The final publication is available at www.degruyter.com

Explicit computation of some families of Hurwitz numbers, II

Carlo Petronio*

August 7, 2018

Abstract

We continue our computation, using a combinatorial method based on Gronthendieck's *dessins d'enfant*, of the number of (weak) equivalence classes of surface branched covers matching certain specific branch data. In this note we concentrate on data with the surface of genus g as source surface, the sphere as target surface, 3 branching points, degree $2k$, and local degrees over the branching points of the form $[2,\ldots,2]$, $[2h+1,3,2,\ldots,2]$, $\pi=[d_i]_{i=1}^{\ell}$. We compute the corresponding (weak) Hurwitz numbers for several values of g and h, getting explicit arithmetic formulae in terms of the d_i's.

MSC (2010): 57M12.

This paper is a continuation of [17], and it is based on the same methods, but the results that we obtain here refer to a topologically more complex situation, so the required arguments are more elaborate. In this introduction we quickly review the subject matter, making the paper independent of [17], and we state our results.

Surface branched covers A surface branched cover is a map

$$f:\widetilde{\Sigma}\to\Sigma$$

where $\widetilde{\Sigma}$ and Σ are closed and connected surfaces and f is locally modeled on maps of the form

$$(\mathbb{C},0)\ni z\mapsto z^m\in(\mathbb{C},0).$$

*Partially supported by INdAM through GNSAGA, by MIUR through PRIN "Real and Complex Manifolds: Geometry, Topology and Harmonic Analysis" and by UniPI through PRA 2018/22 "Geometria e Topologia delle varietà"

If $m > 1$ the point 0 in the target $\mathbb{C}$ is called a *branching point*, and m is called the local degree at the point 0 in the source $\mathbb{C}$. There are finitely many branching points, removing which, together with their pre-images, one gets a genuine cover of some degree d. If there are n branching points, the local degrees at the points in the pre-image of the j-th one form a partition π_j of d of some length ℓ_j, and the following Riemann-Hurwitz relation holds:

$$\chi\left(\widetilde{\Sigma}\right) - (\ell_1 + \ldots + \ell_n) = d\left(\chi\left(\Sigma\right) - n\right).$$

Let us now call *branch datum* a 5-tuple

$$\left(\widetilde{\Sigma}, \Sigma, d, n, \pi_1, \ldots, \pi_n\right)$$

and let us say it is *compatible* if it satisfies the Riemann-Hurwitz relation. (For a non-orientable $\widetilde{\Sigma}$ and/or Σ this relation should actually be complemented with certain other necessary conditions, but we restrict to an orientable Σ in this paper, so we do not spell out these conditions here.)

The Hurwitz problem The very old *Hurwitz problem* asks which compatible branch data are *realizable* (namely, associated to some existing surface branched cover) and which are *exceptional* (non-realizable). Several partial solutions to this problem have been obtained over the time, and we quickly mention here the fundamental [3], the survey [16], and the more recent [13, 14, 15, 2, 18]. In particular, for an orientable Σ the problem has been shown to have a positive solution whenever Σ has positive genus. When Σ is the sphere S, many realizability and exceptionality results have been obtained (some of experimental nature), but the general pattern of what data are realizable remains elusive. One guiding conjecture in this context is that *a compatible branch datum is always realizable if its degree is a prime number*. It was actually shown in [3] that proving this conjecture in the special case of 3 branching points would imply the general case. This is why many efforts have been devoted in recent years to investigating the realizability of compatible branch data with base surface Σ the sphere S and having $n = 3$ branching points. See in particluar [14, 15] for some evidence supporting the conjecture.

Hurwitz numbers Two branched covers

$$f_1 : \widetilde{\Sigma} \to \Sigma \qquad f_2 : \widetilde{\Sigma} \to \Sigma$$

are said to be *weakly equivalent* if there exist homeomorphisms $\widetilde{g}:\widetilde{\Sigma}\to\widetilde{\Sigma}$ and $g:\Sigma\to\Sigma$ such that $f_1\circ\widetilde{g}=g\circ f_2$, and *strongly equivalent* if the set of branching points in Σ is fixed once and forever and one can take $g=\mathrm{id}_\Sigma$. The *(weak or strong) Hurwitz number* of a compatible branch datum is the number of (weak or strong) equivalence classes of branched covers realizing it. So the Hurwitz problem can be rephrased as the question whether a Hurwitz number is positive or not (a weak Hurwitz number can be smaller than the corresponding strong one, but they can only vanish simultaneously). Long ago Mednykh in [10, 11] gave some formulae for the computation of the strong Hurwitz numbers, but the actual implementation of these formulae is rather elaborate in general. Several results were also obtained in more recent years in [4, 7, 8, 9, 12].

Computations In this paper we consider branch data of the form

$$(\heartsuit)\ \left(\widetilde{\Sigma},\Sigma=S,d=2k,n=3,[2,\ldots,2],[2h+1,3,2,\ldots,2],\pi=[d_i]_{i=1}^{\ell}\right)$$

for $h\geqslant 0$. Here we employ square brackets to denote an unordered array of integers with repetitions. A direct calculation shows that such a datum is compatible for $h\geqslant 2g-1$, where g is the genus of $\widetilde{\Sigma}$, and $\ell=h-2g+2$. We compute the weak Hurwitz number of the datum for $g=0,1,2$ and for some of the smallest possible h's, namely in the following cases: for $g=0$ and $h=0,1,2$; for $g=1$ and $h=1,2$; for $g=2$ and $h=3$. More values could be obtained using the same techniques as we employ below, but the complication of the topological and combinatorial situation grows very rapidly, and the arithmetic formulae giving the weak Hurwitz numbers are likely to be rather intricate for larger values of g and/or h.

We will denote by T the torus and by $2T$ the genus-2 surface.

Theorem 0.1.
- $(g=0,\ h=0)$ *The number of weakly inequivalent realizations of*

$$(S,S,2k,3,[2,\ldots,2],[1,3,2,\ldots,2],\pi)$$

(with $\ell(\pi)=2$) is 0 if π contains k, and 1 otherwise.

- $(g=0,\ h=1)$ *The number of weakly inequivalent realizations of*

$$(S,S,2k,3,[2,\ldots,2],[3,3,2,\ldots,2],\pi)$$

(with $\ell(\pi)=3$) is 0 if π contains k, and 1 otherwise.

- $(g = 0,\ h = 2)$ *The number ν of weakly inequivalent realizations of*

$$(S, S, 2k, 3, [2, \ldots, 2], [5, 3, 2, \ldots, 2], \pi)$$

(with $\ell(\pi) = 4$) is as follows:

  - *If $\pi = [p, p, p, p]$ or $\pi = [p, p, q, q]$ for distinct p, q then $\nu = 0$;*
  - *If $\pi = [p, p, p, q]$ for distinct p, q then $\nu = 0$ if k is in π, and $\nu = 1$ otherwise;*
  - *If $\pi = [p, p, q, r]$ for distinct p, q, r then $\nu = 1$ if k is in π or the sum of two entries of π, and $\nu = 3$ otherwise;*
  - *If $\pi = [p, q, r, s]$ for distinct p, q, r, s then $\nu = 2$ if k is the sum of two entries of π, while $\nu = 3$ if k is in π, and $\nu = 6$ otherwise.*

Theorem 0.2.

- $(g = 1,\ h = 1)$ *The number of weakly inequivalent realizations of*

$$(T, S, 2k, 3, [2, \ldots, 2], [3, 3, 2, \ldots, 2], [2k])$$

is $\frac{1}{2}k(k-1)$.

