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Regularity of the Optimal Sets for a Class of Integral Shape Functionals

Giuseppe Buttazzo, Francesco Paolo Maiale, Dario Mazzoleni, Giorgio Tortone & Bozhidar Velichkov

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Abstract

We prove the first regularity theorem for the free boundary of solutions to shape optimization problems involving integral functionals, for which the energy of a domain Ω is obtained as the integral of a cost function $j(u, x)$ depending on the solution u of a certain PDE problem on Ω. The main feature of these functionals is that the minimality of a domain Ω cannot be translated into a variational problem for a single (real or vector valued) state function. In this paper we focus on the case of affine cost functions $j(u, x) = -g(x)u + Q(x)$, where u is the solution of the PDE $-\Delta u = f$ with Dirichlet boundary conditions. We obtain the Lipschitz continuity and the non-degeneracy of the optimal u from the inwards/outwards optimality of Ω and then we use the stability of Ω with respect to variations with smooth vector fields in order to study the blow-up limits of the state function u. By performing a triple consecutive blow-up, we prove the existence of blow-up sequences converging to homogeneous stable solution of the one-phase Bernoulli problem and according to the blow-up limits, we decompose $\partial\Omega$ into a singular and a regular part. In order to estimate the Hausdorff dimension of the singular set of $\partial\Omega$ we give a new formulation of the notion of stability for the one-phase problem, which is preserved under blow-up limits and allows to develop a dimension reduction principle. Finally, by combining a higher order Boundary Harnack principle and a viscosity approach, we prove C^∞ regularity of the regular part of the free boundary when the data are smooth.

1. Introduction

This paper is dedicated to the regularity of the optimal shapes, solutions to shape optimization problems of the form

$$\min\left\{J(A)\ :\ A \in \mathcal{A}\right\},$$

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where $\mathcal{A}$ is an admissible class of open, Lebesgue measurable or quasi-open subsets of $\mathbb{R}^d$, and where $J : \mathcal{A} \to \mathbb{R}$ is a given shape functional, with $J(A)$ usually depending on the solution of a PDE on the domain A. This kind of minimization problems arise in different models in Biology, Engineering and Physics (see for example [6,21] for an overview) and have been extensively studied from both numerical and theoretical points of view. In particular, there are two classes of shape optimization problems of the form above with long history, both leading to overdetermined elliptic PDE problems with Dirichlet boundary conditions.

The first class involves the so-called *spectral functionals*, that is, functionals depending on the eigenvalues of the Dirichlet Laplacian as

$$J(A) = \varphi\big(\lambda_1(A), \dots, \lambda_k(A)\big) + |A|,$$

where $\varphi : \mathbb{R}^k \to \mathbb{R}$ is a real-valued function and $|A|$ denotes the Lebesgue measure of A. The associated shape optimization problems

$$\min\Big\{\varphi\big(\lambda_1(A), \dots, \lambda_k(A)\big) + |A| \ : \ A \subset \mathbb{R}^d\Big\},$$

have a long history and are related to the classical question “Can one hear the shape of the drum?” and, more generally, to the interplay between the geometry of the domains and the spectrum of the Dirichlet Laplacian. The first results on the characterization of the optimal shapes, for the first and the second Dirichlet eigenvalues, go back to the works of Faber–Krahn (1922) and Krahn–Szegö (1923), and consist in finding explicit minimizers (balls and unions of disjoint balls), which is only possible in some special cases as $J = \lambda_1$ and $J = \lambda_2$. Today, thanks to theory developed by Buttazzo and Dal Maso [10], and to the more recent results [5,30], it is well-known that, for monotone functionals φ, minimizers exist in a class of measurable (quasi-open) sets. The regularity of the optimal shapes has also been extensively studied; we refer to [4] and [34] for the case of optimal sets of λ_1 in a box, to [32] for the optimal sets of λ_2 in a box, and to [8,25,26,31] (see also [15,32]) for functionals involving higher eigenvalues λ_k.

The second class of functionals involves *integral shape functionals*, namely, for every bounded open set $A \subset \mathbb{R}^d$ we define

$$J(A) := \int_A j(u_A, x)\, dx,$$

where the cost function

$$j : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$$

is fixed and the state function u_A is the (weak) solution of the PDE

$$-\Delta u = f \quad \text{in} \quad A, \qquad u \in H_0^1(A), \tag{1.1}$$

the force term $f : \mathbb{R}^d \to \mathbb{R}$ being also a prescribed function.

Optimization problems for integral shape functionals arise in Optimal Control and in models from Mechanics, in which the optimization criteria $j(u_A, x)$ takes into account external factors and forces that might appear not immediately, but only after the design is complete and the state function u_A is already fixed. This type of problems pops up also in population dynamics, when one aims to optimize the population size. As in the case of spectral functionals, also for integral functionals with monotone cost, the existence of optimal shapes in bounded domains follows from the general theory of Buttazzo and Dal Maso, and again the solutions belong to the large class of measurable (or quasi-open) sets. This existence problem was studied, in a more general framework, in [13]. A general existence result in the class of open sets, was proved recently in [11] and in [12]. Precisely, it was shown that if D is a bounded open set in $\mathbb{R}^d$ and if the function j satisfies some suitable growth assumptions, then the shape optimization problem

$$\min\Big\{ \int_A j(u_A, x)\,dx \ :\ A \text{ open},\ A \subset D\Big\},$$

has a solution $\Omega \subset D$, Ω-open.
On the other hand, even if the existence theory is quite well understood, there is no regularity theory for the minimizers of integral shape functionals, even in the simplest case

$$j(u, x) = -g(x)u + Q(x), \quad \text{with} \quad g \neq f,$$

the regularity of the optimal sets was out of reach.

In this paper we prove the first general regularity result for the optimal shapes of integral functionals. In order to make our main result (Theorem 1.2) easier to read, we introduce the following definition.

Definition 1.1. Let D be an open set in $\mathbb{R}^d$. For $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in [0, 1]$ and $N \in \mathbb{N}$, we call a set $\Omega \subset D$ (k, α, N)-*regular in D* if the free boundary $\partial\Omega \cap D$ is the disjoint union of a regular part $Reg(\partial\Omega)$ and a (possibly empty) singular part $\mathrm{Sing}(\partial\Omega)$ such that:

- $\mathrm{Reg}(\partial\Omega)$ is a relatively open subset of $\partial\Omega \cap D$ and locally a graph of a $C^{k,\alpha}$-regular function;
- $\mathrm{Sing}(\partial\Omega)$ is a closed subset of $\partial\Omega \cap D$ and has the following properties:
  - If $d < N$, then $\mathrm{Sing}(\partial\Omega)$ is empty.
  - If $d \geq N$, then the Hausdorff dimension of $\mathrm{Sing}(\partial\Omega)$ is at most $d - N$, namely

$$\mathcal{H}^{d-N+\varepsilon}\big(\mathrm{Sing}(\partial\Omega)\big) = 0 \quad \text{for every } \varepsilon > 0.$$

Moreover, if the regular part $\mathrm{Reg}(\partial\Omega)$ is C^∞, then we say that Ω is (∞, N)-*regular in D*.

The main result of the present paper is the following.

Theorem 1.2. *Let D be a bounded open set in $\mathbb{R}^d$, where $d \geq 2$. Let*

$$f : D \to \mathbb{R}, \quad g : D \to \mathbb{R}, \quad Q : D \to \mathbb{R},$$

be given non-negative functions. Suppose that the following conditions hold:

(a) *$f, g \in L^\infty(D) \cap C^2(D)$;*
(b) *there are constants $C_1, C_2 > 0$ such that*

$$0 \le C_1 g \le f \le C_2 g \quad in \quad D. \tag{1.2}$$

(c) *$Q \in C^2(D)$ and there are a positive constants c_Q, C_Q such that*

$$0 < c_Q \le Q(x) \leqq C_Q \quad for\ every \quad x \in D.$$

Then, there is $\alpha \in (0, 1)$ such that every solution $\Omega \subset D$ to the shape optimization problem

$$min\left\{ \int_A \Big(- g(x)u_A + Q(x)\Big)\, dx \ :\ A \subset D,\ A\ open\right\}, \tag{1.3}$$

is $(1, \alpha, 5)$-regular in D. Moreover, if $f, g, Q \in C^\infty(D)$, then Ω is $(\infty, 5)$-regular in D.

Proof. The definitions of the regular and the singular parts, $\mathrm{Reg}(\partial\Omega)$ and $\mathrm{Sing}(\partial\Omega)$, of the free boundary $\partial\Omega \cap D$ are given in Sect. 5. The $C^{1,\alpha}$ regularity of $\mathrm{Reg}(\partial\Omega)$ is proved in Theorem 6.4, while the C^∞ regularity follows from Proposition 6.5. The bounds on the dimension of $\mathrm{Sing}(\partial\Omega)$ are given in Theorem 8.1. □

Remark 1.3. (On the bound on the dimension of the singular set $\mathrm{Sing}(\partial\Omega)$) In Sect. 7 we develop a theory about the regularity of the stable global solutions of the one-phase Bernoulli problem (the definition of global stable solution is given in Definition 7.3); we show that there is a critical dimension d^* (see Definition 7.7), in which a singular global solution appears for the first time (see Theorem 7.8), and we prove that the d^* can be only 5, 6, or 7 (see Theorem 7.9). In Sect. 8 (Theorem 8.1) we use this results to show the following bounds on the singular part $\mathrm{Sing}(\partial\Omega)$ of an optimal set Ω, solution to (1.3):

- if $d < d^*$, then $\mathrm{Sing}(\partial\Omega)$ is empty;
- if $d \ge d^*$, then the Hausdorff dimension of $\mathrm{Sing}(\partial\Omega)$ is at most $d - d^*$.

In particular, since (by Theorem 7.9) $d^* \ge 5$, we get that:

- if $d < 5$, then $\mathrm{Sing}(\partial\Omega)$ is empty;
- if $d \ge 5$, then the Hausdorff dimension of $\mathrm{Sing}(\partial\Omega)$ is at most $d - 5$.

In other words, the precise statement of Theorem 1.2 is that under the conditions (a)–(b)–(c), the optimal sets are $(1, \alpha, d^*)$-regular, where d^* is the critical dimension from Definition 7.7.

Remark 1.4. (On the assumptions (a)–(b)–(c) in Theorem 1.2) The C^2 regularity assumption in (a) is technical and is related to the use we make of the second order variations of the functional J along vector fields (see Sect. 2). The assumption (b) is used in the proofs of the Lipschitz continuity and the non-degeneracy of the state function u_Ω; we notice that (b) is automatically satisfied when f and g are both bounded from above and from below by positive constants. The bounds from above and below on the weight Q in (c) are usual in the class of Bernoulli-type free boundary problems; these bounds are necessary for the Lipschitz continuity and the non-degeneracy of u_Ω (see Sect. 3), which are essential ingredients for the blow-up analysis in Sect. 4. The C^2 regularity of Q, on the other hand, is used again in the computation of the second variation in Sect. 2 and the passage to the blow-up limit in the proof of Theorem 8.1.

Remark 1.5. (On the existence of optimal sets) In [11, Theorem 1.1] it was proved that if D is a bounded open subset of $\mathbb{R}^d$ and if f, g, Q satisfy the following conditions:

- $f, g \in L^\infty(D)$;
- there are positive constants $C_1 \leq C_2$ such that $0 \leq C_1 g \leq f \leq C_2 g$ in D;
- $Q \in L^\infty(D)$, $Q \geq 0$ in D,

then, there is an open set $\Omega \subset D$ solution to the shape optimization problem (1.3).

The presence of the inclusion constraint $\Omega \subset D$ is essential for the existence theory for general shape optimization problems (see for instance [10] and [11]). In the case of integral functionals with affine cost, as the one in (1.3), the inclusion constraint can be removed. In the next theorem, which we prove in Sect. 9, we show that optimal sets exist in $\mathbb{R}^d$. Moreover, we prove that the optimal sets in $\mathbb{R}^d$ are bounded, which implies that they are solutions to (1.3) in some sufficiently large ball $D := B_R$, so the regularity of the free boundary in $\mathbb{R}^d$ is a consequence of Theorem 1.2.

Theorem 1.6. *In $\mathbb{R}^d$, $d \geq 2$, let $f, g, Q : \mathbb{R}^d \to \mathbb{R}$ be non-negative functions. Suppose that the following conditions hold:*

(a) *$f, g \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$ and that*

$$f(x) > 0 \quad \textit{and} \quad g(x) > 0 \quad \textit{for every} \quad x \in \mathbb{R}^d.$$

(b) *$Q \in C^2(\mathbb{R}^d)$ and there are positive constants c_Q, C_Q such that*

$$0 < c_Q \leq Q(x) \leqq C_Q \quad \textit{for every} \quad x \in \mathbb{R}^d.$$

Then, there is an open set $\Omega \subset \mathbb{R}^d$, which is a solution to the shape optimization problem

$$min\left\{ \int_A \Big(- g(x)u_A + Q(x) \Big)\,dx \ : \ A \subset \mathbb{R}^d, \ A \textit{ open}, \ |A| < +\infty \right\}, \quad (1.4)$$

and every solution Ω to (1.4) is bounded and $(1, \alpha, 5)$-regular in $\mathbb{R}^d$. Moreover, if $f, g, Q \in C^\infty(\mathbb{R}^d)$, then Ω is $(\infty, 5)$-regular in $\mathbb{R}^d$.

1.1. Integral Shape Functionals in the Case $f = cg$

In this section we briefly discuss the case in which f and g are proportional, which is the only instance of integral functional studied in the literature. Precisely, we claim that if f and g are such that

$$g = \frac{1}{2\lambda^2} f \quad \text{for some constant} \quad \lambda > 0,$$

the shape optimization problem (1.3) is equivalent to the Bernoulli free boundary problem

$$\min\left\{ \int_D \left(\frac{1}{2}|\nabla u|^2 - f(x)u + \lambda^2 Q(x)\mathbb{1}_{\{u\neq 0\}}\right) dx \ : \ u \in H^1_0(D)\right\}. \tag{1.5}$$

Fix a solution $u \in H^1_0(D)$ to (1.5), for which the set $\{u \neq 0\}$ is open, and fix an optimal set Ω for (1.3) with a state function u_Ω. Since u satisfies

$$-\Delta u = f \quad \text{in} \quad \{u \neq 0\} \qquad u \in H^1_0(\{u \neq 0\}),$$

by integrating by parts, we have that

$$\begin{aligned}\int_D \left(\frac{1}{2}|\nabla u|^2 - f(x)u + \lambda^2 Q(x)\mathbb{1}_{\{u\neq 0\}}\right) dx &= \int_D \left(-\frac{1}{2} f(x)u + \lambda^2 Q(x)\mathbb{1}_{\{u\neq 0\}}\right) dx \\ &= \lambda^2 \int_{\Omega_u} \big(-g(x)u + Q(x)\big)\, dx = \lambda^2 J(\{u \neq 0\}).\end{aligned}$$

Analogously, since $\{u_\Omega \neq 0\} \subset \Omega$ and Q is positive, we have

$$\int_D \left(\frac{1}{2}|\nabla u_\Omega|^2 - f(x)u_\Omega + \lambda^2 Q(x)\mathbb{1}_{\{u_\Omega\neq 0\}}\right) dx \leq \lambda^2 J(\Omega).$$

Thus, if u is a solution to (1.5), then $J(\{u \neq 0\}) \leq J(\Omega)$ and so, $\{u \neq 0\}$ is a solution to (1.3). Conversely, if Ω minimizes (1.3), then u_Ω is a minimizer of (1.5). Thus, for proportional f and g, the problem (1.3) is equivalent to (1.5); moreover, we notice that the argument above does not require the positivity of f and g, so the equivalence of the problems (1.5) and (1.3) holds also when f and g change sign.

The regularity of the solutions to the free boundary problem (1.5) is nowadays well-understood (at least when Q satisfies the condition (c) of Theorem 1.2). When $f \geq 0$, the regularity of $\partial\Omega$ follows from the regularity theory for the one-phase Bernoulli problem (see [2,14,18–20,22,39]). If the right-hand side f changes sign, then (1.5) becomes a two-phase Bernoulli problem, for which the regularity of the free boundary was obtained recently in [35] and [16].

Finally, we notice that when f and g are not proportional, the state function u_Ω of an optimal set Ω (that is, a solution to the shape optimization problem (1.3)) is not a minimizer of a free boundary functional as the one in (1.5). In particular, this implies that one can test the optimality of u_Ω only with functions $\widetilde{u}$ which are themselves state functions of some $\widetilde{\Omega}$. This means that a function $\widetilde{u}$, that differs from u only in a small ball B_r, cannot be used to test the optimality of u_Ω (truncations, harmonic replacements and radial extensions in small balls are not admitted), which makes most of the classical free boundary regularity results impossible to apply.

1.2. Adjoint State and Optimality Condition on the Free Boundary

Let us go back to the general case when f and g are not proportional. We will show that the optimality condition on the boundary $\partial\Omega \cap D$ of an optimal open set Ω for (1.3) leads to a free boundary problem involving the state function u_Ω. In order to see this, we introduce the adjoint state function v_Ω as follows: for every open set $A \subset D$ we will denote by v_A the weak solution to the problem

$$-\Delta v_A = g \quad \text{in} \quad A, \qquad v_A \in H_0^1(A). \tag{1.6}$$

By an integration by parts one can see that

$$\int_D g u_A \, dx = \int_D \nabla u_A \cdot \nabla v_A \, dx = \int_D f v_A \, dx,$$

which means that the two state variables u_A and v_A are interchangeable. Precisely, an open set $\Omega \subset D$ is a solution to (1.3) if and only if it minimizes

$$\min\left\{ \int_A \Big(- f(x) v_A + Q(x) \Big) \, dx \ : \ A \subset D, \ A \text{ open} \right\}.$$

Sometimes, it is more convenient to consider simultaneously the two state functions, by using the following equivalent formulation, which is symmetric in v_Ω and u_Ω

$$\min\left\{ \int_A \Big(\nabla u_A \cdot \nabla v_A - f(x) v_A - g(x) u_A + Q(x) \Big) \, dx \ : \ A \subset D, \ A \text{ open} \right\}.$$

Throughout the paper we will denote the functional from (1.3) by $\mathcal{F}$. Precisely, we set

$$\mathcal{F}(\Omega; D) := \int_D \Big(- g(x) u_\Omega + Q(x) \mathbb{1}_\Omega \Big) \, dx, \tag{1.7}$$

which after an integration by parts has the symmetric form

$$\mathcal{F}(\Omega; D) = \int_D \Big(\nabla u_\Omega \cdot \nabla v_\Omega - g(x) u_\Omega - f(x) v_\Omega + Q(x) \mathbb{1}_\Omega \Big) \, dx,$$

so if $\xi \in C_c^\infty(D; \mathbb{R}^d)$ is a smooth compactly supported vector field in D and $\Omega_t := (Id + t\xi)(\Omega)$, then the first variation of $\mathcal{F}$ along ξ is given by (see Lemma 2.6)

$$\begin{aligned} \delta\mathcal{F}(\Omega; D)[\xi] :&= \frac{\partial}{\partial t}\Big|_{t=0} \mathcal{F}(\Omega_t, D) \\ &= \int_\Omega \Big(\big(\nabla u_\Omega \cdot \nabla v_\Omega + Q(x) \big) \operatorname{div} \xi - \nabla u_\Omega \cdot \big((\nabla \xi) + (D\xi) \big) \nabla v_\Omega \Big) \, dx \\ &\quad - \int_\Omega \Big(u_\Omega \operatorname{div}(g\xi) + v_\Omega \operatorname{div}(f\xi) \Big) \, dx \, . \end{aligned}$$

Moreover, if Ω is a minimizer and $\partial\Omega$ is smooth, then an integration by parts gives that

$$\delta\mathcal{F}(\Omega; D)[\xi] = \int_{\partial\Omega} (\nu \cdot \xi) \big(Q - |\nabla u_\Omega| |\nabla v_\Omega| \big) \, d\mathcal{H}^{d-1} = 0 \quad \text{for every} \quad \xi \in C_c^\infty(D; \mathbb{R}^d),$$

so the state functions u_Ω and v_Ω are (at least formally) solutions to the free boundary system

$$\begin{cases} -\Delta u_\Omega = f \quad \text{in} \quad \Omega, \\ -\Delta v_\Omega = g \quad \text{in} \quad \Omega, \\ u_\Omega = v_\Omega = 0 \quad \text{and} \quad |\nabla u_\Omega||\nabla v_\Omega| = Q \quad \text{on} \quad D \cap \partial\Omega. \end{cases} \tag{1.8}$$

Remark 1.7. An epsilon-regularity theorem for viscosity solutions of the above system was proved recently in [28]. Precisely, it was shown that if:

- $u_\Omega, v_\Omega : D \to \mathbb{R}$ are continuous and non-negative functions such that

$$\Omega = \{u_\Omega > 0\} = \{v_\Omega > 0\};$$

- $(\Omega, u_\Omega, v_\Omega)$ is a viscosity solutions to (1.8);
- u_Ω and v_Ω are ε-*flat* (in a suitable sense) in a ball $B_r(x_0) \subset D$ centered on $\partial\Omega$;

then the free boundary $\partial\Omega$ is $C^{1,\alpha}$-regular in $B_{r/2}(x_0)$.
We notice that an epsilon-regularity theorem for a similar free boundary system, with $f \equiv g \equiv 0$, was studied in [1] in the context of a different shape optimization problem. Precisely, in [1], by using the minimality of Ω, it was shown that it is an NTA domain, so by the Boundary Harnack Principle for harmonic functions, the system reduces to a one-phase problem for the function u_Ω, for which the epsilon-regularity theorem is known by [2] and [18]. In Sect. A we will detail how our results can be applied to the problem of [1].

Remark 1.8. The free boundary system (1.8) emphasizes that, despite the use of the adjoint variable v_Ω, it is not possible to reformulate the shape optimization problem (1.3) in terms of an integral functional depending on u_Ω or the couple (u_Ω, v_Ω). Precisely, no functional of the form

$$\int F(\nabla u, \nabla v, u, v, x) + Q(x)\mathbb{1}_{\{u^2+v^2>0\}}\, dx$$

has Euler-Lagrange equations given by (1.8). The same holds for the equations satisfied by the blow-up limits of (u_Ω, v_Ω). Indeed, since the couple (u_Ω, v_Ω) satisfies (1.8), any 1-homogeneous blow-up limit (u_0, v_0) would be a solution of the same system with $f = g \equiv 0$ and $Q \equiv const > 0$, but this system is not the Euler-Lagrange system of an integral functional of the above type. This peculiarity represents one of the major challenges for studying the regularity of the free boundary and requires rewriting the optimality and stability conditions in terms of inner variations.

1.3. Regularity of the Free Boundary and Dimension of the Singular Set

The key steps in the proof of Theorem 1.2 are the following:

- Prove that if Ω is an optimal set for (1.3), then it is a viscosity solution to (1.8). This, in combination with the epsilon-regularity result cited in Remark 1.7, will imply that the *flat* part of the free boundary is smooth.
- Show that there exists a critical dimension $d^* \in \{5, 6, 7\}$ such that in dimension $d < d^*$ all the points on $\partial\Omega \cap D$ are regular.
- Prove that a Federer-type dimension reduction principle holds for solutions to (1.3).

There are two main difficulties in following the program outlined in the three points above.

The first difficulty is in the fact that the first order optimality condition

$$\delta\mathcal{F}(\Omega; D)[\xi] = 0 \quad \text{for every} \quad \xi \in C_c^\infty(D; \mathbb{R}^d),$$

is not leading to a monotonicity formula for u_Ω and v_Ω; in particular, we do not know if the blow-up limits of u_Ω and v_Ω are in general homogeneous.

The second obstruction comes from the impossibility to make external perturbations of u_Ω and v_Ω (that is, in light of Remark 1.8 no perturbations of the form $\widetilde{u} = u_\Omega + \phi$ and $\widetilde{v} = v_\Omega + \psi$ are allowed), so the only information conserved along blow-up sequences is the one contained in the internal variations of Ω along smooth vector fields.

We overcome these difficulties by using the first and the second variation of $\mathcal{F}$. Indeed, suppose that Ω is optimal in D and consider the flow Φ_t associated to a compactly supported vector field $\xi \in C_c^\infty(D; \mathbb{R}^d)$. Then, setting $\Omega_t := \Phi_t(\Omega)$, we get that the function

$$t \mapsto \mathcal{F}(\Omega_t; D),$$

has a minumum in $t = 0$, so we have

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathcal{F}(\Omega; D) = 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial t^2}\Big|_{t=0}\mathcal{F}(\Omega; D) \geq 0, \tag{1.9}$$

that is, Ω is a stable critical point of the shape functional

$$\Omega \mapsto \mathcal{F}(\Omega; D).$$

We prove that this notion is stable under blow-up limits at free boundary points $x_0 \in \partial\Omega$:

$$u_0(x) = \lim_{n\to\infty} \frac{u_\Omega(x_0 + r_n x)}{r_n} \qquad \text{and} \qquad v_0(x) = \lim_{n\to\infty} \frac{v_\Omega(x_0 + r_n x)}{r_n}.$$

Thus $\Omega_0 = \{u_0 > 0\} = \{v_0 > 0\}$ is a stable critical point for the same functional, but this time with $f \equiv g \equiv 0$ and $Q \equiv Q(x_0)$. Now, since $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ are harmonic on Ω_0, we can apply the Boundary Harnack Principle from [29] to obtain that the ratio

$$\frac{u_0}{v_0} : \Omega_0 \to \mathbb{R}$$

is Hölder continuous in Ω_0, up to $\partial\Omega_0$. After a second blow-up (this time in zero), we get that the positivity set Ω_{00} of the functions

$$u_{00}(x) = \lim_{m\to\infty} \frac{u_0(r_m x)}{r_m} \qquad \text{and} \qquad v_{00}(x) = \lim_{m\to\infty} \frac{v_0(r_m x)}{r_m},$$

is a stable critical point (in every ball $B_R \subset \mathbb{R}^d$) for the functional $\mathcal{F}(\cdot, B_R)$, still with $f \equiv g \equiv 0$ and $Q \equiv Q(x_0)$. Moreover, by the Boundary Harnack Principle, we also have that u_{00} and v_{00} are proportional. Thus, u_{00} is (up to a constant) a stable critical point (in the sense of (1.9)) for the Alt–Caffarelli's one-phase functional

$$\mathcal{G}(u; B_R) := \int_{B_R} |\nabla u|^2\, dx + |\{u > 0\} \cap B_R| \quad \text{in every ball} \quad B_R \subset \mathbb{R}^d,$$

so the third blow-up in zero

$$u_{000}(x) = \lim_{k\to\infty} \frac{u_{00}(r_k x)}{r_k}$$

is a 1-homogeneous stable critical point for the Alt–Caffarelli's functional $\mathcal{G}$.

The existence of a homogeneous blow-up u_{000} is a key element in the proof of Theorem 1.2. Indeed, it allows to prove (see Proposition 6.3) that the state functions u_Ω and v_Ω are viscosity solutions to the system (1.8) by showing that if we take a free boundary point admitting a one-sided tangent ball, then the homogeneous blow-up limits u_{000} and v_{000} constructed above are half-plane solutions. This, in combination with the epsilon-regularity theorem cited in Remark 1.7, implies that, in a neighborhood of any boundary point admitting a half-plane solution as blow-up limit, the free boundary is $C^{1,\alpha}$-regular; we will call these points *regular points* (see Sect. 5), while the remaining part of the free boundary (if any) will be called *singular*. Finally, we notice that the smoothness of the regular part requires only the criticality of Ω (the first part of (1.9)).

