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LIFESPAN ESTIMATES FOR 1D DAMPED WAVE EQUATION WITH ZERO MOMENT INITIAL DATA

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Abstract. In this manuscript, a sharp lifespan estimate of solutions to semilinear classical damped wave equation is investigated in one dimensional case when the Fourier 0th moment of sum of initial position and speed is 0. Especially, it is shown that the behavior of lifespan changes with $p = 3/2$ with respect to the size of the initial data.

1. Introduction

In this manuscript, we study the Cauchy problem of the following classical 1d damped wave equation:

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u = |u|^p, & t \in (0, T_0), \quad x \in \mathbb{R}, \\ u(0) = u_0, & x \in \mathbb{R}, \\ \partial_t u(0) = u_1, & x \in \mathbb{R}. \end{cases} \tag{1}$$

Our goal is to obtain optimal lifespan estimates of solutions to (1) when the 0th moment of sum of initial position and speed is 0.

Extensive studies have been conducted on the existence and nonexistence of time-global solutions to (1) for small initial data in the framework of perturbation argument. Namely, the maximal existence time has been investigated through the examination of quantities that reflect resemblance between the behavior of solutions to (1) and that of their corresponding free linear solutions. We note that the Fujita exponent $p_F(1) = 3$ is known to be the critical exponent to (1), that is, criteria of the existence and nonexistence of time-global solutions for small data in the sense explained below.

In the case where $p > 3$, the global existence of mild solutions is well-known from the article of Todorova and Yordanov [17] if $(u_0, u_1) \in W^{1,2} \times L^2$ is sufficiently small and compactly supported. Here $W^{1,2}$ denotes the usual Sobolev space of 1st order based on L^2 defined by the collection of all L^2 functions f satisyfing $f, f' \in L^2$. We also call $u \in C([0,\infty); W^{1,2}) \cap C^1((0,\infty); L^2)$ as a mild solution to (1) if u satisfies the integral form of (1):

$$u(t) = S(t)(u_0 + u_1) + \partial_t S(t) u_0 + \int_0^t S(t-\tau)|u(\tau)|^p d\tau, \tag{2}$$

where $S(t)$ is given by

$$S(t)f(x) = \frac{1}{2}\int_{-t}^{t} I_0\left(\frac{\sqrt{t^2 - y^2}}{2}\right) f(x-y) dy$$

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and I_0 is the modified Bessel function of 0 order whose Taylor series is

[7, 8.445]
$$I_0(y) = \sum_{k\geq 0} \frac{1}{(k!)^2}\left(\frac{y}{2}\right)^2.$$

We note that the analysis of [17] is based on a modified energy estimate. Moreover, Marcati and Nishara [14] showed the global existence for small $W^{1,2} \times L^2$ data without compactness of support of initial data. Especially, their argument is based on the following heat-like $L^p - L^q$ estimate for fundamental solutions to (1):

Lemma 1.1 ([14, Theorem 1.2])**.** *For $1 \leq q \leq p \leq \infty$ and positive α, β with $1 \leq \alpha + \beta \leq 3$, the following estimates hold:*

$$\|S(t)f\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q},$$
$$\left\|\partial_x^\alpha \partial_t^\beta \left(S(t)f - e^{-t/2}W(t)f\right)\right\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-\frac{\alpha}{2}-\beta}\|f\|_{L^q},$$

where $W(t)$ is the 1-D wave operator defined by

$$W(t)f(x) = \frac{1}{2}\int_{x-t}^{x+t} f(y)dy.$$

Moreover, the estimates

$$\|S(t)f - e^{t\Delta}f\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})-1}\|f\|_{L^q},$$
$$\left\|\partial_x^\alpha \partial_t^\beta \left(S(t)f - e^{t\Delta}f - e^{-t/2}W(t)f\right)\right\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{q})-\frac{\alpha}{2}-\beta-1}\|f\|_{L^q}$$

hold, where $e^{t\Delta}$ is the 1-D heat operator defined by

$$e^{t\Delta}f(x) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} f(y)dy.$$

Applying Lemma 1.1 and employing the standard contraction argument to (2), the small data global existence of mild solutions is established. We note that the three dimensional case is treated in [16] as well. Later, Ikeda, Inui, Okamoto, and Wakasughi extended this approach by Fourier analysis of $S(t)$ in [8]. We also refer [15] as a pioneering work.