- $(g = 1,\ h = 2)$ *The number of weakly inequivalent realizations of*

$$(T, S, 2k, 3, [2, \ldots, 2], [5, 3, 2, \ldots, 2], [p, 2k - p])$$

is 0 for $p = k$, otherwise it is

$$2\left[\frac{1}{4}(k - p - 1)^2\right] + \left[\frac{p}{2}\right] \cdot (k - p - 1) + \left[\frac{1}{4}(p - 1)^2\right].$$

Theorem 0.3. $(g = 2,\ h = 3)$ *The number of weakly inequivalent realizations of*

$$(2T, S, 2k, 3, [2, \ldots, 2], [7, 3, 2, \ldots, 2], [2k])$$

is given by

$$\frac{1}{48}\left(7k^4 - 70k^3 + 290k^2 - 515k + 288\right) - \frac{5}{8}(2k - 5)\left[\frac{k}{2}\right].$$

1 Weak Hurwitz numbers and dessins d'enfant

In this section we quickly recall the machinery described in [17], omitting all the (rather easy) proofs. Our techniques are based on the notion of dessin d'enfant, popularized by Grothendieck in [5] (see also [1]), but actually known before his work and already exploited to give partial answers to the Hurwitz problem (see [6, 16] and the references quoted therein). Here we explain how to employ the dessins d'enfant to compute weak Hurwitz numbers. Let us fix until further notice a branch datum

$$(\spadesuit)\ \left(\widetilde{\Sigma}, \Sigma = S, d, n = 3, \pi_1 = [d_{1i}]_{i=1}^{\ell_1}, \pi_2 = [d_{2i}]_{i=1}^{\ell_2}, \pi_3 = [d_{3i}]_{i=1}^{\ell_3}\right).$$

A graph Γ is *bipartite* if it has black and white vertices, and each edge joins black to white. If Γ is embedded in $\widetilde{\Sigma}$ we call *region* a component R of $\widetilde{\Sigma} \setminus \Gamma$, and *length* of R the number of white (or black) vertices of Γ to which R is incident (with multiplicity). A pair (Γ, σ) is called *dessin d'enfant* representing $(\spadesuit)$ if $\sigma \in \mathfrak{S}_3$ and $\Gamma \subset \widetilde{\Sigma}$ is a bipartite graph such that:

- The black vertices of Γ have valence $\pi_{\sigma(1)}$;
- The white vertices of Γ have valence $\pi_{\sigma(2)}$;
- The regions of Γ have length $\pi_{\sigma(3)}$.

We will also say that Γ *represents* $(\spadesuit)$ *through* σ.

Remark 1.1. Let $f : \widetilde{\Sigma} \to S$ be a branched cover matching $(\spadesuit)$ and take $\sigma \in \mathfrak{S}_3$. If α is a segment in S with a black and a white end at the branching points corresponding to $\pi_{\sigma(1)}$ and $\pi_{\sigma(2)}$, then $\left(f^{-1}(\alpha), \sigma\right)$ represents $(\spadesuit)$, with vertex colours of $f^{-1}(\alpha)$ lifted via f.

Reversing the construction described in the previous remark one gets the following:

Proposition 1.2. *To a dessin d'enfant* (Γ, σ) *representing* $(\spadesuit)$ *one can associate a branched cover* $f : \widetilde{\Sigma} \to S$ *realizing* $(\spadesuit)$*, well-defined up to equivalence.*

We define the equivalence relation $\sim$ on dessins d'enfant generated by:

- $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$ if $\sigma_1 = \sigma_2$ and there is an automorphism $\widetilde{g} : \widetilde{\Sigma} \to \widetilde{\Sigma}$ such that $\Gamma_1 = \widetilde{g}(\Gamma_2)$ matching colours;
- $(\Gamma_1, \sigma_1) \sim (\Gamma_2, \sigma_2)$ if $\sigma_1 = \sigma_2 \circ (1\,2)$ and $\Gamma_1 = \Gamma_2$ as a set but with vertex colours switched;

- $(\Gamma_1,\sigma_1)\sim(\Gamma_2,\sigma_2)$ if $\sigma_1=\sigma_2\circ(2\,3)$ and Γ_1 has the same black vertices as Γ_2 and for each region R of Γ_2 we have that $R\cap\Gamma_1$ consists of one white vertex and disjoint edges joining this vertex with the black vertices on the boundary of R.

Theorem 1.3. *The branched covers associated as in Proposition 1.2 to two dessins d'enfant are equivalent if and only if the dessins are related by* $\sim$.

When the partitions π_1,π_2,π_3 in the branch datum (♠) are pairwise distinct, to compute the corresponding weak Hurwitz number one can stick to dessins d'enfant representing the datum through the identity, namely one can list up to automorphisms of $\widetilde{\Sigma}$ the bipartite graphs with black and white vertices of valence π_1 and π_2 and regions of length π_3. When the partitions are not distinct, however, it is essential to take into account the other moves generating $\sim$. In any case we will henceforth omit any reference to the permutations in $\mathfrak{S}_3$.

Relevant data and repeated partitions We now specialize again to a branch datum of the form (♡). We will compute its weak Hurwitz number ν by enumerating up to automorphisms of $\widetilde{\Sigma}$ the dessins d'enfant Γ representing it through the identity, namely the bipartite graphs Γ with black vertices of valence $[2,\ldots,2]$, the white vertices of valence $[2h+1,3,2,\ldots,2]$, and the regions of length π. Two remarks are in order:

- In all the pictures we will only draw the two white vertices of Γ of valence $(2h+1,3)$, and we will decorate an edge of Γ by an integer $a\geqslant 1$ to understand that the edge contains a black and $a-1$ white valence-2 vertices;
- Enumerating these Γ's up to automorphisms of $\widetilde{\Sigma}$ already gives the right value of ν except if two of the partitions of d in coincide.

Proposition 1.4. *In a branch datum of the form* (♡) *two of the partitions of d coincide precisely in the following cases:*

- $g=0,\ h\geqslant 0,\ k=h+2$, *with partions* $[2,\ldots,2],[2k-3,3],[2,\ldots,2]$;
- *Any* $g,\ h\geqslant 2g,\ k=2h+2-2g$, *with partitions* $[2,\ldots,2],[2h+1,3,2,\ldots,2],[2h+1,3,2,\ldots,2]$.

Proof. The lengths of the partitions π_1,π_2,π in (♡) are $\ell_1=k$, $\ell_2=k-h$ and $\ell=h+2-2g$. We can never have $\pi_1=\pi_2$. Since $k\geqslant h+2$ we can have $\ell_1=\ell$ only if $g=0$ and $k=h+2$, whence the first listed item. We

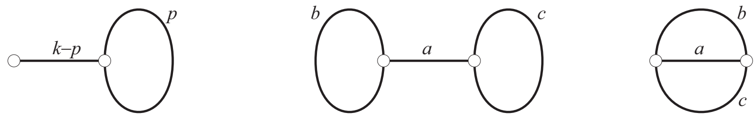

Figure 1: The only $(1,3)$ graph and the two $(3,3)$ graphs in S.

can have $\ell_2 = \ell$ only for $k = 2h + 2 - 2g$, whence $h \geqslant 2g$ and the data in the second listed item. $\square$

This result implies that the data $(\heartsuit)$ relevant to Theorems 0.1 to 0.3 and containing repetitions are precisely

$$(S, S, 4, 3, [2, 2], [1, 3], [2, 2])$$

$$(S, S, 6, 3, [2, 2, 2], [3, 3], [2, 2, 2])$$
$$(S, S, 8, 3, [2, 2, 2, 2], [5, 3], [2, 2, 2, 2])$$
$$(S, S, 4, 3, [2, 2], [1, 3], [1, 3])$$
$$(S, S, 8, 3, [2, 2, 2, 2], [3, 3, 2], [3, 3, 2])$$
$$(S, S, 12, 3, [2, 2, 2, 2, 2, 2], [5, 3, 2, 2], [5, 3, 2, 2])$$
$$(T, S, 8, 3, [2, 2, 2, 2], [5, 3], [5, 3]).$$

Moreover we have $\nu = 0$ in the first and third cases by the *very even data* criterion of [15], while the other cases will be taken into account below.

