# On Integer Partitions and Continued Fraction Type Algorithms

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**Abstract**

Our goal is to show that the additive-slow-Farey version of the Triangle map (a type of multi-dimensional continued fraction algorithm) gives us a method for producing a map from the set of integer partitions of a positive number $n$ into itself. We start by showing that the additive-slow-Farey version of the traditional continued fractions algorithm has a natural interpretation as a method for producing integer partitions of a positive number $n$ into two smaller numbers, with multiplicity. We provide a complete description of how such integer partitions occur and of the conjugation for the corresponding Young shapes via the dynamics of the classical Farey tree. We use the dynamics of the Farey map to get a new formula for $p(2, n)$, the number of ways for partitioning $n$ into two smaller positive integers, with multiplicity. We then turn to the general case, using the the Triangle map to give a natural map from general integer partitions of a positive number $n$ to integer partitions of $n$. This map will still be compatible with conjugation of the corresponding Young shapes. We will close by the observation that it appears few other multi-dimensional continued fraction algorithms can be used to study partitions.

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## 1 Introduction

This paper is an attempt to put two ideas together, that of integer partitions for positive integers and that of the dynamics behind regular and multi-dimensional continued fractions. The theory of partition numbers is one of the richest areas in mathematics, especially in combinatorics. Continued fractions and their generalizations to the multi-dimensional case (all of which can be interpreted as division algorithms) are important to number theory, to dynamical systems as a rich source of examples and a number of other areas.

We have two audiences in mind: the partitions and the dynamical systems communities. Thus there will be a bit more exposition than would be usual in a paper. This is also why we spend so much time on the somewhat special case of partitions into only two parts.

The main idea behind this paper can initially be seen by the relation between the continued fractions expansion of a rational number $m/n$ and the *Farey tree*, a binary tree containing all rational numbers

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in $(0, 1)$. The construction of the Farey tree recalled in Section 3.2 can then be used to generate integer partitions of the denominator $n$. Let us consider for example the number $8/19$, which is studied in details in Examples 3.3 and 4.2. Our method generates the following partitions of 19:

$$\begin{aligned} 19 &= 11 + 8 \\ &= (8 + 3) + 8 = 2 \cdot 8 + 3 \\ &= 2 \cdot (5 + 3) + 3 = 2 \cdot 5 + 3 \cdot 3 \\ &= 2 \cdot (3 + 2) + 3 \cdot 3 = 5 \cdot 3 + 2 \cdot 2 \\ &= 5 \cdot (2 + 1) + 2 \cdot 2 = 7 \cdot 2 + 5 \cdot 1\,. \end{aligned}$$

Notice that at each step we are simply applying a slow version of the Euclidean algorithm starting from the couple $(8, 19)$. In this slow version of the algorithm we subtract the number 8 from 11 just once, and keep doing the same subtraction until it is impossible to continue. Then we pass to the couple $(3, 8)$ and continue. We have thus produced five partitions of 19 into two different parts. We refer to Section 2 for a background on integer partitions.

The first main result of the paper is that given an integer $n$ we generate all its partitions into two coprime different parts with coprime multiplicities by repeating the previous method for all numbers $m/n \in (0, 1)$ with $\gcd(m, n) = 1$. We can then extend this result to show how to generate all the partitions of an integer $n$ into two different parts. The relation of the generated partitions with the Farey tree also leads to a formula for the number of these partitions. In Theorem 4.18, we give a new formula for the number of ways to partition $n$ into two smaller numbers, with multiplicity. This formula is quite different than the formula of Kim [18] and has a dynamical interpretation in terms of the *Farey tree* and the related *Farey map*.

In Section 2, we set up our notation for partitions. Section 3 deals first with the Farey map, which we refer to as the additive-slow-Farey map to remark its role in the generation of the slow version of the additive Euclidean algorithm. In the same section we also recall the construction of the Farey tree and the interpretation of the Farey map via two-by-two matrices in $SL(2, \mathbb{Z})$.

In Section 4, we introduce our method to find integer partitions of a positive number $n$ into two smaller numbers by using the Farey map and prove the main results of the paper. Our method also produces an extremely natural interpretation of the conjugation among partitions. In the theory of integer partition this conjugation is described in terms of the *Young shape* of a partition and of the *flipping* of the shape. In Theorems 4.7 and 4.16 we show that this conjugation comes out from the properties of the *binary sequence* of a rational number in the Farey tree and of its reversed sequence.

All of this is about the quite special case of partitioning $n$ into two parts. It is in Section 5 that we start the discussion of the generalization of our method to partitions into many parts. This is the part of the paper in which we start using a particular multi-dimensional continued fractions algorithm. In particular we use what we call the *additive-slow-Triangle map*, a $m$-dimensional version of the Farey map (whose two dimensional version was introduced and studied in [7] in analogy with the two-dimensional version of the Gauss map defined in [15]). In Section 5 we will see how the Triangle map can be used to produce various partitions of $n$ into $m$ smaller numbers, with multiplicity. The geometry of the Triangle map and of its domain is more complicated than that for the Farey map. We will determine a description of which such integer partitions occur and give description of the conjugation for the corresponding Young shapes via the dynamics of the Triangle map. In Section 6, we will further extend the triangle map acting on partitions, giving us a way for capturing all possible partitions into orbits of other partitions. Now the Triangle map is only one of many possible multi-dimensional continued fractions algorithms that exist. In Section 7, we will look at two other well-known multi-dimensional continued fractions algorithms (the Mönkmeyer map and the Cassaigne map) and show, somewhat surprisingly, that neither can be used to study partition numbers. Further in that section we discuss that there are only a few multi-dimensional continued fractions algorithms that will create orbits of partition numbers. We close with questions in the last section.

## 2 Background on Partition Numbers

There are many sources for background on partition numbers: the classical text is Andrews [1] and a good introduction is Andrews and Eriksson [2]. In this section we recall what we need in the following.

A *partition* of an integer $n \geq 1$ is a non-increasing sequence of positive integers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

such that $\lambda_1 + \cdots + \lambda_r = n$. In this case we shall write $(\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$. The *partition number* $p(n)$ is the number of partitions of $n$, that is the number of ways for adding positive numbers together to get $n$, with order not mattering. For example $p(6) = 11$ because 6 can be obtained in the following 11 different ways:

| | | | |
|---|---|---|---|
| $6$ | $4+1+1$ | $3+1+1+1$ | $2+1+1+1+1$ |
| $5+1$ | $3+3$ | $2+2+2$ | $1+1+1+1+1+1$ |
| $4+2$ | $3+2+1$ | $2+2+1+1$ | |

If a certain $\lambda_i$ is repeated in the sequence, say $k_i$ times, we collapse the repeated values and use the compact notation

$$(n_1^{k_1}, \ldots, n_m^{k_m}) \vdash n$$

to denote the partition $n_1 \cdot k_1 + \cdots + n_m \cdot k_m = n$. We shall call $n_1, \ldots, n_m$ the *parts* and $k_1, \ldots, k_m$ the *multiplicities* of the partition. Thus for $n = 6$, we can rewrite the above partitions as

$$\begin{array}{rcl} (6) & \vdash & 6 \\ (5,1) & \vdash & 6 \\ (4,2) & \vdash & 6 \\ (4,1^2) & \vdash & 6 \\ & \vdots & \\ (2^2,1^2) & \vdash & 6 \\ (2,1^4) & \vdash & 6 \\ (1^6) & \vdash & 6 \end{array}$$

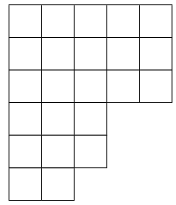
As we will be acting on a partition $(n_1^{k_1}, \ldots, n_m^{k_m})$ via matrices, it will be convenient at some point to use the notation

$$(n_1^{k_1}, \ldots, n_m^{k_m}) = (n_1, \ldots, n_m) \times [k_1, \ldots, k_m],$$

with round brackets for the parts and square brackets for the multiplicities.

To a given partition $(\lambda_1, \ldots, \lambda_r)$ we associate the *Young shape* (also called a *Young diagram*), a left-aligned diagram with $r$ rows such that the $i$-th row contains $\lambda_i$ squares. Equivalently, to a partition of the form $(n_1^{k_1}, \ldots, n_m^{k_m})$ we associate the shape with $k_1 + \cdots + k_m$ rows such that there are $k_1$ rows with $n_1$ squares on top of $k_2$ rows with $n_2$ squares, and so on.

**Example 2.1.** For example, the Young shape for $(5^3, 3^2, 2^1) \vdash 23$ is

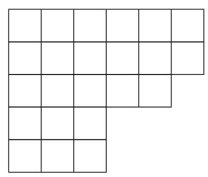


We can always flip any such Young shape, turning the rows into columns, getting a new Young shape which still represents a partition of the same integer. We shall refer to this new partition as the *conjugate partition* and write

$$\lambda \sim_{\mathcal{C}} \mu$$

to indicate that the partitions $\lambda$ and $\mu$ are conjugate.

**Example 2.2.** Flipping the Young shape of the partition $(5^3, 3^2, 2^1) \vdash 23$ of the previous example gives us the Young shape



which represents the conjugate partition $(6^2, 5^1, 3^2) \vdash 23$.

Explicitly we have

$$(n_1^{k_1}, n_2^{k_2}) \sim_{\mathcal{C}} ((k_1 + k_2)^{n_2},\ k_1^{n_1 - n_2}),$$

$$(n_1^{k_1}, n_2^{k_2}, n_3^{k_3}) \sim_{\mathcal{C}} ((k_1 + k_2 + k_3)^{n_3},\ (k_1 + k_2)^{n_2 - n_3},\ k_1^{n_1 - n_2}).$$

and in general

$$(n_1^{k_1}, \ldots, n_m^{k_m}) \sim_{\mathcal{C}} ((k_1 + \ldots + k_m)^{n_m},\ (k_1 + \ldots + k_{m-1})^{n_{m-1} - n_m}, \ldots, k_1^{n_1 - n_2}).$$

# 3 Preliminaries on the additive-slow-Farey map

In this section we first recap some results from the theory of continued fractions, we recall the definition of the Farey map and the construction of the Farey tree, then show the connection between the two worlds. (Partially reflecting that different mathematical communities work on these maps, they are sometimes called slow version or additive version of the Gauss map, which is why we sometimes call this the additive-slow-Farey map).

The basics of continued fractions are in most beginning number theory books. For more in depth treatment, there is the classic work of Khinchin [22]. To see how continued fractions are naturally linked to dynamical systems, see Dajani and Kraakamp [11] or Hensley [17].

### 3.1 Basic properties of continued fractions

For $x \in (0,1)$ we denote by $[a_1, a_2, \ldots]$, its continued fraction expansion, that is

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},$$

where $a_j \geq 1$ for all $j$. The expansion is finite if and only if $x$ is rational and it is also unique provided that for finite expansions such as $[a_1, \ldots, a_k]$ we require $a_k > 1$. The *convergents* of a real number $x \in (0,1)$ are the elements of the sequence $(p_j/q_j)_{j\geq 0}$ recursively defined as follows:

$$\begin{aligned} p_0 &= 0 & q_0 &= 1 \\ p_1 &= 1 & q_1 &= a_1 \\ p_{j+1} &= a_{j+1}p_j + p_{j-1} & q_{j+1} &= a_{j+1}q_j + q_{j-1}. \end{aligned}$$

It is easy to show that

$$\frac{p_j}{q_j} = [a_1, \ldots, a_j].$$

Note that the sequence of convergents is finite or infinite according to whether $x$ is rational or not. In particular, if $x = [a_1, \ldots, a_k]$ is rational then the sequence of convergents stops at $\frac{p_k}{q_k} = x$. On the other hand, if $x$ is irrational then $\frac{p_j}{q_j} \to x$ as $j \to \infty$ and the convergents are the best rational approximations of $x$ in a precise sense. Another classical result which we shall use in the following is the so-called *mirror formula*, that is

$$\frac{q_{j-1}}{q_j} = [a_j, \ldots, a_1]. \tag{1}$$

### 3.2 The Farey map and the Farey tree

Split the unit interval $I = [0,1]$ into two sub-intervals $I_0 = \left[\frac{1}{2}, 1\right]$ and $I_1 = \left[0, \frac{1}{2}\right]$. The *Farey map* is the map $F : I \to I$ defined to be

$$F(x) = \begin{cases} F_0(x) = \frac{1-x}{x}, & \text{if } x \in I_0 \\ F_1(x) = \frac{x}{1-x}, & \text{if } x \in I_1 \end{cases}$$

It has the two local inverses $\Phi_0 := F_0^{-1} : I \to I_0$ and $\Phi_1 := F_1^{-1} : I \to I_1$ given by

$$\Phi_0(x) = \frac{1}{1+x} \quad \text{and} \quad \Phi_1(x) = \frac{x}{1+x}.$$

It is well-known that every rational number in $(0,1)$ can be uniquely described as an element in $\mathcal{F} = \bigcup_{k=0}^{\infty} F^{-k}(\frac{1}{2})$. Furthermore, $\mathcal{F}$ can be ordered as a binary tree, the *Farey tree*. The recursive construction works as follows: the root of the tree is $\frac{1}{2}$ and the two children of the vertex $\frac{p}{q}$ are its backward images under the Farey map, namely $\Phi_1\left(\frac{p}{q}\right) = \frac{p}{p+q}$ and $\Phi_0\left(\frac{p}{q}\right) = \frac{q}{p+q}$. Note that each rational number appears reduced in lowest terms in the tree because the root $\frac{1}{2}$ is. The *levels* of the Farey tree are the sets $\mathcal{L}_k = F^{-k+1}(\frac{1}{2})$ for $k \geq 1$, so that

$$\mathcal{L}_1 = \left\{\frac{1}{2}\right\}, \quad \mathcal{L}_2 = \left\{\frac{1}{3}, \frac{2}{3}\right\}, \quad \mathcal{L}_3 = \left\{\frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}\right\}, \quad \ldots$$

See Figure 1 to see the structure of the first few levels of the Farey tree.

**Remark 3.1.** We are describing here the tree constructed with the inverse branches of the Farey map, which we call the Farey tree. The "Farey " tree $\tilde{\mathcal{F}}$ is the one defined by the mediant between neighboring fractions (see [9] for instance). It can be shown that the levels of $\tilde{\mathcal{F}}$ and those of "our" Farey tree $\mathcal{F}$ coincide, but in $\tilde{\mathcal{F}}$ the fractions of each level appear in ascending order.

Sometimes it is convenient to extend the Farey tree above the root adding $\frac{1}{1}$, which is mapped to $\frac{1}{2}$ by both $\Phi_0$ and $\Phi_1$. In this case we also set $\mathcal{L}_0 = \left\{\frac{1}{1}\right\}$.

**Definition 3.2.** For $\frac{p}{q} \in (0,1)$ in lowest terms we shall call the *depth* of $\frac{p}{q}$ the level of the Farey tree $\frac{p}{q}$ belongs to. That is, we shall write

$$\text{depth}\left(\frac{p}{q}\right) = k \quad \Leftrightarrow \quad \frac{p}{q} \in \mathcal{L}_k.$$

If $\frac{p}{q}$ is not in lowest terms, we define its depth as the depth of the reduced form of $\frac{p}{q}$.