It is natural to expect that the stability of Ω (the second part of (1.9)) leads to an estimate on the dimension of the singular set; in fact, the bounds on the critical dimension (the dimension in which a singularity appears for the first time) for minimizers of the one-phase Alt–Caffarelli functional rely (see [14] and [22]) on the well-known *stability inequality* of Caffarelli-Jerison-Kenig, which was originally obtained in [14] through a particular second order variation of the functional $\mathcal{G}$. On the other hand, this stability inequality is not easy to handle when it comes to passing to blow-up limits and developing a dimension reduction principle. In Sect. 7, we give a different formulation of the stability, which uses the second variation along vector fields, as defined in (1.9). Again in Sect. 7, we show that our notion of stability allows to develop a dimension reduction principle and that there is a critical dimension d^*, which is the smallest dimension admitting one-homogeneous stable solutions with singularity (see Theorem 7.8). Then, in Proposition 7.12, we prove that on smooth cones (that is, cones with isolated singularity) our notion of stability is equivalent to the stability inequality of Caffarelli-Jerison-Kenig. Thus, we obtain the bound $5 \leq d^* \leq 7$ on the critical dimension d^* as a consequence of the results of Jerison-Savin [22] and De Silva-Jerison [19]. Finally, in Theorem

8.1, we prove the bounds on the singular set from Theorem 1.2 by (again) a triple blow-up argument, which allows to transfer the dimension bounds from Theorem 7.8 to the singular set of the solutions to (1.3).

Remark 1.9. The methodology we developed for the analysis of (1.3) can be applied, with some natural adjustments, to various free boundary and shape optimization problems. In fact, once the local behavior of the state variables close to the free boundary is known, the triple blow-up analysis combined with the notion of stability along domain variations can be applied in an almost straightforward manner. We mention some recent contributions where the authors leveraged our methodology, adapting it to different problems, see [7,23,33]. In particular, in [7] the ε-regularity theory was applied to a spectral shape optimization problem, while in [33] and [23] our approach to the stable critical points was applied to free boundary problems.

1.4. Plan of the Paper

In Sect. 2 we compute the first and second variations of the functional $\mathcal{F}$. In Sect. 3 we prove the non-degeneracy and Lipschitz continuity of the state variables. In Sect. 4 we perform a blow-up analysis by proving the existence of triple blow-up sequences converging to a homogeneous limit.

In Sect. 5, we use the blow-up limits from Sect. 4 to define the decomposition of the free boundary into a regular part and a singular part.

In Sect. 6 we prove that the state functions u_Ω, v_Ω of an optimal set Ω for (1.3) are viscosity solutions of the free boundary system (1.8). Using this information, in Theorem 6.4, we prove that the regular part of the free boundary is $C^{1,\alpha}$-smooth.

In Sect. 7 we define the notion of a global stable solution to the one-phase Bernoulli (Alt–Caffarelli) problem and we study the dimension of the singular sets for these global stable solutions. We notice that this section can also be read indipendently and that, together with Sect. 2, it contains the key results for the analysis of the singular set.

In Sect. 8 we use the first and the second variations from Sect. 2, the triple blow-up procedure from Sect. 4 and the theory from Sect. 7 in order to prove the dimension bounds on the singular set. This concludes the proof of Theorem 1.2, which follows from Theorem 8.1, Theorem 6.4 and Proposition 6.5.

In Sect. 9 we address the existence of optimal sets in $\mathbb{R}^d$ and we prove Theorem 1.6. Ultimately, in Appendix A we apply the analysis of Sects. 7 and 8 to optimal sets arising sets arising from the heat conduction problem studied in [1].

1.5. Notations

In the whole paper we use the notation

$$B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\},$$

for the ball of radius r centered at point x_0 and, when $x_0 = 0$, we write $B_r = B_r(0)$; we denote by ω_d the Lebesgue measure of a ball of radius one in $\mathbb{R}^d$. For any set

$A \subset \mathbb{R}^d$, we set

$$\mathbb{1}_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \in \mathbb{R}^d \setminus A. \end{cases}$$

Given a non-negative function u we often denote its positivity set as $\Omega_u := \{u > 0\}$.

We denote by $H^1(\mathbb{R}^d)$ the set of Sobolev functions in $\mathbb{R}^d$, that is, the closure of the smooth functions with bounded support $C_c^\infty(\mathbb{R}^d)$ with respect to the usual Sobolev norm

$$\|\varphi\|_{H^1}^2 = \int_{\mathbb{R}^d} \left(|\nabla\varphi|^2 + \varphi^2\right) dx.$$

Given an open set $\Omega \subset \mathbb{R}^d$, we define the Sobolev space $H_0^1(\Omega)$ as the closure, with respect to $\|\cdot\|_{H^1}$, of the space $C_c^\infty(\Omega)$ of smooth functions compactly supported in Ω. Thus, every $u \in H_0^1(\Omega)$ is identically zero outside Ω and we have the inclusion $H_0^1(\Omega) \subset H^1(\mathbb{R}^d)$.

We will sometimes use the following notation for minimum and maximum of two real numbers:

$$\min\{a, b\} = a \wedge b, \qquad \max\{a, b\} = a \vee b, \qquad a, b \in \mathbb{R}.$$

Given a function $u : \mathbb{R}^d \to \mathbb{R}$ we will denote by ∇u and Du the vectors column and row with components the partial derivatives of u, while D^2u will be the Hessian matrix of u. Given a vector field $F : \mathbb{R}^d \to \mathbb{R}^d$ with components F_k, $k = 1, \ldots, d$ we will denote by ∇F the $d \times d$ matrix with columns ∇F_k, $k = 1, \ldots, d$ and rows $(\partial_j F_1, \partial_j F_2, \ldots, \partial_j F_d)$. By convention $DF := (\nabla F)^T$, where for any matrix $M \in \mathbb{R}^{d\times d}$, we will denote by M^T its transpose. Given a vector field $V : \mathbb{R}^d \to \mathbb{R}^d$ and a matrix with variable coefficients $M = (m_{ij})_{ij} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ we will denote by $(V \cdot \nabla)(M)$ the $d \times d$ matrix with variable coefficients $V \cdot \nabla m_{ij}$.

2. First and Second Variations Under Inner Perturbations

In this section we compute the first and the second variations (with respect to perturbations with compactly supported vector fields) of the functional $\mathcal{F}$ from (1.7). Both variations will be fundamental tools in the study of the blow-up limits of the state variables u_Ω and v_Ω on a domain Ω, which is optimal for (1.3).

2.1. First and Second Variation of the State Function

In Lemma 2.5 we compute the expansion of the state variable u_Ω (solution to (1.1)) with respect to smooth perturbations of a set Ω. We first prove Lemma 2.2 and Lemma 2.3, where we compute the expansion of a one-parameter family of solutions to PDEs on the same domain Ω.

Remark 2.1. In what follows we will denote by $\mathbb{R}^{d\times d}$ the space of $d \times d$ square matrices with real coefficients. Given a real matrix $A = (a_{ij})_{ij} \in \mathbb{R}^{d\times d}$, we define its norm in the space $\mathbb{R}^{d\times d}$ as

$$\|A\|_{\mathbb{R}^{d\times d}} := \left(\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}^2\right)^{1/2},$$

and we notice that for every vector $V \in \mathbb{R}^d$, we have $|AV| \le \|A\|_{\mathbb{R}^{d\times d}}|V|$, where $|V|$ is the usual Euclidean norm of V. Next, let Ω be a measurable set in $\mathbb{R}^d$. Given a matrix $A : \Omega \to \mathbb{R}^{d\times d}$ with variable coefficients $a_{ij} : \Omega \to \mathbb{R}$, we say that

$$A \in L^\infty(\Omega; \mathbb{R}^{d\times d}),$$

if $a_{ij} \in L^\infty(\Omega)$ for every $1 \le i, j \le d$. We define the norm $\|\cdot\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}$ as

$$\|A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} := \big\| \|A\|_{\mathbb{R}^{d\times d}} \big\|_{L^\infty(\Omega)} = \left\| \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}^2 \right\|_{L^\infty(\Omega)}^{1/2}.$$

Lemma 2.2. (First order expansion of solutions to PDEs) *Let Ω be a bounded open set in $\mathbb{R}^d$. Let the functions*

$$f : \mathbb{R} \to L^2(\Omega), \qquad t \mapsto f_t,$$
$$A : \mathbb{R} \to L^\infty(\Omega; \mathbb{R}^{d\times d}), \qquad t \mapsto A_t,$$

be such that:

(a) *$A_t(x)$ is a symmetric matrix for every $(t, x) \in \mathbb{R} \times \Omega$ and there is a symmetric matrix $\delta A \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ such that*

$$A_t = Id + t(\delta A) + o(t) \quad \text{in } L^\infty(\Omega; \mathbb{R}^{d\times d}).$$

(b) *there is $\delta f \in L^2(\Omega)$ such that*

$$f_t = f_0 + t(\delta f) + o(t) \quad \text{in } L^2(\Omega).$$

Then, for every t small enough there is a unique solution u_t to the problem

$$-\operatorname{div}(A_t\nabla u_t) = f_t \quad \text{in } \Omega, \qquad u_t \in H_0^1(\Omega), \tag{2.1}$$

and

$$u_t = u_0 + t(\delta u) + o(t) \quad \text{in } H_0^1(\Omega),$$

where δu is the unique weak solution in $H_0^1(\Omega)$ to the PDE

$$-\Delta(\delta u) - \operatorname{div}((\delta A)\nabla u_0) = \delta f \quad \text{in } \Omega, \qquad \delta u \in H_0^1(\Omega). \tag{2.2}$$

Proof. Clearly $u_0 \in H^1_0(\Omega)$ is the solution to $-\Delta u_0 = f_0$ in Ω. We set $w_t := \frac{1}{t}(u_t - u_0)$. We will prove that w_t converges to δu strongly in $H^1_0(\Omega)$.

We notice that (2.1) can be written as

$$-\operatorname{div}((\operatorname{Id} + (A_t - \operatorname{Id}))\nabla(u_0 + t w_t)) = f_0 + (f_t - f_0) \quad \text{in} \quad \Omega.$$

So, using the equation for u_0 and dividing by t, we get

$$-\Delta w_t - \operatorname{div}\Big(\frac{1}{t}(A_t - \operatorname{Id})\nabla u_0\Big) - \operatorname{div}\Big((A_t - \operatorname{Id})\nabla w_t\Big) = \frac{1}{t}(f_t - f_0) \quad \text{in } \Omega. \tag{2.3}$$

If we fix $\varepsilon > 0$, we can choose t small enough such that

$$\|A_t - \operatorname{Id}\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})} \le \varepsilon \quad \Big\|\frac{1}{t}(A_t - \operatorname{Id}) - \delta A\Big\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}$$
$$\le \varepsilon \quad \Big\|\frac{1}{t}(f_t - f_0) - \delta f\Big\|_{L^2(\Omega)} \le \varepsilon.$$

Thus, by testing (2.3) with w_t, we obtain

$$\begin{aligned}\int_\Omega |\nabla w_t|^2\,dx = &-\int_\Omega \nabla w_t \cdot \frac{1}{t}(A_t - \operatorname{Id})\nabla u_0\,dx \\ &- \int_\Omega \nabla w_t \cdot (A_t - \operatorname{Id})\nabla w_t\,dx + \int_\Omega \frac{1}{t}(f_t - f_0) w_t\,dx \\ \le& \Big(\varepsilon + \|\delta A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\Big)\|\nabla w_t\|_{L^2}\|\nabla u_0\|_{L^2} \\ &+ \varepsilon\|\nabla w_t\|^2_{L^2} + \Big(\varepsilon + \|\delta f\|_{L^2(\Omega)}\Big)\|w_t\|_{L^2}.\end{aligned}$$

Now, by the Poincaré inequality,

$$\|\varphi\|^2_{L^2(\Omega)} \le C_d|\Omega|^{2/d}\|\nabla\varphi\|^2_{L^2(\Omega)} \quad \text{for every} \quad \varphi \in H^1_0(\Omega),$$

and the equation for u_0, we have that

$$\|\nabla u_0\|^2_{L^2(\Omega)} = \int_\Omega f_0 u_0\,dx \le \|f_0\|_{L^2(\Omega)}\|u_0\|_{L^2(\Omega)} \le \|f_0\|_{L^2(\Omega)} C_d|\Omega|^{1/d}\|\nabla u_0\|_{L^2(\Omega)},$$

which gives the bound

$$\|\nabla u_0\|_{L^2(\Omega)} \le C_d|\Omega|^{1/d}\|f_0\|_{L^2(\Omega)}.$$

Thus, we deduce that

$$\begin{aligned}\int_\Omega |\nabla w_t|^2\,dx \le\ & C_d|\Omega|^{1/d}\|f_0\|_{L^2(\Omega)}\Big(\varepsilon + \|\delta A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\Big)\|\nabla w_t\|_{L^2} \\ &+ \varepsilon\|\nabla w_t\|^2_{L^2} + \Big(\varepsilon + \|\delta f\|_{L^2(\Omega)}\Big)C_d|\Omega|^{1/d}\|\nabla w_t\|_{L^2},\end{aligned}$$

and so, for t (and $\varepsilon < 1$) small enough

$$\Big(\int_\Omega |\nabla w_t|^2\,dx\Big)^{1/2}$$

$$\leq \frac{C_d|\Omega|^{1/d}}{(1-\varepsilon)}\Big(1+\|f_0\|_{L^2(\Omega)}+\|f_0\|_{L^2(\Omega)}\|\delta A\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+\|\delta f\|_{L^2(\Omega)}\Big).$$

Thus, for every sequence $t_n \to 0$, there is a subsequence for which w_{t_n} converges as $n \to \infty$, strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, to some function w_∞. Passing to the limit the Eq. (2.3) we get that w_∞ is also a solution to (2.2). Thus $w_\infty = \delta u$. In particular, this implies that w_t converges as $t \to 0$, strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$, to δu. Finally, in order to prove that the convergence is strong, we test again (2.3) with w_t:

$$\begin{aligned}\limsup_{t\to\infty}\int_\Omega |\nabla w_t|^2\,dx &= \lim_{t\to 0}\int_\Omega \nabla w_t\cdot\frac{1}{t}(A_t-Id)\nabla u_0\,dx+\lim_{t\to 0}\int_\Omega \frac{1}{t}(f_t-f_0)w_t\,dx\\ &= \int_\Omega \nabla(\delta u)\cdot\delta A\nabla u_0\,dx+\int_\Omega \delta f\,\delta u\,dx=\int_\Omega |\nabla(\delta u)|^2\,dx\,.\end{aligned}$$

Combining this estimate with the lower semi-continuity of the H^1 norm, we get

$$\lim_{t\to\infty}\int_\Omega |\nabla w_t|^2\,dx=\int_\Omega |\nabla(\delta u)|^2\,dx$$

which implies that w_t converges to δu strongly in $H_0^1(\Omega)$. □

Lemma 2.3. (Second order expansion of solutions to PDEs) *Let Ω be a bounded open set in $\mathbb{R}^d$. Let the functions*

$$f:\mathbb{R}\to L^2(\Omega)\quad and\quad A:\mathbb{R}\to L^\infty(\Omega;\mathbb{R}^{d\times d})$$

be such that:

(a) *$A_t(x)$ is a symmetric matrix for every $(t,x)\in\mathbb{R}\times\Omega$ and there are symmetric matrices*
$\delta A\in L^\infty(\Omega;\mathbb{R}^{d\times d})$ and $\delta^2 A\in L^\infty(\Omega;\mathbb{R}^{d\times d})$ such that

$$A_t = Id+t\,\delta A+t^2\delta^2 A+o(t^2)\quad in\ L^\infty(\Omega;\mathbb{R}^{d\times d});$$

(b) *there are $\delta f\in L^2(\Omega)$ and $\delta^2 f\in L^2(\Omega)$ such that*

$$f_t = f_0+t\,\delta f+t^2\delta^2 f+o(t^2)\quad in\ L^2(\Omega).$$

Then, for every t small enough there is a unique solution u_t to the problem

$$-\operatorname{div}(A_t\nabla u_t)=f_t\quad in\ \Omega,\qquad u_t\in H_0^1(\Omega),$$

and

$$u_t=u_0+t\,\delta u+t^2\delta^2 u+o(t^2)\quad in\ H_0^1(\Omega),$$

where $\delta u\in H_0^1(\Omega)$ is the solution to (2.2) and where $\delta^2 u\in H_0^1(\Omega)$ solves the PDE

$$-\Delta(\delta^2 u)=\operatorname{div}((\delta A)\nabla(\delta u))+\operatorname{div}((\delta^2 A)\nabla u_0)+\delta^2 f\quad in\ \Omega,\quad \delta^2 u\in H_0^1(\Omega). \tag{2.4}$$

Proof. Let $w_t := \frac{1}{t}(u_t - u_0)$ be as in the proof of Lemma 2.2. We set

$$v_t := \frac{1}{t}(w_t - \delta u) \in H_0^1(\Omega).$$

We will prove that v_t converges strongly in $H_0^1(\Omega)$ to $\delta^2 u$. From the equation for w_t, we have

$$-\Delta(\delta u + t v_t) - \operatorname{div}\Big(\frac{1}{t}(A_t - \mathrm{Id})\nabla u_0\Big) - \operatorname{div}\Big((A_t - \mathrm{Id})\nabla(\delta u + t v_t)\Big) = \frac{1}{t}(f_t - f_0).$$

Thus, using the Eq. (2.2) for δu $(-\Delta(\delta u) - \operatorname{div}(\delta A \nabla u_0) = \delta f)$, we get

$$\begin{aligned} &- \Delta v_t - \operatorname{div}\Big(\frac{1}{t^2}(A_t - \mathrm{Id} - t\delta A)\nabla u_0\Big) \\ &\quad - \operatorname{div}\Big(\frac{1}{t}(A_t - \mathrm{Id})\nabla(\delta u + t v_t)\Big) = \frac{1}{t^2}(f_t - f_0 - t\delta f). \end{aligned}$$

Now, reasoning as in Lemma 2.2, we get that the family v_t is uniformly bounded in $H_0^1(\Omega)$ and converges as $t \to 0$ strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$ to the solution $\delta^2 u$ of (2.4). In order to obtain the strong H^1 convergence, we compute

$$\begin{aligned} \limsup_{t\to\infty} \int_\Omega |\nabla v_t|^2\,dx &= -\lim_{t\to 0}\int_\Omega \nabla v_t \cdot \frac{1}{t^2}(A_t - \mathrm{Id} - t\delta A)\nabla u_0\,dx \\ &\quad - \lim_{t\to 0}\int_\Omega \nabla v_t \cdot \frac{1}{t}(A_t - \mathrm{Id})\nabla(\delta u)\,dx \\ &\quad + \lim_{t\to 0}\int_\Omega \frac{1}{t^2}(f_t - f_0 - t\delta f)v_t\,dx \\ &= -\int_\Omega \nabla(\delta^2 u)\cdot(\delta^2 A)\nabla u_0\,dx \\ &\quad - \int_\Omega \nabla(\delta^2 u)\cdot(\delta A)\nabla(\delta u)\,dx + \int_\Omega (\delta^2 f)(\delta^2 u)\,dx \\ &= \int_\Omega |\nabla(\delta^2 u)|^2\,dx\,, \end{aligned}$$

which concludes the proof. □

Remark 2.4. We recall that if $M = (m_{ij})_{1\le i,j\le d} \in \mathbb{R}^{d\times d}$ is a matrix with constant coefficients, then the following Taylor expansions in $\mathbb{R}^{d\times d}$ (with respect to the norm $\|\cdot\|_{\mathbb{R}^{d\times d}}$)

$$\begin{aligned} &(\mathrm{Id} + tM)^{-1} = \mathrm{Id} - tM + t^2M^2 + o(t^2), \\ &\det(\mathrm{Id} + tM) = 1 + t\ \mathrm{tr}(M) + \frac{t^2}{2}\Big(\big(\mathrm{tr}(M)\big)^2 - \mathrm{tr}(M^2)\Big) + o(t^2), \end{aligned}$$

where Id is the identity matrix in $\mathbb{R}^{d\times d}$ and $\mathrm{tr}(M)$ is the trace $\mathrm{tr}(M) := \sum_{i=1}^d m_{ii}$. As a consequence, if $M \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ is a matrix with variable coefficients, then we have the expansions

$$(\mathrm{Id} + tM)^{-1} = \mathrm{Id} - tM + t^2M^2 + o(t^2) \quad \text{in } L^\infty(\Omega;\mathbb{R}^{d\times d}),$$

$$\det(\operatorname{Id} + tM) = 1 + t \operatorname{tr}(M) + \frac{t^2}{2}\Big(\big(\operatorname{tr}(M)\big)^2 - \operatorname{tr}(M^2)\Big) \\ +o(t^2) \quad \text{in } L^\infty(\Omega; \mathbb{R}^{d\times d}).$$

In particular, this implies that given two matrices $M, N \in L^\infty(\Omega; \mathbb{R}^{d\times d})$, we have:

$$\big(\operatorname{Id} + tM + t^2N + o(t^2)\big)^{-1} = \operatorname{Id} - tM + t^2(M^2 - N) + o(t^2)\,;$$

$$\det\big(\operatorname{Id} + tM + t^2N + o(t^2)\big) = \det(\operatorname{Id} + tM)\det(\operatorname{Id} + t^2N) + o(t^2) \\ = 1 + t \operatorname{tr}(M) + \frac{t^2}{2}\Big(\big(\operatorname{tr}(M)\big)^2 - \operatorname{tr}(M^2) + 2\operatorname{tr}(N)\Big) + o(t^2)\,,$$

and so,

$$\big(\operatorname{Id} + tM + t^2N + o(t^2)\big)^{-1}\big(\operatorname{Id} + tM + t^2N + o(t^2)\big)^{-T} \\ \det\big(\operatorname{Id} + tM + t^2N + o(t^2)\big) \\ = \operatorname{Id} + t\Big(-M - M^T + \operatorname{tr}(M)\operatorname{Id}\Big) \\ + t^2\Big(MM^T + M^2 + (M^T)^2 - (N + N^T) - (M + M^T)\operatorname{tr}(M)\Big) \\ + \frac{t^2}{2}\Big(\big(\operatorname{tr}(M)\big)^2 - \operatorname{tr}(M^2) + 2\operatorname{tr}(N)\Big)\operatorname{Id} + o(t^2).$$

Proposition 2.5. *Let D be a bounded open set in $\mathbb{R}^d$ and let $f \in C^2(D)$ with $f \geq 0$ in D. Given an open set $\Omega \subset D$ and a compactly supported vector field $\xi \in C_c^\infty(D; \mathbb{R}^d)$, we consider the associated flow $\Phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$, determined by the family of ODEs (for every $x \in D$)*

$$\begin{cases} \partial_t \Phi_t(x) = \xi\big(\Phi_t(x)\big) & \textit{for every } t \in \mathbb{R} \\ \Phi_0(x) = x. \end{cases} \tag{2.5}$$

We consider the family of open sets $\Omega_t := \Phi_t(\Omega)$ and the corresponding state variables u_{Ω_t} given by (1.1). We define δu_Ω and $\delta^2 u_\Omega$ to be the weak solutions in $H_0^1(\Omega)$ to the PDEs

$$-\Delta(\delta u_\Omega) = \operatorname{div}((\delta A)\nabla u_\Omega) + \delta f \quad \textit{in} \quad \Omega, \qquad \delta u_\Omega \in H_0^1(\Omega),$$

and

$$-\Delta(\delta^2 u_\Omega) = \operatorname{div}((\delta A)\nabla(\delta u_\Omega)) + \operatorname{div}((\delta^2 A)\nabla u_\Omega) + \delta^2 f \quad \textit{in} \quad \Omega, \qquad \delta^2 u_\Omega \in H_0^1(\Omega),$$

where the matrices $\delta A \in L^\infty(D; \mathbb{R}^{d\times d})$ and $\delta^2 A \in L^\infty(D; \mathbb{R}^{d\times d})$ are given by

$$\delta A := -D\xi - \nabla\xi + (\operatorname{div}\xi)\, Id\,, \\ \delta^2 A := (D\xi)(\nabla\xi) + \frac{1}{2}\big(\nabla\xi\big)^2 + \frac{1}{2}(D\xi)^2 - \frac{1}{2}(\xi\cdot\nabla)\big[\nabla\xi + D\xi\big] \\ - \big(\nabla\xi + D\xi\big)\operatorname{div}\xi + Id\frac{(\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\xi)}{2}\,, \tag{2.6}$$

while the variations $\delta f \in L^2(D)$ and $\delta^2 f \in L^2(D)$ of the right-hand side f are:

$$\delta f := \operatorname{div}(f\xi)\,,$$
$$\delta^2 f := \frac{1}{2}\xi\cdot(D^2 f)\xi + \frac{1}{2}\nabla f\cdot D\xi[\xi] + f\,\frac{(\operatorname{div}\xi)^2 + \xi\cdot\nabla[\operatorname{div}\,\xi]}{2} + (\nabla f\cdot\xi)\operatorname{div}\xi\,. \tag{2.7}$$

Then,

$$u_{\Omega_t}\circ\Phi_t = u_\Omega + t(\delta u_\Omega) + t^2(\delta^2 u_\Omega) + o(t^2) \qquad in\ H^1_0(D).$$

Proof. We set $u_t := u_{\Omega_t}\circ\Phi_t$. Then, $u_t \in H^1_0(\Omega)$ and $u_{\Omega_t} = u_t\circ\Phi_t^{-1}$. Moreover, by a change of variables, we have that u_t is satisfies the PDE

$$-\operatorname{div}\big(A_t\nabla u_t\big) = f_t \quad \text{in} \quad \Omega, \qquad u_t \in H^1_0(\Omega),$$

where the matrix A_t and the function f_t are defined as

$$f_t := f(\Phi_t)|\det(D\Phi_t)| \qquad \text{and} \qquad A_t := (D\Phi_t)^{-1}(D\Phi_t)^{-T}|\det(D\Phi_t)|,$$

where for a vector field $F:\mathbb{R}^d\to\mathbb{R}^d$ with components F_k, $k = 1,\dots,d$, we denote by

DF the matrix with rows $DF_k = (\nabla F_k)^T$. We next compute the second order Taylor expansion of $D\Phi_t$ in $t = 0$. By differentiating the equation for the flow Φ_t, we get

$$\begin{cases}\partial_t(D\Phi_t) = D\xi(\Phi_t)D\Phi_t \quad \text{for every} \quad t\in\mathbb{R},\\ D\Phi_0 = \text{Id}.\end{cases}$$

Then, taking another derivative in t, we get

$$\begin{aligned}\partial_{tt}(D\Phi_t) &= \partial_t\big[D\xi(\Phi_t)D\Phi_t\big]\\ &= \partial_t\big[D\xi(\Phi_t)\big]D\Phi_t + D\xi(\Phi_t)\partial_t\big[D\Phi_t\big]\\ &= \big(\partial_t\Phi_t\cdot\nabla\big)\big[D\xi\big](\Phi_t)D\Phi_t + D\xi(\Phi_t)D\xi(\Phi_t)D\Phi_t\\ &= \big(\xi(\Phi_t)\cdot\nabla\big)\big[D\xi\big](\Phi_t)D\Phi_t + D\xi(\Phi_t)D\xi(\Phi_t)D\Phi_t\,,\end{aligned}$$

where for a vector field F and a matrix $M = (m_{ij})_{ij}$, we use the notation $(F\cdot\nabla)[M]$ for the matrix with coefficients $F\cdot\nabla m_{ij}$. Finally, taking $t = 0$, we get

$$\frac{\partial^2}{\partial t^2}\Big|_{t=0}(D\Phi_t) = (\xi\cdot\nabla)[D\xi] + (D\xi)^2,$$

and the Taylor expansion

$$D\Phi_t = \text{Id} + tD\xi + \frac{t^2}{2}\Big((\xi\cdot\nabla)[D\xi] + (D\xi)^2\Big) + o(t^2).$$

By the expansions from Remark 2.4, we get

$$\begin{aligned}A_t &= \text{Id} + t(\delta A) + t^2(\delta^2 A) + o(t^2) \quad \text{in } L^\infty(D;\mathbb{R}^{d\times d}),\\ f_t &= f + t(\delta f) + t^2(\delta^2 f) + o(t^2) \quad \text{in } L^2(D),\end{aligned}$$

where δA, $\delta^2 A$, δf, $\delta^2 f$ are given by (2.6) and (2.7). Thus, the claim follows from Lemmas 2.2 and 2.3. □

2.2. First and Second Variation of $\mathcal{F}$

In the next lemma, we compute the first derivative of the functional $\mathcal{F}$ along inner variations with compact support in D.