In the case where $p \leq 3$, the global non-existence and corresponding lifespan estimate have been studied when

(3)
$$M_0(u_0 + u_1) > 0,$$

where M_0 is the Fourier 0-th moment defined by

$$M_0(f) = \int_{\mathbb{R}^n} f(x)dx$$

for $f \in L^1$. Indeed, let $p \leq 3$ and $f_0, f_1 \in L^1 \cap L^p$ is sufficiently regular and satisfies (3), then there exist positive constants c, C depends on (f_0, f_1) such that for sufficiently small $\varepsilon > 0$, the lifespan $T_0 = T_0(u_0, u_1)$ is estimated by

(4)
$$T_p(c\varepsilon) \leq T_0 \leq T_p(C\varepsilon)$$

with $(u_0, u_1) = (\varepsilon f_0, \varepsilon f_1)$, where

$$T_p(\varepsilon) = \begin{cases} \varepsilon^{-\frac{2(p-1)}{3-p}} & \text{if} \quad p \in (1,3), \\ \exp(\varepsilon^{-2}) & \text{if} \quad p = 3. \end{cases}$$

We note that the second estimate of (4) was obtained by Marcati and Nishihara [14]. The first estimate of (4) was obtained by Li and Zhou [13]. More precisely, they showed that $U(t) = \inf_{|x|\le\sqrt{t}} \sqrt{t}u(x)$ blows up at a finite time by introducing an ordinary differential inequality (ODI) with respect to U. We note that on the domain $\{x; |x| \le \sqrt{t}\}$, the free solutions to (1) behaves like free heat solutions and this property intends the ODI showing the blowup of U. We also note that the coefficient $\sqrt{t}$ in the definition of U is connected with the decay rate of free heat solutions. We revisit their ODI argument in subsection 2.2. We remark that the argument of Li and Zhou [13] covers 2 dimensional case as well. On the other hand, in higher dimensional subcritical and critical cases, where $p \in (1, 1 + 2/n]$ in n dimensional cases, the second estimate of (4) was obtained by Ikeda and Wakasugi [11] and Ikeda and Sobajima [10] by using arguments for the weak form of (1), respectively. However with this weak form argument, the positiveness of $M_0(u_0 + u_1)$ is essential. We also refer [9] for related subjects.

Our interest in this paper is to consider the lifespan when $M_0(u_0 + u_1) = 0$ and $p \le 3$. In this case, by using an ODE argument of [5], the global non-existence is easily seen. However, one cannot apply arguments above directly to estimate the lifespan. It is because in early works, their analysis bases on a perturbation theory and the similarity between the asymptotic behavior of fundamental solutions of (1) and that of free heat solutions. When $M_0(u_0 + u_1) = 0$, the asymptotic behavior of the fundamental solutions of (1) is still similar to that of free heat equations, so they decay faster. This difference of decaying speed causes a problem to apply a classical approach to investigate the lifespan for solutions to (1).

In [4], a partial answer was obtained in the case where

$$u_0(x) + u_1(x) = 0, \text{ for almost every } x. \tag{5}$$

Here $W^{1,p}$ denotes the usual Sobolev space of 1st order based on L^p, namely $f \in W^{1,p} \Leftrightarrow f, f' \in L^p$. Then the following lifespan estimate holds:

Proposition 1.2 ([4, Theorem 1.1])**.** *Let $n = 1$ and $1 < p \le 3$. For any $f \in (W^{1,1} \cap W^{1,p})\backslash\{0\}$, there exist $\varepsilon_f > 0$ and $c_f, C_f > 0$ such that if $0 < \varepsilon < \varepsilon_f$, $u_0 = \varepsilon f$, and $u_1 = -\varepsilon f$, then the corresponding mild solution*

$$u \in C([0, T_0); W^{1,1} \cap W^{1,p}) \cap C^1((0, T_0); L^1 \cap L^p) \tag{6}$$

exists uniquely with

$$T_{1,p}(c_f\varepsilon^p) \le T_0 \le T_{1,p}(C_f\varepsilon^p). \tag{7}$$

In this case, it is shown that solutions $u(t)$ to (1) is approximated by $\partial_t S(t)u_0$ till $T_1 = c\varepsilon^{1-p}$ and

$$S(t - T_1)(u(T_1) + \partial_t u(T_1)) \sim \partial_t S(t - T_1)S(T_1)u_0$$

after T_1. Therefore, solutions decay in the L^1 framework till T_1 and lifespan is extended as in Proposition 1.2. We note that the second estimate of (7) is the longest possible from the view point of [5].