For instance $\text{depth}\left(\frac{2}{3}\right) = \text{depth}\left(\frac{10}{15}\right) = 2$ and $\text{depth}\left(\frac{3}{8}\right) = 4$.

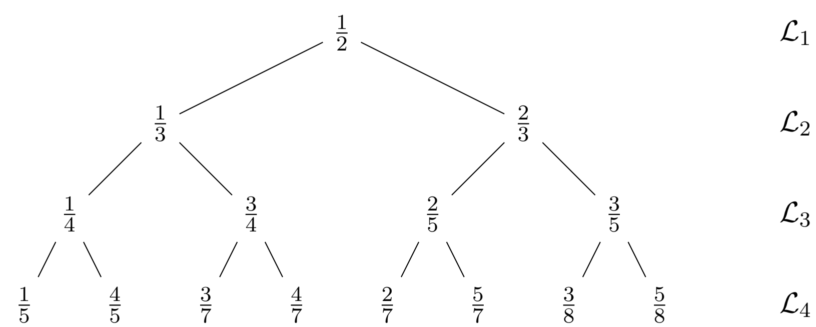



Figure 1: The first four levels of the Farey tree.

Starting from a given $x \in (0,1)$, we can iterate the map $F$, creating an *orbit* of $x$, with the iterations terminating whenever the image is $\frac{1}{2}$. In particular, we can encode a given $x \in (0,1)$ as a sequence of zeros and ones if we keep track of whether the iterations fall in $J_0 = (\frac{1}{2}, 1)$ or in $J_1 = (0, \frac{1}{2})$. More precisely, we associate to $x$ the unique binary word $\sigma(x) = \sigma_1\sigma_2\cdots$ such that

$$\sigma_j \in \{0,1\} \quad \text{and} \quad F^{j-1}(x) \in J_{\sigma_j} \quad \text{for every } j \geq 1.$$

Of course $\sigma(\frac{1}{2})$ is the empty word. We shall refer to $\sigma(x)$ as the *binary sequence* of $x$. We know that $\sigma(x)$ is finite if and only if $x$ is rational: this follows immediately because the Farey tree is constructed by taking backward images of $\frac{1}{2}$ under $F$ and it contains all and only the rational numbers in $(0,1)$. Let $x = \frac{p}{q} \in (0,1)$ be a rational number and let $\sigma(x) = \sigma_1 \cdots \sigma_\ell$. We have $\ell = \text{depth}\,(x) - 1$ and, from the definition of $\sigma$, also

$$F_{\sigma_\ell} \circ \cdots \circ F_{\sigma_1}\left(\frac{p}{q}\right) = \frac{1}{2},$$

which is the same as

$$\frac{p}{q} = \Phi_{\sigma_1} \circ \cdots \circ \Phi_{\sigma_\ell}\left(\frac{1}{2}\right). \tag{2}$$

**Example 3.3.** The orbit of $\frac{8}{19}$ under $F$ is

$$\frac{8}{19} \overset{F_1}{\longmapsto} \frac{8}{11} \overset{F_0}{\longmapsto} \frac{3}{8} \overset{F_1}{\longmapsto} \frac{3}{5} \overset{F_0}{\longmapsto} \frac{2}{3} \overset{F_0}{\longmapsto} \frac{1}{2},$$

so that depth $\left(\frac{8}{19}\right) = 6$, $\sigma\left(\frac{8}{19}\right) = 10100$, and $\frac{8}{19} = \Phi_1 \circ \Phi_0 \circ \Phi_1 \circ \Phi_0 \circ \Phi_0 \left(\frac{1}{2}\right)$.

There is a beautiful connection between the Farey tree and continued fractions. Indeed the Farey map has a simple action on continued fraction expansions. For $x = [a_1, a_2, a_3, \ldots]$ (the expansion may be finite or not) we have

$$F([a_1, a_2, a_3, \ldots]) = \begin{cases} [a_1 - 1, a_2, a_3, \ldots], & \text{if } a_1 > 1 \\ [a_2, a_3, \ldots], & \text{if } a_1 = 1 \end{cases}.$$

In other words, $F$ subtracts off a 1 from the first digit of the expansion if it is greater than 1 and deletes it when it is 1. Note that $a_1 > 1$ if and only if $x = [a_1, a_2, a_3, \ldots]$ is in the interval $(0, \frac{1}{2})$, so that more precisely $F_1$ acts subtracting off a 1 from $a_1$, while $F_0$ acts by deleting it. We now consider again the case $x = \frac{p}{q}$ rational to see equation (2) in a new light. We know that $x = [a_1, \ldots, a_k]$ has a finite continued fraction expansion (which is unique, as long as $a_k > 1$) and thus, considering the action of $F$ on the expansion of $x$, we have

$$\sigma(x) = 1^{a_1-1}01^{a_2-1}0 \cdots 1^{a_k-2}.$$

We remark that if $k = 1$ we have $\sigma(x) = 1^{a_1-2}$. Hence equation (2) can be rewritten as

$$\begin{aligned} \frac{p}{q} &= \Phi_1^{a_1-1}\Phi_0 \circ \cdots \circ \Phi_1^{a_{n-1}-1}\Phi_0 \circ \Phi_1^{a_k-2}\left(\frac{1}{2}\right) = \\ &= \Phi_1^{a_1-1}\Phi_0 \circ \cdots \circ \Phi_1^{a_{k-1}-1}\Phi_0 \circ \Phi_1^{a_k-1}\left(\frac{1}{1}\right). \end{aligned} \tag{3}$$

With the same argument we can also write analogous equations for the convergents $\frac{p_j}{q_j}$. A consequence of equation (3) is

$$\text{depth}\left(\frac{p}{q}\right) = \sum_{j=1}^{k} a_j - 1.$$

For more details the reader can refer to [9, 19].

### 3.3 Rewriting the Farey map as matrix multiplication

We can also interpret the Farey map as not acting on the unit interval $I$ in $\mathbb{R}$ but as acting on the cone

$$\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, y \geq x \geq 0\right\} \setminus \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}.$$

Now the vector $(x, y)$ is representing the real number $\frac{x}{y}$, so that the unit interval is actually in bijection with the lines in the cone. In particular, we represent the rational number $\frac{p}{q}$ with the vector $\left(\begin{smallmatrix} p \\ q \end{smallmatrix}\right)$. Hence the action of $F$ becomes

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} F_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y - x \\ x \end{pmatrix}, & \text{if } x \geq y - x \\ F_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y - x \end{pmatrix}, & \text{if } y - x \geq x \end{cases}.$$

In this way the Farey map can be seen as the action by left multiplication of column vectors by $2 \times 2$ matrices, that is

$$F_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y-x \\ x \end{pmatrix} \quad \text{and} \quad F_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y-x \end{pmatrix}.$$

We highlight the two matrices

$$F_0 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad F_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

The two inverse branches of $F$ can also be expressed in terms of multiplication of two matrices, which we denote by

$$\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \Phi_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

With this convention, equation (2) can be rewritten as

$$\begin{pmatrix} p \\ q \end{pmatrix} = \prod_{j=1}^{\ell} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \prod_{j=1}^{\ell+1} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\ell = \text{depth}\,(p/q) - 1$ and the last digit $\sigma_{\ell+1}$ can be either 0 or 1. Thus each rational $\frac{p}{q}$ is encoded by two matrices, namely

$$\prod_{j=1}^{\ell} \Phi_{\sigma_j} \cdot \Phi_0 \quad \text{and} \quad \prod_{j=1}^{\ell} \Phi_{\sigma_j} \cdot \Phi_1.$$

Note that they have the same columns in the two possible orders, so that in particular one of the two has determinant 1 and the other one has determinant $-1$. We shall refer to the matrix with positive determinant as the *matrix* of $\frac{p}{q}$. We now highlight an important property of these matrices, which we shall use in the proof of Theorem 4.4.

**Lemma 3.4.** *The set of the matrices of the rationals in the interval* $(0,1)$ *is*

$$\left\{ \begin{pmatrix} p' & p'' \\ q' & q'' \end{pmatrix} \in M(2,\mathbb{Z}) \, : \, 1 \le p' \le q', \; 0 \le p'' < q'', \; p'q'' - p''q' = 1 \right\} \subseteq SL(2,\mathbb{Z}).$$

*Proof.* Let $\frac{p}{q}$ be a rational in the interval $(0,1)$ and let

$$\begin{pmatrix} p' & p'' \\ q' & q'' \end{pmatrix}$$

be its matrix. By definition $p'q'' - p''q' = 1$ and an easy induction shows that $1 \le p' \le q'$ and $0 \le p'' < q''$. One may also look at the two columns as representing the two rationals $\frac{p'}{q'} < \frac{p''}{q''}$ with $\frac{p}{q} = \frac{p'+p''}{q'+q''}$. As it is well-known the condition $p'q'' - p''q' = 1$ means that the two rationals are neighbours in the Farey sequence and that the matrix of $\frac{p}{q}$ is nothing but the matrix of the coding described in [9], and the lemma now follows. □

For instance, Example 3.3 can be rewritten in terms of matrices as

$$\begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \Phi_1 \Phi_0 \Phi_0 \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and the matrix of $\frac{8}{19}$ is

$$\Phi_1\Phi_0\Phi_1\Phi_0\Phi_0\Phi_0 = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix}.$$

# 4 Partitions into two different parts

## 4.1 Partitions generated by the Farey map

In this section we show how to generate integer partitions into two parts by using the dynamics of the Farey map and the Farey tree.

Let $n \geq 2$ be an integer and $r$ be such that $1 \leq r < n$ and $(r,n) = 1$, so that $\frac{r}{n}$ appears in some level of the Farey tree. If $\sigma(\frac{r}{n}) = \sigma_1 \cdots \sigma_\ell$ is the binary sequence of $\frac{r}{n}$ introduced in Section 3.2, we have

$$\begin{pmatrix} r \\ n \end{pmatrix} = \prod_{j=1}^{\ell} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \prod_{j=1}^{\ell+1} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where $\sigma_{\ell+1}$ can be either 0 or 1. Recall that $\ell = \text{depth}\,(r/n) - 1$. For each $m = 0, \ldots, \ell+1$ we split the above product as

$$\begin{pmatrix} r \\ n \end{pmatrix} = \prod_{j=1}^{m} \Phi_{\sigma_j} \cdot \prod_{j=m+1}^{\ell+1} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

and we set

$$\begin{pmatrix} h_2(m) & h_1(m) \\ k_2(m) & k_1(m) \end{pmatrix} = \prod_{j=1}^{m} \Phi_{\sigma_j} \quad \text{and} \quad \begin{pmatrix} n_2(m) \\ n_1(m) \end{pmatrix} = \prod_{j=m+1}^{\ell+1} \Phi_{\sigma_j} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Explicitly writing the second component of equation (4), we get

$$n = k_1(m)n_1(m) + k_2(m)n_2(m),$$

with $n_1(m) \geq n_2(m)$. Thus for each $m$ we get a partition of $n$, namely $(n_1(m)^{k_1(m)}, n_2(m)^{k_2(m)}) \vdash n$, and the integer $m$ is called the *generation* of the partition. Note that for $m = 0$ equation (4) reads

$$\begin{pmatrix} r \\ n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r \\ n \end{pmatrix},$$

which induces the partition $(n^1, r^0) = (n^1)$ of generation $m = 0$. On the other hand for $m = \ell+1$, whether $\sigma_{\ell+1} = 0$ or $\sigma_{\ell+1} = 1$, we have $k_1(m) + k_2(m) = n$ and thus the induced partition is $(1^{k_2(m)}, 1^{k_1(m)}) = (1^n)$. Since we are interested in partitions into two distinct parts here, we do not include these two cases in the following definition.

**Definition 4.1.** We shall call the sequence of partitions $(n_1(m)^{k_1(m)}, n_2(m)^{k_2(m)})$ for $m = 1, \ldots, \ell$ the *orbit* of partitions generated by $\frac{r}{n}$.

**Example 4.2.** We again consider Example 3.3 to see how the above construction works in practice. Set $n = 19$ and $r = 8$, so that

$$\begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1\Phi_0\Phi_1\Phi_0\Phi_0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \Phi_1\Phi_0\Phi_1\Phi_0\Phi_0\Phi_i \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with $i = 0$ or $i = 1$. Choosing $i = 0$ we then get the following partitions:

$$
\begin{array}{lll}
m = 0: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = I \begin{pmatrix} 8 \\ 19 \end{pmatrix} & (8^0, 19^1) = (19^1) \\
m = 1: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \begin{pmatrix} 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 11 \end{pmatrix} & (11^1, 8^1) \\
m = 2: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 8 \end{pmatrix} & (8^2, 3^1) \\
m = 3: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \Phi_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} & (5^2, 3^3) \\
m = 4: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \Phi_1 \Phi_0 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} & (3^5, 2^2) \\
m = 5: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \Phi_1 \Phi_0 \Phi_0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} & (2^7, 1^5) \\
m = 6: & \begin{pmatrix} 8 \\ 19 \end{pmatrix} = \Phi_1 \Phi_0 \Phi_1 \Phi_0 \Phi_0 \Phi_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 7 & 12 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & (1^{12}, 1^7) = (1^{19})
\end{array}
$$

In case $i = 1$ the last partition which we get is $(1^7, 1^{12}) = (1^{19})$.

Note that the orbit of partitions of a fraction contains pairwise distinct partitions until the last step. (This is due to that if we have a partition $(2, 1) \times [k_1, k_2]$, then we can act on it either by $F_0$, in which case we have the partition $(1, 1) \times [12, 7]$ or by $F_1$, in which case we get $(1, 1) \times [7, 120.)$ But different fractions can induce orbits of partitions sharing some element and can even induce the same orbit of partitions, as the following lemma shows.

**Lemma 4.3.** *Let $n \geq 2$ be an integer and $r$ be such that $1 \leq r < n$ and $\gcd(r, n) = 1$. Then $\frac{r}{n}$ and $\frac{n-r}{n}$ induce the same orbit of partitions of $n$.*

*Proof.* For $n = 2$ the statement is trivial, so we let $n > 2$. Note that the fractions $\frac{r}{n}$ and $\frac{n-r}{n}$ are the two children of $\frac{r}{n-r}$ in the Farey tree, thus they have the same depth and their forward orbits under $F$ coincide. Without loss of generality we assume $r < \frac{n}{2}$, so that $\frac{r}{n} < \frac{1}{2} < \frac{n-r}{n}$ and

$$
\begin{pmatrix} r \\ n \end{pmatrix} = \Phi_1 \begin{pmatrix} r \\ n-r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n-r \\ n \end{pmatrix} = \Phi_0 \begin{pmatrix} r \\ n-r \end{pmatrix}.
$$

Moreover, their binary sequences are $\sigma(\frac{r}{n}) = 1\sigma_2 \cdots \sigma_\ell$ and $\sigma(\frac{n-r}{n}) = 0\sigma_2 \cdots \sigma_\ell$. As a consequence, the induced partitions at generation $m = 1$ are both equal to $((n-r)^1, r^1)$. Now if $\ell \geq 2$, let $m = 2, \ldots, \ell$ and consider

$$
\begin{pmatrix} r \\ n \end{pmatrix} = \Phi_1 \prod_{j=2}^{m} \Phi_{\sigma_j} \begin{pmatrix} n_2(m) \\ n_1(m) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} n-r \\ n \end{pmatrix} = \Phi_0 \prod_{j=2}^{m} \Phi_{\sigma_j} \begin{pmatrix} n_2(m) \\ n_1(m) \end{pmatrix}.
$$

It is now straightforward to prove that the bottom rows of the two matrices $\Phi_1 \prod_{j=2}^{m} \Phi_{\sigma_j}$ and $\Phi_0 \prod_{j=2}^{m} \Phi_{\sigma_j}$ coincide, and so do the induced partitions of $n$. $\square$

We now characterize precisely the partitions which can be obtained through the Farey tree and the Farey map. We shall give a second proof of this result in Section 4.5.