Lemma 2.6. (First variation of $\mathcal{F}$ along inner perturbations) *Let D be a bounded open set in $\mathbb{R}^d$ and let $f, g, Q \in C^1(D)$. Let $\Omega \subset D$ be open and $\xi \in C_c^\infty(D;\mathbb{R}^d)$ be a vector field with compact support in D. Let Φ_t be the flow of the vector field ξ defined by (2.5) and set $\Omega_t := \Phi_t(\Omega)$. Then*

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathcal{F}(\Omega_t, D) = \int_\Omega (\nabla u_\Omega\cdot\nabla v_\Omega + Q)\operatorname{div}\xi + \xi\cdot\nabla Q - \nabla u_\Omega\cdot((\nabla\xi)+(D\xi))\nabla v_\Omega\,dx - \int_\Omega u_\Omega\operatorname{div}(g\xi) + v_\Omega\operatorname{div}(f\xi)\,dx. \tag{2.8}$$

Moreover, if $\partial\Omega$ is C^2-regular in a neighborhood of the support of ξ, then

$$\frac{\partial}{\partial t}\Big|_{t=0}\mathcal{F}(\Omega_t, D) = \int_{\partial\Omega}(\nu\cdot\xi)\big(Q - |\nabla u_\Omega||\nabla v_\Omega|\big)\,d\mathcal{H}^{d-1}, \tag{2.9}$$

where ν is the outer unit normal to $\partial\Omega$.

Proof. By applying Lemma 2.5 to $u_t = u_{\Omega_t}\circ\Phi_t$ and $v_t = v_{\Omega_t}\circ\Phi_t$, we get that

$$u_t = u_\Omega + t(\delta u_\Omega) + o(t) \quad\text{and}\quad v_t = v_\Omega + t(\delta v_\Omega) + o(t) \quad\text{in}\quad H_0^1(D),$$

where δu_Ω and δv_Ω are the solutions to

$$\begin{aligned} -\Delta(\delta u_\Omega) &= \operatorname{div}((\delta A)\nabla u_\Omega) + \delta f \quad\text{in }\Omega, \qquad \delta u_\Omega\in H_0^1(\Omega),\\ -\Delta(\delta v_\Omega) &= \operatorname{div}((\delta A)\nabla v_\Omega) + \delta g \quad\text{in }\Omega, \qquad \delta v_\Omega\in H_0^1(\Omega),\end{aligned} \tag{2.10}$$

with $\delta f := \operatorname{div}(f\xi)$, $\delta g := \operatorname{div}(g\xi)$, and δA as in Lemma 2.5. Therefore, setting

$$f_t := f(\Phi_t)|\det(D\Phi_t)| \quad g_t := g(\Phi_t)|\det(D\Phi_t)|$$
$$Q_t := Q(\Phi_t)|\det(D\Phi_t)| \quad\text{and}\quad A_t := (D\Phi_t)^{-1}(D\Phi_t)^{-T}|\det(D\Phi_t)|,$$

we get

$$\begin{aligned}\mathcal{F}(\Omega_t, D) &= \int_{\Omega_t}\Big(\nabla u_{\Omega_t}\cdot\nabla v_{\Omega_t} - g u_{\Omega_t} - f v_{\Omega_t} + Q\Big)\,dy\\ &= \int_\Omega\Big(\nabla u_t\cdot A_t\nabla v_t - g_t u_t - f_t v_t + Q_t\Big)\,dx\\ &= \mathcal{F}(\Omega, D) + t\int_\Omega\Big(\nabla(\delta u_\Omega)\cdot\nabla v_\Omega + \nabla u_\Omega\cdot\nabla(\delta v_\Omega) - g(\delta u_\Omega) - f(\delta v_\Omega)\Big)\,dx\\ &\quad + t\int_\Omega\Big(\nabla u_\Omega\cdot(\delta A)\nabla v_\Omega - u_\Omega(\delta g) - v_\Omega(\delta f) + (Q\operatorname{div}\xi + \nabla Q\cdot\xi)\Big)\,dx + o(t)\\ &= \mathcal{F}(\Omega, D) + t\int_\Omega\Big(\nabla u_\Omega\cdot(\delta A)\nabla v_\Omega - u_\Omega(\delta g) - v_\Omega(\delta f) + \operatorname{div}(Q\xi)\Big)\,dx + o(t)\end{aligned}$$

where in the first equality we applied the change of variables $y = \Phi_t(x)$ and in the last one we use the equations $-\Delta u_\Omega = f$ and $-\Delta v_\Omega = g$ in Ω. Substituting with the expression for δA from (2.6), we obtain (2.8).

Suppose now that $\partial\Omega$ is C^2-smooth. Since in Ω we have the identity

$$\begin{aligned}(\nabla u_\Omega \cdot \nabla v_\Omega)\,\mathrm{div}\,\xi - \nabla u_\Omega \cdot ((\nabla\xi) + (D\xi))\nabla v_\Omega \\ = \mathrm{div}\Big(\xi(\nabla u_\Omega \cdot \nabla v_\Omega) - (\nabla u_\Omega \cdot \xi)\nabla v_\Omega - (\nabla v_\Omega \cdot \xi)\nabla u_\Omega\Big) \\ + (\nabla v_\Omega \cdot \xi)\Delta u_\Omega + (\nabla u \cdot \xi)\Delta v_\Omega,\end{aligned}$$

by integrating by parts we get

$$\begin{aligned}&\delta\mathcal{F}(\Omega, D)[\xi] = \\ &= \int_\Omega \mathrm{div}\Big(\xi\big((\nabla u_\Omega \cdot \nabla v_\Omega) + Q\big) - (\nabla u_\Omega \cdot \xi)\nabla v_\Omega - (\nabla v_\Omega \cdot \xi)\nabla u_\Omega\Big)\,dx \\ &\quad - \int_\Omega \Big((\nabla u_\Omega \cdot \xi)g + u_\Omega\,\mathrm{div}\,(g\xi) + (\nabla v_\Omega \cdot \xi)f + v_\Omega\,\mathrm{div}\,(f\xi)\Big)\,dx \\ &= \int_{\partial\Omega} \Big((\nu\cdot\xi)((\nabla u \cdot \nabla v) + Q) - (\nabla u \cdot \xi)(\nu \cdot \nabla v) - (\nabla v \cdot \xi)(\nu \cdot \nabla u)\Big)\,d\mathcal{H}^{d-1}.\end{aligned} \tag{2.11}$$

Since u_Ω and v_Ω are positive in Ω and vanish on $\partial\Omega$, we have that

$$\nabla u_\Omega = -\nu|\nabla u_\Omega| \quad \text{and} \quad \nabla v_\Omega = -\nu|\nabla v_\Omega| \quad \text{on} \quad \partial\Omega,$$

and so

$$\begin{aligned}\delta\mathcal{F}(\Omega, D)[\xi] &= \int_{\partial\Omega} (\nu\cdot\xi)(|\nabla u_\Omega||\nabla v_\Omega| + Q) - |\nabla u_\Omega|(\nu\cdot\xi)|\nabla v_\Omega| \\ &\quad - |\nabla v_\Omega|(\nu\cdot\xi)|\nabla u_\Omega|\,d\mathcal{H}^{d-1} \\ &= \int_{\partial\Omega} (\nu\cdot\xi)(Q - |\nabla u_\Omega||\nabla v_\Omega|)\,d\mathcal{H}^{d-1},\end{aligned}$$

which concludes the proof. □

Remark 2.7. (First variation and stationary domains) Given a bounded open set D, functions f, g and Q on D, and the functional $\mathcal{F}$ defined in (1.7), we will use the notation $\delta\mathcal{F}(\Omega, D)[\xi]$ for the first variation of $\mathcal{F}$ at Ω along a smooth compactly suppported vector field $\xi \in C_c^\infty(D; \mathbb{R}^d)$. Precisely, we set

$$\begin{aligned}\delta\mathcal{F}(\Omega, D)[\xi] := &\int_\Omega (\nabla u_\Omega \cdot \nabla v_\Omega + Q)\,\mathrm{div}\,\xi + \xi\cdot\nabla Q - \nabla u_\Omega \cdot ((\nabla\xi) + (D\xi))\nabla v_\Omega\,dx \\ &- \int_\Omega u_\Omega\,\mathrm{div}(g\xi) + v_\Omega\,\mathrm{div}(f\xi)\,dx.\end{aligned}$$

We will say that an open set $\Omega \subset D$ is stationary (or a critical point) for $\mathcal{F}$ in D if

$$\delta\mathcal{F}(\Omega, D)[\xi] = 0 \quad \text{for every} \quad \xi \in C_c^\infty(D; \mathbb{R}^d).$$

By (2.9), if Ω is stationary in D and the boundary $\partial\Omega \cap D$ is smooth, then

$$|\nabla u_\Omega||\nabla v_\Omega| = Q \quad \text{on} \quad \partial\Omega \cap D.$$

Finally, we notice that, by Lemma 2.5, any minimizer of (1.3) is stationary for $\mathcal{F}$.

Proposition 2.8. (Second variation of $\mathcal{F}$ along inner perturbations) *Let $D \subset \mathbb{R}^d$ be a bounded open set in $\mathbb{R}^d$ and let $f, g, Q \in C^2(D)$. Let $\Omega \subset D$ be an open set and $\xi \in C_c^\infty(D; \mathbb{R}^d)$ be a smooth vector field with compact support. Let Φ_t be the flow of the vector field ξ defined by (2.5) and set $\Omega_t := \Phi_t(\Omega)$. Then*

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\mathcal{F}(\Omega_t, D) = \int_\Omega \Big(\nabla u_\Omega \cdot (\delta^2 A)\nabla v_\Omega - \nabla(\delta u_\Omega)\cdot\nabla(\delta v_\Omega) - (\delta^2 f)v_\Omega - (\delta^2 g)u_\Omega + \delta^2 Q\Big)dx\,, \tag{2.12}$$

where $\delta^2 A$, $\delta^2 f$, $\delta^2 g$, δu_Ω and δv_Ω are the ones defined in Lemma 2.5 and where

$$\delta^2 Q := (\xi\cdot\nabla Q)\operatorname{div}\xi + \frac{1}{2}\xi\cdot D^2 Q\xi + \frac{1}{2}Q(\operatorname{div}\xi)^2 + \frac{1}{2}Q\xi\cdot\nabla(\operatorname{div}\xi).$$

Proof. By applying Lemma 2.5 to $u_t := u_{\Omega_t}\circ\Phi_t$ and $v_t := v_{\Omega_t}\circ\Phi_t$, we get that

$$u_t = u_\Omega + t(\delta u_\Omega) + t^2(\delta^2 u_\Omega) + o(t^2),$$
$$v_t = v_\Omega + t(\delta v_\Omega) + t^2(\delta^2 v_\Omega) + o(t^2) \quad \text{in} \quad H_0^1(\Omega).$$

with δu_Ω, δv_Ω satisfying (2.10) and $\delta^2 u_\Omega$, $\delta^2 v_\Omega \in H_0^1(\Omega)$ such that

$$-\Delta(\delta^2 u_\Omega) = \operatorname{div}((\delta A)\nabla(\delta u_\Omega)) + \operatorname{div}((\delta^2 A)\nabla u_\Omega) + \delta^2 f \quad \text{in } \Omega\,, \qquad \delta^2 u_\Omega \in H_0^1(\Omega)\,,$$
$$-\Delta(\delta^2 v_\Omega) = \operatorname{div}((\delta A)\nabla(\delta v_\Omega)) + \operatorname{div}((\delta^2 A)\nabla v_\Omega) + \delta^2 g \quad \text{in } \Omega\,, \qquad \delta^2 v_\Omega \in H_0^1(\Omega)\,.$$

Thus, by computing the second order (in t) Taylor expansion of

$$\mathcal{F}(\Omega_t, D) = \int_{\Omega_t}\Big(\nabla u_t\cdot A_t\nabla v_t - g_t u_t - f_t v_t + Q_t\Big)\,dx\,,$$

we get that

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\mathcal{F}(\Omega_t, D) =\\
&= \int_\Omega \nabla(\delta^2 u_\Omega)\cdot\nabla v_\Omega + \nabla u_\Omega\cdot(\delta^2 A)\nabla v_\Omega + \nabla u_\Omega\cdot\nabla(\delta^2 v_\Omega)\,dx\\
&\quad + \int_\Omega \nabla(\delta u_\Omega)\cdot\nabla(\delta v_\Omega) + \nabla(\delta u_\Omega)\cdot(\delta A)\nabla v_\Omega + \nabla u_\Omega\cdot(\delta A)\nabla(\delta v_\Omega)\,dx\\
&\quad - \int_\Omega f(\delta^2 v) + (\delta f)(\delta v_\Omega) + (\delta^2 f)v_\Omega + g(\delta^2 u_\Omega) + (\delta g)(\delta u_\Omega) + (\delta^2 g)u_\Omega\,dx\\
&\quad + \int_\Omega \delta^2 Q\,dx.
\end{aligned}$$

Thus, we only have to show that we can write the above expression as in (2.12). By using δv_Ω as a test function in the equation for δu_Ω and vice versa, we obtain

$$\int_\Omega \nabla(\delta v_\Omega)\cdot(\delta A)\nabla u_\Omega\,dx - \int_\Omega(\delta v_\Omega)(\delta f)\,dx = -\int_\Omega\nabla(\delta u_\Omega)\cdot\nabla(\delta v_\Omega)\,dx$$
$$\int_\Omega \nabla(\delta u_\Omega)\cdot(\delta A)\nabla v_\Omega\,dx - \int_\Omega(\delta u_\Omega)(\delta g)\,dx = -\int_\Omega\nabla(\delta u_\Omega)\cdot\nabla(\delta v_\Omega)\,dx.$$

Then, by testing the equations for u_Ω and v_Ω respectively with $\delta^2 v_\Omega$ and $\delta^2 u_\Omega$, we get

$$\int_\Omega \nabla(\delta^2 u_\Omega)\cdot\nabla v_\Omega\,dx = \int_\Omega f\,\delta^2 v\,dx \quad\text{and}\quad \int_\Omega \nabla(\delta^2 v_\Omega)\cdot\nabla u_\Omega\,dx = \int_\Omega g\,\delta^2 u_\Omega\,dx.$$

Using this identities in the expression of the second derivative, we get precisely (2.12). □

Remark 2.9. (Second variation and stable critical domains) Given a bounded open set D, functions f, g and Q on D, and the functional $\mathcal{F}$ from (1.7), we will indicate by $\delta^2\mathcal{F}(\Omega, D)[\xi]$ the second variation of $\mathcal{F}$ at Ω along a smooth compactly supported vector field $\xi \in C_c^\infty(D;\mathbb{R}^d)$. Precisely, we set

$$\begin{aligned}\delta^2\mathcal{F}(\Omega, D)[\xi] := \int_\Omega &\nabla u_\Omega\cdot(\delta^2 A)\nabla v_\Omega - \nabla(\delta u_\Omega)\cdot\nabla(\delta v_\Omega)\\ &- (\delta^2 f)v_\Omega - (\delta^2 g)u_\Omega + \delta^2 Q\,dx\,.\end{aligned}$$

In particular, we notice that for every open set Ω, we have

$$\mathcal{F}(\Omega_t, D) = \mathcal{F}(\Omega, D) + t\delta\mathcal{F}(\Omega, D)[\xi] + t^2\delta^2\mathcal{F}(\Omega, D)[\xi] + o(t^2),$$

where $\Omega_t = \Phi_t(\Omega)$ (with Φ_t the flow associated to the vector field ξ) and $\delta\mathcal{F}(\Omega, D)[\xi]$ is the first variation (2.11).

We will say that an open set $\Omega \subset D$ is a *stable critical point* for $\mathcal{F}$ in D if

$$\delta\mathcal{F}(\Omega, D)[\xi] = 0 \quad\text{and}\quad \delta^2\mathcal{F}(\Omega, D)[\xi] \geq 0 \quad\text{for every}\quad \xi \in C_c^\infty(D;\mathbb{R}^d).$$

By Lemmas 2.5 and 2.8, any minimizer of (1.3) is a stable critical point for $\mathcal{F}$.

3. Lipschitz Regularity and Non-degeneracy of the State Functions

In this section we study the regularity of the state functions u_Ω and v_Ω on an optimal domain Ω, as well as their behavior close to the free boundary $\partial\Omega$. Consequently, we prove that the set Ω satisfies some density estimates.

3.1. Assumptions on D, f, g, and Q

Throughout this subsection we will assume that:

- D is a open subset of $\mathbb{R}^d$, with $d \geq 2$;
- $f, g \in L^\infty(D)$ are two functions such that

  $$\|f\|_{L^\infty} + \|g\|_{L^\infty} \leq M \quad\text{and}\quad 0 \leq C_1 g \leq f \leq C_2 g \quad\text{on } D,$$

  for some positive constants $M, C_1, C_2 > 0$.
- $Q \in L^\infty(D)$ is such that

  $$0 < c_Q \leq Q \leq C_Q \quad\text{on}\quad D.$$

3.2. Inwards and Outwards Minimality Conditions

We will use the following notation. Given a set $A \subset \mathbb{R}^d$, and functions $f \in L^2(A)$ and $\varphi \in H^1(A)$, we set

$$E_f(\varphi, A) := \frac{1}{2}\int_A |\nabla\varphi|^2\,dx - \int_A f(x)\varphi\,dx.$$

Proposition 3.1. *Let $D \subset \mathbb{R}^d$ and $f, g, Q \in L^\infty(D)$ be as in Sect. 3.1 and let $\Omega \subset D$ be an open set that minimizes (1.3) in D. Then, the solution u_Ω to (1.1) has the following properties.*

(i) Outwards minimality. *For every open set $\widetilde{\Omega} \subset D$ such that $\Omega \subset \widetilde{\Omega}$ we have*

$$E_f(u, D) + \frac{C_2 C_Q}{2}|\Omega| \le E_f(\phi, D) + \frac{C_2 C_Q}{2}|\widetilde{\Omega}| \quad \text{for every } \phi \in H^1_0(\widetilde{\Omega}).$$

In particular, for every $B_r(x_0) \subset D$, we have

$$E_f(u, B_r(x_0)) \le E_f(\phi, B_r(x_0)) + \frac{C_2 C_Q}{2}\omega_d r^d, \tag{3.1}$$

for every $\phi \in H^1(B_r(x_0))$ such that $\phi - u_\Omega \in H^1_0(B_r(x_0))$.

(ii) Inwards minimality. *For every open set $\omega \subset \Omega$ we have*

$$E_f(u, D) + \frac{C_1 c_Q}{2}|\Omega| \le E_f(\phi, D) + \frac{C_1 c_Q}{2}|\omega| \quad \text{for every } \phi \in H^1_0(\omega). \tag{3.2}$$

Proof. Suppose that the open set $\widetilde{\Omega} \subset D$ contains Ω. Then, the optimality of Ω implies that

$$\int_\Omega \Big(-g(x)u_\Omega + Q(x)\Big)\,dx \le \int_{\widetilde{\Omega}} \Big(-g(x)u_{\widetilde{\Omega}} + Q(x)\Big)\,dx,$$

which can be written as

$$\int_D g(x)\big(u_{\widetilde{\Omega}} - u_\Omega\big)\,dx \le \int_{\widetilde{\Omega}\setminus\Omega} Q(x)\,dx.$$

Now, the positivity of f implies that $u_{\widetilde{\Omega}} \ge u_\Omega$ on D, so we get

$$\frac{1}{C_2}\int_D f(x)\big(u_{\widetilde{\Omega}} - u_\Omega\big)\,dx \le \int_D g(x)\big(u_{\widetilde{\Omega}} - u_\Omega\big)\,dx \le \int_{\widetilde{\Omega}\setminus\Omega} Q(x)\,dx \le C_Q|\widetilde{\Omega}\setminus\Omega|,$$

which after rearranging the terms gives

$$-\frac{1}{2}\int_\Omega f(x)u_\Omega + \frac{C_2 C_Q}{2}|\Omega| \le -\frac{1}{2}\int_{\widetilde{\Omega}} f(x)u_{\widetilde{\Omega}} + \frac{C_2 C_Q}{2}|\widetilde{\Omega}|,$$

which after an integration by parts on $\widetilde{\Omega}$ and Ω, reads as

$$E_f(u_\Omega, D) + \frac{C_2 C_Q}{2}|\Omega| \le E_f(u_{\widetilde{\Omega}}, D) + \frac{C_2 C_Q}{2}|\widetilde{\Omega}|.$$

Finally, since $u_{\widetilde{\Omega}}$ minimizes the energy $E_f(\cdot, D)$ among all functions in $H^1_0(\widetilde{\Omega})$, we obtain (i).

The proof of (ii) is similar. Let $\omega \subset \Omega$ and let u_ω be the associated state function. Then, using the optimality of Ω and the bounds on f, g and Q, we get

$$\begin{aligned} c_Q\big(|\Omega| - |\omega|\big) &\leq \int_{\Omega\setminus\omega} Q(x)\,dx \leq \int_D g\big(u_\Omega - u_\omega\big)\,dx \\ &\leq \frac{1}{C_1}\int_D f\big(u_\Omega - u_\omega\big)\,dx = \frac{2}{C_1}\Big(E_f(u_\omega, D) - E_f(u_\Omega, D)\Big), \end{aligned}$$

which implies (ii) since u_ω minimizes $E_f(\cdot, D)$ in $H^1_0(\omega)$. □

Remark 3.2. The state variable v_Ω satisfies analogous inwards/outwards minimality conditions for the functional $E_g(\cdot, D)$, where the constants C_1 and C_2 are replaced by $1/C_2$ and $1/C_1$.

3.3. Lipschitz Continuity and Non-degeneracy

As a consequence of Proposition 3.1 we obtain the Lipschitz continuity and the non-degeneracy of u_Ω (and of v_Ω).

Corollary 3.3. *Let $D \subset \mathbb{R}^d$ and $f, g, Q \in L^\infty(D)$ be as in Sect. 3.1. If $\Omega \subset D$ is optimal for (1.3), then the state functions u_Ω and v_Ω are locally Lipschitz in D with Lipschitz constants depending on d, C_1, C_2, M and C_Q.*

Proof. By Proposition 3.1 (i), u_Ω satisfies (3.1) for any $B_r(x_0) \subset D$, so u_Ω is an almost-minimizer in the sense of [8, Definition 3.1]. Thus, by [8, Theorem 3.3], u_Ω is locally Lipschitz continuous in D with Lipschitz constant depending on d and the bounds from above on $\|f\|_{L^\infty}$ and $C_2 C_Q$. □

Lemma 3.4. *Let $D \subset \mathbb{R}^d$ and $f, g, Q \in L^\infty(D)$ be as in Sect. 3.1 and let $\Omega \subset D$ be an optimal set for (1.3) and u_Ω be the associated state function. Then, there are constants $C_0, r_0 > 0$, depending on d, C_1, c_Q and M, such that the following implication holds*

$$\Big(\ \|u_\Omega\|_{L^\infty(B_r(x_0))} \leqq C_0 r\Big) \Rightarrow \Big(u_\Omega \equiv 0 \quad \text{in } B_{r/2}(x_0)\Big),$$

for every $x_0 \in \overline{\Omega} \cap D$ and every $r \in (0, r_0]$. In other words, if $x_0 \in \overline{\Omega}$ and $B_r(x_0) \subset D$, then

$$\sup_{B_r(x_0)} u_\Omega \geq C_0 r.$$

Proof. By Proposition 3.1 (ii), we have that for every $\omega \subset \Omega$

$$E_f(u_\Omega, D) + \frac{c_Q C_1}{2}|\Omega| \leq E_f(u_\omega, D) + \frac{c_Q C_1}{2}|\omega|.$$

Thus, the claim follows by [2, Lemma 4.4], [9, Lemma 3.3], or [20, Lemma 2.8]. □

3.4. Density Estimates on the Boundary of Ω

An a consequence of the Lipschitz continuity and the non-degeneracy of u_Ω and v_Ω, we obtain density estimates for the optimal set Ω.

Proposition 3.5. *Let $D \subset \mathbb{R}^d$ and $f, g, Q \in L^\infty(D)$ be as in Sect. 3.1. Then, there are $\varepsilon_0, r_0 > 0$ (depending on C_1, C_2, M, d, c_Q, C_Q) such that for every set $\Omega \subset D$ optimal for* (1.3)

$$\varepsilon_0|B_r| \leqq |B_r(x_0) \cap \Omega| \leqq (1-\varepsilon_0)|B_r|. \tag{3.3}$$

for every ball $B_r(x_0) \subset D$ of radius $r \leq r_0$ centered on $\partial\Omega$.

Proof. Assume that $x_0 = 0 \in \partial\Omega$. The lower estimate is an immediate consequence of the Lipschitz continuity (Lemma 3.3) and the non-degeneracy (Lemma 3.3) of u_Ω. The upper bound can be obtained as in [2]. Precisely, consider the solution h to

$$-\Delta h = M \quad \text{in } B_r \qquad h = u_\Omega \quad \text{on } D \setminus B_r.$$

Since $\Delta u_\Omega + f \geq 0$ in $\mathbb{R}^d$, we get that $-\Delta(h - u_\Omega) \geq M - f \geqq 0$ in B_r. In particular, we have that $u_\Omega \leqq h$ and $\{u_\Omega > 0\} \subset \{h > 0\}$ in B_r. Thus, testing the optimality (3.1) of u_Ω with h,

$$\begin{aligned}\frac{C_2 C_Q}{2}|B_r \cap \{u_\Omega = 0\}| \geq E_f(u_\Omega, B_r) - E_f(h, B_r)&\\ = \frac{1}{2}\int_{B_r} |\nabla(u_\Omega - h)|^2\,dx&\\ + \int_{B_r}\Big(\nabla h \cdot \nabla(u_\Omega - h) - f(u_\Omega - h)\Big)\,dx&\\ \geq \frac{1}{2}\int_{B_r} |\nabla(u_\Omega - h)|^2\,dx.&\end{aligned}$$

By the Poincaré and Cauchy-Schwarz inequalities, we have

$$\int_{B_r} |\nabla(u_\Omega - h)|^2\,dx \geq \frac{C_d}{|B_r|}\left(\frac{1}{r}\int_{B_r}(h - u_\Omega)\,dx\right)^2,$$

so in order to prove the upper bound in (3.3), we only need to show that $\frac{1}{r^{d+1}}\int_{B_r}(h - u_\Omega)\,dx$ is bounded from below by a positive constant. Notice that, by the non-degeneracy of u_Ω, we have

$$\widetilde{C}r \leqq \sup_{B_{r/2}} u_\Omega \leqq \sup_{B_{r/2}} h.$$

On the other hand, since $h(x) + \frac{M}{2d}|x|^2$ is harmonic in B_R, the Harnack inequality in B_r implies

$$\widetilde{C}r \leq \sup_{B_{r/2}} h \leq C_d\big(h(x) + Mr^2\big) \quad \text{for every} \quad x \in B_{r/2}.$$

Thus, by taking r_0 such that $2C_d r_0 M \le \widetilde{C}$, we get that $h \geqq C_d \widetilde{C} r = \overline{C} r$ in $B_{r/2}$. On the other hand, if L is the Lipschitz constant of u_Ω, then for any $\varepsilon \in (0,1)$, $u_\Omega \leqq L\varepsilon r$ in $B_{\varepsilon r}$. Then

$$\int_{B_r} (h - u_\Omega)\,dx \ge \int_{B_{\varepsilon r}} (h - u_\Omega)\,dx \ge (\bar{C} r - L\varepsilon r)|B_{\varepsilon r}|,$$

which concludes the proof after choosing $\varepsilon \le 1/2$ small enough. □

3.5. An Estimate on the Level Sets of u_Ω

We conclude the section with this auxiliary result that will play a crucial role in Sect. 4.3 in the proof of the existence of homogeneous blow-up limits.