2 Genus 0

In this section we prove Theorem 0.1, starting from the very easy case $h = 0$, for which there is only one homeomorphism type of relevant graph and only one embedding in S, as shown on the left in Fig. 1 —here and below $(2h+1, 3)$ *graph* abbreviates *graph with vertices of valence* $(2h+1, 3)$. This graph gives a unique realization of $\pi = [p, 2k - p]$ for $p < k$, while $[k, k]$ is exceptional. Note that a single graph emerges for the realization of the case with repeated partitions $(S, S, 4, 3, [2, 2], [1, 3], [1, 3])$, so its realization is *a fortiori* unique up to equivalence (and it is also immediate to check that the last move generating $\sim$ leads this graph to itself).

Turning to the case $h = 1$ we note that there are two $(3, 3)$ graphs, both with a unique embedding in S, shown in Fig. 1-center/right and denoted by $\mathrm{I}(a,b,c)$ and $\mathrm{II}(a,b,c)$. Remark that $\mathrm{I}(a,b,c)$ has a $b \leftrightarrow c$ symmetry, while $\mathrm{II}(a,b,c)$ is fully symmetric in a,b,c. Moreover $\mathrm{I}(a,b,c)$ realizes $\pi = [2a+b+c,b,c]$ while $\mathrm{II}(a,b,c)$ realizes $[a+b,a+c,b+c]$.

Now we observe that the partition π satisfies one and only one of the following:

(i) π contains k;

(ii) $\pi = [2k-2q,q,q]$ with (a) $1 \leqslant q < \frac{k}{2}$ or (b) $\frac{k}{2} < q < k$;

(iii) $\pi = [2k-q-r,q,r]$ with

- (a) $1 \leqslant r < \frac{k}{2}$ and $r < q < k-r$, or
- (b) $1 \leqslant r < \frac{k}{2}$ and $k-r < q < k-\frac{r}{2}$, or
- (c) $\frac{k}{2} \leqslant r < \frac{2}{3}k$ and $r < q < k-\frac{r}{2}$

(the conditions $2k-q-r > q > r$ readily imply that $r < \frac{2}{3}k$ and $q < k-\frac{r}{2}$). Since we always have $a+b+c=k$, it is immediate that neither $\mathrm{I}(a,b,c)$ nor $\mathrm{II}(a,b,c)$ can realize (i). We now claim that for all the other listed cases there always is a single realization. For $\pi = [2k-2q,q,q]$ there is a realization as $\mathrm{I}(a,b,c)$ precisely if

$$\begin{cases} 2a+b+c = 2k-2q \\ b=c=q \end{cases} \Leftrightarrow \begin{cases} a = k-2q \\ b=c=q \end{cases} \quad \text{for } 1 \leqslant q < \frac{k}{2}$$

so we have case (ii-a), while there is a realization as $\mathrm{II}(a,b,c)$ if

$$\begin{cases} a+b = 2k-2q \\ a+c = q \\ b+c = q \end{cases} \Leftrightarrow \begin{cases} a = k-q \\ b = k-q \\ c = 2k-q \end{cases} \quad \text{for } \frac{k}{2} < q < k$$

whence case (ii-b). Turning to $[2k-q-r,q,r]$ with $2k-q-r > q > r$ there is a realization as $\mathrm{I}(a,b,c)$ for

$$\begin{cases} 2a+b+c = 2k-q-r \\ b = q \\ c = r \end{cases} \Leftrightarrow \begin{cases} a = k-q-r \\ b = q \\ c = r \end{cases} \quad \begin{array}{l} \text{for } r < q < k-r \\ \text{whence } r < \frac{k}{2} \end{array}$$

and case (iii-a), while there is one as $\mathrm{II}(a,b,c)$ if

$$\begin{cases} a+b = 2k-q-r \\ a+c = q \\ b+c = r \end{cases} \Leftrightarrow \begin{cases} a = k-r \\ b = k-q \\ c = q+r-k \end{cases} \quad \begin{array}{l} \text{for} \\ \max\{r,k-r\} < q < k \end{array}$$

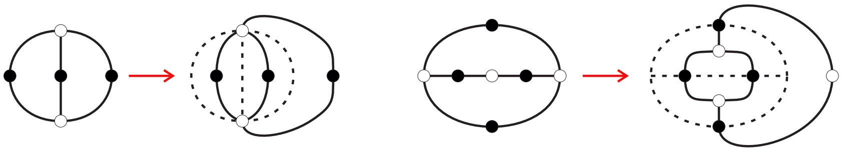

Figure 2: Graphs equivalent under $\sim$.

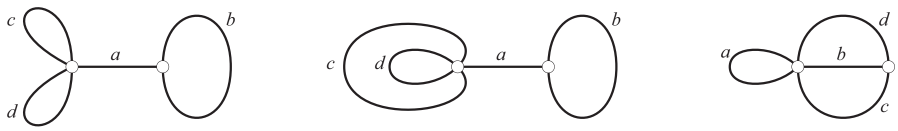

Figure 3: The $(5,3)$ graphs in S.

and depending on whether $\max\{r, k-r\}$ is $k-r$ or not we get (iii-b) and (iii-c). To conclude the case $h=1$, we must deal with the data with repeated partitions, namely

$$(S, S, 6, 3, [2,2,2], [3,3], [2,2,2])$$

$$(S, S, 8, 3, [2,2,2,2], [3,3,2], [3,3,2]).$$

We begin with the former, noting that only one dessin d'enfant Γ arises from the previous argument for its realization, so we must have $\nu = 1$, as in the statement. As a confirmation, we check that of the 6 potentially different graphs equivalent to Γ under $\sim$, for the other one Γ' with black and white vertices of valence $[2,2,2]$ and $[3,3]$ respectively, we actually have $\Gamma' = \widetilde{g}(\Gamma)$ for some $\widetilde{g}: S \to S$, as apparent from Fig. 2-left. For the latter datum with repetitions, the conclusion is analogous: only one graph Γ arises, so $\nu = 1$, and if Γ' is the graph equivalent to Γ under $\sim$ with black and white vertices of valence $[2,2,2,2]$ and $[3,3,2]$ respectively, we have have $\Gamma' = \widetilde{g}(\Gamma)$, as shown in Fig. 2-right.

As an example, we provide in Table 1 an application of Theorem 0.1 for the case $h=1$ and $k=8$, showing that all the listed cases actually occur.

Now we concentrate on the case $h=2$, that requires considerable work. We first note that there are two abstract $(5,3)$ graphs, one of which has two inequivalent embeddings in S, as shown in Fig. 3. Denoting these graphs by $\mathrm{I}(a,b,c,d)$, $\mathrm{II}(a,b,c,d)$, $\mathrm{III}(a,b,c,d)$, we note that they realize the partitions