**Theorem 4.4.** *Let $n \geq 2$ be an integer. A partition $(n_1^{k_1}, n_2^{k_2}) \vdash n$ can be obtained from the dynamics of the Farey map if and only if $\gcd(n_1, n_2) = 1$ and $\gcd(k_1, k_2) = 1$.*

*Proof.* (⇒) If $\gcd(n_1, n_2) > 1$ then the fraction $\frac{n_2}{n_1}$ is not in lowest terms and thus it does not appear on the Farey tree. If $\gcd(k_1, k_2) > 1$ then a matrix of the form

$$\begin{pmatrix} * & * \\ k_2 & k_1 \end{pmatrix}$$

cannot have determinant $\pm 1$, thus it cannot be a finite product of the matrices $\Phi_0$ and $\Phi_1$.

(⇐) We are now given a partition $(n_1^{k_1}, n_2^{k_2}) \vdash n$ with $\gcd(n_1, n_2) = 1$ and $\gcd(k_1, k_2) = 1$. To prove that it is induced from the Farey tree it suffices to show that there exist two integers $h_1$ and $h_2$ such that $0 \le h_1 < k_1$, $1 \le h_2 \le k_2$, and $h_2 k_1 - h_1 k_2 = 1$. Indeed, if this holds then Lemma 3.4 shows that the matrix

$$\begin{pmatrix} h_2 & h_1 \\ k_2 & k_1 \end{pmatrix}$$

is a finite product of the matrices $\Phi_0$ and $\Phi_1$, thus by setting

$$\begin{pmatrix} r \\ n \end{pmatrix} = \begin{pmatrix} h_2 & h_1 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$$

we have that $\frac{r}{n}$ is a fraction on the Farey tree and the partition $(n_1^{k_1}, n_2^{k_2})$ is a member of the orbit of partitions generated by $\frac{r}{n}$.

Since $\gcd(k_1, k_2) = 1$ there exist two integers $\tilde{h}_1$ and $\tilde{h}_2$ such that $\tilde{h}_2 k_1 - \tilde{h}_1 k_2 = 1$. All the solutions to the equation $h_2 k_1 - h_1 k_2 = 1$ are

$$h_1(t) = \tilde{h}_1 - t k_1 \quad \text{and} \quad h_2(t) = \tilde{h}_2 - t k_2,$$

where $t$ could be any integer number. Without loss of generality we can then assume that both $\tilde{h}_1$ and $\tilde{h}_2$ are positive. We choose $t_* = \left[\frac{\tilde{h}_1}{k_1}\right]$, so that $0 \le h_1(t_*) < k_1$. Then

$$\begin{aligned} h_2(t_*) = \tilde{h}_2 - \left[\frac{\tilde{h}_1}{k_1}\right] k_2 = \tilde{h}_2 - \left(\frac{\tilde{h}_1}{k_1} - \left\{\frac{\tilde{h}_1}{k_1}\right\}\right) k_2 = \\ = \frac{\tilde{h}_2 k_1 - \tilde{h}_1 k_2}{k_1} + \left\{\frac{\tilde{h}_1}{k_1}\right\} k_2 = \frac{1}{k_1} + \left\{\frac{\tilde{h}_1}{k_1}\right\} k_2, \end{aligned}$$

so that $1 \le h_2(t_*) \le k_2$. By setting $h_1 = h_1(t_*)$ and $h_2 = h_2(t_*)$, we finish the proof. □

Let $p_F(2, n)$ denote the number of partitions of $n$ into two different parts obtained by the Farey map. Thanks to the previous Theorem 4.4 we can give a formula for $p_F(2, n)$.

**Corollary 4.5.** *For $n \ge 2$ we have*

$$p_F(2, n) = \frac{1}{2} \left( \sum_{\substack{r=1 \\ \gcd(r,n)=1}}^{n-1} \operatorname{depth}\left(\frac{r}{n}\right) - \varphi(n) \right), \tag{5}$$

*where $\varphi(n)$ is the Euler totient function. Moreover, $p_F(2, n) = p(2, n)$ if and only if $n$ is prime or $n = 4$.*

*Proof.* Every $\frac{r}{n}$ in the Farey tree, that is with $1 \le r < n$ and $\gcd(r, n) = 1$, generates an orbit of partitions of $n$. Each of these orbits contains $\operatorname{depth}\left(\frac{r}{n}\right) - 1$ pairwise distinct partitions because there are just as many fractions strictly above $\frac{r}{n}$ in the Farey tree and up to $\frac{1}{2}$. Counting of all of these partitions yields

$$\sum_{\substack{r=1\\ \gcd(r,n)=1}}^{n-1} \left(\text{depth}\left(\frac{r}{n}\right) - 1\right) = \sum_{\substack{r=1\\ \gcd(r,n)=1}}^{n-1} \text{depth}\left(\frac{r}{n}\right) - \varphi(n).$$

Lemma 4.3 shows that each $\frac{r}{n}$ is paired with $\frac{n-r}{n}$, in the sense they induce the same orbit of partitions. Moreover, the orbits of partitions generated by two non-paired fractions are disjoint. Thus in the above counting each partition appears exactly twice.

As for the second part of the theorem we prove the two implications separately.

($\Rightarrow$) If $n$ is prime and $(n_1^{k_1}, n_2^{k_2}) \vdash n$ then necessarily $\gcd(n_1, n_2) = 1$ and $\gcd(k_1, k_2) = 1$, and thus by Theorem 4.4 the partition $(n_1^{k_1}, n_2^{k_2})$ can be obtained from the dynamics of the Farey map. The case $n = 4$ can be verified explicitly.

($\Leftarrow$) Suppose that $n$ is composite and $n \neq 4$, so that $n = ab$ for some $a > b \geq 2$. If $b \geq 3$ then the partition $((b-1)^a, 1^a)$ cannot be obtained from the dynamics of the Farey map because the multiplicities are not relatively prime. If $b = 2$ then $n = 2a$, $a \geq 3$, and the partition $((2(a-1))^1, 2^1)$ cannot be obtained as well. ☐

**Remark 4.6.** Corollary 4.5 can be compared with the following purely number theoretical expression obtained by Kim in [18]:

$$p(2,n) = \frac{1}{2}\left(\sum_{r=1}^{n-1} \sigma_0(r)\sigma_0(n-r) - \sigma_1(n) + \sigma_0(n)\right), \tag{6}$$

where $\sigma_j(n) = \sum_{d|n} d^j$. Note that when $n$ is prime then $\varphi(n) = n - 1$ and $\sigma_1(n) - \sigma_0(n) = n - 1$ do coincide[1]. However the two sums of (5) and (6) does not in general hold termwise, *e.g.* take $n = 11$ and $r = 3$, so that $\text{depth}\left(\frac{3}{11}\right) = 5$ but $\sigma_0(3) \cdot \sigma_0(8) = 2 \cdot 4 = 8$.

## 4.2 Conjugate partitions and continued fractions: Palindromes

Let $n \geq 2$ and consider a partition $(n_1^{k_1}, n_2^{k_2}) \vdash n$. As recalled in Section 2, by flipping its Young shape one obtains a new partition of $n$, the conjugate partition $(\tilde{n}_1^{\tilde{k}_1}, \tilde{n}_2^{\tilde{k}_2})$, with $\tilde{n}_1 = k_1 + k_2$, $\tilde{n}_2 = k_1$, $\tilde{k}_1 = n_2$, and $\tilde{k}_2 = n_1 - n_2$. We showed that partitions into two parts can be generated by the dynamics of the Farey map, thus one may wonder whether there is a way to dynamically characterize also the conjugacy: this is the content of this section.

Consider again equation (2), which expresses every rational in the Farey as a backward image of $\frac{1}{2}$. Since $\frac{1}{2} = \Phi_0(\frac{1}{1}) = \Phi_1(\frac{1}{1})$ we can also rewrite the equation as $\frac{r}{n} = \Phi_{\sigma_1} \circ \cdots \circ \Phi_{\sigma_\ell} \circ \Phi_{\sigma_{\ell+1}}\left(\frac{1}{1}\right)$, where $\sigma_{\ell+1}$ can be either 0 or 1. Here we make the choice $\sigma_{\ell+1} = 1$ and we shall call $\sigma_1 \cdots \sigma_\ell 1$ the *extended binary sequence* of $\frac{r}{n}$. Note that if $1 \leq r < \frac{n}{2}$ then $\sigma_1 = 1$.

**Theorem 4.7** (Palindrome Version 1)**.** *Let $r$ be such that $1 \leq r < \frac{n}{2}$ and $\gcd(r,n) = 1$, and suppose that $\frac{r}{n}$ has extended binary sequence $\sigma_1 \cdots \sigma_\ell 1 = 1\sigma_2 \cdots \sigma_\ell 1$, with $\ell = \text{depth}\left(\frac{r}{n}\right) - 1$. Let $(n_1^{k_1}, n_2^{k_2})$ be a partition of $n$ in the $m$-th generation of the orbit of partitions generated by $\frac{r}{n}$, that is suppose there is $1 \leq m \leq \ell$*

[1]More precisely, when $n > 1$ then $\sigma_1(n) - \sigma_0(n) = \sum_{d|n}(d-1) = n - 1 + \sum_{d|n,\ d<n}(d-1) \geq \varphi(n)$ and the equality holds if and only if $n$ is prime.

*such that*

$$\begin{pmatrix} r \\ n \end{pmatrix} = \prod_{j=1}^{m} \Phi_{\sigma_j} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} t & s \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}.$$

*Then the conjugate partition $(\tilde{n}_1^{\tilde{k}_1}, \tilde{n}_2^{\tilde{k}_2})$ is in the $(\ell+1-m)$-th generation of the orbit generated by $\frac{\tilde{r}}{n}$, where $\frac{\tilde{r}}{n}$ is the fraction with extended binary sequence $1\sigma_\ell \cdots \sigma_1 = 1\sigma_\ell \cdots \sigma_2 1$. In other words,*

$$\begin{pmatrix} \tilde{r} \\ n \end{pmatrix} = \prod_{j=0}^{\ell-m} \Phi_{\sigma_{\ell+1-j}} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 \end{pmatrix} = \begin{pmatrix} \tilde{t} & \tilde{s} \\ \tilde{k}_2 & \tilde{k}_1 \end{pmatrix} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 \end{pmatrix}.$$

*Moreover, the continued fraction expansion of $\frac{\tilde{r}}{n}$ has the same digits of that of $\frac{r}{n}$, but in reversed order.*

*Proof.* As already noted, the extended binary sequence $\sigma_1 \cdots \sigma_\ell 1$ of $\frac{r}{n}$ starts with $\sigma_1 = 1$. Then by reversing it we have another sequence starting by 1 and it represents a fraction $\frac{\tilde{r}}{n}$ in the same level of the Farey tree as $\frac{r}{n}$. By equation (3), the overall effect of this amounts simply to reverse the order of the partial quotients in the continued fractions expansion, that is

$$\frac{r}{n} = [a_1, \ldots, a_k] \quad \text{if and only if} \quad \frac{\tilde{r}}{n} = [a_k, \ldots, a_1].$$

Note that, since $1 \le r < \frac{n}{2}$ then $a_1 > 1$, so that the expansion for $\frac{\tilde{r}}{n}$ is unambiguously defined, and viceversa. Setting $p_{-1} = q_0 = 1$, $q_{-1} = p_0 = 0$, and $\frac{p_j}{q_j} = [a_1, \ldots, a_j]$, by the mirror formula (1) we have $[a_j, \ldots, a_1] = \frac{q_{j-1}}{q_j}$ for every $1 \le j \le k$. In particular $\frac{r}{n} = \frac{p_k}{q_k}$ and $\frac{\tilde{r}}{n} = \frac{q_{k-1}}{q_k}$.
By adapting some well known facts about the slow-additive-Farey continued fraction algorithm to the present context, we start setting

$$\Psi_h = \Phi_1^{h-1}\Phi_0 = \begin{pmatrix} 0 & 1 \\ 1 & h \end{pmatrix}$$

so that thanks to Equation (3) we can write

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \Psi_{a_1} \cdots \Psi_{a_k} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_{k-1} \\ q_k \end{pmatrix} = \Psi_{a_k} \cdots \Psi_{a_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since, as already noted, both $a_1$ and $a_k$ are larger than 1, we also have

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \Psi_{a_1} \cdots \Psi_{a_{k-1}} \Phi_1^{a_k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_{k-1} \\ q_k \end{pmatrix} = \Psi_{a_k} \cdots \Psi_{a_2} \Phi_1^{a_1-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We now show that the depth$(r/n) - 1 = \sum_{j=1}^{k} a_j - 2$ pairs of partitions of $n$ with two different parts generated by the dual pair $\frac{r}{n}$ and $\frac{\tilde{r}}{n}$ can be obtained as

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \Psi_{a_1} \cdots \Psi_{a_{k-j-1}} \Phi_1^{a_{k-j}-r-1} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_{k-1} \\ q_k \end{pmatrix} = \Psi_{a_k} \cdots \Psi_{a_{k-j+1}} \Phi_1^{r} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 \end{pmatrix} \tag{7}$$

for some $j = 0, \ldots, k-1$ and $r = 0, \ldots, a_{k-j} - 1$ (with $r \ge 1$ for $j = 0$ and $r \le a_1 - 2$ for $j = k-1$). We first note that the choice $(j, r) = (0, 0)$ yields the dual pairs $(1^n) \vdash n$ and $(n^1) \vdash n$. For $(j, r) = (0, 1)$, a straightforward calculation gives