In the general case of

$$M_0(u_0 + u_1) = 0,$$

a similar phenomena is expected. However, in this case, not only Fourier 0th moment but also higher order moment determine the decay rate of free solutions $S(t)f$. It is because, Lemma 1.1 implies that $S(t)f$ is approximated by $e^{t\Delta}f$ and the decay rate of $e^{t\Delta}f$ is determined by $M_k(f)$, where $M_k(f)$ is the k-th moment of f defined by

$$M_k(f) = \int_{\mathbb{R}} x^k f(x)dx,$$

for non-negative integer k. By taking account of the decay rate of $S(t)f$, we shall obtain the following lifespan estimate, where $p = 3/2$ is recognized as a critical exponent in the case of $M_0(u_0 + u_1) = 0$ and $M_1(u_0 + u_1) \neq 0$:

Theorem 1.3. *Let*[1] $f_0, f_1 \in \mathcal{S}$ *and* $M_0(f_0 + f_1) = 0$*. Let* $\varepsilon > 0$ *sufficiently small. If* $(u_0, u_1) = \varepsilon(f_0, f_1)$*, then the lifespan* T_0 *is estimated as follows:*

(1) *when* $M_1(f_0 + f_1) \neq 0$ *and* $p < 3/2$*,*

$$c\varepsilon^{-\frac{p-1}{2-p}} \le T_0 \le C\varepsilon^{-\frac{p-1}{2-p}}$$

(2) *when* $M_1(f_0 + f_1) \neq 0$ *and* $p = 3/2$*,*

$$c\varepsilon^{-2/3}e^{2W(c\varepsilon^{-1/2})/3} \le T_0 \le C\varepsilon^{-2/3}e^{2W(C\varepsilon^{-1/2})/3}$$

(3) *when* $M_1(f_0 + f_1) = 0$ *or* $p > 3/2$*,*

$$T_{1,p}(c\varepsilon^p) \le T_0 \le T_{1,p}(C\varepsilon^p).$$

We remark that the third case of Theorem 1.3 corresponds to Proposition 1.2. We note that by canceling more moment of $u_0 + u_1$, the corresponding solutions may decay faster till some time but the lifespan is not extended further because of the viewpoint of [5].

In the next section, we show a proof of Theorem 1.3. The key is the relation between the moments of u_0+u_1 and the decay rate of solutions to (1). In subsection 2.1, we show the lower bound of lifespan by the approach of [4]. In order to estimate the lifespan from above, we modify the functional of [13] so as to use the first moment of $u_0 + u_1$. For details, see subsection 2.2.

2. Proofs

2.1. **Lower bound for lifespan.** We first note that time-local $W^{1,1} \cap W^{1,p}$ mild solutions are constructed for any $u_0 \in W^{1,1} \cap W^{1,p}$ and $u_1 \in L^1 \cap L^p$ by a standard contraction argument and Lemma 1.1 with some $T > 0$ and norm

$$\begin{aligned}\|u\|_{X(T)} = &\sup_{0\le t\le T}\{\|u(t)\|_{L^1} + (1+t)^{\frac{1}{2p'}}\|u(t)\|_{L^p}\}\\ &+ \sup_{0\le t\le T}(1+t)^{1/2}(\|\partial_x u(t)\|_{L^1} + (1+t)^{\frac{1}{2p'}}\|\partial_x u(t)\|_{L^p})\\ &+ \sup_{0\le t\le T}(1+t)(\|\partial_t u(t)\|_{L^1} + (1+t)^{\frac{1}{2p'}}\|\partial_t u(t)\|_{L^p}).\end{aligned}$$

For detail, we refer [14, Section 3] and [4, Section 2], for example.

Here, we recall the estimate of linear solutions without some Fourier moments.

[1]We can take initial data

$$f_0 \in W^{1,1} \cap W^{1,p}, f_1 \in L^1 \cap L^p$$

satisfying $x(f_0 + f_1) \in L^1$ if $M_1(f_0 + f_1) \neq 0$ and $x^2(f_0 + f_1) \in L^1$ if $M_1(f_0 + f_1) = 0$.