π	Case	ν	Realizations
(14,1,1)	(ii-a) $1 \leqslant q = 1 < \frac{k}{2} = 4$	1	I(6,1,1)
(13,2,1)	(iii-a) $1 \leqslant r = 1 < \frac{k}{2} = 4$ $r = 1 < q = 2 < k - r = 7$	1	I(5,2,1)
(12,3,1)	(iii-a) $1 \leqslant r = 1 < \frac{k}{2} = 4$ $r = 1 < q = 3 < k - r = 7$	1	I(4,3,1)
(12,2,2)	(ii-a) $1 \leqslant q = 2 < \frac{k}{2} = 4$	1	I(4,2,2)
(11,4,1)	(iii-a) $1 \leqslant r = 1 < \frac{k}{2} = 4$ $r = 1 < q = 4 < k - r = 7$	1	I(3,4,1)
(11,3,2)	(iii-a) $1 \leqslant r = 2 < \frac{k}{2} = 4$ $r = 2 < q = 3 < k - r = 6$	1	I(3,3,2)
(10,5,1)	(iii-a) $1 \leqslant r = 1 < \frac{k}{2} = 4$ $r = 1 < q = 5 < k - r = 7$	1	I(2,5,1)
(10,4,2)	(iii-a) $1 \leqslant r = 2 < \frac{k}{2} = 4$ $r = 2 < q = 4 < k - r = 6$	1	I(2,4,2)
(10,3,3)	(ii-a) $1 \leqslant q = 3 < \frac{k}{2} = 4$	1	I(2,3,3)
(9,6,1)	(iii-a) $1 \leqslant r = 1 < \frac{k}{2} = 4$ $r = 1 < q = 6 < k - r = 7$	1	I(1,6,1)
(9,5,2)	(iii-a) $1 \leqslant r = 2 < \frac{k}{2} = 4$ $r = 2 < q = 5 < k - r = 6$	1	I(1,5,2)
(9,4,3)	(iii-a) $1 \leqslant r = 3 < \frac{k}{2} = 4$ $r = 3 < q = 4 < k - r = 5$	1	I(1,4,3)
(8,7,1)	(i)	0	
(8,6,2)	(i)	0	
(8,5,3)	(i)	0	
(8,4,4)	(i)	0	
(7,7,2)	(ii-b) $\frac{k}{2} = 4 < q = 7 < k = 8$	1	II(6,1,1)
(7,6,3)	(iii-b) $1 \leqslant r = 3 < \frac{k}{2} = 4$ $k - r = 5 < q = 6 < k - \frac{r}{2} = 6.5$	1	II(5,2,1)
(7,5,4)	(iii-c) $\frac{k}{2} = 4 \leqslant r = 4 < \frac{2}{3}k = 5.\bar{3}$ $r = 4 < q = 5 < k - \frac{r}{2} = 6$	1	II(4,3,1)
(6,6,4)	(ii-b) $\frac{k}{2} = 4 < q = 6 < k = 8$	1	II(4,2,2)
(6,5,5)	(ii-b) $\frac{k}{2} = 4 < q = 5 < k = 8$	1	II(3,3,2)

Table 1: The genus-0 case with $h = 1$ and $k = 8$.

$[2a+b+c+d,b,c,d]$, $[2a+b+c,c+d,b,d]$, $[a+c+d,b+c,b+d,a]$ respectively, and that their only symmetries are $\mathrm{I}(a,b,d,c)=\mathrm{I}(a,b,c,d)$ and $\mathrm{III}(a,b,d,c)=\mathrm{III}(a,b,c,d)$.

The form of the partitions realized by $\mathrm{I},\mathrm{II},\mathrm{III}$ readily shows that $\pi=[p,p,p,p]$ cannot be realized. Moreover, $\pi=[p,p,q,q]$ for $p>q$ of course cannot via I, and it also cannot via II or III, since it only could as

$$\begin{cases} 2a+b+c=p \\ c+d=p \\ b=k-p \\ d=k-p \end{cases} \Rightarrow a=0, \qquad \begin{cases} a+c+d=p \\ b+c=p \\ b+d=k-p \\ a=k-p \end{cases} \Rightarrow d=0.$$

The partition $\pi=[p,p,p,q]$ for $p>q$, so $q=2k-3p$ and $\frac{k}{2}<p<\frac{2}{3}k$, cannot be realized via I or II, while it can via III only as

$$\begin{cases} a+c+d=p \\ b+c=p \\ b+d=p \\ a=2k-3p \end{cases} \Leftrightarrow \begin{cases} a=2k-3p \\ b=k-p \\ c=2p-k \\ d=2p-k \end{cases}$$

which gives positive a,b,c,d, so there is a unique realization. Note that π cannot contain k, so our finding is in agreement with the statement.

Turning to $\pi=[p,q,q,q]$ for $p>q$, so $p=2k-3q$ and $0<q<\frac{k}{2}$, we get realizations via I as

$$\begin{cases} 2a+b+c+d=2k-3q \\ b=c=d=q \end{cases} \Leftrightarrow \begin{cases} a=k-3q \\ b=c=d=q \end{cases} \quad \text{for } q<\frac{k}{3},$$

while there is none via II, and there is one via III as

$$\begin{cases} a+c+d=2k-3q \\ b+c=b+d=a=q \end{cases} \Leftrightarrow \begin{cases} a=q \\ b=3q-k \\ c=k-2q \\ d=k-2q \end{cases} \quad \text{for } q>\frac{k}{3}$$

so there is a unique realization except for $q=\frac{k}{3}$, namely $p=k$, as in the statement.

Before proceeding we prove two facts that we will use a few times. Take $\pi=[p,q,r,s]$ with $p\geqslant q\geqslant r\geqslant s$. Then:

- If π contains k then $p=k$;

- If k is the sum of two entries of π then $p+s=q+r=k$.

The first assertion is obvious. For the second one, note that if $p+r=q+s=k$ then $p=q$ and $r=s$, so we also have $p+s=q+r=k$, while if $p+q=r+s=k$ then $p=q=r=s$, and again $p+s=q+r=k$.

We now study the partitions $\pi=[p,p,q,r]$ with $p>q>r$. Note that $r=2k-2p-q$, so $q<2k-2p$, and $q>2k-2p-q$, so $q>k-p$, and finally $2p>q+r$, so $p>\frac{k}{2}$. No realization via I is possible, while one via II would be only in one of the following ways:

$$\begin{cases} 2a+b+c=p \\ c+d=p \\ b=q \\ d=2k-2p-q \end{cases} \Rightarrow \quad a=k-p-q \quad \Rightarrow \quad q<k-p \text{ (impossible)},$$

$$\begin{cases} 2a+b+c=p \\ c+d=p \\ b=2k-2p-q \\ d=q \end{cases} \Leftrightarrow \begin{cases} a=p+q-k \\ b=2k-2p-q \\ c=p-q \\ d=q. \end{cases}$$

The first way gives nothing, and the second one gives a unique solution without conditions. Similarly, via III we could only have

$$\begin{cases} a+c+d=p \\ b+c=p \\ b+d=q \\ a=2k-2p-q \end{cases} \Leftrightarrow \begin{cases} a=2k-2p-q \\ b=k-p \\ c=2p-k \\ d=p+q-k, \end{cases}$$

$$\begin{cases} a+c+d=p \\ b+c=p \\ a=q \\ b+d=2k-2p-q \end{cases} \Rightarrow \quad d=k-p-q \quad \Rightarrow \quad q<k-p \text{ (impossible)},$$

$$\begin{cases} b+c=p \\ b+d=p \\ a+c+d=q \\ a=2k-2p-q \end{cases} \Leftrightarrow \begin{cases} a=2k-2p-q \\ b=k-q \\ c=p+q-k \\ d=p+q-k \end{cases}$$

where the second way gives nothing and the other two always give acceptable solutions, so we have 2 realizations via III. The total number of realizations of (p,p,q,r) is then always 3. To show that this is in agreement with the statement, we note that we cannot have $p=k$, and we also cannot have $p+r=p+q=k$, otherwise $r=q$.