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \Psi_{a_1} \cdots \Psi_{a_{k-1}} \Phi_1^{a_k-2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} p_k - 2p_{k-1} & p_{k-1} \\ q_k - 2q_{k-1} & q_{k-1} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} q_{k-1} \\ q_k \end{pmatrix} = \Phi_1 \begin{pmatrix} q_{k-1} \\ q_k - q_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_{k-1} \\ q_k - q_{k-1} \end{pmatrix}$$

thus producing the pair of dual partitions $(2^{q_{k-1}}, 1^{q_k - 2q_{k-1}}) \vdash n$ and $(q_k - q_{k-1}, q_{k-1}) \vdash n$. We can now proceed by induction. Suppose that, for some choice of $(j, r)$, (7) produces a pair of dual partitions $(n_1^{k_1}, n_2^{k_2})$ and $(\tilde{n}_1^{\tilde{k}_1}, \tilde{n}_2^{\tilde{k}_2})$, that is

$$\Psi_{a_1} \cdots \Psi_{a_{k-j-1}} \Phi_1^{a_{k-j}-r-1} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \begin{pmatrix} t & s \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \end{pmatrix}$$

and

$$\Psi_{a_k} \cdots \Psi_{a_{k-j+1}} \Phi_1^{r} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 \end{pmatrix} = \begin{pmatrix} \tilde{t} & \tilde{s} \\ \tilde{k}_2 & \tilde{k}_1 \end{pmatrix} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 \end{pmatrix}.$$

Assuming $r < a_{k-j} - 1$ we can make the transition from $(j, r)$ to $(j, r+1)$ and get

$$\Psi_{a_1} \cdots \Psi_{a_{k-j-1}} \Phi_1^{a_{k-j}-r-2} \begin{pmatrix} n_2 \\ n_1 + n_2 \end{pmatrix} = \begin{pmatrix} t - s & s \\ k_2 - k_1 & k_1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 + n_2 \end{pmatrix}$$

and

$$\Psi_{a_k} \cdots \Psi_{a_{k-j+1}} \Phi_1^{r+1} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 - \tilde{n}_2 \end{pmatrix} = \begin{pmatrix} \tilde{t} + \tilde{s} & \tilde{s} \\ \tilde{k}_2 + \tilde{k}_1 & \tilde{k}_1 \end{pmatrix} \begin{pmatrix} \tilde{n}_2 \\ \tilde{n}_1 - \tilde{n}_2 \end{pmatrix} = \begin{pmatrix} \tilde{t} + \tilde{s} & \tilde{s} \\ n_1 & n_2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix},$$

where the last identity takes into account the duality of the previous pair. As it can be easily checked, we obtain a new pair of dual partitions of $n$. A similar argument applies when $r = a_{k-j} - 1$ and the transition is from $(j, r)$ to $(j+1, 0)$. □

**Example 4.8.** We take $n = 11$, a prime number, so that we know that all the $p(2, 11) = 27$ partitions into two parts can be generated from the dynamics of the Farey map. In the following table we show the orbits generated by $\frac{r}{11}$, where we let $r = 1, \ldots, 5$, because Lemma 4.3 implies that for $r = 6, \ldots, 10$ the orbit generated by $\frac{r}{11}$ is pointwise the same as that generated by $\frac{11-r}{11}$.

| $m$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | $r = 5$ |
|---|---|---|---|---|---|
| 1 | $(10, 1)$ | $(9, 2)$ | $(8, 3)$ | $(7, 4)$ | $(6, 5)$ |
| 2 | $(9, 1^2)$ | $(7, 2^2)$ | $(5, 3^2)$ | $(4^2, 3)$ | $(5^2, 1)$ |
| 3 | $(8, 1^3)$ | $(5, 2^3)$ | $(3^3, 2)$ | $(3^3, 1^2)$ | $(4^2, 1^3)$ |
| 4 | $(7, 1^4)$ | $(3, 2^4)$ | $(2^4, 1^3)$ | $(2^3, 1^5)$ | $(3^2, 1^5)$ |
| 5 | $(6, 1^5)$ | $(2^5, 1)$ | | | $(2^2, 1^7)$ |
| 6 | $(5, 1^6)$ | | | | |
| 7 | $(4, 1^7)$ | | | | |
| 8 | $(3, 1^8)$ | | | | |
| 9 | $(2, 1^9)$ | | | | |

This table should clarify the structure of conjugate partitions. For instance, the fractions $\frac{2}{11}$ and $\frac{5}{11}$ generate orbits of dual partitions, in the sense of the previous Theorem 4.7. That is, if we read the column $r = 2$ from top to bottom we find the dual partitions of the column $r = 5$ read from bottom to top. The same holds for $r = 3$ and $r = 4$. For $r = 1$ we have an orbit of partitions which is self-dual, meaning that the dual of the $m$-th of the column partition is the $(10 - m)$-th one in the same column.

### 4.3 The extended Farey map

It will now be more convenient to write partitions as $(n_1^{k_1}, n_2^{k_2}) \vdash n$ with $n_1 > n_2 \geq 1$ as

$$(n_1, n_2) \times [k_1, k_2] \vdash n.$$

In part, this is to some extent placing the numbers $n_1$ and $n_2$ on the same footing as the multiplicities $k_1$ and $k_2$.

We start with extending the definition of the original Farey map $F$ to what we call the *extended Farey map*, which acts on partitions as follows:

$$\begin{aligned}\tilde{F}((n_1, n_2) \times [k_1, k_2]) &= \begin{cases} \tilde{F}_0((n_1, n_2) \times [k_1, k_2]) \,, & \text{if } n_2 \geq n_1 - n_2 \\ \tilde{F}_1((n_1, n_2) \times [k_1, k_2]) \,, & \text{if } n_1 - n_2 \geq n_2 \end{cases} \\ &= \begin{cases} (n_2, n_1 - n_2) \times [k_1 + k_2, k_1] \,, & \text{if } n_2 \geq n_1 - n_2 \\ (n_1 - n_2, n_2) \times [k_1, k_1 + k_2] \,, & \text{if } n_1 - n_2 \geq n_2 \end{cases}\end{aligned}$$

The action of $\tilde{F}$ on $(n_1, n_2)$ is just the action of the Farey map as described in Section 3.3. The corresponding action on the multiplicities $(k_1, k_2)$ is the one obtained as follows. We can write

$$n = k_1 n_1 + k_2 n_2 = \begin{pmatrix} k_2 \\ k_1 \end{pmatrix}^\top \begin{pmatrix} n_2 \\ n_1 \end{pmatrix},$$

so that if the action on $(n_1, n_2)$ is given by $F_i$, with $i = 0$ or 1, then

$$n = \begin{pmatrix} k_2 \\ k_1 \end{pmatrix}^\top \Phi_i F_i \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} = \left( \Phi_i^\top \begin{pmatrix} k_2 \\ k_1 \end{pmatrix} \right)^\top \left( F_i \begin{pmatrix} n_2 \\ n_1 \end{pmatrix} \right)$$

and the action on the multiplicities is that of $\Phi_i^\top$.

**Remark 4.9.** In our work for this paper we quickly arrived at the above map $\tilde{F}$, by simply finding out how the multiplicities should transform. But once we wrote it down, we saw that we were simply reproducing the natural extension of the Farey map, as described, for example, by Arnoux and Nogueira [3]. Natural extensions are a standard tool in dynamical systems which change $n$ to 1 maps into 1 to 1 maps by extending the dimension of the domain. This gives as an interpretation of the extened Farey map $\tilde{F}$ in terms of matrix multiplication if we write the partition $(n_1, n_2) \times [k_1, k_2]$ as the four-dimensional vector

$(n_2, n_1, k_2, k_1)$. Indeed we have

$$\tilde{F}\begin{pmatrix} n_2 \\ n_1 \\ k_2 \\ k_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} F_0 & 0 \\ 0 & \Phi_0^\top \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \\ k_2 \\ k_1 \end{pmatrix}, & \text{if } n_2 \geq n_1 - n_2 \\ \begin{pmatrix} F_1 & 0 \\ 0 & \Phi_1^\top \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \\ k_2 \\ k_1 \end{pmatrix}, & \text{if } n_1 - n_2 \geq n_2 \end{cases}$$

$$= \begin{cases} \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \\ k_2 \\ k_1 \end{pmatrix}, & \text{if } n_2 \geq n_1 - n_2 \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_2 \\ n_1 \\ k_2 \\ k_1 \end{pmatrix}, & \text{if } n_1 - n_2 \geq n_2 \end{cases} = \begin{cases} \begin{pmatrix} n_1 - n_2 \\ n_2 \\ k_1 \\ k_1 + k_2 \end{pmatrix}, & \text{if } n_2 \geq n_1 - n_2 \\ \begin{pmatrix} n_2 \\ n_1 - n_2 \\ k_1 + k_2 \\ k_1 \end{pmatrix}, & \text{if } n_1 - n_2 \geq n_2 \end{cases}$$

We wrote this fully out as this will directly generalize to higher dimensions, as we will see in Section 5.

The extended Farey map $\tilde{F}$ maps a partition of $n$ to a new partition of $n$. Indeed if $(n_1, n_2) \times [k_1, k_2] \vdash n$, a simple calculation shows that both

$$(n_2, n_1 - n_2) \times [k_1 + k_2, k_1] \quad \text{and} \quad (n_1 - n_2, n_2) \times [k_1, k_1 + k_2]$$

are again partitons of $n$. An immediate consequence of the definition given in Section 4.1 and of the present construction is that if $r$ is an integer such that $1 \leq r < n$ and $\gcd(r, n) = 1$ then repeatedly applying $\tilde{F}$ to $(n, r) \times [1, 0]$ yields exactly the orbit of partitions generated by dynamics of the Farey map starting from $\frac{r}{n}$. In particular, the partition $(n, r) \times [1, 0]$ eventually maps to $(1, 1) \times [k_1, k_2]$ for some $k_1$ and $k_2$.

**Example 4.10.** In Example 3.3 we considered the orbit of partitions of 19 generated by $\frac{8}{19}$. We now show the same orbit of partitions as it can be obtained through the map $\tilde{F}$:

$$\begin{aligned} (19, 8) \times [1, 0] &\xmapsto{\tilde{F}_1} (11, 8) \times [1, 1] \xmapsto{\tilde{F}_0} (8, 3) \times [2, 1] \xmapsto{\tilde{F}_1} (5, 3) \times [2, 3] \\ &\xmapsto{\tilde{F}_0} (3, 2) \times [5, 2] \xmapsto{\tilde{F}_0} (2, 1) \times [7, 5] \xmapsto{\tilde{F}_0} (1, 1) \times [12, 7]. \end{aligned}$$

In the last step, exactly as in the construction of Section 4.1, we could have used $\tilde{F}_1$ obtaining the partition $(1, 1) \times [7, 12]$.

In the more general setting of the present section there is no need to restrict ourselves to starting with a vector $(n, r)$ of relatively prime numbers, nor with the vector $[1, 0]$ for the multiplicities. This will allow us to consider also partitions with non-coprime numbers and/or multiplicities and will be key to proving the formula for $p(2, n)$ in Section 4.5. (Again, different formula for $[(2, n)$ was shown by Kim [18].)

**Lemma 4.11.** *Let $n \geq 2$ and $1 \leq r < n$. It holds that $\tilde{F}^m((n,r) \times [1,0]) = (n_1, n_2) \times [k_1, k_2]$ if and only if $\tilde{F}^m((dn, dr) \times [e,0]) = (dn_1, dn_2) \times [ek_1, ek_2]$ for every $d \geq 1$ and $e \geq 1$.*

*Proof.* The map $\tilde{F}$ is linear in all of its arguments. □

As a consequence of the previous lemma, we can now show that the iterations of the extended Farey map stop at some point. Suppose that $d = \gcd(n,r)$, let $n' = \frac{n}{d}$ and $r' = \frac{r}{d}$, so that $\gcd(n', r') = 1$. We know that

$$\tilde{F}^m((n', r') \times [1,0]) = (1,1) \times [k_1, k_2]$$

for some $k_1$ and $k_2$, and $m = \operatorname{depth}(r'/n') = \operatorname{depth}(r/n)$. Thus from Lemma 4.11 it follows that

$$\tilde{F}^m((n,r) \times [e,0]) = (d,d) \times [ek_1, ek_2].$$

In other words, the starting partition $(n,r) \times [e,0]$ is eventually mapped to a partition having equal numbers.

**Definition 4.12.** We shall call the sequence of partitions $F^m((n,r) \times [e,0])$ with $m = 1, \ldots, \operatorname{depth}\left(\frac{r}{n}\right)$ the *orbit of partitions generated by* $(n,r) \times [e,0]$ under the extended Farey map. If $(n_1, n_2) \times [k_1, k_2]$ eventually maps to $(m_1, m_2) \times [l_1, l_2]$ under the extended Farey map, then we say that $(n_1, n_2) \times [k_1, k_2]$ is an *ancestor* of $(m_1, m_2) \times [l_1, l_2]$ and that $(m_1, m_2) \times [l_1, l_2]$ is a *descendant* of $(n_1, n_2) \times [k_1, k_2]$.

We now show how Lemma 4.11 works in practice with some examples.

**Example 4.13.** Consider the partition

$$(10,6) \times [6,9] \vdash 114,$$

which is $(2 \cdot 5, 2 \cdot 3) \times [3 \cdot 2, 3 \cdot 3] \vdash 2 \cdot 3 \cdot 19$. We have

$$\begin{aligned}(38,16) \times [3,0] \xmapsto{\tilde{F}_1} (22,16) \times [3,3] \xmapsto{\tilde{F}_0} (16,6) \times [6,3] \xmapsto{\tilde{F}_1} (10,6) \times [6,9] \\ \xmapsto{\tilde{F}_0} (6,4) \times [15,6] \xmapsto{\tilde{F}_0} (4,2) \times [21,15] \xmapsto{\tilde{F}_0} (2,2) \times [36,21],\end{aligned}$$

ending up with the same orbit as $(19,8) \times [1,0]$ (as we saw in Example 4.10), but now multiplying all the parts by 2 and all the multiplicities by 3.

For orbits of partitions generated with the extended Farey map we have this more general version of Lemma 4.3.

**Proposition 4.14.** *The partitions $(n,r) \times [e,0]$ and $(n, n-r) \times [e,0]$ generate the same orbit of partitions.*

*Proof.* If $r = n - r$ then the result is obvious. So we can assume that $r > n - r$. Thus we apply $\tilde{F}_0$ to $(n,r) \times [e,0]$ and $\tilde{F}_1$ to $(n, n-r) \times [e,0]$ getting

$$\begin{aligned}\tilde{F}_0((n,r) \times [e,0]) &= (r, n-r) \times [e,e] = \\ &= (n - (n-r), n-r) \times [e,e] = \tilde{F}_1((n, n-r) \times [e,0]).\end{aligned}$$

Thus after the first application of the extended Farey map, we have the same partition. □

## 4.4 Conjugation and palindromes, again

The extended Farey map reflects and respects conjugation.