Lemma 2.1. *Let $f \in \mathcal{S}$ and $p \geq 1$. Then the estimate*

$$\|S(t)f\|_{L^p} \leq C(t+1)^{-1/p'}\|f\|_{L^1}$$

holds. Moreover, the following estimate holds:

$$\|S(t)f\|_{L^p} \leq C \begin{cases} (t+1)^{-1/2-1/p'}\|f\|_{L^1} & \text{if} \quad M_0(f)=0, \\ (t+1)^{-1-1/p'}\|f\|_{L^1} & \text{if} \quad M_0(f)=M_1(f)=0. \end{cases}$$

Lemma 2.1 follows from Lemma 1.1 and the following formal expansion of $e^{t\Delta}f$, which follows from the Taylor expansion of heat kernel $g(x)=e^{-x^2/4}$:

$$e^{t\Delta}f(x) = \sum_{k=0}^{\infty} \frac{1}{t^{k/2}} M_k(f) g^{(k)}\left(\frac{x}{\sqrt{t}}\right).$$

For the detail of relation between the decay rate of $e^{t\Delta}f$ and moments of f, we refer the reader [1, 2, 3, 6, 12], for example.

Now, we split the proof into 2 parts.

2.1.1. *case where $M_1(u_0+u_1)\neq 0$.* We put

$$\begin{aligned} \|u\|_{Y(T)} = &\sup_{0\leq t\leq T}(t+1)^{\frac{1}{2}}\{\|u(t)\|_{W^{1,1}} + (t+1)^{\frac{1}{2p'}}\|u(t)\|_{W^{1,p}}\} \\ &+ \sup_{0\leq t\leq T}(1+t)(\|\partial_t u(t)\|_{L^1} + (1+t)^{\frac{1}{2p'}}\|\partial_t u(t)\|_{L^p}). \end{aligned}$$

We claim that mild $W^{1,1}\cap W^{1,p}$ solutions u to (1) satisfies

$$\|u\|_{Y(T)} \lesssim \varepsilon + \|u\|_{Y(T)}^p (T+1)^{1/2} \int_0^T (1+t)^{-\frac{2p-1}{2}}\,dt. \tag{8}$$

(8) implies that if $p>3/2$, then for $0\leq T\leq \widetilde{T}_{2,p}$, mild solution is constructed in $Y(\widetilde{T}_{2,p})$ by a standard contraction argument, where $\widetilde{T}_{2,p}$ satisfies

$$(\widetilde{T}_{2,p}+1)^{1/2}\int_0^{\widetilde{T}_{2,p}}(1+t)^{-\frac{2p-1}{2}}\,dt = C\varepsilon^{-p+1}.$$

We note that $\widetilde{T}_{2,p}$ is given

$$\widetilde{T}_{2,p} = \begin{cases} C\varepsilon^{-2(p-1)} & \text{if} \quad p>3/2, \\ Ce^{2W(C\varepsilon^{-1/2})} & \text{if} \quad p=3/2, \\ C\varepsilon^{-\frac{p-1}{2-p}} & \text{if} \quad p<3/2, \end{cases}$$

where W is the Lambert function defined by $z=W(z)e^{W(z)}$ for $z\geq 0$. Indeed, when $p=3/2$, for $1<T<\widetilde{T}_{2,p}$, we have

$$\begin{aligned} &\sqrt{\widetilde{T}_{2,p}+1}\log(\widetilde{T}_{2,p}+1) = C\varepsilon^{-1/2} \\ &\Rightarrow \sqrt{\widetilde{T}_{2,p}+1}\log\sqrt{\widetilde{T}_{2,p}+1} = C\varepsilon^{-1/2} \\ &\Rightarrow \widetilde{T}_{2,p}+1 = e^{2W(C\varepsilon^{-1/2})}. \end{aligned}$$

Now we estimate $u(\widetilde{T}_{2,p})$ as

$$\|u(\widetilde{T}_{2,p})\|_{W^{1,1}\cap W^{1,p}} + \|\partial_t u(\widetilde{T}_{2,p})\|_{L^1\cap L^p} \le \begin{cases} C\varepsilon^p & \text{if} \quad p > 3/2, \\ C\varepsilon e^{-W(C\varepsilon^{-1/2})} & \text{if} \quad p = 3/2, \\ C\varepsilon^{\frac{3-p}{2(2-p)}} & \text{if} \quad p < 3/2. \end{cases}$$