Lemma 3.6. *Let $D \subset \mathbb{R}^d$ and $f, g, Q \in L^\infty(D)$ be as in Sect. 3.1; let Ω be a solution to (1.3) and let u_Ω be the associated state function. Then, there are constants $C > 0$ and $r_0 > 0$, depending only on d, M, C_1, C_2, c_Q, C_Q, such that*

$$|\{0 < u < rt\} \cap B_r(x_0)| \leqq Ct|B_r|, \tag{3.4}$$

for every $B_r(x_0) \subset D$ centered on $\partial\Omega$ and every $t \in (0,1)$.

Proof. The estimate is contained in the proof of [11, Theorem 1.10]; we sketch the idea for the sake of completeness. Let $x_0 = 0 \in \partial\Omega$ and $t > 0$. We fix a function $\eta \in C_c^\infty(B_{2r})$ such that $0 \le \eta \le 1$ in B_{2r} and $\eta \equiv 1$ in B_r, and we use the competitor

$$\phi = \eta(u_\Omega - rt)^+ + (1-\eta)u_\Omega.$$

to test the optimality condition (3.2) in B_{2r}. Since

$$\phi = u_\Omega - rt\eta \quad \text{in } \{u_\Omega > rt\} \qquad \text{and} \qquad \phi = (1-\eta)u_\Omega \quad \text{in } \{0 \leqq u_\Omega \leqq rt\},$$

we get

$$\begin{aligned}
E_f(\phi, B_{2r}) &= \int_{\{u_\Omega > rt\}\cap B_{2r}} \left(\frac{1}{2}|\nabla u_\Omega|^2 + \frac{(rt)^2}{2}|\nabla\eta|^2 - rt\nabla u_\Omega\cdot\nabla\eta - fu_\Omega + frt\eta\right)dx\\
&\quad + \int_{\{0\leqq u_\Omega\leqq rt\}\cap B_{2r}} \left(\frac{(1-\eta)^2}{2}|\nabla u_\Omega|^2 + \frac{u_\Omega^2}{2}|\nabla\eta|^2 - (1-\eta)u_\Omega\nabla u_\Omega\cdot\nabla\eta - fu_\Omega + fu_\Omega\eta\right)dx\\
&\leqq E_f(u_\Omega, B_{2r}) - rt\int_{\{u_\Omega > rt\}\cap B_{2r}} \nabla u_\Omega\cdot\nabla\eta\,dx + \int_{B_{2r}}\left(\frac{(rt)^2}{2}|\nabla\eta|^2 + frt\eta\right)dx\\
&\quad + \int_{\{0\leqq u_\Omega\leqq rt\}\cap B_{2r}} \left(\frac{(1-\eta)^2 - 1}{2}|\nabla u_\Omega|^2 - (1-\eta)u_\Omega\nabla u_\Omega\cdot\nabla\eta\right)dx\\
&\leqq E_f(u_\Omega, B_{2r}) + C_d\Big(t\|\nabla u_\Omega\|_{L^\infty} + rt\,\|f\|_{L^\infty} + t^2\Big)|B_r|.
\end{aligned}$$

Thus (3.2) implies

$$\begin{aligned}
\frac{C_1 c_Q}{2}|\{0 < u_\Omega \le rt\}\cap B_r| &\le \frac{C_1 c_Q}{2}\Big(|\{u_\Omega > 0\}\cap B_{2r}| - |\{\varphi > 0\}\cap B_{2r}|\Big)\\
&\le E_f(\phi, D) - E_f(u_\Omega, D)
\end{aligned}$$

$$\leq C_d\Big(t\|\nabla u\|_{L^\infty} + rt\,\|f\|_{L^\infty} + t^2\Big)|B_r|.$$

Therefore, we get

$$\frac{C_1 c_Q}{2}|B_r \cap \{0 \leqq u \leqq rt\}| \leqq Ct|B_r|$$

with $C > 0$ depending on d, $\|u_\Omega\|_{L^\infty(B_{2r})}$, M, C_1 and c_Q. □

4. Compactness and Convergence of Blow-Up Sequences

Take an optimal set Ω for (1.3) in some $D \subset \mathbb{R}^d$, and consider the corresponding state functions $u = u_\Omega$ and $v = v_\Omega$. For any $x_0 \in \partial\Omega \cap D$ and any sequence $r_k \to 0^+$, we set

$$u_{x_0,r_k}(x) := \frac{1}{r_k}u(x_0 + r_k x), \quad v_{x_0,r_k}(x) := \frac{1}{r_k}v(x_0 + r_k x), \quad \Omega_{x_0,r_k} := \frac{\Omega - x_0}{r_k}.$$

Since u and v vanish in x_0 and are Lipschitz (in a neighborhood of x_0), we have that u_{x_0,r_k} and v_{x_0,r_k} vanish in 0 and (for large k) are uniformly Lipschitz in any ball B_R. Thus, there are functions $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ and subsequences of u_{x_0,r_k} and v_{x_0,r_k} that converge locally uniformly in $\mathbb{R}^d$ respectively to u_0 and v_0. As usual we say that u_0 and v_0 are blow-up limits of u and v in x_0; and we recall that they might depend on the sequence r_k. We notice that the blow-up limits of u and v will always be taken along the same sequence $r_k \to 0$.

The main results are Propositions 4.3 and 4.7. In Proposition 4.3 we list the properties of any couple of functions u_0, v_0 obtained as blow-up limits of u, v, while in Proposition 4.7 we show that there is at least one sequence $r_k \to 0$ that provides blow-up limits which are 1-homogeneous and stationary for the one-phase Alt–Caffarelli functional.

4.1. A General Lemma About the Convergence of Blow-Up Sequences

The construction of the blow-up limit from Proposition 4.7 will require taking three consecutive blow-ups. We give here a general lemma, which we will use several times in this section.

Lemma 4.1. *Let B_{2R} be a ball in $\mathbb{R}^d$. Let $u_n : \overline{B}_{2R} \to \mathbb{R}$ be a sequence of non-negative Lipschitz functions converging uniformly to a Lipschitz function $u_\infty : \overline{B}_{2R} \to \mathbb{R}$, and suppose that there is a constant $L > 0$ such that*

$$\|\nabla u_n\|_{L^\infty(B_{2R})} \leq L \quad \textit{for every} \quad n \geq 1.$$

Then, the following holds.

(i) *Suppose that there is a constant* $\tilde{C} > 0$ *such that, for every* $n \geq 1$,

$$\sup_{B_r(x_0)} u_n \geq \tilde{C} r \quad \textit{for every} \quad x_0 \in B_R \cap \overline{\{u_n > 0\}} \quad \textit{and every} \quad r \in (0, R). \tag{4.1}$$

Then

$$\sup_{B_r(x_0)} u_\infty \geq \tilde{C} r \quad \textit{for every} \quad x_0 \in B_R \cap \overline{\{u_\infty > 0\}} \quad \textit{and every} \quad r \in (0, R), \tag{4.2}$$

and

$$\mathbb{1}_{\{u_n > 0\}} \to \mathbb{1}_{\{u_\infty > 0\}} \quad \textit{pointwise a.e. in } B_R. \tag{4.3}$$

(ii) *Suppose that there is a constant* $\tilde{M} > 0$ *such that, for every* $n \geq 1$, *we have the bound*

$$\left| \int_{B_{2R}} \nabla u_n \cdot \nabla \varphi \, dx \right| \leq \tilde{M} \|\varphi\|_{L^\infty(B_{2R})} \quad \textit{for every } \varphi \in C_c^{0,1}(B_{2R}), \tag{4.4}$$

where $C_c^{0,1}(B_{2R})$ *is the space of Lipschitz functions with compact support in* B_{2R}.
Then u_n *converges to* u_∞ *strongly in* $H^1(B_R)$.

Proof. We first prove (i). Suppose that $x_0 \in B_R \cap \overline{\{u_\infty > 0\}}$. Then, there is a sequence $x_n \to x_0$ of points $x_n \in B_R \cap \overline{\{u_\infty > 0\}}$. Let $r_n := r - |x_n - x_0|$. By (4.1), there is a point $y_n \in B_{r_n}(x_n) \subset B_r(x_0)$ such that $u_n(y_n) \geq \tilde{C} r_n$. Thus, by the uniform convergence of u_n, we get (4.2). In order to prove (4.3), we first notice that by the pointwise convergence of u_n to u_∞, we have that

$$\mathbb{1}_{\{u_\infty > 0\}}(x_0) = 1 \;\Rightarrow\; \mathbb{1}_{\{u_n > 0\}}(x_0) = 1 \quad \text{for large } n,$$

so it is sufficient to prove that the set

$$S := \left\{ x_0 \in B_R \;:\; \mathbb{1}_{\{u_\infty > 0\}}(x_0) = 0 \;\text{ and }\; \limsup_{n \to \infty} \mathbb{1}_{\{u_n > 0\}}(x_0) = 1 \right\},$$

is of measure zero. Now, by (4.1), we get that for every $r \in (0, R)$ there is a sequence $y_n \in \overline{B}_r(x_0)$ such that $u_n(y_n) \geq \tilde{C} r$. Then, by the uniform convergence of u_n, we have that there is $y_\infty \in \overline{B}_r(x_0)$ such that $u_\infty(y_\infty) \geq \tilde{C} r$. Then, by the Lipschitz continuity of u_∞, the ball $B_{\tilde{C}r/L}(y_\infty)$ is contained in $\{u_\infty > 0\}$. Since r is arbitrary, we get that the Lebesgue density of $\{u_\infty > 0\}$ in x_0 cannot be zero, so the set S has zero measure.

We next prove (ii). Since u_n is uniformly bounded in $H^1(B_{2R})$, we have that u_n converges to u_∞ weakly in $H^1(B_{2R})$. Thus, the estimate (4.4) holds also for u_∞, that is

$$\left| \int_{B_{2R}} \nabla u_\infty \cdot \nabla \varphi \, dx \right| \leq \tilde{M} \|\varphi\|_{L^\infty(B_{2R})} \quad \text{for every } \varphi \in C_c^{0,1}(B_{2R}).$$

Choose a function $\varphi \in C_c^\infty(B_{2R})$ such that $\varphi \equiv 1$ in B_R. Then,

$$\begin{aligned}\int_{B_R} |\nabla(u_n - u_\infty)|^2\,dx &\le \int_{B_{2R}} |\nabla(\varphi(u_n - u_\infty))|^2\,dx\\ &= \int_{B_{2R}} |\nabla\varphi|^2(u_n - u_\infty)^2\,dx\\ &\quad+ \int_{B_{2R}} \nabla(u_n - u_\infty)\cdot\nabla\big(\varphi^2(u_n - u_\infty)\big)\,dx\\ &\le \int_{B_{2R}} |\nabla\varphi|^2(u_n - u_\infty)^2\,dx + 2\tilde{M}\|\varphi^2(u_n - u_\infty)\|_{L^\infty(B_{2R})}.\end{aligned}$$

Since the right-hand side converges to zero, we get the claim. □

Finally, we notice that the above lemma can be applied to the state functions u_Ω and v_Ω of an optimal domain. This is a consequence of the following lemma.

Lemma 4.2. *Let $B_{2R} \subset \mathbb{R}^d$ and $u \in H^1(B_{2R})$ be a non-negative function with the following properties.*

(a) *There is a function $f \in L^\infty(B_{2R})$ such that*

$$-\Delta u = f \quad \text{in } \Omega_u := \{u > 0\}, \qquad u = 0 \quad \text{on } \partial\Omega_u \cap B_{2R},$$

in the sense that

$$\int_{B_{2R}} \nabla u\cdot\nabla\varphi\,dx = \int_{B_{2R}} \varphi f\,dx \quad \text{for every } \varphi \in H_0^1(B_{2R}) \quad \text{with} \quad \varphi = 0 \quad \text{in } B_{2R}\setminus\Omega_u.$$

(b) *There is $\Lambda > 0$ such that, for every non-negative $\varphi \in H_0^1(B_{2R})$,*

$$E_f(u, B_{2R}) + \frac{\Lambda}{2}|B_{2R}\cap\{u > 0\}| \le E_f(u+\varphi, B_{2R}) + \frac{\Lambda}{2}|B_{2R}\cap\{u+\varphi > 0\}|.$$

Then, for every $\varphi \in C_c^{0,1}(B_R)$, we have

$$\left|\int_{B_R} \nabla u\cdot\nabla\varphi\,dx\right| \le C_d\Big(1 + \Lambda + R\|f\|_{L^\infty(B_{2R})}\Big)R^{d-1}\|\varphi\|_{L^\infty(B_R)}.$$

Proof. We only sketch the proof and we refer to [38, Chapter 3] for the details. By (a) we have that

$$\mu := \Delta u + |f|$$

is a positive Radon measure on B_{2R}, where

$$\int_{B_{2R}} \varphi\,d\mu := \int_{B_{2R}} \Big(-\nabla u\cdot\nabla\varphi + \varphi|f|\Big)\,dx \qquad \text{for every } \varphi \in C_c^{0,1}(B_{2R}).$$

By testing the optimality condition in (b) with a function $\varphi \in C_c^{0,1}(B_{2R})$, we get

$$\int_{B_{2R}} \varphi\,d\mu = \int_{B_{2R}} \Big(-\nabla u\cdot\nabla\varphi + \varphi f\Big)\,dx \le \int_{B_{2R}} |\nabla\varphi|^2\,dx + \Lambda|B_{2R}|,$$

and choosing $\varphi = R\phi$ with $\phi \equiv 1$ in B_R, we obtain

$$\mu(B_R) \leq \int_{B_{2R}} \phi \, d\mu \leq C_d(1+\Lambda)R^{d-1}.$$

As a consequence, for every $\varphi \in C_c^{0,1}(B_R)$, we have

$$\left| \int_{B_R} \nabla u \cdot \nabla \varphi \, dx \right| \leq \int_{B_R} |\varphi| \, d\mu$$
$$+ \int_{B_R} |\varphi||f| \, dx \leq C_d\Big(1 + \Lambda + R\|f\|_{L^\infty}\Big) R^{d-1} \|\varphi\|_{L^\infty(B_R)}.$$

which concludes the proof. □

4.2. First Blow-Up

In this subsection we list the properties of any couple of blow-ups

$$u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$$

of the state functions u_Ω and v_Ω at a boundary point $x_0 \in \partial\Omega \cap D$. The qualitative properties (Lipschitz continuity, non-degeneracy, density estimates) of u_Ω and v_Ω are conserved under blow-up limits; the stationarity condition also passes to the limit; the main difference is that u_0 and v_0 are harmonic where they are positive.

Proposition 4.3. *Let $D \subset \mathbb{R}^d$ be a bounded open set, let Ω be a solution to (1.3) with f, g, Q as in Theorem 1.2, and let $u := u_\Omega$ and $v := v_\Omega$ be the state functions on Ω defined in (1.1) and (1.6). We consider a point $x_0 \in \partial\Omega \cap D$ and blow-up sequences u_{x_0,r_k}, v_{x_0,r_k} of u, v converging locally uniformly in $\mathbb{R}^d$ to blow-up limits $u_0, v_0 \in C^{0,1}(\mathbb{R}^d)$. Then*

(1) *taking C_1 and C_2 to be the constants from (1.2), we have*

$$C_1 v_0 \leq u_0 \leq C_2 v_0 \quad \text{on } \mathbb{R}^d;$$

(2) *the functions u_0 are v_0 are harmonic in the open set $\Omega_0 := \{u_0 > 0\} = \{v_0 > 0\}$.*

(3) *there are constants $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\varepsilon_0|B_r| \leq |B_r(x_0) \cap \Omega_0| \leq (1-\varepsilon_0)|B_r|, \tag{4.5}$$

for every $x_0 \in \partial\Omega_0$, $r > 0$, and

$$|\{0 < u_0 < rt\} \cap B_r(x_0)| \leqq Ct|B_r|, \tag{4.6}$$

for every $x_0 \in \partial\Omega_0, r > 0, t > 0$;

(4) *$0 \in \partial\Omega_0$ and there is a constant $C_0 > 0$ such that*

$$\sup_{B_r(x_0)} u_0 \geq C_0 r \quad \text{and} \quad \sup_{B_r(x_0)} v \geq C_0 r \quad \text{for every} \quad x_0 \in \overline{\Omega}_0 \quad \text{and every} \quad r > 0;$$

(5) *there is a constant* $\Lambda > 0$ *depending on* C_1, C_2, C_Q *and* d *such that, for every* $R > 0$,

$$\left|\int_{B_R} \nabla u_0 \cdot \nabla\varphi\,dx\right| + \left|\int_{B_R} \nabla v_0 \cdot \nabla\varphi\,dx\right| \le \Lambda R^{d-1}\|\varphi\|_{L^\infty(B_R)};$$

(6) *for every compactly supported smooth vector field* $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d}\Big(-\nabla u_0\cdot\big((\nabla\xi)+(D\xi)\big)\nabla v_0+\big(\nabla u_0\cdot\nabla v_0+Q(x_0)\mathbb{1}_{\Omega_0}\big)\,\mathrm{div}\,\xi\Big)\,dx=0.$$

Proof. For simplicity, we set

$$u_k := u_{x_0,r_k},\ \ v_k := v_{x_0,r_k},\ \ f_k := f_{x_0,r_k},\ \ g_k := g_{x_0,r_k}\ \text{ and }\ \Omega_k := (\Omega - x_0)/r_k,$$

and we notice that

$$-\Delta u_k = r_k^2 f_k \quad\text{and}\quad -\Delta v_k = r_k^2 g_k \quad \text{in } \Omega_k.$$

The first two claims are just a consequence of the locally uniform convergence of u_k and v_k to u_0 and v_0. By Proposition 3.1, Corollary 3.3 and Lemma 3.4, we already know that u and v fulfill the assumptions of Lemma 4.2. Therefore, (4) and (5) follow from Lemma 4.1, while (6) follows from Lemma 2.6 and Remark 2.7, and the strong H^1 convergence of u_k and v_k.

We next prove (3). By Proposition 3.5 and rescaling, we know that, for every $k > 0$,

$$\varepsilon_0|B_r| \le |B_r(x_0)\cap\{u_k>0\}| \le (1-\varepsilon_0)|B_r|,\quad \text{for } r<r_0/r_k,\ x_0\in\partial\Omega_k,$$

so, by the strong convergence of $\mathbb{1}_{\Omega_k}$ to $\mathbb{1}_{\Omega_0}$ in $L^1_{loc}(\mathbb{R}^d)$ (see Lemma 4.1), we get the density estimate (4.5) for Ω_0. Similarly, by rescaling (3.4) we obtain

$$|\{0<u_k<rt\}\cap B_r(x_0)| \leqq Ct|B_r| \quad\text{for}\quad r<\frac{r_0}{r_k},\ x_0\in\partial\Omega_k,\ t>0,$$

which, passing to the limit as $k\to\infty$, gives (4.6). □

4.3. Second Blow-Up

Consider blow-up limits u_0, v_0 as in the previous subsection and let

$$u_{00}, v_{00} : \mathbb{R}^d \to \mathbb{R}.$$

be blow-up limits of u_0 and v_0 in zero. Then u_{00} and v_{00} are still blow-up limits of the state functions u_Ω and v_Ω at $x_0 \in \partial\Omega\cap D$ and Proposition 4.3 still applies. On the other hand, [29, Theorem 1.2] applies to the domain Ω_0 and the function u_0, so the Boundary Harnack Principle (see [29, Definition 1.1]) holds on Ω_0; since u_0 and v_0 are harmonic in Ω_0, we get that the ratio u_0/v_0 is Hölder continuous up to the boundary $\partial\Omega_0$. This, in particular means that the second blow-ups u_{00} and v_{00} at any boundary point (and thus in zero) are proportional.

Lemma 4.4. *Let $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ be non-negative Lipschitz functions on $\mathbb{R}^d$ with the same positivity set and let $\Omega_0 := \{u_0 > 0\} = \{v_0 > 0\}$. Suppose that u_0 and v_0 satisfy the conditions (1)-(6) from Proposition 4.3 and let $u_{00}, v_{00} : \mathbb{R}^d \to \mathbb{R}$ be blow-ups of u_0, v_0 at zero. Then*

(1) there is a constant $\lambda \in (C_1, C_2)$ such that $u_{00} = \lambda v_{00}$ on $\mathbb{R}^d$;
(2) the function u_{00} is harmonic in the open set $\Omega_{00} := \{u_{00} > 0\}$;
(3) there are constants $\varepsilon_0 > 0$ and $C > 0$ such that

$$\varepsilon_0 |B_r| \leqq |B_r(x_0) \cap \Omega_{00}| \leqq (1 - \varepsilon_0)|B_r|,$$

for every $x_0 \in \partial\Omega_{00}$, $r > 0$, and

$$|\{0 < u_{00} < rt\} \cap B_r(x_0)| \leqq Ct|B_r|,$$

for every $x_0 \in \partial\Omega_{00}, r > 0, t > 0$;
(4) $0 \in \partial\Omega_{00}$ and there is a constant $C_0 > 0$ such that

$$\sup_{B_r(x_0)} u_{00} \geq C_0 r \quad \textit{for every} \quad x_0 \in \overline{\Omega}_{00} \quad \textit{and every} \quad r > 0;$$

(5) there is a constant $\Lambda > 0$ such that, for every $R > 0$,

$$\left| \int_{B_R} \nabla u_{00} \cdot \nabla \varphi \, dx \right| \leq \Lambda R^{d-1} \|\varphi\|_{L^\infty(B_R)};$$

(6) for every compactly supported smooth vector field $\xi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \Big(- \nabla u_{00} \cdot \big((\nabla \xi) + (D\xi)\big)\nabla u_{00} + \Big(|\nabla u_{00}|^2 + \lambda Q(x_0)\mathbb{1}_{\Omega_{00}}\Big) \operatorname{div} \xi \Big) \, dx = 0. \quad (4.7)$$

Proof. Let $R > 0$ be fixed and let $\phi := u_0$. In order to show that the Boundary Harnack Principle holds on Ω_0, we check that Ω_0 and ϕ satisfy the list of assumptions (a)–(g) from [29, Theorem 1.2] in the ball B_R:

(a) by definition of Ω_0, we have $\phi > 0$ in Ω_0 and $\phi \equiv 0$ on $B_R \setminus \Omega_0$;
(b) by hypothesis ϕ is Lipschitz continuous in $\mathbb{R}^d$ and so, in B_R;
(c) since ϕ is harmonic in Ω_0 and satisfies the condition (4) from Proposition Proposition 4.3, we can apply [38, Lemma 6.8]; thus, there is a constant $\kappa > 0$ such that

$$\phi \geq \kappa \ \operatorname{dist}_{B_R \setminus \Omega_0} \quad \text{in } B_{R/2};$$

(d) since $\phi \geqq 0$ and $\Delta\phi = 0$ in Ω_0, we have that $\Delta\phi \geq 0$ in $\mathbb{R}^d$;
(e) for every $x_0 \in \partial\Omega \cap B_R$, we have

$$|B_r(x_0) \setminus \Omega_0| \geq \varepsilon_0 |B_r(x_0)| \qquad \text{for every } r \in (0, R - |x_0|);$$

(f) for every $x_0 \in \partial\Omega_0 \cap B_R$ and every $r \in (0, R - |x_0|)$, we have

$$\Big|\{0 < \phi < rt\} \cap B_r(x_0)\Big| \leq Ct|B_r| \qquad \text{for every } t > 0; .$$

(g) by (4) of Proposition 4.3, for every $x_0 \in \partial\Omega_0 \cap B_R$ and every $r \in (0, R - |x_0|)$, we have

$$\sup_{B_r(x_0)} \phi \geq C_0 r.$$

Therefore, all the assumptions of [29, Theorem 1.2] are fulfilled and so, the ratio

$$\frac{u_0}{v_0} : \Omega_0 \to \mathbb{R},$$

can be extended to a Hölder continuous function on $\overline{\Omega}_0 \cap B_{R/2}$. Thus, the blow-ups u_{00} and v_{00} are proportional and all the other claims follow as in Proposition 4.3. □

4.4. Third Blow-Up

Take the state functions u_Ω and v_Ω on an optimal domain Ω. Let

$$u_{00}, v_{00} : \mathbb{R}^d \to \mathbb{R},$$

be the second blow-up limits of u_Ω and v_Ω at a free boundary point $x_0 \in \partial\Omega \cap D$. Then, u_{00} and v_{00} are proportional and satisfy the conditions listed in Lemma 4.4. We will show that if we perform a further blow-up in zero, then we obtain functions $u_{000}, v_{000} : \mathbb{R}^d \to \mathbb{R}$ that still satisfy the conditions from Lemma 4.4 but are also 1-homogeneous. Before we give the precise statement, we notice that the stationarity condition (4.7) implies the monotonicity of the associated Weiss' boundary adjusted energy from [39].

Lemma 4.5. (Monotonicity formula) *Let $B_R \subset \mathbb{R}^d$, $u \in H^1(B_R)$ be a continuous non-negative function, and let $\Omega := \{u > 0\}$. Suppose that for every smooth compactly supported vector field $\xi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$, we have:*

$$\int_\Omega \Big(\operatorname{div} \xi \left(|\nabla u|^2 + \Lambda \right) - \nabla u \cdot ((\nabla \xi) + (D\xi))\nabla u \Big)\, dx = 0.$$

Then, for every $x_0 \in \partial\Omega$, $r > 0$ the map

$$r \mapsto W_\Lambda(u_{x_0,r}) := \int_{B_1} |\nabla u_{x_0,r}|^2\, dx + \Lambda |\{u_{x_0,r} > 0\} \cap B_1| - \int_{\partial B_1} u_{x_0,r}^2\, d\mathcal{H}^{d-1},$$

is non-decreasing in $(0, +\infty)$ and

$$\frac{\partial}{\partial r} W_\Lambda(u_{x_0,r}) \geqq \frac{2}{r} \int_{\partial B_1} |x \cdot \nabla u_{x_0,r} - u_{x_0,r}|^2\, d\mathcal{H}^{d-1},$$

In particular, if $r \mapsto W_\Lambda(u_{x_0,r})$ is constant, then u is 1-homogeneous.

Proof. See for instance [38, Proposition 9.9]. □

Lemma 4.6. *Let $u_{00}, v_{00} : \mathbb{R}^d \to \mathbb{R}$ be non-negative Lipschitz functions on $\mathbb{R}^d$ with the same positivity set and let $\Omega_{00} := \{u_{00} > 0\} = \{v_{00} > 0\}$. Suppose that u_{00} and v_{00} satisfy the conditions (1)-(6) from Lemma 4.4 and let $u_{000}, v_{000} : \mathbb{R}^d \to \mathbb{R}$ be blow-ups of u_{00}, v_{00} at zero. Then:*

(1) $u_{000} = \lambda v_{000}$ *on* $\mathbb{R}^d$*;*
(2) for every compactly supported smooth vector field $\xi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d} \Big(-\nabla u_{000} \cdot \big((\nabla\xi) + (D\xi)\big)\nabla u_{000} + \Big(|\nabla u_{000}|^2 + \lambda Q(x_0)\mathbb{1}_{\Omega_{000}}\Big) \operatorname{div}\xi \Big)\, dx = 0; \tag{4.8}$$

(3) u_{000} *is* 1*-homogeneous in* $\mathbb{R}^d$*.*

Proof. Set $\Lambda := \lambda Q(x_0)$ and for simplicity, let $u := u_{00}$. By Lemma 4.5, the function

$$r \mapsto W_\Lambda\big(u_r\big),$$

is non-decreasing in r and so, it admits a limit Θ as $r \to 0$; moreover, by the Lipschitz continuity of u, Θ is finite. Let $r_k \to 0$ be such that $u_{r_k} \to u_{000}$. Then, by Lemma 4.1, u_{r_k} converges to u_{000} strongly in H^1_{loc} and the level sets $\{u_{r_k} > 0\}$ converge in L^1_{loc} to Ω_{000}. Thus, for any $s > 0$

$$\Theta := \lim_{k\to+\infty} W_\Lambda\big(u_{sr_k}\big) = W_\Lambda\big((u_{000})_s\big).$$

Moreover, using again the strong convergence of u_{r_k} and their level sets, we get that u_{000} satisfies (4.8). Thus, using again Lemma 4.5 and the fact that $s \mapsto W_\Lambda\big((u_{000})_s\big)$ is constantly equal to Θ, we get that u_{000} is homogeneous. □

As an immediate consequence, we obtain the following proposition.