**Proposition 4.15.** *The diagram*

$$\begin{array}{rcl}
(n_1, n_2) \times [k_1, k_2] & \sim_{\mathcal{C}} & (k_1 + k_2, k_1) \times [n_2, n_1 - n_2] \\
\tilde{F}_0 \big\downarrow & & \big\uparrow \tilde{F}_0 \\
(n_2, n_1 - n_2) \times [k_1 + k_2, k_1] & \sim_{\mathcal{C}} & (2k_1 + k_2, k_1 + k_2) \times [n_1 - n_2, 2n_2 - n_1]
\end{array}$$

*when $n_2 \geq n_1 - n_2$, and the diagram*

$$\begin{array}{rcl}
(n_1, n_2) \times [k_1, k_2] & \sim_{\mathcal{C}} & (k_1 + k_2, k_1) \times [n_2, n_1 - n_2] \\
\tilde{F}_1 \big\downarrow & & \big\uparrow \tilde{F}_1 \\
(n_1 - n_2, n_2) \times [k_1, k_1 + k_2] & \sim_{\mathcal{C}} & (2k_1 + k_2, k_1) \times [n_2, n_1 - 2n_2]
\end{array}$$

*when $n_2 \leq n_1 - n_2$, are both commutative.*

*Proof.* It is a simple verification using the definition of the map $\tilde{F}$ and the conjugation rule. □

Commutative diagrams as the ones shown in the above proposition can be glued together following an orbit of partitions. As an example, consider

$$\begin{array}{rcl}
(19, 15) \times [1, 0] & \sim_{\mathcal{C}} & (1, 1) \times [15, 4] \\
\tilde{F}_0 \big\downarrow & & \big\uparrow \tilde{F}_0 \\
(15, 4) \times [1, 1] & \sim_{\mathcal{C}} & (2, 1) \times [4, 11] \\
\tilde{F}_1 \big\downarrow & & \big\uparrow \tilde{F}_1 \\
(11, 4) \times [1, 2] & \sim_{\mathcal{C}} & (3, 1) \times [4, 7] \\
\tilde{F}_1 \big\downarrow & & \big\uparrow \tilde{F}_1 \\
(7, 4) \times [1, 3] & \sim_{\mathcal{C}} & (4, 1) \times [4, 3] \\
\tilde{F}_0 \big\downarrow & & \big\uparrow \tilde{F}_0 \\
(4, 3) \times [4, 1] & \sim_{\mathcal{C}} & (5, 4) \times [3, 1] \\
\tilde{F}_0 \big\downarrow & & \big\uparrow \tilde{F}_0 \\
(3, 1) \times [5, 4] & \sim_{\mathcal{C}} & (9, 5) \times [1, 2] \\
\tilde{F}_1 \big\downarrow & & \big\uparrow \tilde{F}_1 \\
(2, 1) \times [5, 9] & \sim_{\mathcal{C}} & (14, 5) \times [1, 1] \\
\tilde{F}_0 \big\downarrow & & \big\uparrow \tilde{F}_0 \\
(1, 1) \times [14, 5] & \sim_{\mathcal{C}} & (19, 14) \times [1, 0]
\end{array}$$

We remark that the last two lines could also have been

$$\begin{array}{ccc}(2,1)\times[5,9] & \sim_{\mathcal{C}} & (14,5)\times[1,1]\\ \tilde{F}_1\big\downarrow & & \big\uparrow\tilde{F}_1\\ (1,1)\times[5,14] & \sim_{\mathcal{C}} & (19,5)\times[1,0].\end{array}$$

Hence the act of conjugation can be viewed as simply reversing the arrows of the extended map, and this can be described by another version of Theorem 4.7.

**Theorem 4.16** (palindromes Version 2)**.** *Suppose that $\tilde{F}_{\sigma_1},\ldots,\tilde{F}_{\sigma_\ell}$ is a sequence of extended Farey maps such that*

$$\tilde{F}_{\sigma_\ell}\circ\cdots\circ\tilde{F}_{\sigma_1}((n_1,n_2)\times[k_1,k_2])=(\bar{n}_1,\bar{n}_2)\times[\bar{k}_1,\bar{k}_2].$$

*Then*

$$\tilde{F}_{\sigma_1}\circ\cdots\circ\tilde{F}_{\sigma_\ell}((\bar{k}_1+\bar{k}_2,\bar{k}_1)\times[\bar{n}_2,\bar{n}_1-\bar{n}_2])=(k_1+k_2,k_1)\times[n_2,n_1-n_2].$$

*Proof.* It is a repeated application of Proposition 4.15. To shorten the notation, for $j=1,\ldots,\ell$ denote by $\lambda^{(j)}$ the partition $\tilde{F}_{\sigma_j}\circ\cdots\circ\tilde{F}_{\sigma_1}((n_1,n_2)\times[k_1,k_2])$ and by $\lambda_{\mathcal{C}}^{(j)}$ its conjugate. Now let $1\le j<\ell$ and suppose that

$$\tilde{F}_{\sigma_1}\circ\cdots\circ\tilde{F}_{\sigma_{j-1}}(\lambda_{\mathcal{C}}^{(j)})=\lambda_{\mathcal{C}}^{(1)}.$$

Since $\lambda^{(j+1)}=\tilde{F}_{\sigma_j}(\lambda^{(j)})$, Proposition 4.15 yields $\lambda_{\mathcal{C}}^{(j)}=\tilde{F}_{\sigma_j}(\lambda_{\mathcal{C}}^{(j+1)})$, so that

$$\lambda_{\mathcal{C}}^{(1)}=\tilde{F}_{\sigma_1}\circ\cdots\circ\tilde{F}_{\sigma_{j-1}}(\lambda_{\mathcal{C}}^{(j)})=\tilde{F}_{\sigma_1}\circ\cdots\circ\tilde{F}_{\sigma_j}(\lambda_{\mathcal{C}}^{(j+1)}),$$

proving the inductive step. □

As an application of the previous result about conjugation, we give a new version of Theorem 4.4.

**Theorem 4.17.** *Suppose that $(n_1,n_2)\times[k_1,k_2]\vdash n$ with $\gcd(n_1,n_2)=1$ and $\gcd(k_1,k_2)=1$. Then there exists some positive integer $r$ relatively prime to $n$ and such that $(n,r)\times[1,0]$ is an ancestor of $(n_1,n_2)\times[k_1,k_2]$.*

*Proof.* We start with the partition $(n_1,n_2)\times[k_1,k_2]\vdash n$ and consider its conjugate partition, namely $(k_1+k_2,k_1)\times[n_2,n_1-n_2]$, which is still a partition of $n$. Since $(k_1+k_2,k_1)=1$ we know that the fraction $\frac{k_1}{k_1+k_2}$ appears in some level of the Farey tree, thus there is a sequence of Farey matrices $F_{\sigma_1},\ldots,F_{\sigma_\ell}$ such that

$$F_{\sigma_\ell}\cdots F_{\sigma_1}\begin{pmatrix}k_1\\k_1+k_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}.$$

Hence for the extended maps we have

$$\tilde{F}_{\sigma_\ell}\circ\cdots\circ\tilde{F}_{\sigma_1}((k_1+k_2,k_1)\times[n_2,n_1-n_2])=(1,1)\times[r,s]$$

with $(1,1)\times[r,s]\vdash n$. Note that $r+s=n$ and that Lemma 4.11 implies that $(r,n)=1$. Now, the partition $(1,1)\times[r,s]$ is conjugate to $(r+s,r)\times[1,0]=(n,r)\times[1,0]$, thus by the Power of Palindromes we have

$$\tilde{F}_{\sigma_1}\cdots\tilde{F}_{\sigma_\ell}((n,r)\times[1,0])=(n_1,n_2)\times[k_1,k_2]$$

and we are done. □

### 4.5 Formula for $p(2,n)$

We now use the previous results to describe the partitions which can be obtained through the extended Farey map and derive the formula for $p(2,n)$.

**Theorem 4.18.** *Let $n \geq 2$ be an integer. Every partition of $n$ can be obtained from the dynamics of the extended Farey map $\tilde{F}$.*

*Proof.* Consider a partition $(n_1,n_2)\times[k_1,k_2]$ of $n$ and let $d=\gcd(n_1,n_2)$ and $e=\gcd(k_1,k_2)$. By setting $n_1'=\frac{n_1}{d}$, $n_2'=\frac{n_2}{d}$, $k_1'=\frac{k_1}{e}$, $k_2'=\frac{k_2}{e}$ we have that

$$(n_1',n_2')\times[k_1',k_2']\vdash\frac{n}{de}$$

is by construction a partition with relatively prime numbers and multiplicities. Thus there exists $h$ such that $\left(\frac{n}{de},h\right)\times[1,0]$ is an ancestor of $(n_1',n_2')\times[k_1',k_2']$ . Hence by Lemma 4.11 we have that

$$\left(\frac{n}{e},hd\right)\times[e,0]$$

is an ancestor of $(dn_1',dn_2')\times[ek_1',ek_2']=(n_1,n_2)\times[k_1,k_2]$. □

**Theorem 4.19.** *Let $n\geq 2$ be an integer number. Then*

$$p(2,n)=\frac{1}{2}\sum_{r=1}^{n-1}\left(\text{depth}\left(\frac{r}{n}\right)-1\right)\sigma_0((r,n)).$$

*Proof.* To shorten the notation, we denote by $\mathcal{O}(\lambda)$ the orbit of partition generated by the extended Farey map starting from the partition $\lambda$. We claim that the set of the partitions of $n$ into two parts is

$$P=\bigcup_{1\leq r\leq\frac{n}{2}}\ \bigcup_{e|\gcd(r,n)}\mathcal{O}\left(\left(\frac{n}{e},\frac{r}{e}\right)\times[e,0]\right).$$

It is clear that each partition of this set is a partition of $n$ into two parts. For the converse we consider a partition $(n_1,n_2)\times[k_1,k_2]$ of $n$. By arguing as in the proof of Theorem 4.18 we have that there exists $h$ such that $\left(\frac{n}{e},hd\right)\times[e,0]$ is an ancestor of our partition. Furthermore, it is possible to choose $h$ such that $1\leq h\leq\frac{n}{2de}$. Thus our partition $(n_1,n_2)\times[k_1,k_2]$ is in the orbit of $\left(\frac{n}{e},\frac{r}{e}\right)\times[e,0]$, where $r=hde$. We have that $de\leq r\leq\frac{n}{2}$ and also that $e$ is a divisor of $(r,n)$ since it divides both $n$ and $r$, thus our partition is in the set $P$.

To prove the formula for $p(2,n)$ it now suffices to count the elements of the above set $P$. For each $1\leq r\leq\frac{n}{2}$ we consider the $\sigma_0((r,n))$ disjoint orbits generated by $\left(\frac{n}{e},\frac{r}{e}\right)\times[e,0]$. Each of them contains

$$\text{depth}\left(\frac{r/e}{n/e}\right)-1=\text{depth}\left(\frac{r}{n}\right)-1$$

pairwise distinct partitions of $n$. Thus the number of partitions in $P$ is

$$\sum_{1\leq r\leq\frac{n}{2}}\ \sum_{e|\gcd(r,n)}\left(\text{depth}\left(\frac{r}{n}\right)-1\right)=\sum_{1\leq r\leq\frac{n}{2}}\left(\text{depth}\left(\frac{r}{n}\right)-1\right)\sigma_0((r,n)).$$

If we extend the first sum over $1\leq r\leq n-1$ we are counting each partition twice due to Proposition 4.14, for which $(n,r)\times[e,0]$ gives rise to the same descendants as $(n,n-r)\times[e,0]$. □

**Example 4.20.** Here are all the partitions of $n = 12$ into two numbers:

$$\begin{array}{cccc}
(11,1)\times[1,1] & (10,2)\times[1,1] & (10,1)\times[1,2] & (9,3)\times[1,1] \\
(9,1)\times[1,3] & (8,4)\times[1,1] & (8,2)\times[1,2] & (8,1)\times[1,4] \\
(7,5)\times[1,1] & (7,1)\times[1,5] & (6,3)\times[1,2] & (6,2)\times[1,3] \\
(6,1)\times[1,6] & (5,2)\times[2,1] & (5,1)\times[2,2] & (5,1)\times[1,7] \\
(4,2)\times[2,2] & (4,1)\times[2,4] & (4,2)\times[1,4] & (4,1)\times[1,8] \\
(3,1)\times[3,3] & (3,2)\times[2,3] & (3,1)\times[2,6] & (3,1)\times[1,9] \\
(2,1)\times[5,2] & (2,1)\times[4,4] & (2,1)\times[3,6] & (2,1)\times[2,8] \\
(2,1)\times[1,10] & & &
\end{array}$$

Thus $p(2, 12) = 29$. Indeed we have

$$\begin{aligned}
29 = &\left(\operatorname{depth}\left(\frac{1}{12}\right) - 1\right)\sigma_0(\gcd(1,12)) + \left(\operatorname{depth}\left(\frac{2}{12}\right) - 1\right)\sigma_0(\gcd(2,12)) + \\
&+ \left(\operatorname{depth}\left(\frac{3}{12}\right) - 1\right)\sigma_0(\gcd(3,12)) + \left(\operatorname{depth}\left(\frac{4}{12}\right) - 1\right)\sigma_0(\gcd(4,12)) + \\
&+ \left(\operatorname{depth}\left(\frac{5}{12}\right) - 1\right)\sigma_0(\gcd(5,12)) + \left(\operatorname{depth}\left(\frac{6}{12}\right) - 1\right)\sigma_0(\gcd(6,12)).
\end{aligned}$$

Let us see how each fraction $\frac{r}{12}$ with $1 \le r \le 6$ will produce $\left(\operatorname{depth}\left(\frac{r}{12}\right) - 1\right)\sigma_0((r, 12))$ distinct partitions of 12 into two parts. We start with the two values of $r$ for which $\sigma_0((r, 12)) = 1$ namely $r = 1$ and $r = 5$. We have

$$\begin{aligned}
(12,1)\times[1,0] &\xmapsto{\tilde{F}_1} (11,1)\times[1,1] \xmapsto{\tilde{F}_1} (10,1)\times[1,2] \\
&\xmapsto{\tilde{F}_1} (9,1)\times[1,3] \xmapsto{\tilde{F}_1} (8,1)\times[1,4] \xmapsto{\tilde{F}_1} (7,1)\times[1,5] \\
&\xmapsto{\tilde{F}_1} (6,1)\times[1,6] \xmapsto{\tilde{F}_1} (5,1)\times[1,7] \xmapsto{\tilde{F}_1} (4,1)\times[1,8] \\
&\xmapsto{\tilde{F}_1} (3,1)\times[1,9] \xmapsto{\tilde{F}_1} (2,1)\times[1,10]
\end{aligned}$$

This accounts for precisely $\operatorname{depth}(1/12) - 1 = 10$ of the desired partitions. Further, each of these will occur uniquely. Similarly, for $r = 5$ we have

$$(12,5)\times[1,0] \xmapsto{\tilde{F}_1} (7,5)\times[1,1] \xmapsto{\tilde{F}_0} (5,2)\times[2,1] \xmapsto{\tilde{F}_1} (3,2)\times[2,3] \xmapsto{\tilde{F}_0} (2,1)\times[5,2]$$

giving us $\operatorname{depth}(5/12) - 1 = 4$ additional partitions. Now we consider $r = 2$ and see the partitions linked to the fraction $2/12$. In this case $\gcd(r, n) = 2$, so that we have two different choices for $e$, namely $e = 1$ and $e = 2$. Choosing $e = 1$ we start with

$$(12,2)\times[1,0] \xmapsto{\tilde{F}_1} (10,2)\times[1,1] \xmapsto{\tilde{F}_1} (8,2)\times[1,2] \xmapsto{\tilde{F}_1} (6,2)\times[1,3] \xmapsto{\tilde{F}_1} (4,2)\times[1,4]$$

giving us $\operatorname{depth}(2/12) - 1 = 4$ additional partitions. Choosing $e = 2$ we also have

$$(6,1)\times[2,0] \xmapsto{\tilde{F}_1} (5,1)\times[2,2] \xmapsto{\tilde{F}_1} (4,1)\times[2,4] \xmapsto{\tilde{F}_1} (3,1)\times[2,6] \xmapsto{\tilde{F}_1} (2,1)\times[2,8]$$

giving us another 4 partitions. Now we consider $r = 3$, that is the fraction $3/12$. For $e = 1$ we have

$$(12,3) \times [1,0] \overset{\tilde{F}_1}{\longmapsto} (9,3) \times [1,1] \overset{\tilde{F}_1}{\longmapsto} (6,3) \times [1,2]$$

giving us $\mathrm{depth}\,(3/12) - 1 = 2$ partitions. But we also have the choice $e = 3$, yielding

$$(4,1) \times [3,0] \overset{\tilde{F}_1}{\longmapsto} (3,1) \times [3,3] \overset{\tilde{F}_1}{\longmapsto} (2,1) \times [3,6]$$

giving us 2 more partitions. The remaining cases $r = 4$, $r = 5$ and $r = 6$ work in the same way and complete the list of partitions of 12.