We note that

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{3/2}}{\varepsilon e^{-W(C\varepsilon^{-1/2})}} = 0.$$

Then by a reconstruction of solutions from $t = \widetilde{T}_{2,p}$ with $X(t - \widetilde{T}_{2,p})$ norm, T_0 is estimated by

$$T_0 \ge \begin{cases} C\varepsilon^{\frac{-2p(p-1)}{3-p}} & \text{if} \quad p > 3/2, \\ C\varepsilon^{-2/3} e^{2W(C\varepsilon^{-1/2})/3} & \text{if} \quad p = 3/2, \\ C\varepsilon^{-\frac{p-1}{2-p}} & \text{if} \quad p < 3/2. \end{cases}$$

Now we show (8). Lemmas 1.1 and 2.1 imply that the estimate

$$\|S(t)(u_0 + u_1) + \partial_t S(t) u_0\|_{L^1\cap L^p} \le C(1+t)^{-1/2}$$

holds. Moreover, we have

$$\begin{aligned} \left\| \int_0^t S(t-\tau)|u(\tau)|^p \right\|_{L^1\cap L^p} &\le C \int_0^t \|u(\tau)\|^p_{L^p\cap L^{p^2}} d\tau \\ &\le C\|u\|^p_{Y(t)} \int_0^t (1+\tau)^{-\frac{2p-1}{2}} d\tau. \end{aligned}$$

Here we note that L^{p^2} is an interpolation space of L^p and $W^{1,p}$. Since derivatives of u are estimated similarly, the estimates above imply (8). For the detail of estimates with derivatives, we refer [14, Proof of Theorem 1.2], for example.

2.1.2. *case where* $M_1(u_0 + u_1) = 0$. In this case, we put

$$\begin{aligned} \|u\|_{Z(T)} = &\sup_{0\le t\le T} (1+t)\{\|u(t)\|_{W^{1,1}} + (1+t)^{\frac{1}{2p'}} \|u(t)\|_{W^{1,p}}\} \\ &+ \sup_{0\le t\le T} (1+t)\{\|\partial_t u(t)\|_{L^1} + (1+t)^{\frac{1}{2p'}} \|\partial_t u(t)\|_{L^p})\}. \end{aligned}$$

A similar computation implies that for $1 < p \le 3$, mild solutions are constructed in $Z(C\varepsilon^{1-p})$ by a standard contraction argument. Then we estimate

$$\|u(C\varepsilon^{1-p})\|_{W^{1,1}\cap W^{1,p}} + \|\partial_t u(C\varepsilon^{1-p})\|_{L^1\cap L^p} \lesssim \varepsilon^p.$$

Therefore, the estimate $T_0 \ge T_{1,p}(C\varepsilon^p)$.

2.2. **Upper bound for lifespan.** In this subsection, we deploy a modified approach of [13]. The following lemma plays an essential role.

Lemma 2.2. *Let* $F : [0,\infty)\times\mathbb{R} \to [0,\infty)$. *Let* $t_0 \ge 4$ *and let*

$$D_\pm(t) = \{x \in \mathbb{R}; \sqrt{t}/2 < \pm x < \sqrt{t}\}.$$

Then

$$\inf_{x\in D_\pm(t)} \sqrt{t}\int_0^t e^{-\frac{t-\tau}{2}} \int_{|x-y|\le t-\tau} I_0(\frac{\sqrt{(t-\tau)^2-(x-y)^2}}{2})F(\tau,y)dy$$
$$\gtrsim \int_{t-1}^t (t-\tau)G_\pm(\tau)d\tau + \int_{t_0}^{t-1} G_\pm(\tau)d\tau,$$

where $G_\pm(t) = \inf_{x\in D_\pm(t)} \sqrt{t}F(t,x)$

Remark 2.1. *In* [13], $D_\pm(t)$ *is replaced by* $\{x, |x| < \sqrt{t}\}$. *Since we shall use the asymptotic form of linear solutions with first moment of initial data, the sign restriction of* x *is required.*