Turning to (p,q,q,r) for $p>q>r$, we first note that $r=2k-p-2q$, so $q<k-\frac{p}{2}$, and $q>2k-p-2q$, so $q>\frac{1}{3}(2k-p)$. Moreover $p>\frac{k}{2}$, whence $3k-3p<2k-p$ and $k-p<\frac{1}{3}(2k-p)$, therefore $q>k-p$, which we will need below. The realizations via I come from

$$\begin{cases} 2a+b+c+d=p \\ c=q \\ d=q \\ b=2k-p-2q \end{cases} \Leftrightarrow \begin{cases} a=p-k \\ b=2k-p-2q \\ c=q \\ d=q \end{cases} \quad \text{for } p>k,$$

$$\begin{cases} 2a+b+c+d=p \\ c=q \\ d=2k-p-2q \\ b=q \end{cases} \Leftrightarrow \begin{cases} a=p-k \\ b=q \\ c=q \\ d=2k-p-2q \end{cases} \quad \text{for } p>k$$

whence two solutions for $p>k$ and none otherwise. Via II we can have

$$\begin{cases} 2a+b+c=p \\ b=q \\ c+d=q \\ d=2k-p-2q \end{cases} \Leftrightarrow \begin{cases} a=k-2q \\ b=q \\ c=p+3q-2k \\ d=2k-p-2q \end{cases} \quad \text{for } q<\frac{k}{2},$$

$$\begin{cases} c+d=p \\ 2a+b+c=q \\ d=q \\ b=2k-p-2q \end{cases} \Leftrightarrow \begin{cases} a=2q-k \\ b=2k-p-2q \\ c=p-q \\ d=q \end{cases} \quad \begin{array}{l} \text{for } q>\frac{k}{2} \\ (\text{so } p<k), \end{array}$$

therefore, no solution for $q=\frac{k}{2}$ and one otherwise. Finally, from III we get

$$\begin{cases} a+c+d=p \\ b+c=q \\ b+d=q \\ a=2k-p-2q \end{cases} \Leftrightarrow \begin{cases} a=2k-p-2q \\ b=k-p \\ c=p+q-k \\ d=p+q-k \end{cases} \quad \text{for } p<k$$

$$\begin{cases} a+c+d=p \\ b+c=q \\ b+d=2k-p-2q \\ a=q \end{cases} \Leftrightarrow \begin{cases} a=q \\ b=k-p \\ c=p+q-k \\ d=k-2q \end{cases} \quad \text{for } p<k \text{ and } q<\frac{k}{2}$$

$$\begin{cases} a+c+d=q \\ b+c=p \\ b+d=q \\ a=2k-p-2q \end{cases} \Leftrightarrow \begin{cases} a=2k-p-2q \\ b=k-q \\ c=p+q-k \\ d=2q-k \end{cases} \quad \begin{array}{l} \text{for } q>\frac{k}{2} \\ (\text{so } p<k). \end{array}$$

Summing up, for (p,q,q,r) we have 1 realization if $p=k$ or $q=\frac{k}{2}$, and 3 otherwise, in accordance with the statement.

In the last case with repetitions, namely $\pi=[p,q,r,r]$ with $p>q>r$, we note that $p-q$ is even, and we denote it by $2j$, so $q=p-2j$ and $r=k-p+j$, with $p-2j>k-p+j$, so $p-k<j<\frac{1}{3}(2p-k)$. We also note that $p>\frac{k}{2}$, which readily implies that $\frac{1}{3}(2p-k)<p-\frac{k}{2}$, therefore $j<p-\frac{k}{2}$, which we will need soon. The realizations of π via I come as

$$\begin{cases} 2a+b+c+d=p \\ b=p-2j \\ c=d=k-p+j \end{cases} \Leftrightarrow \begin{cases} a=p-k \\ b=p-2j \\ c=d=k-p+j \end{cases} \text{for } p>k,$$

$$\begin{cases} 2a+b+c+d=p \\ b=d=k-p+j \\ c=p-2j \end{cases} \Leftrightarrow \begin{cases} a=p-k \\ b=d=k-p+j \\ c=p-2j \end{cases} \text{for } p>k$$

so there are 2 of them for $p>k$ and none otherwise. Now II gives

$$\begin{cases} 2a+b+c=p \\ c+d=p-2j \\ b=d=k-p+j \end{cases} \Leftrightarrow \begin{cases} a=j \\ b=d=k-p+j \\ c=2p-k-3j, \end{cases}$$

$$\begin{cases} 2a+b+c=p-2j \\ c+d=p \\ b=d=k-p+j \end{cases} \Rightarrow \quad a=-j \text{ (impossible)}$$

whence always a unique realization. From III we get instead

$$\begin{cases} a+c+d=p \\ b+c=p-2j \\ b+d=a=k-p+j \end{cases} \Leftrightarrow \begin{cases} a=k-p+j \\ b=k-p \\ c=2p-2j-k \\ d=j \end{cases} \text{for } p<k,$$

$$\begin{cases} a+c+d=p \\ a=p-2j \\ b+c=b+d=k-p+j \end{cases} \Leftrightarrow \begin{cases} a=p-2j \\ b=k-p \\ c=d=j \end{cases} \text{for } p<k,$$

$$\begin{cases} a+c+d=p-2j \\ a=b+d=k-p+j \\ b+c=p \end{cases} \Rightarrow \quad d=-j$$

whence 2 realizations for $p<k$ and none otherwise. Summarizing the case $\pi=[p,q,r,r]$, we have one realization for $p=k$ and 3 otherwise, which

agrees with the statement, since for such a π the conditions $p+r=q+r=k$ implies $p=q$.

We are only left to deal with the general case $\pi=[p,q,r,s]$ for $p>q>r>s$, so $s=2k-p-q-r$ with $0<2k-p-q-r<r$, so $k-\frac{1}{2}(p+q)<r<2k-p-q$. Note that we also have $p+q>p+r>k$ and $r+s<q+s<k$. From I we get

$$\begin{cases} 2a+b+c+d=p \\ b=q \\ c=r \\ d=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=p-k \\ b=q \\ c=r \\ d=2k-p-q-r \end{cases} \qquad \text{for } p>k$$

plus two more instances with $b=r$, $\{c,d\}=\{q,s\}$ and $b=s$, $\{c,d\}=\{q,r\}$, always with an acceptable solution with $a=k-p$, so I gives 3 realizations of π if $p>k$ and none otherwise. From II we get instead

$$\begin{cases} 2a+b+c=p \\ b=q \\ c+d=r \\ d=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=k-q-r \\ b=q \\ c=p+q+2r-2k \\ d=2k-p-q-r \end{cases} \qquad \text{for } q+r<k$$

$$\begin{cases} 2a+b+c=p \\ c+d=q \\ b=r \\ d=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=k-q-r \\ b=r \\ c=p+2q+r-2k \\ d=2k-p-q-r \end{cases} \qquad \text{for } q+r<k,$$

(note that $p+2q+r-2k>p+q+2r-2k>0$),

$$\begin{cases} 2a+b+c=p \\ c+d=q \\ d=r \\ b=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=p+r-k \\ b=2k-p-q-r \\ c=q-r \\ d=r, \end{cases}$$

$$\begin{cases} c+d=p \\ 2a+b+c=q \\ b=r \\ d=2k-p-q-r \end{cases} \Rightarrow \quad a=k-p-r \quad \Rightarrow \quad \begin{array}{l} p+r<k \\ \text{(impossible)}, \end{array}$$

$$\begin{cases} c+d=p \\ 2a+b+c=q \\ d=r \\ b=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=q+r-k \\ b=2k-p-q-r \\ c=p-r \\ d=r \end{cases} \qquad \text{for } q+r>k,$$

$$\begin{cases} c+d=p \\ d=q \\ 2a+b+c=r \\ b=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=q+r-k \\ b=2k-p-q-r \\ c=p-q \\ d=q \end{cases} \quad \text{for } q+r>k$$

whence 1 realization of π for $q+r=k$ and 3 otherwise. Finally, from Ⅲ we get

$$\begin{cases} a+c+d=p \\ a=q \\ b+c=r \\ b+d=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=q \\ b=k-p \\ c=p+r-k \\ d=k-q-r \end{cases} \quad \begin{array}{l} \text{for } p<k \\ \text{and } q+r<k, \end{array}$$

$$\begin{cases} a+c+d=p \\ b+c=q \\ a=r \\ b+d=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=r \\ b=k-p \\ c=p+q-k \\ d=k-q-r \end{cases} \quad \begin{array}{l} \text{for } p<k \\ \text{and } q+r<k, \end{array}$$

$$\begin{cases} a+c+d=p \\ b+c=q \\ b+d=r \\ a=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=2k-p-q-r \\ b=k-p \\ c=p+q-k \\ d=p+r-k \end{cases} \quad \text{for } p<k,$$

$$\begin{cases} b+c=p \\ a+c+d=q \\ a=r \\ b+d=2k-p-q-r \end{cases} \Rightarrow \quad d=k-p-r \quad \Rightarrow \quad \begin{array}{l} p+r<k \\ \text{(impossible)}, \end{array}$$

$$\begin{cases} b+c=p \\ a+c+d=q \\ b+d=r \\ a=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=2k-p-q-r \\ b=k-q \\ c=p+q-k \\ d=q+r-k \end{cases} \quad \text{for } q+r>k,$$

$$\begin{cases} b+c=p \\ b+d=q \\ a+c+d=r \\ a=2k-p-q-r \end{cases} \Leftrightarrow \begin{cases} a=2k-p-q-r \\ b=k-r \\ c=p+r-k \\ d=q+r-k \end{cases} \quad \text{for } q+r>k.$$