# 5 On partitions into many parts

In this section we begin the extension of the construction explained in Sections 3 and 4. The Farey map may be defined to act on a cone in $\mathbb{R}^2$ as in Section 3.3, and we used it to generate partitions into two parts. To generate partitions into $n$ different parts it is necessary to consider a map acting on a subset of $\mathbb{R}^N$. To this aim we consider the Triangle map and its slow version studied in [7, 15].

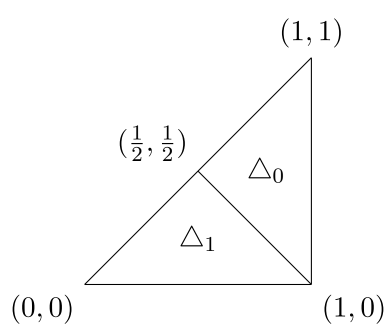
## 5.1 Background on the additive-slow-Triangle map

The Triangle map has been introduced in [15] to define a type of multidimensional continued fraction algorithm. Multidimensional continued fractions have been developed and studied over the years for many reasons (for a background see Schweiger [28] or Karpenkov [21]). Historically, the first such algorithm (now called the Jacobi-Perron algorithm) was created to answer a question of Hermite, which was to find an analogue of the classical fact that a real number has an eventually periodic continued fraction expansion if and only if the number is a quadratic irrational. This problem is still open. Another motivation was to find methods for good simultaneous Diophantine approximations of $n$-tuples of real numbers (for example, see Lagarias [23]). By now, multidimensional continued fractions provide a rich source of examples in dynamical systems, automata theory and many other areas. For background on and applications of the Triangle map for multidimensional continued fractions, see [4, 6–8, 14, 15, 20, 21, 27, 29]. Almost all of these papers are concerned with the three dimensional case.

Set

$$\begin{aligned}
\triangle &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 1 > x_1 > \cdots > x_n > 0\} \\
\triangle_0 &:= \{x_1, \ldots, x_n) \in \triangle : x_1 + x_n > 1\} \\
\triangle_1 &:= \{x_1, \ldots, x_n) \in \triangle : x_1 + x_n < 1\}
\end{aligned}$$

When $n = 2$, we have



The slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ is

$$\begin{aligned} T(x_1, \ldots, x_n) &= \begin{cases} T_0(x_1, \ldots, x_n), & \text{if } x_1 + x_n > 1 \\ T_1(x_1, \ldots, x_n), & \text{if } x_1 + x_n < 1 \end{cases} \\ &= \begin{cases} \left(\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}, \frac{1-x_1}{x_1}\right), & \text{if } x_1 + x_n > 1 \\ \left(\frac{x_1}{1-x_n}, \ldots, \frac{x_n}{1-x_n}\right), & \text{if } x_1 + x_n < 1 \end{cases} \end{aligned}$$

It can be checked that $T_i : \triangle_i \to \triangle$ is one-to-one and onto. In analogy with the construction in Section 3.2, any point $\bar{x} \in \triangle$ is associated to a binary sequence of zeros and ones $i(\bar{x}) = (i_0, i_1, i_2, \ldots)$ by encoding whether its iterations fall in $\triangle_0$ or $\triangle_1$, that is by the rule $T^n(\bar{x}) \in \triangle_{i_n}$. We call $i(\bar{x})$ the additive-slow-Triangle sequence of $\bar{x}$. If we concatenate the 1's we can associate $\bar{x}$ to a sequence of nonnegative integers, a sequence that is called the multiplicative-fast-Triangle sequence and is the analogue of the continued fraction expansion of a real number. Either of these sequences tells us a lot about the point $\bar{x}$. For example, when $n = 3$, if the sequence is eventually periodic, then both $x_1$ and $x_2$ are no worse than cubic irrationals, both in the same number field of degree less than or equal to three.

**Remark 5.1.** With respect to [7, 15] we have not defined the map $T$ on the boundary of $\triangle$ and on the hyperplane $x_1 + x_n = 1$. In earlier work, such points are a set of measure zero and hence are ignored. This creates a problem, though, when we start to link these maps with partitions, as we will discuss in Section 6.

It is natural, and in analogy to Section 3.3 for the Farey map, to pass from points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ to vectors $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ (or points $(x_0, \ldots, x_n)$ in $\mathbb{RP}^{n+1}$) via sending $(x_1, \ldots, x_n)$ to $(1, x_1, \ldots, x_n)$ with inverse map $(x_0, \ldots, x_n) \to (x_1/x_0, \ldots, x_n/x_0)$. Then, by an abuse of notation, we set

$$\begin{aligned} \triangle &:= \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0 > x_1 > \cdots > x_n > 0\} \\ \triangle_0 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n > x_0\} \\ \triangle_1 &:= \{(x_0, \ldots, x_n) \in \triangle : x_1 + x_n < x_0\} \end{aligned}$$

and define the slow-Triangle map $T : \triangle_0 \cup \triangle_1 \to \triangle$ by

$$\begin{aligned} T(x_0, \ldots, x_n) &= \begin{cases} T_0(x_0, \ldots, x_n), & \text{if } x_1 + x_n > x_0 \\ T_1(x_0, \ldots, x_n), & \text{if } x_1 + x_n < x_0 \end{cases} \\ &= \begin{cases} (x_1, x_2, \ldots, x_n, x_0 - x_1), & \text{if } x_1 + x_n > x_0 \\ (x_0 - x_n, x_1, x_2, \ldots, x_n), & \text{if } x_1 + x_n < x_0 \end{cases} \end{aligned}$$

By writing the row vector $(x_0, \ldots, x_n)$ instead as the column vector $\overline{x}$, the action of $T$ is given by left multiplication by $(n+1) \times (n+1)$ matrices:

$$T(\overline{x}) = \begin{cases} T_0(\overline{x}), & \text{if } x_1 + x_n > x_0 \\ T_1(\overline{x}), & \text{if } x_1 + x_n < x_0 \end{cases}$$

where

$$T_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

For later use, note that

$$\tau_0 := T_0^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau_1 := T_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

As a technical aside, it is important that both the matrices $\tau_0$ and $\tau_1$ have nonnegative entries, as it is in the case for $\Phi_0$ and $\Phi_1$ in Section 3.3. As we will see, this is what allows the Triangle map to be used to understand partitions. This is not necessarily the case for all multidimensional continued fraction algorithms.

Following the construction in Section 4, we jump to the definition of the extended map given in Section 4.3. The *extended slow-Triangle map* $\tilde{T}$, which can be thought of as the natural extension of $T$, is defined by

$$\begin{aligned} \tilde{T}((n_1,\dots,n_m)\times[k_1,\dots,k_m]) &= \begin{cases} \tilde{T}_0((n_1,\dots,n_m)\times[k_1,\dots,k_m]), & \text{if } n_2+n_m>n_1 \\ \tilde{T}_1((n_1,\dots,n_m)\times[k_1,\dots,k_m]), & \text{if } n_2+n_m<n_1 \end{cases} \\ &= \begin{cases} (n_2,n_3,\dots,n_m,n_1-n_2),\times[k_1+k_2,k_3,k_4,\dots,k_m,k_1], \\ \qquad \text{if } n_2+n_m>n_1 \\ (n_1-n_m,n_2,n_3,\dots,n_m)\times[k_1,\dots,k_{m-1},k_1+k_m], \\ \qquad \text{if } n_2+n_m<n_1 \end{cases} \end{aligned}$$

which can be read as the action of two $m\times m$ matrices on column vectors in $\mathbb{R}^{2m}$, with the matrices

$$\begin{pmatrix} T_0 & 0 \\ 0 & \tau_0^\top \end{pmatrix}, \begin{pmatrix} T_1 & 0 \\ 0 & \tau_1^\top \end{pmatrix}.$$

## 5.2 Link with integer partitions

We now simply repeat what we did in Section 4, but now for the slow-Triangle map. Consider a partition $(n_1^{k_1},\dots,n_m^{k_m}) \vdash n$ with $n_1 > \cdots > n_m$, also written as $(n_1,\dots,n_m)\times[k_1,\dots,k_m] \vdash n$. As $n_1 > \cdots > n_m > 0$, we can act on by the extended map. The key, both in the definition of the natural extension and for our use in partition theory, is that if

$$(n_1^{k_1},\dots,n_m^{k_m}) \vdash n,$$

then

$$(n_2^{k_1+k_2}, n_3^{k_3},\dots,n_m^{k_m},(n_1-n_2)^{k_1}) \vdash n$$

and

$$((n_1 - n_m)^{k_1}, n_2^{k_2}, \ldots, n_{m-1}^{k_m - 1}, n_m^{k_1 + k_m}) \vdash n.$$

This proves the following result.

**Proposition 5.2.** *The extended slow-Triangle map $\tilde{T}$ sends a partition of $n$ to a new partition of $n$. Thus if $(n_1, \ldots, n_m) \times [k_1, \ldots, k_m] \vdash n$, then*

$$\tilde{T}((n_1, \ldots, n_m) \times [k_1, \ldots, k_m]) \vdash n.$$

We can iterate the extended map $\tilde{T}$ and create an orbit of partitions. As before, start with some

$$(n_1, \ldots, n_m) \times [1, 0, \ldots, 0] \vdash n.$$

This a 0th generation partition, which we write as

$$(n_1(0), \ldots, n_m(0)) \times [k_1(0), \ldots, k_m(0)].$$

Acting on this vector by $\tilde{T}$ gets us the first generation partition

$$\tilde{T}((n_1(0), \ldots, n_m(0)) \times [k_1(0), \ldots, k_m(0)]) = (n_1(1), \ldots, n_m(1)) \times [k_1(1), \ldots, k_m(1)].$$

and recursively we obtain $(n_1(a), \ldots, n_m(a)) \times [k_1(a), \ldots, k_m(a)]$ for integers $a \geq 1$. This relates the dynamics of the map $T$ with the sequence of partitions obtained by $\tilde{T}$.

An example is :

| $a$ | $(x, y)$ | $n_1(a)$ | $n_2(a)$ | $n_3(a)$ | $k_1(a)$ | $k_2(a)$ | $k_3(a)$ | $\tilde{T}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $(9/11, 4/11)$ | 11 | 9 | 4 | 1 | 0 | 0 | $\tilde{T}_0$ |
| 1 | $(4/9, 2/9)$ | 9 | 4 | 2 | 1 | 0 | 1 | $\tilde{T}_1$ |
| 2 | $(4/7, 2/7)$ | 7 | 4 | 2 | 1 | 0 | 2 | $\tilde{T}_1$ |
| 3 | $(4/5, 2/5)$ | 5 | 4 | 2 | 1 | 0 | 3 | $\tilde{T}_0$ |
| 4 | $(2/4, 1/4)$ | 4 | 2 | 1 | 1 | 3 | 1 | $\tilde{T}_1$ |
| 5 | $(2/3, 1/3)$ | 3 | 2 | 1 | 1 | 3 | 2 | |

We stop here, for now, as $(2/3, 1/3)$ is on the line $x + y = 1$. We will deal with these boundary type points in Section 6.

Another example, in one dimension higher, is

| $a$ | $(x, y, z)$ | $n_1(a)$ | $n_2(a)$ | $n_3(a)$ | $n_4(a)$ | $k_1(a)$ | $k_2(a)$ | $k_3(a)$ | $k_4(a)$ | $\tilde{T}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $(7/14, 6/14, 5/14)$ | 14 | 7 | 6 | 5 | 1 | 0 | 0 | 0 | $\tilde{T}_1$ |
| 1 | $(6/9, 5/9, 2/9)$ | 9 | 7 | 6 | 5 | 1 | 0 | 0 | 1 | $\tilde{T}_0$ |
| 2 | $(6/7, 5/7, 2/7)$ | 7 | 6 | 5 | 2 | 1 | 0 | 1 | 1 | $\tilde{T}_0$ |
| 3 | $(5/6, 2/6, 1/6)$ | 6 | 5 | 2 | 1 | 1 | 1 | 1 | 1 | |

Here we stop, as $5/6 + 1/6 = 1$ and is hence on the line $x + z = 1$. Again, in Section 6 we will see how to continue this orbit.

In analogue to Proposition 4.14 we have:

**Proposition 5.3.** *For $n_2 + n_m > n_1$, the partitions*

$$(n_1, \ldots, n_m) \times [1, 0, \ldots, 0] \quad \textit{and} \quad (n_1, n_3, \ldots, n_m, n_1 - n_2) \times [1, 0, \ldots, 0]$$

*will have the same generations after generation* 0.

*Proof.* As $n_2 + n_m > n_1$, we first apply $\tilde{T}_0$ to $(n_1, \ldots, n_m) \times [1, 0, \ldots, 0]$ and $\tilde{T}_1$ to $(n_1, n_3, \ldots, n_m, n_1 - n_2) \times [1, 0, \ldots, 0]$ to get

$$\begin{aligned} \tilde{T}_0((n_1, \ldots, n_m) \times [1, 0, \ldots, 0] &= (n_2, n_3, \ldots, n_m, n_1 - n_2) \times [1, 0, \ldots, 0, 1] \\ &= \tilde{T}_1((n_1, n_3, \ldots, n_m, n_1 - n_2) \times [1, 0, \ldots, 0]). \end{aligned}$$

Thus after the first application of the extended slow-Triangle map, we have the same partition. □

### 5.3 Conjugation and Palindromes for the slow-additive-Triangle map

Here we extend the results of Section 4.4 to the slow-Triangle map. A partition $(n_1, \ldots, n_m) \times [k_1 \ldots, k_m]$ is conjugate to

$$(k_1 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots k_1) \times [n_m, n_{m-1} - n_m, \ldots, n_1 - n_2]$$

(see Section 2). We again have that the extended slow-Triangle map reflects and respects conjugation.