Proof. For $x \in D_\pm(t)$, we obtain

$$\sqrt{t}\int_0^t e^{-\frac{t-\tau}{2}} \int_{|x-y|\le t-\tau} I_0(\frac{\sqrt{(t-\tau)^2-(x-y)^2}}{2})F(\tau,y)dy\,d\tau$$
$$\gtrsim \int_{t-1}^t G(\tau)\int_{\substack{|x-y|\le t-\tau\\ y\in D_\pm(\tau)}} dy\,d\tau + \int_{t_0}^{t-1} \frac{1}{\sqrt{t-\tau}}G_\pm(\tau)\int_{\substack{|x-y|\le\sqrt{t-\tau}\\ y\in D_\pm(\tau)}} dy\,d\tau$$
$$\gtrsim \int_{t-1}^t (t-\tau)G_\pm(\tau)d\tau + \int_{t_0}^{t-1} G_\pm(\tau)d\tau,$$

where in order to estimate the second term of the LHS of last estimate, we have used the asymptotic behavior of I_0:

$$I_0(y) \sim \sqrt{\frac{1}{2\pi y}}e^y.$$

For the asymptotic behavior of I_0, we refer [7, 8.451 5]. □

Now we remind a lifespan estimate of some integral inequalities introduced in [13, Corollary 3.2].

Lemma 2.3 ([13, Corollary 3.2])**.** *Let* $p > 1$ *and* $0 \le \beta \le 1$ *and* $t_0 > 4$. *Let* $\varepsilon > 0$ *small enough,* $T > t_0 + 1$, *and* $v : [t_0, T) \to [0,\infty)$ *satisfy*

$$v(t) \gtrsim \varepsilon + \int_{t-1}^t (t-\tau)v(\tau)^p\tau^{-\beta}d\tau + \int_{t_0}^{t-1} v(\tau)^p\tau^{-\beta}d\tau.$$

Then

$$T \le \begin{cases} C\varepsilon^{-\frac{p-1}{1-\beta}}, & \text{if} \quad 0\le\beta<1,\\ e^{-C\varepsilon^{p-1}}, & \text{if} \quad \beta=1. \end{cases}$$

Proof of the upper bound of Theorem 1.3. For simplicity, we abbreviate $M_1(f_0+f_1)$ as M_1. We note that when $M_1 = 0$ or $p > 3/2$, the lifespan is estimated as [4]. Therefore we consider the case where $M_1 \ne 0$ and $p \le 3/2$. Put $w_\pm(t) = \inf_{x\in D_\pm(t)} tu(t,x)$. When $\pm M_1 \ne 0$, we have

$$\inf_{x\in D_\pm(t)}\{S(t)(u_0+u_1)(x) + \partial_t S(t)u_0(x)\} \gtrsim t^{-1}\varepsilon|M_1|.$$

Therefore, Lemma 2.2 implies that we have,

$$\begin{aligned} w_\pm(t) &\gtrsim \varepsilon|M_1| + \sqrt{t}\int_{t-1}^{t} (t-\tau) w_\pm(\tau)^p \tau^{-p+\frac{1}{2}} d\tau + \sqrt{t}\int_{t_0}^{t-1} w_\pm(\tau)^p \tau^{-p+\frac{1}{2}} d\tau \\ &\gtrsim \varepsilon + \varepsilon^p \sqrt{t}\int_{t_0}^{t-1} \tau^{-p+1/2} d\tau. \end{aligned}$$

Then we have

$$\inf_{t\in[t_0,\widetilde{T}_{2,p}]} w_\pm(t) \gtrsim \varepsilon.$$

Now we repeat the argument of [13] from $t = \widetilde{T}_{2,p}$ with $v(t) = \inf_{x\in D_\pm(t)} \sqrt{t} u(t)$. Since

$$\begin{aligned} v(\widetilde{T}_{2,p}) &= \widetilde{T}_{2,p}^{-1/2} w(\widetilde{T}_{2,p}) \\ &\geq \begin{cases} C\varepsilon e^{-W(C\varepsilon^{-1/2})} & \text{if} \quad p = 3/2, \\ C\varepsilon^{\frac{3-p}{2(2-p)}} & \text{if} \quad p < 3/2, \end{cases} \end{aligned}$$

Lemmas 2.2 and 2.3 imply that the estimate

$$T_0 \leq C v(\widetilde{T}_{2,p})^{-\frac{2(p-1)}{3-p}},$$

holds. This coincides the first estimates of (2) and (3) in Theorem 1.3. □

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