Since $q+r>k$ implies $p<k$, we conclude that Ⅲ yields 3 realizations of π for $p<k$ and $q+r \neq k$, only 1 if $p<k$ and $q+r=k$, and none if $p \geqslant k$. Noting that the conditions $q+r=k$ and $p=k$ are mutually exclusive, we conclude that we have the number of realizations of π is as follows: 2 if $q+r=k$ (1 from Ⅱ and 1 from Ⅲ), then 3 if $p=k$ (from Ⅱ), and 6 otherwise

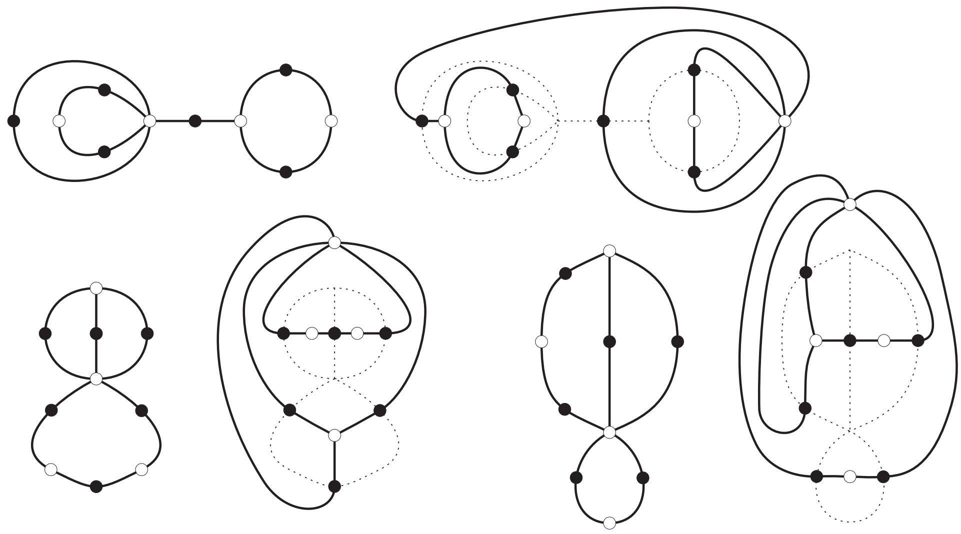

Figure 4: Graphs mapped to themselves by the move generating $\sim$.

(always 3 from $\mathbb{II}$, plus 3 from I is $p > k$ and 3 from $\mathbb{III}$ if $p < k$. This is precisely what the statement says.

To conclude the case $h = 2$ we must take into account the datum with repeated partitions $(S, S, 12, 3, [2, 2, 2, 2, 2, 2], [5, 3, 2, 2], [5, 3, 2, 2])$, for which the statement gives $\nu = 3$, coming from the realizations of the datum via the graphs $\mathbb{II}(1, 2, 1, 2)$, $\mathbb{III}(3, 1, 1, 1)$, $\mathbb{III}(2, 1, 2, 1)$ according to the above argument. So to verify that our computation of $\nu = 3$ is correct also in this case we must show that these graphs are pairwise inequivalent under $\sim$, which amounts to showing that by applying to each of them the last move generating $\sim$ we always get the same graph again, and not one of the other two. This is done in Fig. 4.

3 Genus 1

In this section we prove Theorem 0.2, starting from the case $h = 1$. We have to consider the embeddings in T of the $(3, 3)$ graphs of Fig. 1 with a single disc as a region. For the first graph, of course neither of the closed edges can be trivial in T, so the situation must be as in Fig. 5-left, but then the complement has disconnected boundary. One also easily sees that the second graphs embeds only as shown in Fig. 5-right and with a full $\mathfrak{S}_3 \times \mathbb{Z}/_2$

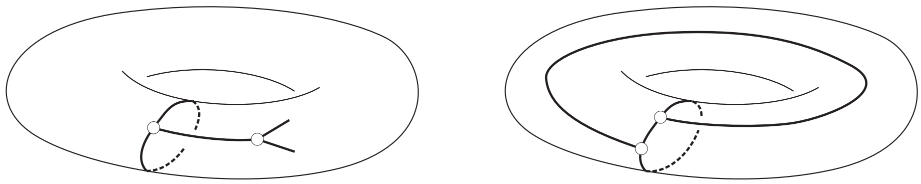

Figure 5: The only $(3,3)$ graph that embeds in T with a single disc as region.

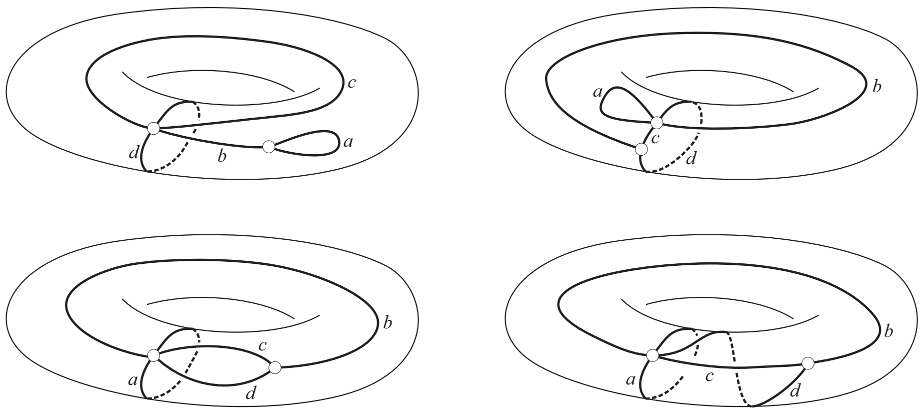

Figure 6: The $(5,3)$ graphs in T with two regions.

symmetry, so ν is the number of ways to write k as the sum of 3 unordered positive integers, which is $\binom{k-1}{2}$ as claimed.

Let us now consider the case $h=2$, whence $\ell=2$. We first recall that there are two abstract $(5,3)$ graphs, already shown in Fig. 3 (we can ignore here the graph in the centre, that represents a different embedding in S of that on the left). We now claim that they have exactly the inequivalent embeddings in T shown in Fig. 6 and denoted by $\mathrm{I}(a,b,c,d)$, $\mathrm{II}(a,b,c,d)$, $\mathrm{III}(a,b,c,d)$ and $\mathrm{IV}(a,b,c,d)$. To this end, we first note that the graph of Fig. 3-left cannot embed with the loop b being non-trivial in T, otherwise one of the regions would not be a disc. Knowing that b is trivial we easily get the only embedding $\mathrm{I}(a,b,c,d)$. Turning to the graph of Fig. 3-right, we ask ourselves whether the loop a is trivial in T or not. If it is, we easily get $\mathrm{II}(a,b,c,d)$. If a is non-trivial, the edges a,b,c cannot leave the vertex of a from the same side, so assume b leaves from one side and c,d from the opposite side. It is then easy to see that the only possibilities are $\mathrm{III}(a,b,c,d)$

and $\mathrm{IV}(a,b,c,d)$.