Proposition 4.7. (Existence of stationary 1-homogeneous blow-ups) *Let* $D \subset \mathbb{R}^d$ *be a bounded open set and* Ω *be a solution to* (1.3) *with* f, g, Q *as in Theorem 1.2. Let* $u := u_\Omega$ *and* $v := v_\Omega$ *be the state functions on* Ω *defined in* (1.1) *and* (1.6) *and let* $x_0 \in \partial\Omega \cap D$*. Then, there is a sequence* $r_k \to 0$ *such that the corresponding blow-up limits*

$$u_0 := \lim_{k\to\infty} u_{x_0,r_k} \quad \text{and} \quad v_0 := \lim_{k\to\infty} v_{x_0,r_k}$$

satisfy the following conditions:

(i) u_0 *and* v_0 *are* 1*-homogeneous in* $\mathbb{R}^d$ *and* $u_0 = \lambda v_0$ *for some constant* $\lambda \in (C_1, C_2)$*;*
(ii) *for every compactly supported smooth vector field* $\xi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$*, we have*

$$\int_{\mathbb{R}^d} \Big(-\nabla u_0 \cdot \big((\nabla\xi) + (D\xi)\big)\nabla u_0 + \Big(|\nabla u_0|^2 + \lambda Q(x_0)\mathbb{1}_{\Omega_0}\Big) \operatorname{div}\xi \Big)\, dx = 0.$$

Remark 4.8. The blow-up sequence arising from Proposition 4.7 is obtained by a diagonal argument between the three different blow-ups defined respectively in Sects. 4.2, 4.3 and 4.4. Therefore, by combining all the previous result, we can see that u_0 and v_0 fulfill the conditions (a), (b), (c), (d) and (e) in Definition 7.3.

5. Regular and Singular Parts of the Free Boundary

Let $D \subset \mathbb{R}^d$ be a bounded open set and let Ω be a solution to the problem (1.3). As in Sect. 4, we denote by u, v the associated state variables.

Definition 5.1. (*Regular and singular points*) We will say that a boundary point $x_0 \in \partial\Omega \cap D$ is *regular* if there is a blow-up limit (u_0, v_0) of (u, v) at x_0 such that

$$u_0(x) = \alpha(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta(x \cdot \nu)_+$$

for some unit vector $\nu \in \mathbb{R}^d$ and some $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta = Q(x_0)$. If such a blow-up limit does not exist, then we will say that x_0 is *singular*.

We will denote by $\text{Reg}(\partial\Omega)$ the set of all regular points on $\partial\Omega \cap D$ and by $\text{Sing}(\partial\Omega)$ the set of all singular points on $\partial\Omega \cap D$. Clearly, we have that

$$\text{Reg}(\partial\Omega) \cap \text{Sing}(\partial\Omega) = \emptyset \qquad \text{and} \qquad \text{Reg}(\partial\Omega) \cup \text{Sing}(\partial\Omega) = \partial\Omega \cup D.$$

In Sect. 6 we will show that the regular part $\text{Reg}(\partial\Omega)$ is in fact, locally, a smooth manifold (and in particular, a relatively open subset of $\partial\Omega \cap D$), while in Sect. 8 we will give an estimate on the dimension of the singular set $\text{Sing}(\partial\Omega)$.

5.1. Regular and Singular Parts in Dimension Two

One can easily show that in dimension $d = 2$ the free boundary $\partial\Omega \cap D$ is composed only of regular points; in particular, in dimension two the proof of Theorem 1.2 is concluded already in Sect. 6, while the results from Sect. 8 are needed only when $d \geq 3$.

Lemma 5.2. *Let D be a bounded open set in $\mathbb{R}^2$ and let Ω be a solution to* (1.3). *Then, every point $x_0 \in \partial\Omega \cap D$ is a regular point in the sense of Definition 5.1.*

Proof. Let $r_k \to 0$ be a sequence such that the blow-up limits

$$u_0 := \lim_{k\to\infty} u_{r_k,x_0} \qquad \text{and} \qquad v_0 := \lim_{k\to\infty} v_{r_k,x_0},$$

are, as in Proposition 4.7, proportional ($u_0 = \lambda v_0$), non-negative and 1-homogeneous functions on $\mathbb{R}^d$, which are harmonic on the positivity set $\Omega_0 := \{u_0 > 0\} = \{v_0 > 0\}$. Now, reasoning as in [38, Proposition 9.13], we write u_0 and v_0 in polar coordinates as

$$u_0(r, \theta) = r\phi(\theta) \qquad \text{and} \qquad v_0(r, \theta) = r\psi(\theta),$$

where (since u_0 and v_0 are harmonic) ϕ and ψ are solutions to

$$-\phi''(\theta) = \phi(\theta) \quad \text{and} \quad -\psi''(\theta) = \psi(\theta) \quad \text{in} \quad \partial\Omega_0 \cap \partial B_1.$$

Since the only solutions of this equations (up to a rotation) are multiples of $\sin\theta$, and since $\mathcal{H}^1(\Omega_0 \cap \partial B_1) < 2\pi$ (this follows from the density estimate in Proposition 4.3), we get that

$$u_0(x) = \alpha(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta(x \cdot \nu)_+$$

for some unit vector $\nu \in \mathbb{R}^d$ and some $\alpha > 0$ and $\beta > 0$. Moreover, by Proposition 4.7, the function u_0 is critical point for the one-phase problem (that is, (4.8) holds). Thus,

$$|\nabla u_0|^2 = \lambda Q(x_0) \qquad \text{on } \partial\Omega_0 = \{x \in \mathbb{R}^2 : x \cdot \nu = 0\},$$

where $\lambda = \alpha/\beta$. Since $|\nabla u_0|^2 = \alpha^2$, we get that $\alpha\beta = Q(x_0)$, which concludes the proof. □

5.2. A Geometric Condition for the Regularity in Every Dimension

By an argument similar to the one in Lemma 5.2, we have that the free boundary points admitting one-sided tangent ball are regular points. This result holds in every dimension and will be useful in Sect. 6.

Lemma 5.3. *Let D be a bounded open set in $\mathbb{R}^d$, $d \geq 2$, and let Ω be a solution to* (1.3)*. Suppose that there is a one-sided tangent ball at the boundary point $x_0 \in \partial\Omega \cap D$ in the sense that:*

$$\text{there is} B_r(y_0) \subset \Omega \text{with} x_0 \in \partial B_r(y_0) \text{or there is} B_r(z_0) \subset \mathbb{R}^d \setminus \overline{\Omega} \text{with} x_0 \in \partial B_r(z_0). \tag{5.1}$$

Then, x_0 is a regular point in the sense of Definition 5.1.

Proof. As in Lemma 5.2, we consider a sequence $r_k \to 0$ for which the blow-up limits

$$u_0 := \lim_{k\to\infty} u_{r_k,x_0} \qquad \text{and} \qquad v_0 := \lim_{k\to\infty} v_{r_k,x_0},$$

are 1-homogeneous, non-negative, harmonic on their positivity set $\Omega_0 := \{u_0 > 0\}$ and are such that $u_0 = \lambda v_0$. The one-sided ball condition implies that there is a unit vector $\nu \in \mathbb{R}^d$ such that

$$\Omega_0 \subset \{x \in \mathbb{R}^d : x \cdot \nu > 0\} \qquad \text{or} \qquad \Omega_0 \supseteq \{x \in \mathbb{R}^d : x \cdot \nu > 0\}.$$

Since Ω_0 satisfies an exterior density estimate (by Proposition 4.3, claim Item (3)), the only possibility is that $\Omega_0 = \{x \in \mathbb{R}^d : x \cdot \nu > 0\}$ and

$$u_0(x) = \alpha(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) = \beta(x \cdot \nu)_+$$

for some $\alpha > 0$ and $\beta > 0$ with $\alpha/\beta = \lambda$. As in Lemma 5.2, since u_0 satisfies (4.8), we get that

$$|\nabla u_0|^2 = \frac{\alpha}{\beta} Q(x_0) \qquad \text{on } \partial\Omega_0 = \{x \in \mathbb{R}^2 : x \cdot \nu = 0\},$$

which gives that $\alpha\beta = Q(x_0)$. □

6. Regularity of Reg$(\partial\Omega)$

In this section we prove that the regular part Reg$(\partial\Omega)$ of the boundary of an optimal set Ω is locally the graph of a smooth function.

6.1. Viscosity Formulation

In this subsection we prove that on the free boundary $\partial\Omega \cap D$ of an optimal set Ω, solution to (1.3), we have the following optimality condition

$$|\nabla u_\Omega||\nabla v_\Omega| = Q \qquad \text{on } \partial\Omega,$$

in viscosity sense, as in [28, Section 2], in terms of the blow-up limit of the state functions u_Ω and v_Ω at free boundary points (see Remark 6.2).

Definition 6.1. (*Viscosity solutions*) Let D be a bounded open set in $\mathbb{R}^d$ and let $f, g \in L^\infty(\mathbb{R}^d)$. Let $u, v : D \to \mathbb{R}$ be two non-negative continuous functions with the same support

$$\Omega := \{u > 0\} = \{v > 0\},$$

on which they satisfy the PDE

$$-\Delta u = f \quad \text{and} \quad -\Delta v = g \quad \text{in } \Omega \cap D. \tag{6.1}$$

We say that the boundary condition

$$|\nabla u||\nabla v| = Q \qquad \text{on } \partial\Omega \cap D,$$

holds in viscosity sense if at any point $x_0 \in \partial\Omega \cap D$, at which Ω admits a one-sided tangent ball in the sense of (5.1), there exist:

(a) a decreasing sequence $r_k \to 0$;
(b) two positive constants $\alpha, \beta > 0$ such that $\alpha\beta = Q(x_0)$;
(c) a unit vector $\nu \in \mathbb{R}^d$;

such that the rescalings

$$u_{x_0,r_k}(x) := \frac{u(x_0 + r_k x)}{r_k} \qquad \text{and} \qquad v_{x_0,r_k}(x) := \frac{v(x_0 + r_k x)}{r_k},$$

converge uniformly in every ball $B_R \subset \mathbb{R}^d$ respectively to the blow-up limits

$$u_0(x) := \alpha\,(x \cdot \nu)_+ \qquad \text{and} \qquad v_0(x) := \beta\,(x \cdot \nu)_+ \,. \tag{6.2}$$

Remark 6.2. In [28] the authors addressed the ε-regularity theory for viscosity solutions of (1.8). In particular, in [28, Lemma 2.9] they proved that if the free boundary condition is satisfied in the sense of Definition 6.1, then for any smooth function $\varphi \in C^\infty(D)$ the following holds:

(i) If φ_+ touches $\sqrt{uv}$ from below at a point $x_0 \in D \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \le \sqrt{Q(x_0)}$.
(ii) If φ_+ touches $\sqrt{uv}$ from above at a point $x_0 \in D \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \ge \sqrt{Q(x_0)}$.

(iii) If a and b are constants such that

$$a > 0, \quad b > 0 \qquad \text{and} \qquad ab = Q(x_0),$$

and if φ_+ touches $w_{ab} := \frac{1}{2}(au + bv)$ from above at $x_0 \in D \cap \partial\Omega$, then $|\nabla\varphi(x_0)| \geq \sqrt{Q(x_0)}$.

Proposition 6.3. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $f, g, Q : D \to \mathbb{R}$ be as in Theorem 1.2. Let Ω be a solution to (1.3). Then the state variables $u := u_\Omega$ and $v := v_\Omega$ satisfy*

$$|\nabla u||\nabla v| = Q \qquad \text{on } \partial\Omega \cap D, \tag{6.3}$$

in the sense of Definition 6.1.

Proof. It follows as in the proof of Lemma 5.3. □

We can now prove that $\text{Reg}(\partial\Omega)$ is $C^{1,\alpha}$-regular for some $\alpha \in (0, 1)$ by exploiting the ε-regularity theory developed in [28].

Theorem 6.4. (Regularity of $Reg(\partial\Omega)$) *Let D be a bounded open set in $\mathbb{R}^d$, where $d \geq 2$. Let*

$$f : D \to \mathbb{R}, \quad g : D \to \mathbb{R}, \quad Q : D \to \mathbb{R},$$

be given non-negative functions. Suppose that the following conditions hold:

(a) $f, g \in L^\infty(D)$;
(b) *there are constants $C_1, C_2 > 0$ such that*

$$0 \leq C_1 g \leq f \leq C_2 g \qquad \text{in } D.$$

(c) *$Q \in C^{0,\alpha_Q}(D)$, for some $\alpha_Q > 0$, and there are a positive constants c_Q, C_Q such that*

$$0 < c_Q \leq Q(x) \leqq C_Q \qquad \text{for every } x \in D.$$

Let Ω be a solution to (1.3). Then, the regular part $Reg(\partial\Omega)$, defined in Sect. 5, is locally the graph of a $C^{1,\alpha}$ function, for some $\alpha > 0$.

Proof. The proof follows by the epsilon-regularity theory developed in [28]. Indeed, if $x_0 \in \text{Reg}(\partial\Omega)$ and $u_{x_0,\rho_k}, v_{x_0,\rho_k}$ are the blow-up sequences from the definition of $\text{Reg}(\partial\Omega)$. Then, there are $\alpha, \beta > 0$, $\nu \in \mathbb{R}^d$ such that $\alpha\beta = Q(x_0)$, $|\nu| = 1$ and

$$\lim_{k\to\infty} \left\| u_{x_0,\rho_k} - \alpha(x\cdot\nu)_+ \right\|_{L^\infty(B_1)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \left\| v_{x_0,\rho_k} - \beta(x\cdot\nu)_+ \right\|_{L^\infty(B_1)} = 0.$$

Moreover, the Lipschitz continuity and the non-degeneracy imply that for every $\varepsilon > 0$ there exists $k_0 > 0$ such that for every $k \geqq k_0$ we have

$$\begin{aligned} \alpha(x\cdot\nu-\varepsilon)_+ \leq & u_{x_0,\rho_k} \leq \alpha(x\cdot\nu+\varepsilon)_+ \qquad \text{for every } x \in B_1, \\ \beta(x\cdot\nu-\varepsilon)_+ \leq & v_{x_0,\rho_k} \leq \beta(x\cdot\nu+\varepsilon)_+ \qquad \text{for every } x \in B_1, \end{aligned}$$

that is, $u_{x_0,\rho_k}, v_{x_0,\rho_k}$ are ε-flat in the direction ν (see [28, Definition 1.2]). By rescaling the associated state equations, we have

$$-\Delta u_{x_0,\rho_k} = \rho_k^2 f_{x_0,\rho_k} \quad \text{and} \quad -\Delta v_{x_0,\rho_k} = \rho_k^2 g_{x_0,\rho_k} \quad \text{in} \quad B_1 \cap \{u_{x_0,\rho_k} > 0\},$$

where

$$\left\|\Delta u_{x_0,\rho_k}\right\|_{L^\infty(B_1)} + \left\|\Delta v_{x_0,\rho_k}\right\|_{L^\infty(B_1)} \leqq \rho_k \left(\|f\|_{L^\infty(B_1)} + \|g\|_{L^\infty(B_1)}\right).$$

On the other hand, since both u_{x_0,ρ_k} and v_{x_0,ρ_k} still satisfy (6.3) in viscosity sense, by applying [28, Theorem 3.1], $\partial\{u_{x_0,\rho_k} > 0\}$ is $C^{1,\alpha}$ in $B_{1/2}$. Finally, the result follows by rescaling back to the original problem. □

6.2. Higher Regularity

We can pass from $C^{1,\alpha}$ to C^∞-regularity of Reg$(\partial\Omega)$ by exploiting the higher order Boundary Harnack Principle for solutions to (1.8).

Proposition 6.5. (Higher regularity of Reg$(\partial\Omega)$) *Let D be a bounded open set in $\mathbb{R}^d$ and let f, g, Q be as in Theorem 6.4. If $f, g, Q \in C^{k,\alpha}(D)$ for some $k \geq 1$ and $\alpha > 0$, then the regular part Reg$(\partial\Omega)$ of the free boundary $\partial\Omega \cap D$ of any solution Ω to (1.3) is locally the graph of a $C^{1+k,\alpha}$ function for some $\alpha > 0$. In particular, if f, g, Q are C^∞, then Reg$(\partial\Omega)$ is locally the graph of a C^∞ function.*

Proof. We use a bootstrap argument as in [31, Section 5.4]. Suppose that Reg$(\partial\Omega)$ is locally the graph of a $C^{k,\alpha}$-regular function, for some $k \geqq 1$ (when $k = 1$ the claim follows from Theorem 6.4). Let $x_0 \in \partial\Omega$ and $r > 0$ be such that $\partial\Omega \cap B_r(x_0) =$ Reg$(\partial\Omega)$. Since u and v satisfy (6.1) in Ω and since v is non-degenerate, we can apply the higher order Boundary Harnack Principle (for PDEs with right-hand side) from [27, Theorem 1.3] and [36, Theorem 1.3], obtaining a non-negative $C^{k,\alpha}$-regular function $w\colon B_r(x_0) \cap \overline{\Omega} \to \mathbb{R}$ satisfying $u = wv$ in $B_r(x_0) \cap \overline{\Omega}$. Then, we have:

$$|\nabla u||\nabla v| = Q \quad \text{and} \quad |\nabla u| = w|\nabla v| \quad \text{on} \quad \partial\Omega \cap B_r(x_0).$$

Thus, u is a solution to the problem

$$-\Delta u = f \quad \text{in } \Omega \cap B_r(x_0) \qquad |\nabla u||\nabla u| = \sqrt{wQ} \quad \text{on } \partial\Omega \cap B_r(x_0),$$

which by [24, Theorem 2] implies that Reg$(\partial\Omega)$ is locally the graph of a $C^{k+1,\alpha}$ function. □

7. Stable Homogeneous Solutions of the One-Phase Bernoulli Problem

In this section we study the singular set of the stable global solutions of the one-phase Bernoulli problem (see Definition 7.3 for the definition of global stable solution). The main results are Theorems 7.8 and 7.9, which we will use in Theorem 8.1 in order to estimate the dimension of the singular set of the free boundary of the optimal sets for (1.3). The present section can be read separately from the rest of the paper; we will use only the Taylor expansions from Sect. 2 and the general results from Sect. 4.

7.1. Solutions of PDEs in Unbounded Domains

In $\mathbb{R}^d$, $d \geq 3$, we define

$$\dot{H}^1(\mathbb{R}^d) := \Big\{ u \in L^{2^*}(\mathbb{R}^d) \ : \ \nabla u \in L^2(\mathbb{R}^d; \mathbb{R}^d) \Big\} \quad \text{where} \quad 2^* := \frac{2d}{d-2}.$$

It is well known that $\dot{H}^1(\mathbb{R}^d)$ is a Hilbert space equipped with the norm $\|u\|_{\dot{H}^1(\mathbb{R}^d)} := \|\nabla u\|_{L^2(\mathbb{R}^d;\mathbb{R}^d)}$ and that $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot{H}^1(\mathbb{R}^d)$. Moreover, given an open (bounded or unbounded) set $\Omega \subset \mathbb{R}^d$, we define the space $\dot{H}^1_0(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{\dot{H}^1(\mathbb{R}^d)}$.

Let Ω be an open (bounded or unbounded) subset of $\mathbb{R}^d$ and let $F \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ be a given vector field. We say that w is a weak solution of the PDE

$$-\Delta w = \operatorname{div} F \quad \text{in } \Omega, \qquad w \in \dot{H}^1_0(\Omega), \tag{7.1}$$

if $w \in \dot{H}^1_0(\Omega)$ and

$$\int_{\mathbb{R}^d} \nabla w \cdot \nabla \varphi \, dx = -\int_{\mathbb{R}^d} \nabla \varphi \cdot F \, dx \qquad \text{for every } \varphi \in \dot{H}^1_0(\Omega).$$

It is standard to check that w is a solution to (7.1) if and only if w minimizes the functional

$$J(\varphi) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dx + \int_{\mathbb{R}^d} \nabla \varphi \cdot F \, dx, \tag{7.2}$$

among all functions $\varphi \in \dot{H}^1_0(\Omega)$. Since it is immediate to check that a minimizer of (7.2) in $\dot{H}^1_0(\Omega)$ exists and is unique, we get that also the solution to (7.1) exists and is unique. Finally, we notice that if $w \in \dot{H}^1_0(\Omega)$ is the solution to (7.1), then

$$\int_{\mathbb{R}^d} |\nabla w|^2 \, dx = -\int_{\mathbb{R}^d} \nabla w \cdot F \leq \|F\|_{L^2} \|\nabla w\|_{L^2},$$

which gives

$$\int_{\mathbb{R}^d} |\nabla w|^2 \, dx \leq \int_{\mathbb{R}^d} |F|^2 \, dx. \tag{7.3}$$

Remark 7.1. (Exterior density estimate and the space $\dot{H}^1_0$) We say that an open set $\Omega \subset \mathbb{R}^d$ satisfies a uniform exterior density estimate with a constant $c > 0$ if

$$|B_r(x) \setminus \Omega| \geq c|B_r| \quad \text{for every} \quad r \in (0,1) \quad \text{and every} \quad x \in \mathbb{R}^d \setminus \Omega. \tag{7.4}$$

It is known (see for example [17]) that if an open set $\Omega \subset \mathbb{R}^d$ satisfies (7.4) then the space $\dot{H}^1_0(\Omega)$ can be characterized as:

$$\dot{H}^1_0(\Omega) = \Big\{ u \in \dot{H}^1(\mathbb{R}^d) \ : \ u = 0 \ \text{ a.e. on } \ \mathbb{R}^d \setminus \Omega \Big\}. \tag{7.5}$$

Lemma 7.2. (Convergence of solutions) *Let Ω_n be a sequence of open sets in $\mathbb{R}^d$, $d \geq 3$ such that:*

(a) *there is a constant* $c > 0$ *such that, for every* $n \geq 1$, Ω_n *satisfies the exterior density estimate* (7.4)*;*
(b) *there is an open set* $\Omega_\infty \subset \mathbb{R}^d$ *satisfying the exterior density estimate* (7.4) *with the constant* $c > 0$ *and such that the sequence of characteristic functions* $\mathbb{1}_{\Omega_n}$ *converges pointwise almost-everywhere in* $\mathbb{R}^d$ *to* $\mathbb{1}_{\Omega_\infty}$.

Let $F_n \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ *be a sequence of vector fields converging strongly in* $L^2(\mathbb{R}^d; \mathbb{R}^d)$ *to the vector field* $F_\infty \in L^2(\mathbb{R}^d; \mathbb{R}^d)$. *For every* $n \geq 1$, *let* w_n *be the solution to the PDE*

$$-\Delta w_n = \operatorname{div}\, F_n \quad in \quad \Omega_n \qquad w_n \in \dot{H}^1_0(\Omega_n).$$

Then, w_n *converges strongly in* $\dot{H}^1(\mathbb{R}^d)$ *to the solution* w_∞ *of*

$$-\Delta w_\infty = \operatorname{div}\, F_\infty \quad in \quad \Omega_\infty \qquad w_\infty \in \dot{H}^1_0(\Omega_\infty).$$

Proof. First of all, we notice that the sequence w_n is bounded in $\dot{H}^1(\mathbb{R}^d)$ (by (7.3)). Thus, we can extract a subsequence, that we still denote by w_n, which converges weakly in $\dot{H}^1(\mathbb{R}^d)$ and pointwise almost-everywhere on $\mathbb{R}^d$ to a function $w \in \dot{H}^1(\mathbb{R}^d)$. We will show that $w = w_\infty$.

First, notice that for almost every $x \in \mathbb{R}^d \setminus \Omega_\infty$ we have that

$$w_n(x) \to w(x) \qquad \text{and} \qquad \mathbb{1}_{\Omega_n}(x) \to \mathbb{1}_{\Omega_\infty}(x) = 0.$$

But then $w_n(x) = 0$ (by (7.5)) and so $w(x) = 0$. Thus, using again (7.5), we get $w \in \dot{H}^1_0(\Omega_\infty)$.

Now, let $\varphi \in C^\infty_c(\Omega_\infty)$. We will show that for large enough n, $\varphi \in C^\infty_c(\Omega_n)$. Indeed, let $\delta > 0$ be a constant such that $B_\delta(x) \subset \Omega_\infty$ for every x in the support of φ. Suppose by contradiction that there is a sequence $x_n \in \overline{\{\varphi \neq 0\}}$ such that $x_n \notin \Omega_n$. Then, by the density estimate for Ω_n, $|B_\delta(x_n) \cap \Omega_n| \leq (1-c)|B_\delta|$. Now, up to a subsequence, $x_n \to x_\infty \in \overline{\{\varphi \neq 0\}}$. But then,

$$(1-c)|B_\delta| \geq \lim_{n\to\infty} |B_\delta(x_n) \cap \Omega_n| = |B_\delta(x_\infty) \cap \Omega_\infty| = |B_\delta|,$$

which is a contradiction. Thus, for large n, $\overline{\{\varphi \neq 0\}} \subset \Omega_n$. Now, using the equation for w_n,

$$\int_{\mathbb{R}^d} \nabla w_n \cdot \nabla \varphi \, dx = -\int_{\mathbb{R}^d} \nabla \varphi \cdot F_n \, dx$$

and passing to the limit, we get that

$$\int_{\mathbb{R}^d} \nabla w_\infty \cdot \nabla \varphi \, dx = -\int_{\mathbb{R}^d} \nabla \varphi \cdot F_\infty \, dx,$$

that is $w = w_\infty$.

Finally, in order to prove that the convergence is strong, we use the equations for w_n and w_∞ and the strong convergence of F_n to F_∞:

$$\int_{\mathbb{R}^d} |\nabla w_n|^2 \, dx = -\int_{\mathbb{R}^d} \nabla w_n \cdot F_n \, dx \to -\int_{\mathbb{R}^d} \nabla w_\infty \cdot F_\infty \, dx = \int_{\mathbb{R}^d} |\nabla w_\infty|^2 \, dx,$$

which concludes the proof. □

7.2. Global Stable Solutions of the One-Phase Problem

We define the functionals

$$\delta\mathcal{G} : C^{0,1}(\mathbb{R}) \times C_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R} \quad \text{and} \quad \delta^2\mathcal{G} : C^{0,1}(\mathbb{R}) \times C_c^\infty(\mathbb{R}^d;\mathbb{R}^d) \to \mathbb{R},$$

as follows. Given:

- a Lipschitz function $u : \mathbb{R}^d \to \mathbb{R}$,
- a smooth compactly supported vector field $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

we set:

$$\delta\mathcal{G}(u)[\xi] := \int_{\mathbb{R}^d} \Big(\nabla u \cdot \delta A \nabla u + \mathbb{1}_{\Omega_u} \operatorname{div}\xi\Big)\,dx,$$

$$\begin{aligned} \delta^2\mathcal{G}(u)[\xi] := &\int_{\mathbb{R}^d} 2\nabla u \cdot (\delta^2 A)\nabla u - 2|\nabla(\delta u)|^2\,dx \\ &+ \int_{\mathbb{R}^d} \mathbb{1}_{\Omega_u}\Big((\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\xi)\Big)\,dx, \end{aligned} \tag{7.6}$$

where $\Omega_u := \{u > 0\}$, and where δA, $\delta^2 A$ are defined as in (2.6) and where $\delta u \in \dot H^1_0(\Omega_u)$ is the weak solution to the PDE

$$-\Delta(\delta u) = \operatorname{div}\big((\delta A)\nabla u\big) \quad \text{in } \Omega_u\,, \qquad \delta u \in \dot H^1_0(\Omega_u). \tag{7.7}$$

Definition 7.3. (*Global stable solutions of the one-phase problem*) We say that a function $u : \mathbb{R}^d \to \mathbb{R}$ is a global stable solution of the one-phase problem if, for every compactly supported smooth vector field $\xi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^d)$, we have:

$$\delta\mathcal{G}(u)[\xi] = 0 \quad \text{and} \quad \delta^2\mathcal{G}(u)[\xi] \ge 0, \tag{7.8}$$

and if the following conditions hold:

(a) u is globally Lipschitz continuous and non-negative on $\mathbb{R}^d$;
(b) u is harmonic in the open set $\Omega_u := \{u > 0\}$;
(c) there is a constant $c > 0$ such that

$$|B_r(x_0) \cap \Omega_u| \leqq (1-c)|B_r|,$$

for every $x_0 \in \mathbb{R}^d \setminus \Omega_u$ and every $r > 0$;
(d) $0 \in \partial\Omega_u$ and there is a constant $\eta > 0$ such that

$$\sup_{B_r(x_0)} u \ge \eta r \quad \text{for every} \quad x_0 \in \overline{\Omega}_u \quad \text{and every} \quad r > 0;$$

(e) there is a constant $C > 0$ such that, for every $R > 0$,

$$\left|\int_{B_R} \nabla u \cdot \nabla\varphi\,dx\right| \le C R^{d-1}\|\varphi\|_{L^\infty(B_R)} \quad \text{for every} \quad \varphi \in C_c^\infty(B_R).$$

Remark 7.4. The functionals $\delta\mathcal{G}$ and $\delta^2\mathcal{G}$ correspond to the first and the second variation along vector fields of the one-phase Alt–Caffarelli functional[1]

$$\mathcal{G}(u) = \int_{\mathbb{R}^d} \Big(|\nabla u|^2 + \mathbb{1}_{\{u>0\}}\Big)\, dx,$$

so one may expect that the natural definition of a global stable solution is a function $u : \mathbb{R}^d \to \mathbb{R}$ that satisfies (7.8). Unfortunately, the condition (7.8) alone seems to be quite weak. For instance,

$$u(x, y) = 1 + |x| \quad \text{e} \quad u(x, y) = 1 + |xy|$$

satisfy (7.8), but they are not even harmonic in $\{u > 0\}$, while the function

$$u(x, y) = |xy|,$$

is harmonic in $\{u > 0\}$, but the free boundary $\partial\{u > 0\}$ has a cross-like singularity in zero; more generally, the solutions of optimal partition problems satisfy (7.8) and the structure of their nodal sets can be very different from the one of the one-phase free boundaries.