**Proposition 5.4.** *The diagram*

$$\begin{array}{ccc} (n_1, \ldots, n_m) \times [k_1 \ldots, k_m] & \sim_{\mathcal{C}} & (k_1 + \ldots + k_m, \ldots k_1) \times [n_m, n_{m-1} - n_m, \ldots, n_1 - n_2] \\ \tilde{T}_0 \big\downarrow & & \big\uparrow \tilde{T}_0 \\ (n_2, \ldots, n_1 - n_2) \times [k_1 + k_2, k_3, \ldots, k_m, k_1] & \sim_{\mathcal{C}} & (2k_1 + k_2 + \ldots + k_m, k_1 + k_2 + \ldots + k_m, k_1 + k_2) \\ & & \times [n_1 - n_2, n_{m-1} + n_2 - n_1, \ldots, n_2 - n_3] \end{array}$$

*when* $n_2 + n_m > n_1$ *and the diagram*

$$\begin{array}{ccc} (n_1, \ldots, n_m) \times [k_1 \ldots, k_m] & \sim_{\mathcal{C}} & (k_1 + \ldots + k_m, \ldots k_1) \times [n_m, n_{m-1} - n_m, \ldots, n_1 - n_2] \\ \tilde{T}_1 \big\downarrow & & \big\uparrow \tilde{T}_1 \\ (n_1 - n_m, n_2, \ldots, n_m) \times [k_1, \ldots, k_{m-1}, k_1 + k_m] & \sim_{\mathcal{C}} & (2k_1 + k_2 + \ldots + k_m, k_1 + \ldots + k_{m-1}, \ldots, k_1) \\ & & \times [n_m, n_{m-1} - n_m, \ldots, n_1 - n_m - n_2] \end{array}$$

*when* $n_2 + n_m < n_1$ *are both commutative.*

As before, the proof is a simple calculation. Thus for the slow-Triangle map we still have that the act conjugation can be viewed as simply reversing the arrows of the extended map, giving us the following result.

**Theorem 5.5** (The Power of Palindromes: Higher Dimension)**.** *Suppose that* $\tilde{T}_{i_1}, \ldots, \tilde{T}_{i_N}$ *is a sequence of extended slow-Triangle maps such that*

$$\tilde{T}_{i_N} \circ \cdots \circ \tilde{T}_{i_1}((n_1, \ldots, n_m) \times [k_1 \ldots, k_m]) = (\bar{n}_1, \ldots, \bar{n}_m) \times [\bar{k}_1 \ldots, \bar{k}_m]$$

*Then*

$$\begin{aligned} &\tilde{T}_{i_1} \circ \cdots \circ \tilde{T}_{i_N}((\bar{k}_1 + \ldots + \bar{k}_m, \ldots \bar{k}_1) \times [\bar{n}_m, \bar{n}_{m-1} - \bar{n}_m, \ldots, \bar{n}_1 - \bar{n}_2]) \\ &= (k_1 + \ldots + k_m, \ldots k_1) \times [n_m, n_{m-1} - n_m, \ldots, n_1 - n_2] \end{aligned}$$

# 6 The map $\tilde{T}_D$ on the boundary and "Allowable Partitions"

## 6.1 The definition and naturalness of $\tilde{T}_D$

The original additive-slow-Triangle map $T$ acting on the space $x_0 > x_1 > \cdots > x_m > 0$ is simply not defined on the hyperplane $x_0 = x_1 + x_m$. Before now, much of the work on this map has been concerned with it as a dynamical system. As this hyperplane has measure zero, it could be conveniently ignored. But when applied to partitions, this means we would be ignoring many perfectly reasonable partitions. In this section we will show how to extend the map $\tilde{T}$ to the hyperplane $x_0 = x_1 + x_m$. (This map was originally defined in work done in parallel to this paper in [5].)

We start with an example. Consider the partition $(6, 5, 2, 1) \times [1, 1, 1, 1] \vdash 14$. If we naively apply $\tilde{T}_0$ and $\tilde{T}_0$, we have

$$(6, 5, 2, 1) \times [1, 1, 1, 1] \xrightarrow{\tilde{T}_0} (5, 2, 1, 1) \times [2, 1, 1, 1]$$
$$(6, 5, 2, 1) \times [1, 1, 1, 1] \xrightarrow{\tilde{T}_1} (5, 5, 2, 1) \times [1, 1, 1, 2]$$

Both of these images would be a strange way for writing the partitions, as the natural notation would be to concatenate common terms. But after concatenation, we get the same partition:

$$(6, 5, 2, 1) \times [1, 1, 1, 1] \to (5, 2, 1) \times [2, 1, 2].$$

This leads to the following map $\tilde{T}_D$:

$$\tilde{T}_D((n_1, \ldots, n_m) \times [k_1, \ldots, k_m]) = (n_2, \ldots, n_m) \times [k_1 + k_2, k_3, \ldots, k_{m-1}, k_1 + k_m]$$

whenever

$$n_1 = n_2 + n_m.$$

This allows us to extend the orbits in 5.2:

| $a$ | $(x, y)$ | $n_1(a)$ | $n_2(a)$ | $n_3(a)$ | $k_1(a)$ | $k_2(a)$ | $k_3(a)$ | $\tilde{T}$ |
|---|---|---|---|---|---|---|---|---|
| 0 | (9/11, 4/11) | 11 | 9 | 4 | 1 | 0 | 0 | $\tilde{T}_0$ |
| 1 | (4/9, 2/9) | 9 | 4 | 2 | 1 | 0 | 1 | $\tilde{T}_1$ |
| 2 | (4/7, 2/7) | 7 | 4 | 2 | 1 | 0 | 2 | $\tilde{T}_1$ |
| 3 | (4/5, 2/5) | 5 | 4 | 2 | 1 | 0 | 3 | $\tilde{T}_0$ |
| 4 | (2/4, 1/4) | 4 | 2 | 1 | 1 | 3 | 1 | $\tilde{T}_1$ |
| 5 | (2/3, 1/3) | 3 | 2 | 1 | 1 | 3 | 2 | $\tilde{T}_D$ |
| 6 | (1/2) | 2 | 1 | | 4 | 3 | | |

and

| $a$ | $(x,y,z)$ | $n_1(a)$ | $n_2(a)$ | $n_3(a)$ | $n_4(a)$ | $k_1(a)$ | $k_2(a)$ | $k_3(a)$ | $k_4(a)$ | $\tilde{T}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $(7/14, 6/14, 5/14)$ | 14 | 7 | 6 | 5 | 1 | 0 | 0 | 0 | $\tilde{T}_1$ |
| 1 | $(7/9, 6/9, 5/9)$ | 9 | 7 | 6 | 5 | 1 | 0 | 0 | 1 | $\tilde{T}_0$ |
| 2 | $(6/7, 5/7, 2/7)$ | 7 | 6 | 5 | 2 | 1 | 0 | 1 | 1 | $\tilde{T}_0$ |
| 3 | $(5/6, 2/6, 1/6)$ | 6 | 5 | 2 | 1 | 1 | 1 | 1 | 1 | $\tilde{T}_D$ |
| 4 | $(2/5, 1/5)$ | 5 | 2 | 1 | | 2 | 1 | 2 | | $\tilde{T}_1$ |
| 5 | $(2/4, 1/4)$ | 4 | 2 | 1 | | 2 | 1 | 4 | | $\tilde{T}_1$ |
| 6 | $(2/3, 1/3)$ | 3 | 2 | 1 | | 2 | 1 | 6 | | $\tilde{T}_D$ |
| 7 | $(1/2)$ | 2 | 1 | | | 3 | 8 | | | |

Thus changing the dimension allows us to create longer trees.

### 6.2 Allowable partitions

When first starting to explore the link between the slow-Triangle map and partitions, we did many examples. We set up charts of all possible orbits for various $(n_1, n_2, n_3) \times [1, 0, 0]$ and we quickly realized that the multiplicity vectors $[k_1, k_2, k_3]$ in these orbits could only fit certain patterns. These difficulties remain in higher dimensions.

**Definition 6.1.** A vector of multiplicities $[k_1, \ldots, k_m]$ is *allowable* if there are integers $n_1 > \cdots > n_m > 0$ with $n_1 \neq n_2 + n_m$ and integers $a_1 > \cdots > a_m > 0$ such that the partition $(a_1, \ldots, a_m) \times [k_1, \ldots, k_m]$ is a descendant of $(n_1, \ldots, n_m) \times [1, 0, \ldots, 0]$.

This leads to the natural question of which partitions into $m$ parts stem from an iteration of the extended slow-Triangle map starting from a root $(n_1, \ldots, n_m) \times [1, 0, \ldots, 0]$. What matters here, as we will see, is the multiplicity vector $[k_1, \ldots, k_m]$. Under $\tilde{T}_0$,

$$[k_1, \ldots, k_m] \to [k_1 + k_2, k_3, \ldots, k_m, k_1]$$

and under $\tilde{T}_1$,

$$[k_1, \ldots, k_m] \to [k_1, k_2, \ldots, k_{m-1}, k_m + k_1].$$

For notation, suppose we are iterating a partition. At the *pth* step, label the corresponding multiplicity as

$$[k_1(p), \ldots, k_m(p)].$$

Further, recall that

$$[k_1(0), \ldots, k_m(0)] = [1, 0, \ldots, 0].$$

**Proposition 6.2.** *For any orbit with root* $(n_1, \ldots, n_m) \times [1, 0, \ldots, 0]$ *we have*

(i) *For all* $p$, $k_1(p) > 0$.

(ii) *If there is an* $i > 1$ *so that*

$$k_i(p), k_{i+1}(p), \ldots, k_m(p) > 0,$$

*then*

$$k_i(p+1), k_{i+1}(p+1), \ldots, k_m(p+1) > 0.$$

(iii) *If* $k_1(p) < k_1(p+1)$, *then* $k_2(p) > 0$.

(iv) *A vector* $[k_1, 0, k_3, \ldots, k_m]$ *with* $k_1 > 1$ *is not an allowable multiplicity.*

(v) *A vector* $[k_1, \ldots, k_{m-1}, 0]$ *with* $k_2 > 0$, *is not an allowable multiplicity.*

(vi) *A vector* $[k, k_2, \ldots, k_{m-1}, k]$ *with* $k > 0$ *is not an allowable multiplicity.*

*Proof.* By simply looking at the maps $\tilde{T}_0$ and $\tilde{T}_1$, we can see that the first condition is true.

Noting that $\tilde{T}_0$ sends the first term in the multiplicity to the last and shifts to the left the others, and that $\tilde{T}_1$ leave all the terms alone, save for adding the first term to the last, we see that the second and third conditions are true.

The first three conditions give us the fourth condition, since the initial first term must be 1.

The fifth condition is true since applying either $\tilde{T}_0$ or $\tilde{T}_1$ to the initial $[1, 0, \ldots, 0]$ will give us

$$[1, 0, \ldots, 1],$$

meaning that the last term can never return to zero.

To show the sixth condition, suppose that $[k, k_2, \ldots, k_{m-1}, k,]$ is allowable. Then it is the image of an allowable multiplicity from $\tilde{T}_0$ or from $\tilde{T}_1$. Now, the inverse of this multiplicity by $\tilde{T}_0$ is

$$[k, 0, k_2, \ldots, k_{m-1}]$$

which is not allowable. The inverse of this multiplicity by $\tilde{T}_1$ is

$$[k, k_2, \ldots, k_{m-1}, 0]$$

which is also not allowable.

□

As an example, this means that the partition of 20 given by

$$(5, 4, 3) \times [2, 1, 2]$$

cannot be obtained from any possible sequences of $\tilde{T}_0$ and $\tilde{T}_1$ from any

$$(20, n_2, n_3) \times [1, 0, 0].$$

### 6.3 Capturing non-allowable partitions

We just saw that the partition $(5, 2, 1) \times [2, 1, 2] \vdash 14$ is the image of $(6, 5, 2, 1) \times [1, 1, 1, 1]$ under the map $\tilde{T}_D$. But Proposition 6.2 seems to state that $(5, 2, 1) \times [2, 1, 2]$ is nonallowable. Of course this is not at all a contradiction, as Proposition 6.2 is only about possible images of $\tilde{T}_0$ and $\tilde{T}_1$. This "merging dimensions" allows us to capture all partitions.

Part of this comes down to the inverses of the three maps $\tilde{T}_0$, $\tilde{T}_1$, which are $\tilde{T}_D$. As described in [5], the maps $\tilde{T}_0$ and $\tilde{T}_1$ are one-to-one, each having the following inverses: for $k_1 > k_m$,

$$(n_1, \ldots, n_m) \times [k_1, \ldots, k_m] \xrightarrow{\tilde{T}_0^{-1}} (n_1 + n_m, n_1, n_2, n_3, \ldots, n_{m-1}) \times [k_m, k_1 - k_m, k_2, k_3, \ldots, k_{m-1}]$$

and for $k_1 < k_m$ ,

$$(n_1, \ldots, n_m) \times [k_1, \ldots, k_m] \xrightarrow{\tilde{T}_1^{-1}} (n_1 + n_m, n_2, \ldots, \ldots, n_m) \times [k_1, \ldots, \ldots, k_{m-1}, k_m - k_1].$$

The map $\tilde{T}_D$ that changes dimensions though is not one-to-one. If

$$K = \min(k_1 + k_2, k_1 + k_m) - 1,$$

then $T_D$ is a $K$ to one map. Explicitly, for any $k = 1, \ldots, \min(k_1, k_{m-1}) - 1$, an inverse is

$$(n_1, \ldots, n_m) \times [k_1, \ldots, k_m] \xrightarrow{\tilde{T}_D^{-1}} (n_1 + n_m, n_1, \ldots, n_{m-1}) \times [k, k_1 - k, k_2, \ldots, k_{m-2}, k_{m-1} - k].$$

This allows us to fit any partition into an eventual image of some $(n, n_2, \ldots, n_m) \times [1, \ldots, 0]$. This creates a network of interrelations among partitions, a network that overall remains to be fully understood.

# 7 On other multidimensional continued fraction algorithms

There are many different multidimensional continued fraction algorithms. (To get a feel of how many there are, see Schweiger [28] and Karpenkov [21].) Each has their own strengths and weaknesses. All are trying to generalize the many wonderful properties of traditional continued fractions to higher dimensional analogs. Historically, the two main sources of inspiration have been trying to find good Diophantine approximation properties and trying to find methods for understanding algebraic numbers via periodicity properties (generalizing the classical fact that a real number is a quadratic irrational if and only if its continued fraction expansion is eventually periodic.)

When we started this project, we assumed that each of the other well known multidimensional continued fraction algorithms would provide their own dynamical interpretation of partitions. This is quite false, as we will now see. To give a flavor of other multidimensional continued fractions algorithms and why they are not useful at all for studying partition numbers, we will look at the Mönkemeyer algorithm and then the Cassaigne algorithm. We then turn to the language of triangle partition maps, which is an attempt to put various multidimensional continued fraction algorithms into a single framework. We will see in terms of this framework, hardly any multidimensional continued fraction algorithm can be used in partition theory. Thus in the context of linking dynamics with partition numbers, it seems that the triangle map is unusual (though as we will also see not unique).

