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Remarks on the "Onsager Singularity Theorem" for Leray–Hopf Weak Solutions: The Hölder Continuous Case

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Abstract: In this paper, we first present an overview of the results related to energy conservation in spaces of Hölder-continuous functions for weak solutions to the Euler and Navier–Stokes equations. We then consider families of weak solutions to the Navier–Stokes equations with Hölder-continuous velocities with norms uniformly bound in terms of viscosity. We finally provide the proofs of our original results that extend the range of allowed exponents for inviscid limits producing solutions to the Euler equations satisfying the energy equality, and improve the so-called "Onsager singularity" theorem.

Keywords: energy conservation; Onsager conjecture; Navier–Stokes equations

MSC: 35Q30

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1. Introduction

The aim of this paper is first to make a review of some results about the energy conservation for incompressible fluids. After having collected relevant known results, we provide some original improvements of two recent results, also simplifying the proofs, at the price of working in the setting of classical Hölder continuous functions (instead of Besov spaces, as done in most of the literature). Results are, in fact, proved without resorting to Paley–Littlewood decomposition or other tools from harmonic analysis and are then accessible to the wider audience of graduate students or researchers working in problems of applied fluid dynamics.

The main original results we prove concern: (i) the emergence of solutions to the Euler equations satisfying the energy equality (cf. Theorem 4), as inviscid limits of Leray–Hopf weak solutions, plus some additional assumptions of Hölder regularity (with a range of exponents $\sigma \in]1/3, 1[$) balanced by a proper integrability in time; (ii) A so-called Onsager singularity theorem (cf. Theorem 5), which implies emergence of the quasi-singularities for Leray–Hopf weak solutions, even if the total energy dissipation vanishes in the limit $\nu \to 0$, as long as it does so sufficiently slowly.

The original results proved here are inspired by a recent work of Drivas and Eyink [1], and are obtained by using a functional setting with classical Hölder continuous spaces. This allows us to obtain some improvements in the extra assumptions on the velocity, especially relaxing the mutual constraints between β-integrability in time and σ-Hölder exponent with respect to the space variables.

To properly set the problem, we consider the space-periodic setting with $\mathbb{T}^3 := (\mathbb{R}/2\pi\mathbb{Z})^3$. The functions $v^E : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p^E : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ (velocity and pressure, respectively) solve the Euler equations for incompressible homogeneous fluids

$$\begin{aligned} \partial_t v^E + (v^E \cdot \nabla)\, v^E + \nabla p^E &= 0 && (t,x) \in (0,T) \times \mathbb{T}^3, \\ \operatorname{div} v^E &= 0 && (t,x) \in (0,T) \times \mathbb{T}^3, \\ v^E(0,x) &= v_0^E(x) && x \in \mathbb{T}^3, \end{aligned} \tag{1}$$

and we consider only weak solutions; for a precise description, see Definition 1.

We also denote by $v^\nu : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$ and $p^\nu : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3$, the velocity and pressure, which solve the Navier–Stokes equations (denoted simply as NSE in the sequel)

$$\begin{aligned} \partial_t v^\nu + (v^\nu \cdot \nabla)\, v^\nu - \nu\Delta v^\nu + \nabla p^\nu = 0 & \qquad (t,x) \in (0,T) \times \mathbb{T}^3, \\ \operatorname{div} v^\nu = 0 & \qquad (t,x) \in (0,T) \times \mathbb{T}^3, \\ v^\nu(0,x) = v_0^\nu(x) & \qquad x \in \mathbb{T}^3, \end{aligned} \tag{2}$$

for fixed $\nu > 0$. In this case, we consider (Leray–Hopf) weak solutions, see the Definition 2.

For both Equations (1) and (2), the definition of a weak solution concerns only the velocity since the pressure can be eliminated by a projection operator. We will consider for both ideal and viscous fluids velocities satisfying the additional assumption of being $C^{0,\sigma}(\mathbb{T}^3)$ (σ-Hölder continuity with respect to the space variables). This is suggested by Onsager conjecture [2] (which nowadays could be called Onsager theorem) concerning the energy conservation for weak solutions of the Euler equations, such that $v^E \in L^\infty(0,T; C^{1/3}(\mathbb{T}^3))$.

The problem of understanding vanishing viscosity limits and the construction of distributional (dissipative) solutions to the Euler equations has a long history and is interlaced with energy conservation. We mainly refer to Duchon and Robert [3] for results similar to those we will prove. Additionally, the recent Fourier-based approach developed by Chen and Glimm [4,5] uses spectral properties to deduce fractional regularity results suitable for proving inviscid limits. For simplicity, we will assume $f = 0$, but the results can easily be adapted to include smooth non-zero external forces.