We add the conditions (a), (b), (c), (d), (e) in order to have a regularity theory for one-phase stable solutions, which is similar to the one available for minimizers of the Alt–Caffarelli functional. The Lipschitz continuity (a) and the non-degeneracy (d) guarantee the existence of non-trivial blow-up limits obtained by 1-homogeneous rescalings of u. The condition (e) is needed for the strong convergence of the blow-up sequences (see Lemma 4.1), which together with the exterior density estimate (c) allows to transfer the stability condition (7.8) to the blow-up limits of u (thanks to Lemma 7.2). We also notice that the conditions (e) and (b) imply the Lipschitz continuity of u, so we could actually avoid adding (a) to the list.

Finally, we highlight that the conditions (a)–(b)–(c)-(d)–(e) are satisfied by the blow-ups of numerous one-phase problems, for instance by the state functions on the domains minimizing (1.3) (see Sect. 4), and of course, by the global minimizers of the classical one-phase Bernoulli problem.

Remark 7.5. (Blow-ups of global stable solutions) We notice that if $u : \mathbb{R}^d \to \mathbb{R}$ satisfies (a), (b), (c), (d) and (e), then any blow-up $u_0 : \mathbb{R}^d \to \mathbb{R}$ of u at $x_0 \in \partial\Omega_u$,

$$u_0 = \lim_{n\to\infty} u_{x_0,r_n} \quad \text{with} \quad u_{x_0,r_n}(x) := \frac{u(x_0 + r_n x)}{r_n} \quad \text{and} \quad \lim_{n\to\infty} r_n = 0,$$

still satisfies (a), (b), (c), (d) and (e). In particular, by Lemma 4.1, this means that the convergence $u_{x_0,r_n} \to u_0$ is strong in H^1_{loc} and thus, by Lemmas 7.2 and 4.5, if u is a global stable solution, then u_0 is a 1-homogeneous global stable solution.

[1] Precisely, since $\mathcal{G}(u) = \infty$ as soon as u has positivity set of infinite measure, we will work with a localized version of $\mathcal{G}$, in which the integration is over compact sets.

Remark 7.6. (Decomposition of the free boundary) Let $u : \mathbb{R}^d \to \mathbb{R}$ be a global stable solution in the sense of Definition 7.3 and let $\Omega_u := \{u > 0\}$. We decompose the free boundary $\partial\Omega_u$ as

$$\partial\Omega_u = \text{Reg}(\partial\Omega_u) \cup \text{Sing}(\partial\Omega_u),$$

where the regular part $\text{Reg}(\partial\Omega_u)$ consists of all points $x_0 \in \partial\Omega_u$ at which there is a blow-up limit u_0 which is a half-space solution, that is,

$$u_0(x) = (x \cdot \nu)_+ \quad \text{for some unit vector} \quad \nu \in \mathbb{R}^d,$$

while the singular part is given by $\text{Sing}(\partial\Omega_u) = \partial\Omega_u \setminus \text{Reg}(\partial\Omega_u)$. As in Sect. 6, it is immediate to check that the global stable solutions satisfy the optimality condition

$$|\nabla u| = 1 \quad \text{on} \quad \partial\Omega_u$$

in viscosity sense, so the ε-regularity theorem of [18] holds and we have that the regular part is a relatively open subset of $\partial\Omega_u$ and a C^∞ manifold. We notice that this decomposition is precisely the one from Sect. 5 with $\alpha = \beta = Q = 1$, $u = v$, and $f = g = 0$.

Definition 7.7. (*Critical dimension for stable solutions*) We define d^* to be the smallest dimension admitting a 1-homogeneous global stable solution (in the sense of Definition 7.3) $u : \mathbb{R}^d \to \mathbb{R}$ with $0 \in \text{Sing}(\partial\Omega_u)$.

Theorem 7.8. (Dimension reduction for stable solutions) *Suppose that $u : \mathbb{R}^d \to \mathbb{R}$ is a 1-homogeneous global stable solution of the one-phase problem (in the sense of Definition 7.3).*

(i) *If $d < d^*$, then u is a half-plane solution.*
(ii) *If $d = d^*$, then $\text{Sing}(\partial\Omega_u) = \{0\}$ or u is a half-plane solution.*
(iii) *If $d > d^*$, then the Hausdorff dimension of $\text{Sing}(\partial\Omega_u)$ is at most $d - d^*$, that is,*

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\text{Sing}(\partial\Omega_u)\big) = 0 \quad \textit{for every} \quad \varepsilon > 0,$$

where d^* is the critical dimension from Definition 7.7.

Proof. The proof follows from a classical argument that can be found for instance in [38] (in particular, [38, Proposition 10.13]). □

Theorem 7.9. (Bounds on the critical dimension for stable solutions) $5 \le d^* \le 7$, *where d^* is the critical dimension from Definition 7.7.*

Proof. The claim follows from Propositions 7.11 and 7.12 below. □

7.3. Global Minimizers and Global Stable Solutions

Given an open set $D \subset \mathbb{R}^d$ and a function $u \in H^1(D)$, we define:

$$\mathcal{G}(u, D) := \int_D \Big(|\nabla u|^2 + \mathbb{1}_{\{u>0\}}\Big)\, dx.$$

Definition 7.10. (*Global minimizers*) We say that a function $u : \mathbb{R}^d \to \mathbb{R}$ is a global minimizer of the Alt–Caffarelli functional, if:

- u is non-negative and $u \in H^1_{loc}(\mathbb{R}^d)$;
- $\mathcal{G}(u, B_R) \leq \mathcal{G}(v, B_R)$, for every $B_R \subset \mathbb{R}^d$ and every $v : \mathbb{R}^d \to \mathbb{R}$ such that $v - u \in H^1_0(B_R)$.

Proposition 7.11. *Suppose that $u : \mathbb{R}^d \to \mathbb{R}$ is a global minimizer in the sense of Definition 7.10. Then, u is a global stable solution in the sense of Definition 7.3. In particular, $d^* \leq 7$, where d^* is the critical dimension from Definition 7.7.*

Proof. It is well-known that the global minimizers satisfy the conditions (a)-(b)-(c)-(d)-(e) of Definition 7.3 (see for instance [38]). Moreover, by [38, Lemma 9.5 and Lemma 9.6], the global minimizers are critical points, that is,

$$\delta\mathcal{G}(u)[\xi] = 0 \qquad \text{for every} \qquad \xi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d).$$

Thus, it only remains to prove the positivity of the second variation:

$$\delta^2\mathcal{G}(u)[\xi] \geq 0 \qquad \text{for every} \qquad \xi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d).$$

Let $\xi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$, Φ_t be the associated defined by (2.5) and let $\Omega_t := \Phi_t(\Omega)$, for $t \in \mathbb{R}$. In B_R, we consider the solution u_t to the problem

$$-\Delta u_t = 0 \quad \text{in} \quad \Omega_t \cap B_R, \qquad u_t = 0 \quad \text{on} \quad \partial\Omega_t \cap B_R, \qquad u_t = u \quad \text{on} \quad \partial B_R.$$

By the optimality of u_t, we have that

$$\mathcal{G}(u_t, B_R) \geq \mathcal{G}(u, B_R) \quad \text{for every} \quad t \in \mathbb{R},$$

so,

$$\frac{d}{dt}\Big|_{t=0}\mathcal{G}(u_t, B_R) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}\Big|_{t=0}\mathcal{G}(u_t, B_R) \geq 0.$$

By Lemma 2.8, we have that

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{G}(u_t, B_R) := \int_{B_R} 2\nabla u \cdot (\delta^2 A)\nabla u - 2|\nabla w_R|^2 + \mathbb{1}_{\Omega_u}\Big((\operatorname{div}\xi)^2 + \xi \cdot \nabla(\operatorname{div}\xi)\Big)\, dx,$$

where $\Omega_u = \{u > 0\}$ and w_R is the solution to the PDE

$$-\Delta w_R = \operatorname{div}\big((\delta A)\nabla u\big) \quad \text{in} \quad \Omega_u \cap B_R \qquad w_R \in H^1_0(\Omega_u \cap B_R).$$

Thus, for every $R > 0$,

$$\int_{\mathbb{R}^d} |\nabla w_R|^2 \, dx \geq \int_{B_R} \nabla u \cdot (\delta^2 A) \nabla u + \frac{1}{2} \mathbb{1}_{\Omega_u} \Big((\operatorname{div} \xi)^2 + \xi \cdot \nabla (\operatorname{div} \xi) \Big) \, dx \, .$$

Since by Lemma 7.2 the sequence $w_R \to \delta u$ strongly in $\dot{H}^1(\mathbb{R}^d)$ as $R \to \infty$, we get that

$$\int_{\mathbb{R}^d} |\nabla (\delta u)|^2 \, dx \geq \int_{B_R} \nabla u \cdot (\delta^2 A) \nabla u + \frac{1}{2} \mathbb{1}_{\Omega_u} \Big((\operatorname{div} \xi)^2 + \xi \cdot \nabla (\operatorname{div} \xi) \Big) \, dx \, ,$$

which is precisely the inequality $\delta^2 \mathcal{G}(u)[\xi] \geq 0$. Finally, the bound $d^* \leq 7$ follows by the example of a singular 1-homogeneous global minimizer constructed by De Silva and Jerison in [19]. □

7.4. Global Stable Solutions and the Stability Inequality of Caffarelli–Jerison–Kenig

Let $u : \mathbb{R}^d \to \mathbb{R}$ be a 1-homogeneous global stable solution of the one-phase problem with an isolated singularity in zero, that is,

$$\operatorname{Sing}(\partial \Omega_u) = \{0\}.$$

In particular, we have that the regular part

$$\operatorname{Reg}(\partial \Omega_u) := \partial \Omega_u \setminus \{0\},$$

is a smooth C^∞ manifold and the function u is C^∞ in $\overline{\Omega_u} \setminus \{0\}$, up to the boundary $\partial \Omega_u \setminus \{0\}$. Thus, u is a classical solution to the PDE

$$\Delta u = 0 \quad \text{in } \Omega_u, \qquad |\nabla u| = 1 \quad \text{on } \partial \Omega_u \setminus \{0\}. \tag{7.9}$$

Together with the homogeneity of u this implies (see for instance [22]) that

$$H > 0 \quad \text{on } \partial \Omega_u \setminus \{0\},$$

where H is the mean curvature of $\partial \Omega_u$ oriented towards the complement of Ω_u.

We will say that Ω_u supports the *stability inequality of Caffarelli-Jerison-Kenig* if

$$\int_{\Omega_u} |\nabla \varphi|^2 \, dx \geq \int_{\partial \Omega_u} H \varphi^2 \, d\mathcal{H}^{d-1} \qquad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^d \setminus \{0\}). \tag{7.10}$$

In [14] and [22] it was shown that if:

- $d = 3$ (see [14]) or $d = 4$ (see [22]);
- $u : \mathbb{R}^d \to \mathbb{R}$ is a 1-homogeneous non-negative Lipschitz function;
- $\partial \Omega_u \setminus \{0\}$ is C^∞ smooth;
- u is a solution of the one-phase Bernoulli problem (7.9);
- Ω_u supports the stability inequality (7.10);

then u is a half space solution, that is,

$$u(x) = (x \cdot \nu)_+ \quad \text{for some unit vector} \quad \nu \in \mathbb{R}^d.$$

Thus, in order to show that in dimension 3 and 4 there are no global stable solutions (in the sense of Definition 7.3) with singularities, it is sufficient to prove the following proposition.

Proposition 7.12. (The global stable cones satisfy the stability inequality) *Let $u : \mathbb{R}^d \to \mathbb{R}$ be a 1-homogeneous global stable solution (in the sense of Definition 7.3) with $Sing(\partial\Omega_u) = \{0\}$. Then, Ω_u supports the stability inequality* (7.10). *In particular, $d^* \geq 5$, where d^* is the critical dimension from Definition 7.7.*

Remark 7.13. Following the proof of Proposition 7.12, it is immediate to check that also the converse is true. Precisely, if $u : \mathbb{R}^d \to \mathbb{R}$ is a 1-homogeneous function, if $\partial\Omega_u \setminus \{0\}$ is smooth and if u is a solution to (7.9) such that Ω_u supports the stability inequality (7.10), then u is a global stable solution in the sense of Definition 7.3.

Proof. For any bounded open set $D \subset \mathbb{R}^d$ and any function $u \in H^1(D)$, we define:

$$\mathcal{G}(u, D) := \int_D |\nabla u|^2\, dx + |D \cap \{u > 0\}|.$$

Moreover, for any $R > 1$, we call the annulus

$$A_R := B_R \setminus \overline{B}_{1/R}.$$

We fix a smooth vector field $\xi \in C_c^\infty(A_R, \mathbb{R}^d)$ and we define the open set

$$\Omega_t := \Phi_t(\Omega_u) \quad \text{for every} \quad t \in \mathbb{R},$$

where Φ_t is the flow of ξ defined by (2.5). Let $u_t : A_R \to \mathbb{R}$ be the solution of the PDE

$$\Delta u_t = 0 \quad \text{in} \quad \Omega_t \cap A_R, \qquad u_t = 0 \quad \text{on} \quad \partial\Omega_t \cap A_R, \qquad u_t = u \quad \text{in} \quad \Omega_t \cap \partial A_R.$$

Step 1. We will show that

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{G}(u_t, A_R) \geq \delta^2 \mathcal{G}(u)[\xi], \tag{7.11}$$

where $\delta^2\mathcal{G}$ is defined in (7.6). Following Lemma 2.8 we have that

$$\begin{aligned} &\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{G}(u_t, A_R) \\ &= \int_{A_R \cap \Omega_u} 2\nabla u \cdot (\delta^2 A)\nabla u + \Big((\operatorname{div} \xi)^2 + \xi \cdot \nabla(\operatorname{div}\, \xi)\Big) - 2|\nabla w_R|^2\, dx. \end{aligned}$$

where δA and $\delta^2 A$ are defined in (2.6) and w_R is the solution of the PDE

$$-\Delta w_R = \operatorname{div}\big((\delta A)\nabla u\big) \quad \text{in } \Omega_u \cap A_R \qquad w_R \in H_0^1(\Omega_u \cap A_R). \tag{7.12}$$

Now, let δu be the solution to (7.7). By the variational characterization of (7.7) in $\dot{H}^1_0(\Omega_u)$, the fact that $w_R \in \dot{H}^1_0(\Omega_u)$, and an integration by parts, we have that

$$
\begin{aligned}
-\frac{1}{2}\int_{B_R} |\nabla(\delta u)|^2\,dx &= \frac{1}{2}\int_{B_R} |\nabla(\delta u)|^2\,dx + \int_{B_R} \nabla(\delta u)\cdot(\delta A)\nabla u\,dx \\
&\le \frac{1}{2}\int_{B_R} |\nabla w_R|^2\,dx + \int_{B_R} \nabla w_R\cdot(\delta A)\nabla u\,dx \\
&= -\frac{1}{2}\int_{B_R} |\nabla w_R|^2\,dx,
\end{aligned}
$$

which gives (7.11).
Step 2. We set u' to be the solution of the PDE

$$
\begin{cases}
\Delta u' = 0 & \text{in} \quad \Omega_u \cap A_R,\\
u' = \xi\cdot\nu & \text{on} \quad \partial\Omega_u \cap A_R,\\
u' = 0 & \text{on} \quad \Omega_u \cap \partial A_R,
\end{cases}
$$

where ν is the outer normal to $\partial\Omega_u$. We will first show that

$$
u' = w_R - \xi\cdot\nabla u.
$$

Indeed, since ξ is supported in A_R and since $\nabla u = -\nu$ on $\partial\Omega_u \cap A_R$, we have:

$$
u' = w_R - \xi\cdot\nabla u \quad \text{on} \quad \partial(\Omega_u \cap A_R).
$$

In order to show that u' is harmonic in Ω_u, we compute (using the repeated index summation convention)

$$
\begin{aligned}
\operatorname{div}\big((\delta A)\nabla u\big) &= \partial_j\big[-\partial_i\xi_j\partial_i u - \partial_j\xi_i\partial_i u + \partial_i\xi_i\partial_j u\big] \\
&= -\partial_{ij}\xi_j\partial_i u - \partial_i\xi_j\partial_{ij}u - \partial_{jj}\xi_i\partial_i u - \partial_j\xi_i\partial_{ij}u + \partial_{ij}\xi_i\partial_j u \\
&= -2\partial_i\xi_j\partial_{ij}u - \partial_{jj}\xi_i\partial_i u = -\partial_j\big[\partial_j\xi_i\partial_i u + \xi_i\partial_{ij}u\big] = -\Delta(\xi\cdot\nabla u),
\end{aligned}
$$

and then we use the equation (7.12) for w_R.
Step 3. We next compute the second derivative of $\mathcal{G}(u_t, A_R)$ in terms of u'. For the sake of simplicity, through the rest of the proof we use the notations introduced in Sect. 1.5.

$$
\begin{aligned}
-\int_{\Omega_u\cap A_R} |\nabla w_R|^2\,dx &= \int_{A_R} -|\nabla(u' + \xi\cdot\nabla u)|^2\,dx \\
&= \int_{\Omega_u\cap A_R} -|\nabla u'|^2 - 2\nabla u'\cdot\nabla(\xi\cdot\nabla u) - |\nabla(\xi\cdot\nabla u)|^2\,dx \\
&= -\int_{\Omega_u\cap A_R}\Big(|\nabla u'|^2 + |\nabla(\xi\cdot\nabla u)|^2\Big)\,dx \\
&\quad - 2\int_{\partial(\Omega_u\cap A_R)} \frac{\partial u'}{\partial\nu}(\xi\cdot\nabla u)\,d\mathcal{H}^{d-1} \\
&= -\int_{\Omega_u\cap A_R}\Big(|\nabla u'|^2 + |\nabla(\xi\cdot\nabla u)|^2\Big)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_{\partial(\Omega_u\cap A_R)} \frac{\partial u'}{\partial \nu} u' \, d\mathcal{H}^{d-1}\\
&= \int_{\Omega_u\cap A_R} \Big(|\nabla u'|^2 - |\nabla(\xi\cdot\nabla u)|^2\Big)\, dx.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\int_{\Omega_u\cap A_R} \nabla u\cdot(\delta^2 A)\nabla u\, dx &= \int_{\Omega_u\cap A_R} |D\xi(\nabla u)|^2 + \nabla u\cdot(D\xi)^2(\nabla u)\, dx\\
&\quad - \int_{\Omega_u\cap A_R} \nabla u\cdot\Big((\xi\cdot\nabla)[D\xi]\Big)\nabla u\, dx\\
&\quad - \int_{\Omega_u\cap A_R} 2 \text{ div } \xi\, \nabla u\cdot D\xi\nabla u\, dx\\
&\quad + \int_{\Omega_u\cap A_R} \frac{1}{2}|\nabla u|^2 \text{div}\Big(\xi \text{ div } \xi\Big)\, dx\\
&= \int_{\Omega_u\cap A_R} |D\xi(\nabla u)|^2 + \nabla u\cdot(D\xi)^2(\nabla u)\, dx\\
&\quad - \int_{\Omega_u\cap A_R} \nabla u\cdot\Big((\xi\cdot\nabla)[\nabla\xi]\Big)\nabla u\, dx\\
&\quad - \int_{\Omega_u\cap A_R} 2 \text{ div } \xi\, \nabla u\cdot\nabla\xi\nabla u\, dx\\
&\quad - \int_{\Omega_u\cap A_R} D^2u(\nabla u)\cdot\xi(\text{div } \xi)\, dx\\
&\quad + \int_{\partial\Omega_u\cap A_R} \frac{1}{2}|\nabla u|^2 \text{ div } \xi(\xi\cdot\nu)\, d\mathcal{H}^{d-1}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
-\nabla u\cdot\Big((\xi\cdot\nabla)[\nabla\xi]\Big)\nabla u &= -\partial_i u\, \xi_k \partial_{ki}\xi_j\, \partial_j u\\
&= -\partial_k\Big[\partial_i u\, \xi_k \partial_i \xi_j\, \partial_j u\Big] + \partial_{ki}u\, \xi_k\partial_i\xi_j\, \partial_j u\\
&\quad + \partial_i u\, \xi_k\partial_i\xi_j\, \partial_{kj}u + (\text{div } \xi)\partial_i u\, \partial_i\xi_j\, \partial_j u
\end{aligned}
$$

and

$$
\begin{aligned}
-D^2u(\nabla u)\cdot\xi(\text{div } \xi) &= -\partial_{ij}u\, \partial_j u\xi_i(\text{div } \xi)\\
&= -\partial_j\Big[\partial_i u\, \partial_j u\xi_i(\text{div } \xi)\Big]\\
&\quad + \partial_i u\, \partial_j u\partial_j\xi_i(\text{div } \xi) + \partial_i u\, \partial_j u\xi_i\partial_j(\text{div } \xi),
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&-\nabla u\cdot\Big((\xi\cdot\nabla)[\nabla\xi]\Big)\nabla u - 2\,\text{div } \xi\, \nabla u\cdot\nabla\xi(\nabla u) - D^2u(\nabla u)\cdot\xi(\text{div } \xi)\\
&\quad = -\partial_k\Big[\partial_i u\, \xi_k\partial_i\xi_j\, \partial_j u\Big] - \partial_j\Big[\partial_i u\, \partial_j u\xi_i(\text{div } \xi)\Big]\\
&\qquad + \partial_{ki}u\, \xi_k\partial_i\xi_j\, \partial_j u + \partial_i u\, \xi_k\partial_i\xi_j\, \partial_{kj}u + \xi_i\, \partial_i u\, \partial_j u\, \partial_j(\text{div } \xi).
\end{aligned}
$$

Similarly, we can compute

$$\begin{aligned} -|\nabla(\xi\cdot\nabla u)|^2 = -\partial_k(\xi_i\partial_i u)\partial_k(\xi_j\partial_j u) &= -(\partial_k\xi_i\partial_i u+\xi_i\partial_{ki}u)(\partial_k\xi_j\partial_j u+\xi_j\partial_{kj}u)\\ &= -|D\xi(\nabla u)|^2-\xi_i\partial_{ki}u\xi_j\partial_{kj}u-2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u\,, \end{aligned}$$

and so

$$\begin{aligned} &|\nabla\xi(\nabla u)|^2+\nabla u\cdot(\nabla\xi)^2(\nabla u)-|\nabla(\xi\cdot\nabla u)|^2\\ &\quad=\partial_i u\,\partial_i\xi_j\partial_j\xi_k\partial_k u-2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u-\xi_i\partial_{ki}u\xi_j\partial_{kj}u\\ &\quad=\partial_i u\,\partial_i\xi_j\partial_j\xi_k\partial_k u-2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u\\ &\qquad-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u+\xi_i\partial_i u\partial_k\xi_j\partial_{kj}u\\ &\quad=\partial_i u\,\partial_i\xi_j\partial_j\xi_k\partial_k u-2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u\\ &\qquad-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u\\ &\qquad+\partial_j\Big[\xi_i\partial_i u\partial_k\xi_j\partial_k u\Big]-\partial_j\xi_i\partial_i u\partial_k\xi_j\partial_k u-\xi_i\partial_{ij}u\partial_k\xi_j\partial_k u-\xi_i\partial_i u\partial_k(\operatorname{div}\xi)\partial_k u. \end{aligned}$$

Finally, by collecting the previous computations, we obtain the identity

$$\begin{aligned} &|\nabla\xi[\nabla u]|^2+\nabla u\cdot(\nabla\xi)^2(\nabla u)-|\nabla(\xi\cdot\nabla u)|^2\\ &\qquad-\nabla u\cdot\Big((\xi\cdot\nabla)[\nabla\xi]\Big)\nabla u-2\operatorname{div}\,\xi\,\nabla u\cdot\nabla\xi(\nabla u)-D^2u(\nabla u)\cdot\xi(\operatorname{div}\,\xi)\\ &\quad=\cancel{\partial_i u\,\partial_i\xi_j\partial_j\xi_k\partial_k u}-2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u\\ &\qquad+\partial_j\Big[\xi_i\partial_i u\partial_k\xi_j\partial_k u\Big]-\cancel{\partial_j\xi_i\partial_i u\partial_k\xi_j\partial_k u}-\xi_i\partial_{ij}u\partial_k\xi_j\partial_k u-\cancel{\xi_i\partial_i u\partial_k(\operatorname{div}\xi)\partial_k u}\\ &\qquad-\partial_k\Big[\partial_i u\,\xi_k\partial_i\xi_j\,\partial_j u\Big]-\partial_j\Big[\partial_i u\,\partial_j u\xi_i(\operatorname{div}\,\xi)\Big]\\ &\qquad+\partial_{ki}u\,\xi_k\partial_i\xi_j\,\partial_j u+\partial_i u\,\xi_k\partial_i\xi_j\,\partial_{kj}u+\cancel{\xi_i\,\partial_i u\,\partial_j u\,\partial_j(\operatorname{div}\,\xi)}\\ &\quad=-\cancel{2\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u}-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\cancel{\partial_k\xi_i\partial_i u\xi_j\partial_{kj}u}\\ &\qquad+\partial_j\Big[\xi_i\partial_i u\partial_k\xi_j\partial_k u\Big]-\xi_i\partial_{ij}u\partial_k\xi_j\partial_k u-\partial_k\Big[\partial_i u\,\xi_k\partial_i\xi_j\,\partial_j u\Big]\\ &\qquad-\partial_j\Big[\partial_i u\,\partial_j u\xi_i(\operatorname{div}\,\xi)\Big]+\cancel{\partial_{ki}u\,\xi_k\partial_i\xi_j\,\partial_j u}+\partial_i u\,\xi_k\partial_i\xi_j\,\partial_{kj}u\\ &\quad=-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\partial_j\Big[\xi_i\partial_i u\partial_k\xi_j\partial_k u\Big]-\cancel{\xi_i\partial_{ij}u\partial_k\xi_j\partial_k u}\\ &\qquad-\partial_k\Big[\partial_i u\,\xi_k\partial_i\xi_j\,\partial_j u\Big]\\ &\qquad-\partial_j\Big[\partial_i u\,\partial_j u\xi_i(\operatorname{div}\,\xi)\Big]+\cancel{\partial_i u\,\xi_k\partial_i\xi_j\,\partial_{kj}u}\\ &\quad=-\partial_k\Big[\xi_i\partial_i u\xi_j\partial_{kj}u\Big]+\partial_j\Big[\xi_i\partial_i u\partial_k\xi_j\partial_k u\Big]-\partial_k\Big[\partial_i u\,\xi_k\partial_i\xi_j\,\partial_j u\Big]\\ &\qquad-\partial_j\Big[\partial_i u\,\partial_j u\xi_i(\operatorname{div}\,\xi)\Big]\\ &\quad=\operatorname{div}\Big(-(\xi\cdot\nabla u)D^2u(\xi)+(\xi\cdot\nabla u)D\xi(\nabla u)\\ &\qquad-(\nabla u\cdot\nabla\xi(\nabla u))\xi-(\xi\cdot\nabla u)(\operatorname{div}\xi)\nabla u\Big). \end{aligned}$$

Thus, integrating by parts and using that $\nabla u = -\nu$ on $\partial\Omega_u \cap A_R$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{G}(u_t, A_R) &= \int_{A_R\cap\Omega_u} \nabla u\cdot(\delta^2 A)\nabla u \\
&\quad + \frac{1}{2}\Big((\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\,\xi)\Big) - |\nabla w_R|^2\,dx \\
&= \int_{\Omega_u\cap A_R} |\nabla u'|^2\,dx + \cancel{\frac{1}{2}\int_{\partial\Omega_u\cap A_R} |\nabla u|^2 \operatorname{div}\xi(\xi\cdot\nu)\,d\mathcal{H}^{d-1}} \\
&\quad + \cancel{\frac{1}{2}\int_{\partial\Omega_u\cap A_R} \operatorname{div}\xi(\xi\cdot\nu)\,d\mathcal{H}^{d-1}} \\
&\quad + \int_{\partial\Omega_u\cap A_R}\Big(-(\xi\cdot\nabla u)D^2u(\xi) + (\xi\cdot\nabla u)D\xi(\nabla u) - (\nabla u\cdot\nabla\xi(\nabla u))\xi \\
&\quad - \cancel{(\xi\cdot\nabla u)(\operatorname{div}\xi)}\nabla u\Big)\cdot\nu\,d\mathcal{H}^{d-1} \\
&= \int_{\Omega_u\cap A_R} |\nabla u'|^2\,dx + \int_{\partial\Omega_u\cap A_R}(\xi\cdot\nu) \\
&\quad \Big(\nu\cdot D^2u(\xi) + \cancel{\nabla u\cdot D\xi(\nabla u)} - \cancel{\nabla u\cdot\nabla\xi(\nabla u)}\Big)\,d\mathcal{H}^{d-1} \\
&= \int_{\Omega_u\cap A_R} |\nabla u'|^2\,dx + \int_{\partial\Omega_u\cap A_R}(\xi\cdot\nu)^2(\nu\cdot D^2u(\nu))\,d\mathcal{H}^{d-1} \\
&= \int_{\Omega_u\cap A_R} |\nabla u'|^2\,dx - \int_{\partial\Omega_u\cap A_R}(\xi\cdot\nu)^2 H\,d\mathcal{H}^{d-1} \\
&= \int_{\Omega_u\cap A_R} |\nabla u'|^2\,dx - \int_{\partial\Omega_u\cap A_R}(u')^2 H\,d\mathcal{H}^{d-1},
\end{aligned}$$

where H is the mean curvature of $\partial\Omega_u\cap A_R$ oriented towards the complement of Ω_u.