We next claim that all four graphs have the only symmetry $c \leftrightarrow d$. Checking that there cannot be any other one is immediate, while we prove that $c \leftrightarrow d$ exists for $\mathrm{IV}(a,b,c,d)$, which is the hardest case to visualize. Note that, with suitable orientations, the attaching maps of the complementary discs to $\mathrm{IV}(a,b,c,d)$ are described by the words $bc^{-1}db^{-1}a^{-1}$ and acd^{-1}. Now we consider the automorphism of $\mathrm{IV}(a,b,c,d)$ that maps a,b,c,d to a^{-1},b,d,c. This transforms the attaching words into $bd^{-1}cb^{-1}a$ and $a^{-1}dc^{-1}$, whose inverses are $a^{-1}bc^{-1}db^{-1}$ and $cd^{-1}a$, which are cyclically identical to the initial ones. So the automorphism of $\mathrm{IV}(a,b,c,d)$ extends to T, and we are done.

To count the realizations of a partition $(p, 2k-p)$ with $p \leqslant k$, we note that $\mathrm{I}(a,b,c,d)$ and $\mathrm{II}(a,b,c,d)$ both realize $[a+2b+2c+2d, a]$, while $\mathrm{III}(a,b,c,d)$ realizes $[2a+2b+c+d, c+d]$, and $\mathrm{IV}(a,b,c,d)$ realizes $[a+2b+c+d, a+c+d]$. The conclusion is now easy. Of course $\nu = 0$ if $p = k$, while for $p < k$ in I and II we must have $a = p$, so they both contribute with

$$\sum_{b=1}^{k-p-2} \left[\frac{k-p-b}{2}\right] = \left[\frac{1}{4}(k-p-1)^2\right];$$

in III we must have $c + d = p$ and $a + b = k - p$, whence

$$\left[\frac{p}{2}\right] \cdot (k-p-1);$$

finally, in IV we must have $a + c + d = p$ and $b = k - p$, whence

$$\sum_{a=1}^{p-2} \left[\frac{p-a}{2}\right] = \left[\frac{1}{4}(p-1)^2\right].$$

To conclude the case $h = 2$ we need to consider the datum with repetitions

$$(T, S, 8, 3, [2,2,2,2], [5,3], [5,3]),$$

but the previous discussion implies that only the graph $\mathrm{IV}(1,1,1,1)$ can realize it, so the value $\nu = 1$ already obtained is correct. In particular, the graph must be mapped to itself by the last move generating $\sim$, which one can verify rather easily.

4 Genus 2

We start by showing in Fig. 7 the only two abstract $(7,3)$ graphs, Γ_1 and

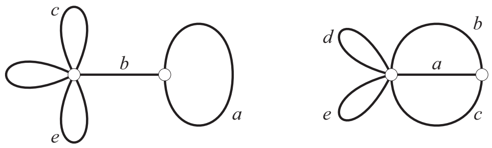

Figure 7: The abstract $(7,3)$ graphs.

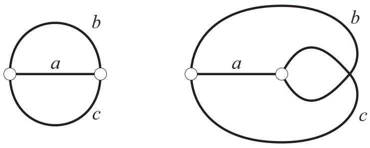

Figure 8: Fattenings of the graph θ.

Γ_2. Next, we note that Γ_1 cannot embed in the genus-2 surface $2T$ with a single disc discal region: if a bounds in $2T$ a disc disjoint from the rest of Γ_1, then there is more than one region, otherwise a regions is non-discal.

We must then enumerate up to symmetry the embeddings of Γ_2 in $2T$ with a single region. We do so by describing the fattenings of Γ_2 to a ribbon having a single boundary component attaching a disc to which we get $2T$. Note that a fattening of a graph can be described by an immersion in the plane. There are only two fattenings of the subgraph $\theta = a \cup b \cup c$ of Γ_2, shown in Fig. 8 and both seen to be totally symmetric. We now concentrate on the fattenings of the subgraph Δ of Γ_2 given by a regular neighbourhood of the 7-valent vertex union $d \cup e$. One easily sees that there are 12 fattenings of Δ, but 7 of them cannot give rise to fattenings of Γ_2 with a connected boundary, because a small boundary circle is already created near Δ. The other 5 fattenings of Δ are shown in Fig. 9. We must now combine the 2 fattenings of θ with the 5 of Δ, and thanks to the stated symmetry of those

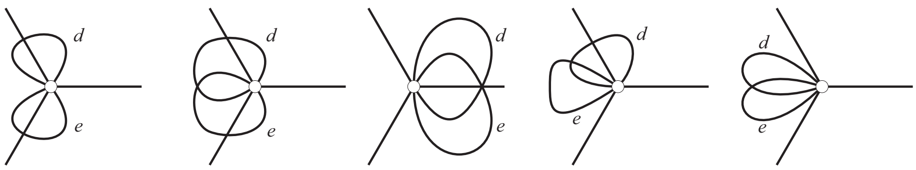

Figure 9: Fattenings of Δ without small boundary circles.

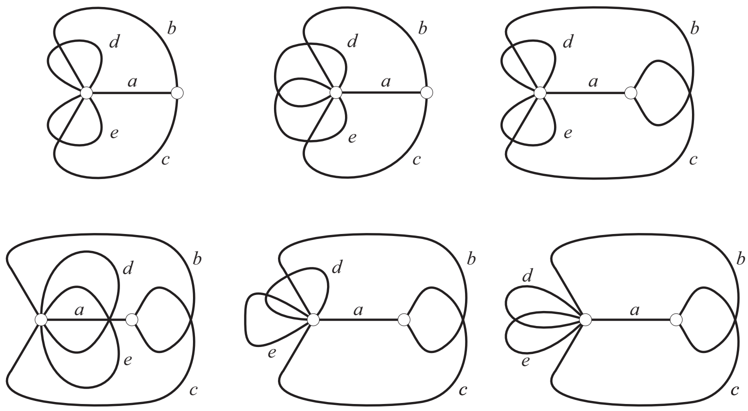

Figure 10: Fattenings of Γ_2 with a single boundary circle, capping off which we get $2T$.

of θ there is only one way to combine any given pair of fattenings. Of the resulting 10 ribbons, 4 turn out to have disconnected boundary, and the other 6 are shown in Fig. 10.

Since 5 of the graphs of Fig. 10 have a symmetry of type $(b \leftrightarrow c, d \leftrightarrow e)$ and one has no symmetries, the number ν of realizations of the datum is given by 5 times the number x of ways to express k as $a+b+c+d+e$ up to $(b \leftrightarrow c, d \leftrightarrow e)$, plus the number y of ways to express k as $a+b+c+d+e$ with no symmetries to take into account. Of course $y = \binom{k-1}{4}$, while

$$x = \sum_{a=1}^{k-4} z(k-a) = \sum_{h=4}^{k-1} z(h),$$

where $z(h)$ is the number of ways to express h as $b+c+d+e$ up to $(b \leftrightarrow c, d \leftrightarrow e)$.