## 7.1 Mönkemeyer

The Mönkemeyer map is a particularly good multidimensional continued fraction algorithm for generalizing the classical Minkowksi Question Mark function, as shown by Panti [25]. For ease of notation, we will only treat the case of $m = 3$.

Here we will simply write down the map. Set

$$\begin{aligned} \triangle &= \{(x, y) \in \mathbb{R}^2 : 1 > x > y > 0\} \\ \triangle_0 &= \{(x, y) \in \triangle : x + y > 1\} \\ \triangle_1 &= \{(x, y) \in \triangle : x + y < 1\} \end{aligned}$$

exactly the same as in Section 5. The slow-Mönkemeyer map $M : \triangle_0 \cup \triangle_1 \to \triangle$ is

$$\begin{aligned} M(x, y) &= \begin{cases} M_0(x, y), & \text{if } x + y > 1 \\ M_1(x, y), & \text{if } x + y < 1 \end{cases} \\ &= \begin{cases} \left(\frac{1-y}{x}, \frac{x-y}{x}\right), & \text{if } x + y > 1 \\ \left(\frac{x}{1-y}, \frac{x-y}{1-y}\right), & \text{if } x + y < 1 \end{cases} \end{aligned}$$

As in Section 3.3 and Section 5, we pass from points $(x, y)$ in $\mathbb{R}^2$ to vectors $(z, x, y)$ in $\mathbb{R}^3$ (or points $(z : x : y)$ in $\mathbb{RP}^2$) via sending $(x, y)$ to $(1, x, y)$ with inverse map $(z, x, y) \to (x/z, y/z)$. Then, by an abuse of notation, as before, we again set

$$\begin{aligned} \triangle &:= \{(z,x,y) \in \mathbb{R}^3 : z > x > y > 0\} \\ \triangle_0 &:= \{(z,x,y) \in \triangle : x + y > z\} \\ \triangle_1 &:= \{(z,x,y) \in \triangle : x + y < z\}. \end{aligned}$$

This allows us to define the slow-Mönkemeyer map $T : \triangle_0 \cup \triangle_1 \to \triangle$ by

$$\begin{aligned} M(z,x,y) &= \begin{cases} M_0(z,x,y), & \text{if } x+y>z \\ M_1(z,x,y), & \text{if } x+y<z \end{cases} \\ &= \begin{cases} (x, z-y, x-y), & \text{if } x+y>z \\ (z-y, x, x-y), & \text{if } x+y<z \end{cases} \end{aligned}$$

By writing the row vector $(z,x,y)$ as a column vector, the action of $M$ is given by left multiplication by $3 \times 3$ matrices:

$$\begin{aligned} M \begin{pmatrix} z \\ x \\ y \end{pmatrix} &= \begin{cases} M_0 \begin{pmatrix} z \\ x \\ y \end{pmatrix}, & \text{if } x+y>z \\ M_1 \begin{pmatrix} z \\ x \\ y \end{pmatrix}, & \text{if } x+y<z \end{cases} \\ &= \begin{cases} \begin{pmatrix} x \\ z-y \\ x-y \end{pmatrix}, & \text{if } x+y>z \\ \begin{pmatrix} z-y \\ x \\ x-y \end{pmatrix}, & \text{if } x+y<z \end{cases} \end{aligned}$$

where

$$M_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad M_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

and

$$m_0 := M_0^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad m_1 := M_1^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Note that the entries for both $m_0$ and $m_1$ have negative entries. This is the technical reason why the Mönkemeyer map will not be good to understand partitions.

As seen in the earlier sections, we need to look at the *extended slow-Mönkemeyer map* $\tilde{M}$ (the natural extension of $M$), which is

$$\tilde{M}((n_1,n_2,n_3)\times[k_1,k_2,k_3]) = \begin{cases} \tilde{M}_0((n_1,n_2,n_3)\times[k_1,k_2,k_3]), & \text{if } n_2+n_3>n_1 \\ \tilde{M}_1((n_1,n_2,n_3)\times[k_1,k_2,k_3]), & \text{if } n_2+n_3<n_1 \end{cases}$$
$$= \begin{cases} (n_2, n_1-n_3, n_2-n_3)\times[k_1+k_2+k_3, k_1, -k_1-k_3], & \text{if } n_2+n_3>n_1 \\ (n_1-n_3, n_2, n_2-n_3)\times[k_1, k_1+k_2+k_3, -k_1-k_3], & \text{if } n_2+n_3<n_1 \end{cases}$$

which can be read as the action of two $6\times 6$ matrices on column vectors in $\mathbb{R}^6$, with the matrices

$$\begin{pmatrix} M_0 & 0 \\ 0 & m_0^\top \end{pmatrix}, \begin{pmatrix} M_1 & 0 \\ 0 & m_1^\top \end{pmatrix}.$$

If we try to see, for example, what the partition

$$(7,5,4)\times[3,2,4]\vdash 47$$

would map to under $\tilde{M}$, we get

$$\tilde{M}((7,5,4)\times[3,2,4]) = (5,3,2)\times[9,3,-7].$$

That $-7$ for one of the multiplicities means that this dynamical system will not generate partitions.

## 7.2 Cassaigne

This algorithm is of fairly recent vintage. Back in the early 1940s, Morse and Hedlund [24] started investigating infinite words. They showed that any infinite word made up from an alphabet of two letters whose linear complexity is bounded by $n$ (meaning that there are no more than $n$ subwords of length $n$) must actually be eventually periodic. An infinite word $w$ whose linear complexity is exactly $n+1$ is called Sturmian, meaning that there are exactly $n+1$ subwords of length $n$ in $w$. Morse and Hedlund not only showed that Sturmian words exist but more so that all such words stem from continued fraction expansions of real numbers. This is quite amazing. (For more see Arnoux's work in Chapter 6 of N. Pytheas Fogg 's *Substitutions in Dynamics, Arithmetics, and Combinatorics* [13].

The natural question then arises for methods to generalize this link of continued fractions with infinite words of low complexity. The Cassaigne algorithm, as described by Cassaigne, Labbé and Leroy [10], is a multidimensional continued fraction algorithm that produces infinite words on three letters whose linear complexity is exactly $2n+1$. We could now describe how this algorithm acts on vectors in the cone $\triangle$ via matrix multiplication, etc. We will instead simply write down the *extended slow-Cassaigne map* $\tilde{M}$ , which is

$$\tilde{C}((n_1,n_2,n_3)\times[k_1,k_2,k_3]) = \begin{cases} (n_2, n_3, n_2+n_3-n_1)\times[k_1+k_2, k_1+k_3, -k_1], & \text{if } n_2+n_3>n_1 \\ (n_1-n_3, n_2, n_2-n_3)\times[k_1, k_1+k_2+k_3, -k_1-k_3], & \text{if } n_2+n_3<n_1 \end{cases}$$

Looking at the partition

$$(7,5,4)\times[3,2,4]\vdash 47$$

again, we see that

$$\tilde{C}((7,5,4)\times[3,2,4]) = (5,4,2)\times[5,7,-3].$$

That $-3$ for one of the multiplicities means that this dynamical system will also not generate partitions.

### 7.3 Triangle Partition Maps

There are many multidimensional continued fractions. For each of these we can look at the corresponding extended map and see if the maps on the multiplicities keeps all the terms positive. This seems like an almost endless process. In [12], a family of multidimensional continued fractions was created which seems to capture all known multidimensional continued fractions (in a certain well-defined sense). The method is to first create a list of 216 different multidimensional continued fractions, parameterized by $S_3 \times S_3 \times S_3$, where $S_3$ is the set of permutations of three elements. Thus given some $(\sigma, \tau_0, \tau_1) \in S_3 \times S_3 \times S_3$ we associate a multidimensional continued fraction which we will denote by $T(\sigma, \tau_0, \tau_1)$, which in turn stems from two maps

$$T_0(\sigma, \tau_0, \tau_1),\ T_1(\sigma, \tau_0, \tau_1)$$

and corresponding extended versions

$$\tilde{T}(\sigma, \tau_0, \tau_1),\ \tilde{T}_0(\sigma, \tau_0, \tau_1),\ \tilde{T}_1(\sigma, \tau_0, \tau_1).$$

It can be calculated that the Triangle map is $T(e, e, e)$, that Mönkemeyer is $T(e, 132, 23)$ and that Cassaigne is $T(e, 23, 23)$.

For the generating list of 216 different multidimensional continued fraction algorithms, we can explicitly compute for each the corresponding extended version. What struck us initially as a somewhat surprising fact is that only four of these maps can be used to study partitions, meaning that for all but four some of the multiplicities will be negative. These four are

$$T(e, e, e),\ T(13, 12, 12),\ T(12, e, 12),\ T(132, 12, e).$$

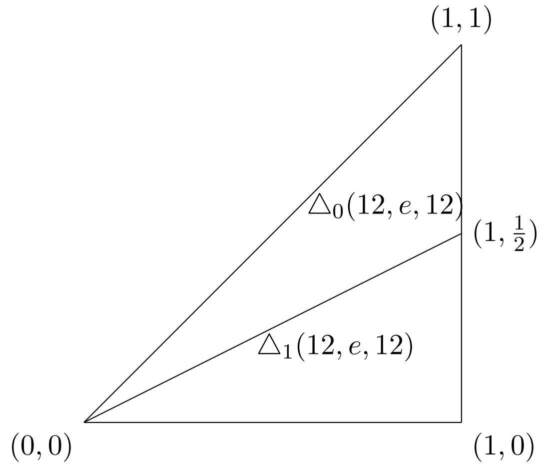
Though the first is the Triangle map, the other three do not have names and have never really been studied before.

While all four of these maps give rise to different dynamical systems, there is the phenomenon of "twinning" (see Section 8.1 in [16]), which is that

$$\begin{array}{rclcrcl} \tilde{T}_0(e,e,e) & = & \tilde{T}_1(13,12,12), & & \tilde{T}_0(12,e,12) & = & \tilde{T}_1(132,12,e) \\ \tilde{T}_1(e,e,e) & = & \tilde{T}_0(13,12,12), & & \tilde{T}_1(12,e,12) & = & \tilde{T}_0((132,12,e) \end{array}$$

This means really there are only two different $T(\sigma, \tau_0, \tau_1)$ maps that can be used to study partitions. As $T(12, e, 12)$ has never really been studied before, we will briefly describe it here. Set

$$\begin{array}{rcl} \triangle & := & \{(x, y) \in \mathbb{R}^2 : 1 > x > y > 0\} \\ \triangle_0(12, e, 12) & := & \{(x, y) \in \triangle : 2y > x\} \\ \triangle_1(12, e, 12) & := & \{(x, y) \in \triangle : 2y < x\} \end{array}$$




The map $T(12,e,12) : \triangle_0(12,e,12) \cup \triangle_1(12,e,12) \to \triangle$ is

$$\begin{aligned} T(x,y) &= \begin{cases} T_0(x,y), & \text{if } 2y > x \\ T_1(x,y), & \text{if } 2y < x \end{cases} \\ &= \begin{cases} \left(\frac{y}{1+y-x}, \frac{x-y}{1+y-x}\right), & \text{if } 2y > x \\ \left(\frac{x-y}{1-y}, \frac{y}{1-y}\right), & \text{if } 2y < x \end{cases} \end{aligned}$$

The extended $\tilde{T}(12,e,12)$, which can be thought of as the natural extension of $T(12,e,12)$, is

$$\begin{aligned} \tilde{T}(12,e,12)((n_1,n_2,n_3)\times[k_1,k_2,k_3]) &= \begin{cases} \tilde{T}_0(12,e,12)((n_1,n_2,n_3)\times[k_1,k_2,k_3]), & \text{if } 2n_3 > n_2 \\ \tilde{T}_1(12,e,12)((n_1,n_2,n_3)\times[k_1,k_2,k_3]), & \text{if } 2n_3 < n_2 \end{cases} \\ &= \begin{cases} (n_1+n_3-n_2, n_3, n_2-n_3)\times[k_1, k_2+k_3, k_1+k_2], & \text{if } 2n_3 > n_2 \\ (n_1-n_3, n_2-n_3, n_3)\times[k_1, k_2, k_1+k_2+k_3], & \text{if } 2n_3 < n_2 \end{cases} \end{aligned}$$

As the multiplicities will never be negative, we have orbits of partition numbers. But these orbits will be different than the corresponding orbits for the Triangle map, as can be seen by comparing the orbit of

$$(11,9,4)\times[1,0,0]$$

via $\tilde{T}(12,e,12)$, which is

| $m$ | $(x,y)$ | $n_1(m)$ | $n_2(m)$ | $n_3(m)$ | $k_1(m)$ | $k_2(m)$ | $k_3(m)$ | $\tilde{T}_i(12,e,12)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | (9/11, 4/11) | 11 | 9 | 4 | 1 | 0 | 0 | $\tilde{T}_1(12,e,12)$ |
| 1 | (5/7, 4/7) | 7 | 5 | 4 | 1 | 0 | 1 | $\tilde{T}_0(12,e,12)$ |
| 2 | (4/6, 1/6) | 6 | 4 | 1 | 1 | 1 | 1 | $\tilde{T}_1(12,e,12)$ |
| 3 | (3/5, 1/5) | 5 | 3 | 1 | 1 | 1 | 3 | $\tilde{T}_1(12,e,12)$ |
| 4 | (2/4, 1/4) | 4 | 2 | 1 | 1 | 1 | 5 | |

(stopping here, for now, as $(2/4,1/4)$ is on the line $2y = x$) to the orbit under the triangle map $\tilde{T}(e,e,e)$ given in the table in Section 5.2.

Unfortunately, while $\tilde{T}(12,e,12)$ does provide a map of partitions to partitions, almost any example will show that this map does not respect conjugation, and hence the analog of Proposition 5.4 is false.

### 7.4 Selmer and Brun

Recently Matthew Phang [26] has shown that two classical multi-dimensional continued fractions, the Selmer algorithm (see Chapter 7 in Schweiger [28] ) and the Brun algorithm (Chapter 4 in [28] ), both can be used to produce maps from partitions to partitions. Neither though respect conjugation and hence for both of these the analog of Proposition 5.4 are also false.

## 8 Questions

We view this paper as only a start. For example, in [5] many new identities among partitions are given using the dynamics of the triangle map applied to partitions. This work also gives a process to discover many new partition identities. This strikes as quite promising.

In this paper, we have spent a lot of time on the special case of partitioning a number into two numbers, with multiplicity, in large part due to the richness behind the classical Farey map and its corresponding Farey tree. This correspondence is what is key to our formula for $p(2, n)$. Recently, two of the authors, with Sara Munday, [7] have developed an analogous tree structure for the slow-Triangle map (see also [8]). Thus suggests that there will be a rich analog of the Farey tree, a possible Triangle graph. This is underlying Subsection 6.3. We hope to pursue this in future work.

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