Step 4. Conclusion. Given any $\varphi \in C_c^\infty(\mathbb{R}^d\setminus\{0\})$, we consider an annulus A_R containing the support of φ and the vector field $\xi := \varphi\nabla u$. Thus, by the minimality of u' in $\Omega_u \cap A_R$, we get

$$\begin{aligned}
\int_{\Omega_u} |\nabla\varphi|^2\,dx - \int_{\partial\Omega_u}\varphi^2 H\,d\mathcal{H}^{d-1} &\geq \int_{\Omega_u\cap A_R}|\nabla u'|^2\,dx - \int_{\partial\Omega_u\cap A_R}(u')^2 H\,d\mathcal{H}^{d-1} \\
&= \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{G}(u_t, A_R) \geq \frac{1}{2}\delta^2\mathcal{G}(u)[\xi] \geq 0,
\end{aligned}$$

where the last inequality follows from (7.11). □

8. Hausdorff Dimension of Sing$(\partial\Omega)$

In this section we will estimate the dimension of the singular set Sing$(\partial\Omega)$ of a solution Ω to the shape optimization problem (1.3); our main result is the following.

Theorem 8.1. (Dimension of the singular set) *Let D be a bounded open set in $\mathbb{R}^d$ and let f, g, Q be as in Theorem 1.2, namely:*

(a) $f, g \in C^2(D) \cap L^\infty(D)$;
(b) *there are constants* $C_1, C_2 > 0$ *such that* $0 \le C_1 g \le f \le C_2 g$ *in* D;
(c) $Q \in C^2(D)$ *and there are constants* c_Q, C_Q *such that* $0 < c_Q \le Q \leqq C_Q$ *on* D.

Let Ω *be a solution to* (1.3) *and let the singular part* $Sing(\partial\Omega)$ *of the free boundary* $\partial\Omega \cap D$ *be as in Sect. 5. Then, the following holds:*

(i) *If* $d < d^*$, *then* $Sing(\partial\Omega) = \emptyset$.
(ii) *If* $d \ge d^*$, *then the Hausdorff dimension of* $Sing(\partial\Omega)$ *is at most* $d - d^*$, *that is,*

$$\mathcal{H}^{d-d^*+\varepsilon}\big(Sing(\partial\Omega)\big) = 0 \quad \text{for every} \quad \varepsilon > 0.$$

In order to prove Theorem 8.1 above, we will first show that the stability of Ω, expressed in terms of the state functions u_Ω and v_Ω as in Lemma 2.8, passes to a blow-up limit.

Lemma 8.2. (Stability of the blow-up limits) *Let* D *be a bounded open set in* $\mathbb{R}^d$ *and let* f, g, Q *be as in Theorems 1.2 and 8.1. Let* Ω *be an optimal set for* (1.3) *and let* $u := u_\Omega$ *and* $v := v_\Omega$ *be the state functions defined in* (1.1) *and* (1.6). *Suppose that the couple* $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ *is a blow-up limit of* u, v *at a point* $x_0 \in \partial\Omega \cap D$. *Then,*

$$\int_{\Omega_0} \Big(\nabla u_0 \cdot (\delta^2 A)\nabla v_0 - \nabla(\delta u_0)\cdot\nabla(\delta v_0) + Q(x_0)\frac{(\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\xi)}{2}\Big)\,dx \ge 0, \tag{8.1}$$

where $\Omega_0 := \{u_0 > 0\} = \{v_0 > 0\}$, *and* δu_0 *and* δv_0 *are the solutions respectively to the PDEs*

$$\begin{aligned} -\Delta(\delta u_0) &= \operatorname{div}\big((\delta A)\nabla u_0\big) \quad \text{in } \Omega_0\,, \qquad \delta u_0 \in \dot H^1_0(\Omega_0)\,,\\ -\Delta(\delta v_0) &= \operatorname{div}\big((\delta A)\nabla u_0\big) \quad \text{in } \Omega_0\,, \qquad \delta v_0 \in \dot H^1_0(\Omega_0)\,, \end{aligned} \tag{8.2}$$

in the sense explained in Sect. 7.1. In particular, if u_0 *and* v_0 *are proportional, then there is a constant* $\lambda > 0$ *such that* λu_0 *is a global stable solution of the one-phase Bernoulli problem in the sense of Definition 7.3.*

Proof. Let $r_k \to 0$, be a sequence such that

$$u_k(x) := \frac{1}{r_k}u(x_0 + r_k x) \qquad \text{and} \qquad v_k(x) := \frac{1}{r_k}v(x_0 + r_k x),$$

converge respectively to u_0 and v_0 locally uniformly and (by Proposition 4.3) strongly in $H^1_{loc}(\mathbb{R}^d)$. We define:

$$f_k(x) := r_k f(x_0 + r_k x), \quad g_k(x) := r_k g(x_0 + r_k x), \quad Q_k(x) := Q(x_0 + r_k x).$$

Then, the functions u_k and v_k satisfy the PDEs

$$-\Delta u_k = f_k \quad \text{in } \Omega_k \qquad u_k \in H^1_0(\Omega_k),$$

$$-\Delta v_k = g_k \quad \text{in } \Omega_k \qquad v_k \in H^1_0(\Omega_k),$$

where

$$\Omega_k := \frac{1}{r_k}(-x_0 + \Omega).$$

Moreover, Ω_k is optimal in the rescaled domain

$$D_k := \frac{1}{r_k}(-x_0 + D),$$

for the functional

$$\mathcal{F}_k(A) := \int_A \Big(\nabla u_A \cdot \nabla v_A - u_A g_k - v_A f_k + Q_k\Big)\, dx$$

where, this time, by u_A and v_A we denote the solutions to

$$\begin{aligned} -\Delta u_A &= f_k \quad \text{in} \quad A \qquad u_A \in H^1_0(A),\\ -\Delta v_A &= g_k \quad \text{in} \quad A \qquad v_A \in H^1_0(A). \end{aligned}$$

We fix a compactly supported smooth vector field $\xi \in C^\infty_c(\mathbb{R}^d;\mathbb{R}^d)$. Since $r_k \to 0$, for k large enough the support of ξ is contained in D_k and the stability of Ω_k (Lemma 2.8) reads as

$$\int_{\mathbb{R}^d} \Big(\nabla u_k \cdot (\delta^2 A)\nabla v_k - \nabla(\delta u_k)\cdot\nabla(\delta v_k) - (\delta^2 f_k)v_k - (\delta^2 g_k)u_k + \delta^2 Q_k\Big)\, dx \ge 0, \tag{8.3}$$

where δu_k and δv_k are the solutions to

$$\begin{aligned} -\Delta(\delta u_k) &= \operatorname{div}\big((\delta A)\nabla u_k\big) + \delta f_k = \operatorname{div}\big((\delta A)\nabla u_k + f_k\xi\big) \quad \text{in} \quad \Omega_k \qquad \delta u_k \in H^1_0(\Omega_k),\\ -\Delta(\delta v_k) &= \operatorname{div}\big((\delta A)\nabla v_k\big) + \delta g_k = \operatorname{div}\big((\delta A)\nabla v_k + g_k\xi\big) \quad \text{in} \quad \Omega_k \qquad \delta v_k \in H^1_0(\Omega_k), \end{aligned}$$

and where we used the following notation:

- δA and $\delta^2 A$ are the matrices defined in (2.6) (we notice that δA and $\delta^2 A$ are defined in terms of ξ only);
- the variations δf_k and $\delta^2 f_k$ (δg_k and $\delta^2 g_k$ are defined analogously) are given by;

$$\begin{aligned} \delta f_k(x) &:= \operatorname{div}(f_k\xi) = r_k^2\nabla f(x_0 + r_k x)\cdot\xi + r_k f(x_0 + r_k x)\operatorname{div}\xi\,,\\ \delta^2 f_k(x) &:= \frac{r_k^3}{2}\xi\cdot(D^2 f(x_0 + r_k x))\xi + \frac{r_k^2}{2}\nabla f(x_0 + r_k x)\cdot D\xi[\xi]\\ &\qquad + r_k f(x_0 + r_k x)\frac{(\operatorname{div}\xi)^2 + \xi\cdot\nabla[\operatorname{div}\xi]}{2} + r_k^2(\nabla f(x_0 + r_k x)\cdot\xi)\operatorname{div}\xi\,, \end{aligned}$$

- $\delta^2 Q_k$ is given by

$$\delta^2 Q_k(x) := (\xi\cdot\nabla Q_k)\,\mathrm{div}\,\xi + \frac{1}{2}\xi\cdot D^2Q_k\xi + Q_k\frac{1}{2}\Big((\mathrm{div}\,\xi)^2 + \xi\cdot\nabla(\mathrm{div}\,\xi)\Big)$$
$$= r_k(\xi\cdot\nabla Q(x+r_kx))\,\mathrm{div}\,\xi + \frac{r_k^2}{2}\xi\cdot D^2Q(x_0+r_kx)\xi$$
$$+\frac{1}{2}Q(x_0+r_k)\Big((\mathrm{div}\,\xi)^2 + \xi\cdot\nabla(\mathrm{div}\,\xi)\Big),$$

where in all the formulas above, the field ξ and its derivatives are all computed in x.

We notice that δf_k and $\delta^2 f_k$ vanish outside the support of ξ. Moreover, since f and g are C^2, we get that δf_k and $\delta^2 f_k$ converge to zero uniformly. Similarly, since Q is C^2, we get that

$$\delta^2 Q_k \to \frac{1}{2}Q(x_0)\Big((\mathrm{div}\,\xi)^2 + \xi\cdot\nabla(\mathrm{div}\,\xi)\Big)\quad\text{strongly in}\quad L^1(\mathbb{R}^d).$$

Next, we notice that by Proposition 4.3, u_k and v_k converge strongly in $H^1_{loc}(\mathbb{R}^d)$ to the blow-up limits u_0, v_0. Thus, since the support of $\delta^2 A$ is compact, we get

$$\lim_{k\to\infty}\int_{\mathbb{R}^d}\nabla u_k\cdot(\delta^2 A)\nabla v_k\,dx = \int_{\mathbb{R}^d}\nabla u_0\cdot(\delta^2 A)\nabla v_0\,dx.$$

Similarly, we have that

$$(\delta A)\nabla u_k + f_k\xi \to (\delta A)\nabla u_0 \qquad\text{and}\qquad (\delta A)\nabla v_k + g_k\xi \to (\delta A)\nabla v_0,$$

strongly in $L^2(\mathbb{R}^d;\mathbb{R}^d)$. Thus, by Lemma 7.2, we get that δu_k and δv_k converge respectively to the solutions δu_0 and δv_0 of (8.2). Now, (8.1) follows by passing to the limit (8.3). $\square$

Proof of Theorem 8.1. The strategy is similar to the one for minimizers of the one-phase problem. We split the proof in two different cases.

Case 1: $d < d^*$. Let $x_0 \in \partial\Omega_u \cap D$ and $r_k \to 0^+$ be the infinitesimal sequence such that the rescalings u_{x_0,r_k} and v_{x_0,r_k} converge to the 1-homogeneous blow-up limits u_0 and v_0 given by Proposition 4.7. Then, given $\lambda \in (C_1, C_2)$ as in Proposition 4.7, the blow-up limits

$$u_0 := \lim_{k\to\infty}\frac{1}{\sqrt{\lambda}\,Q(x_0)}u_{x_0,r_k} \qquad\text{and}\qquad v_0 := \lim_{k\to\infty}\sqrt{\lambda}\,v_{x_0,r_k} \tag{8.4}$$

coincide up to a multiplicative constant (that is $v_0 = Q(x_0)v_0$) and, by Lemma 8.2 and Proposition 4.7, u_0 is a 1-homogeneous stable solutions of the one-phase problem in the sense of Definition 7.3. Therefore, by definition of d^*, $\mathrm{Sing}(\partial\Omega_{u_0}) = \emptyset$ and u_0 is a half space solution:

$$u_0(x) = (x\cdot\nu)_+ \quad\text{for some unit vector}\quad \nu\in\mathbb{R}^d.$$

Finally, by rewriting the result in terms of the blow-up sequence of Proposition 4.7, we deduce the existence $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta = Q(x_0)$ such that

$$\frac{1}{r_k}u(x_0 + r_k x) \to \alpha(x \cdot \nu)_+, \quad \text{and} \quad \frac{1}{r_k}v(x_0 + r_k x) \to \beta(x \cdot \nu)_+$$

as $k \to \infty$. Thus, by definition, $x_0 \in \operatorname{Reg}(\partial\Omega)$. Since this is true at every free boundary point $x_0 \in \partial\Omega \cap D$, we get that $\operatorname{Sing}(\partial\Omega) = \emptyset$.

Case 2: $d \geqq d^*$. Given $\varepsilon > 0$, let us prove that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega)\big) = 0.$$

We will apply consecutively the three blow-ups from Sect. 4. By contradiction, assume that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega)\big) \geq C > 0.$$

Therefore, by [37, Lemma 10.5], there are $x_0 \in \operatorname{Sing}(\partial\Omega)$ and a sequence $r_k \to 0$ such that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega) \cap B_{r_k}(x_0)\big) \geq C r_k^{d-d^*+\varepsilon}.$$

By the first blow-up analysis from Sect. 4.2 (see Proposition 4.3), we deduce that the blow-up sequences u_{x_0,r_k}, v_{x_0,r_k} and Ω_{x_0,r_k} converge to some limits u_0, v_0 and Ω_0 such that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega_0) \cap B_1\big) \geq C,$$

where u_0, v_0 and Ω_0 are as in Proposition 4.3 and where u_0 and v_0 satisfy the stability condition (8.1) from Lemma 8.2. By applying again [37, Lemma 10.5], there exists a point $x_{00} \in \operatorname{Sing}(\partial\Omega_0)$ and another sequence $r_k \to 0^+$ such that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega_0) \cap B_{r_k}(x_{00})\big) \geq C r_k^{d-d^*+\varepsilon}.$$

Hence, by applying the second blow-up analysis of Sect. 4.3 to the sequences

$$\frac{1}{r_k\sqrt{\lambda}\,Q(x_0)}u_0(x_{00} + r_k x), \qquad \frac{\sqrt{\lambda}}{r_k}v_0(x_{00} + r_k x) \quad \text{and} \quad \frac{1}{r_k}(\Omega_0 - x_{00}),$$

we get that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega_{00}) \cap B_1\big) \geq C,$$

where $v_{00} = Q(x_0)u_{00}$ and u_{00} satisfies (2)-(3)-(4)-(5)-(6) in Lemma 4.4. Moreover, by the strong convergence of the blow-up sequence, u_{00} and v_{00} still satisfy (8.1), so u_{00} is a global stable solution of the one-phase problem in the sense of Definition 7.3. Finally, by applying for the last time [37, Lemma 10.5], we deduce that

$$\mathcal{H}^{d-d^*+\varepsilon}\big(\operatorname{Sing}(\partial\Omega_{000}) \cap B_1\big) \geq C, \tag{8.5}$$

in which $\Omega_{000} = \{u_{000} > 0\}$ and u_{000} is a blow-up limit of u_{00} at some free boundary point $x_{000} \in \partial\Omega_{00}$. But now u_{000} is a 1-homogeneous stable solution of the one-phase problem in the sense of Definition 7.3, so (8.5) contradicts Theorem 7.8. $\square$

9. Existence of Optimal Sets in $\mathbb{R}^d$ and Proof of Theorem 1.6

In this section we prove Theorem 1.6; we prove the existence of an optimal set for (1.4) and then we show how to obtain the regularity of the optimal sets as a consequence from Theorem 1.2.

9.1. Statement of the Problem in the Class of Measurable Sets

One can not obtain the existence of optimal sets (for (1.4) or (1.3)) directly in the class of open sets. Thus, we extend the definition of the shape functional to the class of measurable sets. Precisely, if Ω is a Lebesgue measurable set of finite measure in $\mathbb{R}^d$, we define the space $\widetilde{H}_0^1(\Omega)$ of all $H^1(\mathbb{R}^d)$ functions vanishing (Lebesgue-)almost-everywhere outside Ω. We will say that u is a (weak) solution to the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u \in \widetilde{H}_0^1(\Omega), \tag{9.1}$$

if $u \in \widetilde{H}_0^1(\Omega)$ and

$$\int_\Omega \nabla v \cdot \nabla u \, dx = \int_\Omega vf \, dx \quad \text{for every} \quad v \in \widetilde{H}_0^1(\Omega). \tag{9.2}$$

It is immediate to check that u satisfies (9.2) if and only if

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \int_\Omega uf(x) \, dx \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla v|^2 \, dx - \int_\Omega vf(x) \, dx \quad \text{for every } v \in \widetilde{H}_0^1(\Omega). \tag{9.3}$$

From now on, we will denote the unique solution to (9.1) by $\widetilde{u}_\Omega$.

We will first prove the following lemma, which in particular implies that for every open set $\Omega \subset \mathbb{R}^d$ of finite measure and non-negative functions $f, g \in L^2(\Omega)$, we have

$$\int_\Omega \Big(- g(x)\widetilde{u}_\Omega + Q(x)\Big) \, dx \leq \int_\Omega \Big(- g(x)u_\Omega + Q(x)\Big) \, dx,$$

where u_Ω is the weak solution to (1.1).

Lemma 9.1. *Let Ω be a bounded open set in $\mathbb{R}^d$ and $f \in L^2(\Omega)$ a non-negative function. Then*

$$0 \leq u_\Omega \leq \widetilde{u}_\Omega, \tag{9.4}$$

where $\widetilde{u}_\Omega \in \widetilde{H}_0^1(\Omega)$ is the solution to (9.1) and $u_\Omega \in H_0^1(\Omega)$ is the weak solution to (1.1).

Moreover, if Ω satisfies the following exterior density estimate:

$$\begin{aligned} &\text{there are constants } r_0 > 0 \text{ and } c > 0 \text{ such that} \\ &\quad \text{for every } x_0 \in \partial\Omega \text{ and every } r \leq r_0 \quad |B_r(x_0) \setminus \Omega| \geq c|B_r|, \end{aligned} \tag{9.5}$$

then $H_0^1(\Omega) = \widetilde{H}_0^1(\Omega)$ and $u_\Omega = \widetilde{u}_\Omega$.

Proof. First we notice that u_Ω and $\widetilde{u}_\Omega$ are the unique minimizers of the functional (9.3) in $H^1_0(\Omega)$ and $\widetilde{H}^1_0(\Omega)$. Since $f \geq 0$, by testing the optimality of u_Ω with $u_\Omega \vee 0 \in H^1_0(\Omega)$ and the optimality of $\widetilde{u}_\Omega$ with $\widetilde{u}_\Omega \vee 0 \in \widetilde{H}^1_0(\Omega)$, we get that $u_\Omega \geq 0$ and $\widetilde{u}_\Omega \geq 0$ in Ω. Now, a standard argument (see for instance [38, Lemma 2.6]) gives that $\Delta u_\Omega + f \geq 0$ on $\mathbb{R}^d$ in the sense of distributions, precisely:

$$-\int_{\mathbb{R}^d} \nabla u_\Omega \cdot \nabla \varphi \, dx + \int_{\mathbb{R}^d} \varphi f \, dx \geq 0 \quad \text{for every } \varphi \geq 0,\ \varphi \in H^1(\mathbb{R}^d). \tag{9.6}$$

Next, we notice that for every open set Ω, we have the inclusion $H^1_0(\Omega) \subset \widetilde{H}^1_0(\Omega)$. Thus, if we take φ to be the negative part of $\widetilde{u}_\Omega - u_\Omega$,

$$\varphi := -\Big((\widetilde{u}_\Omega - u_\Omega) \wedge 0\Big),$$

we have that $\varphi \in \widetilde{H}^1_0(\Omega)$. So, using the (weak) equation for $\widetilde{u}_\Omega$, we get

$$\int_{\mathbb{R}^d} \nabla \varphi \cdot \nabla \widetilde{u}_\Omega \, dx = \int_\Omega \varphi \, f(x) \, dx$$

On the other hand, by the positivity of φ and (9.6),

$$\int_{\mathbb{R}^d} \nabla \varphi \cdot \nabla u_\Omega \, dx \leq \int_\Omega \varphi \, f(x) \, dx.$$

Combining the two, we obtain

$$\int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dx = -\int_{\mathbb{R}^d} \nabla \varphi \cdot \nabla (\widetilde{u}_\Omega - u_\Omega) \, dx \leq 0,$$

which gives that $\varphi \equiv 0$ and so (9.4) holds. Finally, it is known (see for instance [17, Proposition 4.7]) that in the presence of the density estimate (9.5), we have that $H^1_0(\Omega) = \widetilde{H}^1_0(\Omega)$. Thus, the equality $u_\Omega = \widetilde{u}_\Omega$ follows by the fact that the minimizer of the functional in (9.3) is unique. □

9.2. Existence of Optimal Measurable Sets in $\mathbb{R}^d$

For all measurable set $\Omega \subset \mathbb{R}^d$, we set

$$\widetilde{J}(\Omega) := \int_\Omega -g(x) u_\Omega + Q(x) \, dx.$$

We can prove the following existence result for the minimization of $\widetilde{J}$.

