_2, \tilde{E}_3)$ of Ω by setting $\tilde{E}_i := F_i \cap \Omega'$ for any $i = 1, 2, 3$.

It remains to exhibit a local paired calibration for \tilde{E} . Along any edge that is not connected to an endpoint, we locally perform the same construction of Figure 5 for the network Γ'_* . More precisely, at any midpoint M_{ij} between any two triple junctions O^i and O^j connected by an edge L , we consider two lines through M_{ij} forming an angle of 30° with the segment connecting O^i and O^j . By the choice of δ , such lines intersect $\partial\Omega'$ at points whose projections along the direction determined by L lie in the interior of L . Hence, such lines divide Ω' into open regions where we can set $\Psi_{12}, \Psi_{23}, \Psi_{31}$ to be the suitable rotation of $\pm \frac{2\pi}{3}$ of the vector $(0,1)$ like done in the simpler case in Figure 5.

The only issue that we need to check is the fact that a choice of these fields is coherent along an arbitrary cycle of the network, see Figure 7. As we proved that a cycle in Γ_* is a hexagon, this is easily established by directly prescribing a choice of $\Psi_{12}, \Psi_{23}, \Psi_{31}$, depending on the preassigned $\tilde{E}'_1, \tilde{E}'_2, \tilde{E}'_3$ as done before, along a cycle as in Figure 7, which form a partition of Ω' . Up to relabeling, we have that $\tilde{E}'_i \cap \text{int}(D) = \tilde{E}_i$ for $i = 1, 2, 3$. Hence, taking pointwise restrictions on D of the fields defining the calibration for \tilde{E}' on Ω' , we get a calibration for \tilde{E} on Ω . \square

Remark 3.10. In the construction in Theorem 3.9 of a neighborhood relative to D of a minimal network Γ_* together with a Caccioppoli partition minimizing for \mathcal{P} , it is *not* possible to choose the neighborhood arbitrarily large. More precisely, recall that in the proof of Theorem 3.9, Ω is

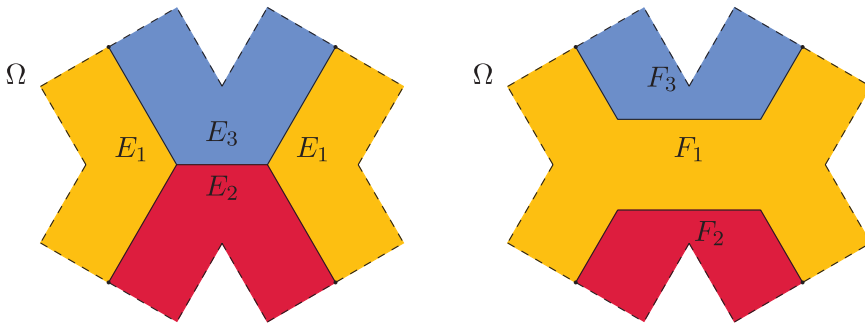


FIGURE 8 Partitions **E** and **F** in Remark 3.10.

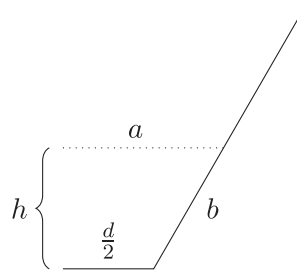


FIGURE 9 Continuous lines: Upper-right quarter of Γ_* in Remark 3.10. Dotted line: half of the upper horizontal edge of the interfaces of **F** in Remark 3.10.

essentially given by a tubular neighborhood of Γ_* of width $\delta < \sqrt{3}d/8$ where d is the length of the shortest edge, intersected with the given domain D . Here we construct an example showing that for a choice of a bigger δ , the partition associated to a minimal network Γ_* is no longer a minimizer for \mathcal{P} in Ω .

Consider the minimal network Γ_* depicted in Figure 8 on the left, together with a δ -tubular neighborhood (orthogonally truncated at endpoints) Ω and the associated partition $\mathbf{E} = (E_1, E_2, E_3)$, for δ to be chosen (comparing to the notation of Theorem 3.9, here we can take D equal to a suitable C^1 topological disk containing Ω). In this case, Γ_* is composed of five curves joining at two triple junction whose distance equals d , and we assume that the four edges connected to endpoints have length strictly bigger than d . We consider a comparison partition \mathbf{F} (not induced by a network!) in the class \mathcal{A} of partitions having the same trace of \mathbf{E} on the boundary, as depicted in Figure 8 on the right. The interfaces determining the partition \mathbf{F} are obtained by deleting the central curve of Γ_* , by shortening the four edges connected to endpoints of the same amount, and then by joining with two horizontal segments the four new endpoints of the shortened edges.

We claim that if $\delta > \frac{\sqrt{3}}{4}d$, then \mathbf{F} can be chosen so that $\mathcal{P}(\mathbf{F}) < \mathcal{P}(\mathbf{E})$, giving an upper bound on the width of the tubular neighborhood where the partition induced by Γ_* may be minimizing for \mathcal{P} .

Indeed, let a be half of the length of one of the horizontal edges of the interfaces of \mathbf{F} and let b be the length of the deleted portion of one of the four external edges of Γ_* (see Figure 9). By symmetry, if $\frac{d}{2} + 2b > 2a$ then $\mathcal{P}(\mathbf{F}) < \mathcal{P}(\mathbf{E})$. Letting h be the distance between the horizontal edge of Γ_* and the horizontal part of the interfaces of \mathbf{F} , then $b = \frac{2}{\sqrt{3}}h$ and $a = \frac{d}{2} + \frac{b}{2} = \frac{d}{2} + \frac{h}{\sqrt{3}}$.

Hence, the condition $\frac{d}{2} + 2b > 2a$ is equivalent to $h > \frac{\sqrt{3}}{4}d$. Hence, whenever $\delta > \frac{\sqrt{3}}{4}d$, then such a partition \mathbf{F} satisfying $\mathcal{P}(\mathbf{F}) < \mathcal{P}(\mathbf{E})$ can be constructed within Ω .

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ORCID

Alessandra Pluda  <https://orcid.org/0000-0003-4714-4119>

Marco Pozzetta  <https://orcid.org/0000-0002-2757-0826>

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