We can now compute $z(h)$ by distinguishing the case $b < c$ from $b = c$, $d < e$ and from $b = c$, $d = e$. For the case $b < c$ we can choose $j = b+c+d$ between 4 and $h-1$, then $i = b+c$ between 3 and $j-1$, and we are left with $\left[\frac{i-1}{2}\right]$ choices for b and c, so we have a contribution to $z(h)$ equal to

$$\sum_{j=4}^{h-1}\sum_{i=3}^{j-1}\left[\frac{i-1}{2}\right] = \sum_{j=4}^{h-1}\sum_{i=2}^{j-2}\left[\frac{i}{2}\right] = \sum_{j=4}^{h-1}\left[\left(\frac{j-2}{2}\right)^2\right] = \sum_{j=4}^{h-1}\left[\left(\frac{j}{2}-1\right)^2\right].$$

We can now distinguish between the odd case $h = 2m+1$ and the even case $h = 2m$, splitting the sum between the odd and the even values of j, and getting respectively, after easy calculations,

$$\frac{1}{6}m(m-1)(4m-5) \qquad \text{and} \qquad \frac{1}{6}(m-1)(m-2)(4m-3).$$

For the case $b=c,\ d<e$ we can choose $b=c$ between 1 and $\left[\frac{h-3}{2}\right]$, so we have a contribution to $z(h)$ equal to

$$\begin{aligned}
&\sum_{b=1}^{\left[\frac{h-3}{2}\right]}\left[\frac{h-2b-1}{2}\right] = \sum_{b=1}^{\left[\frac{h-3}{2}\right]}\left(\left[\frac{h-1}{2}\right]-b\right)\\
=\ & \left[\frac{h-1}{2}\right]\cdot\left[\frac{h-3}{2}\right]-\frac{1}{2}\left[\frac{h-3}{2}\right]\left(\left[\frac{h-3}{2}\right]+1\right)\\
=\ & \frac{1}{2}\left[\frac{h-3}{2}\right]\left(2\left(\left[\frac{h+3}{2}\right]+1\right)-\left(\left[\frac{h-3}{2}\right]+1\right)\right)\\
=\ & \frac{1}{2}\left[\frac{h-3}{2}\right]\left(\left[\frac{h-3}{2}\right]+1\right).
\end{aligned}$$

Therefore we have a contribution to $z(h)$ for $h=2m+1$ and for $h=2m$ given respectively by

$$\frac{1}{2}m(m-1) \qquad \text{and} \qquad \frac{1}{2}(m-2)(m-1).$$

Finally, for $b=c,\ d=e$ we have a contribution of 0 for odd h and of $m-1$ for $h=2m$.

We can now plug these contributions to $z(h)$ in the formula for x. Again we distinguish between the case $k=2p+1$ and the case $k=2p$, splitting the sum between the odd $h=2m+1$ and the even $h=2m$, getting respectively

$$\begin{aligned}
&\sum_{m=2}^{p-1}\left(\frac{1}{6}m(m-1)(4m-5)+\frac{1}{2}m(m-1)\right)\\
&+\sum_{m=2}^{p}\left(\frac{1}{6}(m-1)(m-2)(4m-3)+\frac{1}{2}(m-1)(m-2)+(m-1)\right)\\
=\ & \frac{1}{6}(2p^4-6p^3+7p^2-3p)
\end{aligned}$$

and

$$
\begin{aligned}
& \sum_{m=2}^{p-1}\left(\frac{1}{6}m(m-1)(4m-5)+\frac{1}{2}m(m-1)\right) \\
& \quad +\sum_{m=2}^{p-1}\left(\frac{1}{6}(m-1)(m-2)(4m-3)+\frac{1}{2}(m-1)(m-2)+(m-1)\right) \\
= & \frac{1}{6}(2p^4-10p^3+19p^2-17p+6).
\end{aligned}
$$

Replacing $p=\frac{1}{2}(k-1)$ in the first formula and $p=\frac{k}{2}$ in the second one we get respectively

$$
\begin{aligned}
x_{\mathrm{odd}} &= \frac{1}{48}\left(k^4-10k^3+38k^2-62k+33\right), \\
x_{\mathrm{even}} &= \frac{1}{48}\left(k^4-10k^3+38k^2-68k+48\right).
\end{aligned}
$$

Recalling that $\nu=5x+y$ and replacing the expressions just found for x and $y=\binom{k-1}{4}$ we get

$$
\begin{aligned}
\nu_{\mathrm{odd}} &= \frac{1}{48}\left(7k^4-70k^3+260k^2-410k+213\right) \\
\nu_{\mathrm{even}} &= \frac{1}{48}\left(7k^4-70k^3+260k^2-440k+288\right).
\end{aligned}
$$

Thus

$$
\nu=\nu_{\mathrm{even}}+2\left(\frac{k}{2}-\left[\frac{k}{2}\right]\right)(\nu_{\mathrm{odd}}-\nu_{\mathrm{even}})
$$

and the stated formula easily follows.

References

[1] P. B. Cohen (now P. Tretkoff), *Dessins denfant and Shimura varieties*, In: "The Grothendieck Theory of Dessins dEnfants," (L. Schneps, ed.), London Math. Soc. Lecture Notes Series, Vol. 200, Cambridge University Press, 1994, pp. 237-243.

[2] P. Corvaja – C. Petronio – U. Zannier, *On certain permutation groups and sums of two squares*, Elem. Math. **67** (2012), 169-181.

[3] A. L. Edmonds – R. S. Kulkarni – R. E. Stong, *Realizability of branched coverings of surfaces*, Trans. Amer. Math. Soc. **282** (1984), 773-790.

[4] I. P. Goulden – J. H. Kwak – J. Lee, *Distributions of regular branched surface coverings*, European J. Combin. **25** (2004), 437-455.

[5] A. Grothendieck, *Esquisse d'un programme (1984)*. In: "Geometric Galois Action" (L. Schneps, P. Lochak eds.), 1: "Around Grothendieck's Esquisse d'un Programme," London Math. Soc. Lecture Notes Series, Vol. 242, Cambridge Univ. Press, 1997, pp. 5-48.

[6] S. K. Lando – A. K. Zvonkin, "Graphs on Surfaces and their Applications," Encyclopaedia Math. Sci. Vol. 141, Springer, Berlin, 2004.

[7] J. H. Kwak, A. Mednykh, *Enumeration of branched coverings of closed orientable surfaces whose branch orders coincide with multiplicity*, Studia Sci. Math. Hungar. **44** (2007), 215-223.

[8] J. H. Kwak, A. Mednykh, *Enumerating branched coverings over surfaces with boundaries*, European J. Combin. **25** (2004), 23-34.

[9] J. H. Kwak, A. Mednykh, V. Liskovets, *Enumeration of branched coverings of nonorientable surfaces with cyclic branch points*, SIAM J. Discrete Math. **19** (2005), 388-398.

[10] A. D. Mednykh, *On the solution of the Hurwitz problem on the number of nonequivalent coverings over a compact Riemann surface* (Russian), Dokl. Akad. Nauk SSSR **261** (1981), 537-542.

[11] A. D. Mednykh, *Nonequivalent coverings of Riemann surfaces with a prescribed ramification type* (Russian), Sibirsk. Mat. Zh. 25 (1984), 120-142.

[12] S. Monni, J. S. Song, Y. S. Song, *The Hurwitz enumeration problem of branched covers and Hodge integrals*, J. Geom. Phys. **50** (2004), 223-256.

[13] F. Pakovich, *Solution of the Hurwitz problem for Laurent polynomials*, J. Knot Theory Ramifications **18** (2009), 271-302.

[14] M. A. Pascali – C. Petronio, *Surface branched covers and geometric 2-orbifolds*, Trans. Amer. Math. Soc. **361** (2009), 5885-5920

[15] M. A. Pascali – C. Petronio, *Branched covers of the sphere and the prime-degree conjecture*, Ann. Mat. Pura Appl. **191** (2012), 563-594.

[16] E. Pervova – C. Petronio, *Realizability and exceptionality of candidate surface branched covers: methods and results*, Seminari di Geometria 2005-2009, Università degli Studi di Bologna, Dipartimento di Matematica, Bologna 2010, pp. 105-120.

[17] C. Petronio, *Explicit computation of some families of Hurwitz numbers*, preprint `arXiv:1805.00317`.

[18] J. Song – B. Xu, *On rational functions with more than three branch points*, `arXiv:1510.06291`

Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 PISA – Italy
`petronio@dm.unipi.it`