Lemma 9.2. *If $d \geq 3$, let $f, g \in L^2(\mathbb{R}^d)$, while if $d = 2$, let $f, g \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be nonnegative functions, and let $Q : \mathbb{R}^d \to \mathbb{R}$ be a measurable function bounded from below by a positive constant $c_Q > 0$. Then, the following variational problem has a solution*

$$min\left\{ \int_\Omega \Big(-g(x)\widetilde{u}_\Omega + Q(x) \Big) \, dx \ :\ \Omega \subset \mathbb{R}^d,\ \Omega \text{ measurable},\ |\Omega| < +\infty \right\}. \tag{9.7}$$

Proof. Let $d \geq 3$, let Ω_n be a minimizing sequence of sets of finite Lebesgue measure for (9.7). We set $u_n := \widetilde{u}_{\Omega_n}$. By the Poincaré inequality, the equation for u_n and the Hölder inequality, there is a dimensional constant C_d such that

$$\begin{aligned} \|u_n\|^2_{L^2(\Omega_n)} &\leq C_d|\Omega_n|^{2/d} \int_{\Omega_n} |\nabla u_n|^2 \, dx \\ &= C_d|\Omega_n|^{2/d} \int_{\Omega_n} u_n f \, dx \leq C_d|\Omega_n|^{2/d} \|f\|_{L^2(\mathbb{R}^d)} \|u_n\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Thus,

$$\|u_n\|_{L^2(\Omega_n)} \leq C_d|\Omega_n|^{2/d} \|f\|_{L^2(\mathbb{R}^d)},$$

and so (since we can suppose that $\widetilde{J}(\Omega_n) \leq \widetilde{J}(\emptyset) = 0$), we get

$$\begin{aligned} 0 \geq \widetilde{J}(\Omega_n) &= \int_{\Omega_n} \Big(- g(x)u_n + Q(x) \Big) \, dx \\ &\geq -C_d|\Omega_n|^{2/d} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)} + c_Q|\Omega_n|, \end{aligned}$$

which implies that the sequence of measures $|\Omega_n|$ is bounded

$$|\Omega_n|^{\frac{d-2}{d}} \leq \frac{C_d}{c_Q} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}. \tag{9.8}$$

If $d = 2$, to obtain a bound on the measure analogous to (9.8), we need to use a generalized Poincaré-Sobolev (or generalized Faber–Krahn) inequality (see for example [3, equation (1.2)]) instead of the classical one, namely, for $2 < q < +\infty$, there exists a dimensional constant $\widetilde{C}_2$ such that (using also Hölder inequality)

$$\begin{aligned} \|u_n\|^2_{L^q(\Omega_n)} &\leq \widetilde{C}_2|\Omega_n|^{1+\frac{2-q}{q}} \int_{\Omega_n} |\nabla u_n|^2 \, dx = \widetilde{C}_2|\Omega_n|^{1+\frac{2-q}{q}} \int_{\Omega_n} u_n f \, dx \\ &\leq \widetilde{C}_2|\Omega_n|^{1+\frac{2-q}{q}} \|f\|_{L^{q'}(\mathbb{R}^2)} \|u_n\|_{L^q(\Omega_n)}. \end{aligned}$$

We immediately deduce that

$$\|u_n\|_{L^q(\Omega_n)} \leq \widetilde{C}_2|\Omega_n|^{1+\frac{2-q}{q}} \|f\|_{L^{q'}(\mathbb{R}^2)}, \tag{9.9}$$

and again supposing that $\widetilde{J}(\Omega_n) \leq \widetilde{J}(\emptyset) = 0$, we obtain, using (9.9)

$$\begin{aligned} 0 \geq \widetilde{J}(\Omega_n) &= \int_{\Omega_n} \Big(- g(x)u_n + Q(x) \Big) \, dx \\ &\geq -\widetilde{C}_2|\Omega_n|^{1+\frac{2-q}{q}} \|f\|_{L^{q'}(\mathbb{R}^2)} \|g\|_{L^{q'}(\mathbb{R}^2)} + c_Q|\Omega_n|, \end{aligned}$$

which, since $q > 2$ implies that the sequence of measures $|\Omega_n|$ is bounded, namely

$$|\Omega_n|^{\frac{q-2}{q}} \leq \frac{\widetilde{C}_2}{c_Q} \|f\|_{L^{q'}(\mathbb{R}^2)} \|g\|_{L^{q'}(\mathbb{R}^2)}.$$

Now the proof continues in the same way both for $d = 2$ and $d \geqq 3$. Using again the equation for u_n we get that the sequence u_n is bounded in $H^1(\mathbb{R}^d)$. Thus, up to pass to a subsequence, we have that u_n converges weakly in $H^1_{\rm loc}(\mathbb{R}^d)$ and strongly in $L^2_{\rm loc}(\mathbb{R}^d)$ (and, up to another subsequence, also pointwise a.e. in $\mathbb{R}^d$) to a certain $u \in H^1_{\rm loc}(\mathbb{R}^d)$. Now, for all $R > 0$

$$|\{u \neq 0\} \cap B_R| \leq \liminf_{n\to\infty} |\{u_n \neq 0\} \cap B_R| \leq \liminf_{n\to\infty} |\Omega_n \cap B_R| \leq \liminf_{n\to\infty} |\Omega_n|.$$

Taking a supremum over $R > 0$, we obtain that

$$|\{u \neq 0\}| \leqq \liminf_{n\to\infty} |\Omega_n|,$$

and as a consequence (using Fatou Lemma),

$$\int_{\{u\neq 0\}} Q(x)\,dx \leqq \liminf_{n\to+\infty} \int_{\Omega_n} Q(x)\,dx,$$

Then one can prove also that $u \in H^1(\mathbb{R}^d) \cap \widetilde{H}^1_0(\{u \neq 0\})$, as, for all $R > 0$,

$$\int_{B_R} |\nabla u|^2\,dx \leqq \liminf_{n\to+\infty} \int_{B_R\cap\Omega_n} |\nabla u_n|^2\,dx \leqq \liminf_{n\to+\infty} \int_{\Omega_n} |\nabla u_n|^2\,dx,$$

and then we can take the supremum over $R > 0$. We note that from the pointwise a.e. convergence and the fact that $u_n \geqq 0$ (since $f \geqq 0$), we deduce $u \geqq 0$. As a consequence, using also that $g \geqq 0$, for all $R > 0$ we have

$$-\int_{\{u\neq 0\}} gu\,dx \leqq -\int_{\{u\neq 0\}\cap B_R} gu\,dx \leqq \liminf_{n\to+\infty} -\int_{\Omega_n\cap B_R} gu_n\,dx.$$

We then conclude that $\Omega = \{u \neq 0\}$ is a solution to (9.7). □

Lemma 9.3. *Let $d \geq 2$, $f, g \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be non-negative, and let $Q : \mathbb{R}^d \to \mathbb{R}$ be a measurable function bounded from below by a positive constant $c_Q > 0$. If the measurable set $\Omega \subset \mathbb{R}^d$, $|\Omega| < +\infty$, is a solution to (9.7), then Ω is bounded.*

Proof. For any set $E \subset \mathbb{R}^d$ of finite measure and any function $h \in L^2(E)$, we will denote by $\widetilde{R}_E(h)$ the unique solution to the problem

$$-\Delta u = h \quad \text{in} \quad E \qquad u \in \widetilde{H}^1_0(E).$$

We have that $\widetilde{R}_E$ is linear, $\widetilde{R}_E(h_1 + h_2) = \widetilde{R}_E(h_1) + \widetilde{R}_E(h_2)$ and positive: if $h \geq 0$, then $\widetilde{R}_E(h) \geq 0$. Moreover, by the weak maximum principle, if $E_1 \subset E_2$ and $h \geq 0$, then $\widetilde{R}_{E_2}(h) \geq \widetilde{R}_{E_1}(h)$.

Let ω be any measurable set contained in Ω. Then, the minimality of Ω gives that

$$\int_\Omega \Big(- g(x)\widetilde{R}_\Omega(f) + Q(x)\Big)\,dx \leq \int_\omega \Big(- g(x)\widetilde{R}_\omega(f) + Q(x)\Big)\,dx.$$

So, rearranging the terms and using the positivity of f and g, and the inequality

$$0 \le \widetilde{R}_\Omega(f) - \widetilde{R}_\omega(f) \le \|f\|_{L^\infty}\Big(\widetilde{R}_\Omega(1) - \widetilde{R}_\omega(1)\Big),$$

we get

$$\begin{aligned} c_Q|\Omega \setminus \omega| &\le \int_\Omega Q(x)\,dx - \int_\omega Q(x)\,dx \le \int_\Omega g(x)\Big(\widetilde{R}_\Omega(f) - \widetilde{R}_\omega(f)\Big)\,dx \\ &\le \|f\|_{L^\infty}\int_\Omega g(x)\Big(\widetilde{R}_\Omega(1) - \widetilde{R}_\omega(1)\Big)\,dx = \|f\|_{L^\infty}\int_\Omega \Big(\widetilde{R}_\Omega(g) - \widetilde{R}_\omega(g)\Big)\,dx \\ &\le \|f\|_{L^\infty}\|g\|_{L^\infty}\int_\Omega \Big(\widetilde{R}_\Omega(1) - \widetilde{R}_\omega(1)\Big)\,dx\,. \end{aligned}$$

Finally, rearranging the terms again, we get that

$$-\frac{1}{2}\int_\Omega \widetilde{R}_\Omega(1)\,dx + \frac{c_Q}{2\|f\|_{L^\infty}\|g\|_{L^\infty}}|\Omega| \le -\frac{1}{2}\int_\omega \widetilde{R}_\omega(1)\,dx + \frac{c_Q}{2\|f\|_{L^\infty}\|g\|_{L^\infty}}|\omega|,$$

so the set Ω is inwards minimizing (or, in terms of [9], a shape subsolution) for the functional

$$\mathcal{E}(\Omega) = -\frac{1}{2}\int_\Omega \widetilde{R}_\Omega(1)\,dx + \frac{c_Q}{2\|f\|_{L^\infty}\|g\|_{L^\infty}}|\Omega|.$$

Thus, applying [9, Theorem 3.13], for every

$$0 < \eta \le \frac{c_Q}{2\|f\|_{L^\infty}\|g\|_{L^\infty}},$$

the set Ω is contained in an open set $A \subset \mathbb{R}^d$ obtained as a finite union of N balls $B_\rho(x_i)$, $i = 1, \dots, N$, with N and ρ depending only on the dimension d and η. In particular, A is bounded and the diameter of any connected component of A is at most $N\rho$. □

9.3. Proof of Theorem 1.6

The existence of an optimal set is a consequence of Proposition 9.4 below, while the regularity follows from Theorem 1.2.

Proposition 9.4. *In $\mathbb{R}^d$, $d \ge 2$, let $f, g, Q : \mathbb{R}^d \to \mathbb{R}$ be non-negative functions. Suppose that:*

(a) *$f, g \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ and that $f > 0$ and $g > 0$ on $\mathbb{R}^d$;*
(b) *there is a constant $c_Q > 0$ such that $c_Q \le Q$ on $\mathbb{R}^d$.*

Then, there exists a solution $\Omega \subset \mathbb{R}^d$ to the shape optimization problem (1.4). Moreover, every solution to (1.4) is also a solution to (9.7).

Proof. By Lemma 9.2, there is a measurable set Ω (of finite measure) that minimizes (9.7). By Lemma 9.3, Ω is contained in some ball $B_R \subset \mathbb{R}^d$. Since f and g are continuous and strictly positive on $\overline{B}_R$, we can find positive constants C_1, C_2 such that $C_1 g \le f \le C_2 g$ on $\overline{B}_R$. Thus, reasoning as in Proposition 3.1 and Corollary 3.3 we get that the set

$$A := \{\widetilde{u}_\Omega > 0\}$$

is open. Now, the optimality of Ω gives that $|\Omega \Delta A| = 0$, so we have $\widetilde{u}_A = \widetilde{u}_\Omega$. Moreover, by Proposition 3.5, A satisfies the exterior density estimate from Lemma 9.1. Thus, $u_A = \widetilde{u}_A$. In order to check that the open set A minimizes (1.4), we notice that, for any open set $E \subset \mathbb{R}^d$,

$$\begin{aligned}\int_A \Big(- g(x) u_A + Q(x) \Big)\, dx &= \int_A \Big(- g(x) \widetilde{u}_A + Q(x) \Big)\, dx \\ &\le \int_E \Big(- g(x) \widetilde{u}_E + Q(x) \Big)\, dx \\ &\le \int_E \Big(- g(x) u_E + Q(x) \Big)\, dx\,.\end{aligned}$$

Moreover, by the same chain of inequalities, we obtain that if the open set E is a solution to (1.4), then it also minimizes (9.7). □

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Declarations

Conflict of interest The authors have no conflict of interest to declare.

Appendix A. An Optimization Problem in Heat Conduction

As a consequence of our analysis from Sects. 2, 4 and 7, we obtain an estimate on the dimension of the singular part of the boundary of the optimal sets arising in a heat conduction problem studied by Aguilera, Caffarelli and Spruck in [1].

A.1 Statement of the Problem and Known Results

Let D be a smooth bounded open set in $\mathbb{R}^d$. Let $\phi : \partial D \to \mathbb{R}$ be a smooth function on ∂D bounded from below and above by positive constants $0 < c \le C$. For every open set $\Omega \subset D$, for which the set $K := D \setminus \Omega$ is a compact subset of D, we consider the state function $u_\Omega \in H^1(D)$ solution to the problem

$$\Delta u_\Omega = 0 \quad \text{in } \Omega, \qquad u_\Omega = \phi \quad \text{on } \partial D, \qquad u_\Omega \equiv 0 \quad \text{on } K = D \setminus \Omega, \tag{A.1}$$

and we define the functional

$$\mathcal{F}(\Omega) = \int_{\partial D} \frac{\partial u_\Omega}{\partial \nu}\, d\mathcal{H}^{d-1} + \Lambda|\Omega|$$

where $\Lambda > 0$ is a real number and ν is the outer unit normal to ∂D. In [1] it was shown that there is a solution to the following shape optimization problem:

$$\min\Big\{\mathcal{F}(\Omega)\ \ \Omega \subset D;\ \ \Omega\ -\ \text{open};\ \ D \setminus \Omega\ -\ \text{compact subset of } D\Big\}, \tag{A.2}$$

and that for every solution Ω to (A.2) the following holds.

- The state function u_Ω is Lipschitz continuous in D and smooth in a neighborhood of the fixed boundary ∂D.
- u_Ω is non-degenerate in the sense that there is a constant $\eta > 0$, for which

$$\frac{1}{r^{d-1}} \int_{\partial B_r(x)} u_\Omega\, d\mathcal{H}^{d-1} \ge \eta r \quad \text{whenever} \quad x \in \overline{\Omega} \quad \text{and} \quad B_r(x) \subset D.$$

- There is $\varepsilon > 0$ such that, for every x on the boundary of the set $K := D \setminus \Omega$, we have:

$$\varepsilon_0|B_r| \le |\Omega \cap B_r(x)| \le (1-\varepsilon_0)|B_r| \quad \text{whenever} \quad B_r(x) \subset D.$$

- There is a constant $\tilde{M} > 0$ such that, for every $B_{2r}(x_0) \subset D$, we have the bound

$$0 \le \left|\int_{B_{2r}(x_0)} \nabla u_\Omega \cdot \nabla \varphi\, dx\right| \le \tilde{M}\|\varphi\|_{L^\infty(B_{2r}(x_0))} \quad \text{for every} \quad \varphi \in C_c^{0,1}(B_{2r}(x_0)),$$

  where $C_c^{0,1}(B_{2r}(x_0))$ is the space of Lipschitz functions with compact support in $B_{2r}(x_0)$.

Then, they showed that the reduced boundary $\partial^*\Omega \cap D$ is C^∞ smooth and that

$$|\nabla u_\Omega||\nabla v_\Omega| = \Lambda \quad \text{on } \partial^*\Omega \cap D, \tag{A.3}$$

where v_Ω is the solution to the PDE

$$\Delta v_\Omega = 0 \quad \text{in } \Omega, \qquad v_\Omega = 1 \quad \text{on } \partial D, \qquad v_\Omega \equiv 0 \quad \text{on } K = D \setminus \Omega. \tag{A.4}$$

In Theorem A.4, we will improve this result in low dimension ($d \geq 4$) by showing that the whole free boundary $\partial\Omega \cap D$ is smooth.

Remark A.1. In the problem originally considered by Aguilera, Caffarelli and Spruck in [1], the minimization of $\mathcal{F}$ is among all domains with prescribed measure. Thus, all the results from [1] apply to (A.2) with the only difference that the constant Λ from (A.3) is an unknown positive Lagrange multiplier. In this last section of the present paper, we choose to work with the penalized version (A.2) since it allows to apply all the results from Sects. 2, 7 and 8 directly, without the need to add technical details related to the Lagrange multiplier.

A.2 First and Second Variation

By an integration by parts, we can rewrite $\mathcal{F}(\Omega)$ as

$$\mathcal{F}(\Omega) := \int_\Omega \nabla u_\Omega \cdot \nabla v_\Omega \, dx + \Lambda|\Omega|,$$

where u_Ω and v_Ω are given by (A.1) and (A.4). Now, let $\xi \in C_c^\infty(D;\mathbb{R}^d)$ be a smooth compactly supported vector field in D; let $\Phi_t : D \to D$, $t \in \mathbb{R}$, be the flow associated to ξ and let $\Omega_t := \Phi_t(\Omega)$. Reasoning as in Sect. 7, we get that

$$\frac{\partial}{\partial t}\Big|_{t=0} \mathcal{F}(\Omega_t) = \int_\Omega \Big(\nabla u_\Omega \cdot (\delta A)\nabla v_\Omega + \Lambda \operatorname{div}\xi\Big)\, dx\,,$$

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\Big|_{t=0} \mathcal{F}(\Omega_t) = \int_\Omega \Big(\nabla u_\Omega \cdot (\delta^2 A)\nabla v_\Omega - \nabla(\delta u)\cdot\nabla(\delta v) + \frac{\Lambda}{2}\big((\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\,\xi)\big)\Big) dx\,,$$

where δA and $\delta^2 A$ are given by (2.6), and where δu and δv are the solutions to the PDEs

$$-\Delta(\delta u) = \operatorname{div}\big((\delta A)\nabla u_\Omega\big) \quad \text{in } \Omega\,, \qquad \delta u \in H_0^1(\Omega)\,;$$
$$-\Delta(\delta v) = \operatorname{div}\big((\delta A)\nabla v_\Omega\big) \quad \text{in } \Omega\,, \qquad \delta v \in H_0^1(\Omega)\,.$$

Using the blow-up analysis from Sect. 4 and the argument from Lemma 8.2, we get

Lemma A.2. *Let Ω be an optimal set for* (A.2) *and let $u := u_\Omega$ and $v := v_\Omega$ be the state functions from* (A.1) *and* (A.4)*. Suppose that $x_0 \in \partial\Omega \cap D$ and that $r_k \to 0$ are such that*

$$\lim_{k\to\infty} u_{x_0,r_k} = u_0 \quad \text{and} \quad \lim_{k\to\infty} v_{x_0,r_k} = v_0$$

where u_0, v_0 are non-negative Lipschitz functions on $\mathbb{R}^d$ and both limits are locally uniform in $\mathbb{R}^d$, and where, as usual, we set:

$$u_{x_0,r_k}(x) := \frac{1}{r_k}u(x_0 + r_k x) \quad \text{and} \quad v_{x_0,r_k}(x) := \frac{1}{r_k}v(x_0 + r_k x).$$

Then, u_0 and v_0 are proportional and there is a constant $\lambda > 0$ such that λu_0 is a global stable solution of the one-phase Bernoulli problem in the sense of Definition 7.3.

Proof. By the analysis of the blow-up sequences from Sect. 4, we get that u_{x_0,r_k} and v_{x_0,r_k} converge strongly in $H^1_{loc}(\mathbb{R}^d)$ respectively to u_0 and v_0, and also that the sets $\frac{1}{r_k}(\Omega - x_0)$ converge (in the local Hausdorff distance) to $\Omega_0 = \{u_0 > 0\} = \{v_0 > 0\}$. Then, as in Lemma 8.2, for every smooth compactly supported vector field $\xi \in C^\infty_c(\mathbb{R}^d; \mathbb{R}^d)$, the first and the second variation of $\mathcal{F}$ pass to the limit, that is,

$$\int_{\Omega_0} \Big(\nabla u_0 \cdot (\delta A)\nabla v_0 + \Lambda \operatorname{div} \xi\Big)\, dx = 0\,,$$
$$\int_{\Omega_0} \Big(\nabla u_0 \cdot (\delta^2 A)\nabla v_0 - \nabla(\delta u_0)\cdot\nabla(\delta v_0) + \frac{\Lambda}{2}\big((\operatorname{div}\xi)^2 + \xi\cdot\nabla(\operatorname{div}\,\xi)\big)\Big)dx \geqq 0\,, \tag{A.5}$$

where δA and $\delta^2 A$ are as in (2.6), and where δu_0 and δv_0 solve the PDE (see Sect. 7.1)

$$-\Delta(\delta u_0) = \operatorname{div}\big((\delta A)\nabla u_0\big) \quad \text{in } \Omega_0\,, \qquad \delta u_0 \in \dot H^1_0(\Omega_0)\,;$$
$$-\Delta(\delta v_0) = \operatorname{div}\big((\delta A)\nabla v_0\big) \quad \text{in } \Omega_0\,, \qquad \delta v_0 \in \dot H^1_0(\Omega_0)\,.$$

On the other hand, since by [1] the Boundary Harnack Principle holds on the optimal set Ω, we get that the blow-up limits are proportional, that is, $v_0 = Cu_0$ for some $C > 0$. Finally, we notice that, up to multiplying u_0 by a constant, the inequalities (A.5) correspond precisely to the stability condition (7.8) in Definition 7.3, which concludes the proof of the lemma. □

A.3 Main Theorem

Let now Ω be an optimal set for (A.2). We define the regular and the singular parts of the free boundary $\partial\Omega \cap D$ as follows. The regular part, $\operatorname{Reg}(\partial\Omega)$, is the set of points $x_0 \in \partial\Omega \cap D$ at which there exists a blow up limit $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$ of the form

$$u_0(x) = \alpha(x\cdot\nu)_+ \quad \text{and} \quad v_0(x) = \beta(x\cdot\nu)_+,$$

for some unit vector $\nu \in \mathbb{R}^d$ and some constants $\alpha > 0$ and $\beta > 0$ such that $\alpha\beta = \Lambda$. The remaining part of the free boundary is the singular set:

$$\operatorname{Sing}(\partial\Omega) := (\partial\Omega \cap D) \setminus \operatorname{Reg}(\partial\Omega).$$

Then, in terms of the decomposition into $\partial\Omega \cap D = \operatorname{Reg}(\partial\Omega) \cup \operatorname{Sing}(\partial\Omega)$, we can rewrite the regularity theorem of Aguilera–Caffarelli–Spruck (for the penalized problem) as follows:

Theorem A.3. (Aguilera–Caffarelli–Spruck [1]) *If Ω is a solution to (A.2) in a domain $D \subset \mathbb{R}^d$, then $\operatorname{Reg}(\partial\Omega)$ is a relatively open subset of $\partial\Omega \cap D$ and (locally) a C^∞ manifold.*

For what concerns the singular set, by reasoning as in Theorem 8.1 and by applying Theorems 7.9 and 7.8, we obtain the following result, which in particular implies that in the physically relevant dimension $d = 3$ (and also in $d = 2$ and $d = 4$) the free boundary $\partial\Omega \cap D$ of a solution Ω to (A.2) is C^∞ smooth.

Theorem A.4. (Dimension of the singular set for solutions to (A.2)) *If Ω is a solution to (A.2) in a domain $D \subset \mathbb{R}^d$, then the following holds:*

(i) *if $d < d^*$, then $\operatorname{Sing}(\partial\Omega) = \emptyset$;*
(ii) *if $d \geq d^*$, then the Hausdorff dimension of $\operatorname{Sing}(\partial\Omega)$ is at most $d - d^*$;*

where d^* is the critical dimension from Definition 7.7.

References

1. Aguilera, N.E., Caffarelli, L.A., Spruck, J.: An optimization problem in heat conduction. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **14**(3), 355–387, 1987
2. Alt, H.W., Caffarelli, L.A.: Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325**, 105–144, 1981
3. Brasco, L., Mazzoleni, D.: On principal frequencies, volume and inradius in convex sets. *NoDEA Nonlinear Differ. Equ. Appl.* **27**(2), 26, 2020
4. Briançon, T., Lamboley, J.: Regularity of the optimal shape for the first eigenvalue of the Laplacian with volume and inclusion constraints. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**(4), 1149–1163, 2009
5. Bucur, D.: Minimization of the k-th eigenvalue of the Dirichlet Laplacian. *Arch. Ration. Mech. Anal.* **206**(3), 1073–1083, 2012
6. Bucur, D., Buttazzo, G.: Variational Methods in Shape Optimization Problems. Progress in Nonlinear Differential Equations, vol. 65. Birkhäuser Verlag, Basel (2005)
7. Bucur, D., Lamboley, J., Nahon, M., Prunier, R.: Sharp quantitative stability of the Dirichlet spectrum near the ball. arXiv:2304.10916.
8. Bucur, D., Mazzoleni, D., Pratelli, A., Velichkov, B.: Lipschitz regularity of the eigenfunctions on optimal domains. *Arch. Ration. Mech. Anal.* **216**(1), 117–151, 2015
9. Bucur, D., Velichkov, B.: Multiphase shape optimization problems. *SIAM J. Control Optim.* **52**(6), 3556–3591, 2014
10. Buttazzo, G., Dal Maso, G.: An existence result for a class of shape optimization problems. *Arch. Ration. Mech. Anal.* **122**, 183–195, 1993
11. Buttazzo, G., Maiale, F.P., Velichkov, B.: Shape optimization problems in control form. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* **32**(3), 413–435, 2021
12. Buttazzo, G., Shrivastava, H.: Optimal shapes for general integral functionals. *Ann. H. Lebesgue* **3**, 261–272, 2020
13. Buttazzo, G., Velichkov, B.: A shape optimal control problem with changing sign data. *SIAM J. Math. Anal.* **50**(3), 2608–2627, 2018
14. Caffarelli, L.A., Jerison, D., Kenig, C.E.: Global energy minimizers for free boundary problems and full regularity in three dimensions. *Contemp. Math.* **350**, 83–97, 2004
15. Caffarelli, L.A., Shahgholian, H., Yeressian, K.: A minimization problem with free boundary related to a cooperative system. *Duke Math. J.* **167**(10), 1825–1882, 2018
16. De Philippis, G., Spolaor, L., Velichkov, B.: Regularity of the free boundary for the two-phase Bernoulli problem. *Invent. Math.* **225**, 347–394, 2021
17. De Philippis, G., Velichkov, B.: Existence and regularity of minimizers for some spectral optimization problems with perimeter constraint. *Appl. Math. Optim.* **69**(2), 199–231, 2014
18. De Silva, D.: Free boundary regularity for a problem with right hand side. *Interfaces Free Bound.* **13**(2), 223–238, 2011
19. De Silva, D., Jerison, D.: A singular energy minimizing free boundary. *J. Reine Angew. Math.* **635**, 1–21, 2009
20. Gustafsson, B., Shahgholian, H.: Existence and geometric properties of solutions of a free boundary problem in potential theory. *J. Reine Angew. Math.* **473**, 137–180, 1995
21. Henrot, A., Pierre, M.: Shape Variation and Optimization. A Geometrical Analysis. EMS Tracts in Mathematics, vol. 28. European Mathematical Society (EMS), Zürich (2018)
22. Jerison, D., Savin, O.: Some remarks on stability of cones for the one phase free boundary problem. *Geom. Funct. Anal.* **25**, 1240–1257, 2015
23. Kamburov, N.: Nondegeneracy and stability in the limit of a one-phase singular perturbation problem. *Discrete Contin. Dyn. Syst. Ser. A* **43**(12), 4328–4360, 2023
24. Kinderlehrer, D., Nirenberg, L.: Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa* **4**, 373–391, 1977

25. Kriventsov, D., Lin, F.: Regularity for shape optimizers: the degenerate case. *Commun. Pure Appl. Math.* **72**(8), 1678–1721, 2019
26. Kriventsov, D., Lin, F.: Regularity for shape optimizers: the nondegenerate case. *Commun. Pure Appl. Math.* **71**(8), 1535–1596, 2018
27. Kukuljan, T.: Higher order parabolic boundary Harnack inequality in C^1 and $C^{k,\alpha}$ domains. *Discrete Contin. Dyn. Syst.* **42**(6), 2667–2698, 2022
28. Maiale, F.P., Tortone, G., Velichkov, B.: Epsilon-regularity for the solutions of a free boundary system. *Rev. Mat. Iberoam.* **39**(5), 1947–1972, 2023
29. Maiale, F.P., Tortone, G., Velichkov, B.: The boundary Harnack principle on optimal domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **25** (1), 127-149, 2024
30. Mazzoleni, D., Pratelli, A.: Existence of minimizers for spectral problems. *J. Math. Pures Appl.* **100**(3), 433–453, 2013
31. Mazzoleni, D., Terracini, S., Velichkov, B.: Regularity of the optimal sets for some spectral functionals. *Geom. Funct. Anal.* **27**, 373–426, 2017
32. Mazzoleni, D., Trey, B., Velichkov, B.: Regularity of the optimal sets for the second Dirichlet eigenvalue. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **39**(3), 529–573, 2022
33. Mazzoleni, D., Tortone, G., Velichkov, B.: On the dimension of the singular set in optimization problems with measure constraint. J. Convex Anal. **31**(2), 689-708, 2024
34. Russ, E., Trey, B., Velichkov, B.: Existence and regularity of optimal shapes for elliptic operator with drift. *Calc. Var. Partial Differ. Equ.* **58**(6), 199, 2019
35. Spolaor, L., Velichkov, B.: An epiperimetric inequality for the regularity of some free boundary problems: the 2-dimensional case. *Commun. Pure Appl. Math.* **72**(2), 375–421, 2019
36. Terracini, S., Tortone, G., Vita, S.: Higher order boundary Harnack principle via degenerate equations. *Arch. Ration. Mech. Anal.,* **248**(2), 29 2024.
37. Velichkov, B.: Existence and regularity results for some shape optimization problems. Edizioni della Normale 19, Pisa, 2015
38. Velichkov, B.: Regularity of the one-phase free boundaries. Lecture Notes of the Unione Matematica Italiana, Springer Cham, 2023. https://doi.org/10.1007/978-3-031-13238-4
39. Weiss, G.S.: Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.* **9**(2), 317–326, 1999

Giuseppe Buttazzo, Giorgio Tortone & Bozhidar Velichkov
Dipartimento di Matematica,
Università di Pisa,
Largo Bruno Pontecorvo, 5,
56127 Pisa
Italy.
e-mail: bozhidar.velichkov@unipi.it

and

Giuseppe Buttazzo
e-mail: giuseppe.buttazzo@unipi.it

and

Giorgio Tortone
e-mail: giorgio.tortone@dm.unipi.it

and

Francesco Paolo Maiale
Gran Sasso Science Institute,
Viale F. Crispi 7,
67100 L'Aquila
Italy.
e-mail: francescopaolo.maiale@gssi.it

and

Dario Mazzoleni
Dipartimento di Matematica,
University of Pavia,
Via Ferrata 5,
27100 Pavia
Italy.
e-mail: dario.mazzoleni@unipv.it

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