The Onsager conjecture, which was recently fully solved, suggested the value $\sigma = 1/3$ for the Hölder regularity in space variables, especially for the case of the Euler equations. In this work, we consider a combination of space–time conditions, identifying families of criteria. The first rigorous results about the Onsager conjecture were probably those of Eyink [6,7] in the Fourier setting and Constantin, E, Titi [8] in Besov spaces, which are slightly larger than Hölder spaces as increments are only $\mathcal{O}(h^\sigma)$ in the L^3-norm. A well-known result is that if v^E is a weak solution to the Euler equations such that

$$v^E \in L^3(0,T; B_3^{\sigma,\infty}(\mathbb{T}^3)) \cap C(0,T; L^2(\mathbb{T}^3)) \qquad \text{with } \sigma > \frac{1}{3},$$

then $\|v^E(t)\| = \|v_0^E\|$, for all $t \in [0,T]$. As explained in [8], "...This is basically the content of Onsager's conjecture, except Onsager stated his conjecture in Hölder spaces rather than Besov spaces. Obviously the above theorem implies similar results in Holder spaces..." For the Euler equations the results recalled here (Theorem 1) in the Hölder case follow the same methods used in [8]. Nevertheless, we are here trying to focus on the hypotheses in the time variable (more than in the spatial ones), showing how a slightly more stringent assumption on the space variables (Hölder instead of Besov), produces some improvement in the time variable.

Remark 1. *The focus of this paper is to consider the Hölder regularity case, as it keeps the results simple and understandable for an audience familiar with classical spaces of mathematical analysis. This approach also aligns with recent developments in the theory, particularly after the work of De Lellis and Székelyhidi [9], devoted to the construction of counterexamples when the Hölder continuity condition is violated. Their work sparked an intense effort to prove the Onsager conjecture, specifically the non-conservation below the critical space of $C^{1/3}$, as demonstrated in the works of Buckmaster et al. [10] and Isett [11]. These works proved the existence of space-periodic weak solutions v^E of the Euler equations that do not conserve kinetic energy and belong to the space $L^1(0,T; C^{1/3-\epsilon}(\mathbb{T}^3))$. Recently, Daneri, Runa, and Székelyhidi [12] have generalized this result, demonstrating non-uniqueness for a dense set of initial data.*

The situation for the energy conservation for the Navier–Stokes equations (NSE) is slightly different. Recent results, reviewed in [13], are those of Cheskidov [14], Cheskidov and Luo [15], and Farwig and Taniuchi [16]. The proofs of the results for the NSE take advantage of the bound $\sqrt{\nu}\nabla v^{\nu} \in L^2((0,T)\times\mathbb{T}^3)$. The sharpest result proven is the following: Suppose that $1 \leq \beta < p \leq \infty$ are such that $\frac{2}{p}+\frac{1}{\beta} < 1$. If v^{ν} is a Leray–Hopf weak solution such that

$$v^{\nu} \in L^{\beta}_{w}(0,T;B^{\frac{2}{p}+\frac{2}{\beta}-1}_{p,\infty}(\mathbb{T}^3)), \tag{3}$$

then v^{ν} satisfies the energy equality. The proof of this result, which is based on Littlewood–Paley decomposition, is too lengthy to be presented here. However, it is worth noting that this result includes, as a special case, the classical $v^{\nu} \in L^4(0,T;L^4)$ result by Prodi and Lions [17,18]. Note that the limiting case of (3) is $v^{\nu} \in L^{\beta}_{w}(0,T;B^{1/3}_{3,\infty}(\mathbb{T}^3))$, for all $\beta > 3$, just to compare with results in Hölder spaces. Recent extensions are those based on sufficient conditions on the gradient of v^{ν} as in [19] and in Beirão da Veiga and Yang [20], which are not-optimal, but valid also for the Dirichlet problem. The results in [19,20] can be "measured" in terms of Hölder spaces as follows

$$v \in L^{\frac{5}{3+2\sigma}}(0,T;C^{0,\sigma}(\overline{\Omega})) \qquad \text{with } \sigma \in]0,1[. \tag{4}$$

The results are obtained by embedding and—even if valid also for the boundary value problem—are far away from being sharp. Hence, this is why we consider here results directly in the class of Hölder continuous functions. Up to our knowledge, the only (optimal) result available for the NSE in the Hölder space is Theorem 3 below, which has been proved in [13,21] and which has the same scaling as (3).

Remark 2. *We also wish to mention that uniqueness is also believed to be connected with (local) conservation of energy. In this context, see the recent result of Cheskidov and Luo [22] where they proved non-uniqueness for a class of solutions, which are not Leray–Hopf but produce anomalous scaling for $v^{\nu} \in L^{3/2-\alpha}(0,T;C^{1/3}(\mathbb{T}^3)$, which is at the same level of scaling of the results in our Theorem 3. Additionally, for the NSE, we can recall that Albritton, Brué, and Colombo [23] produced a non-uniqueness result in the class of Leray–Hopf solutions. Furthermore, Maremonti [24] recently proved that for any initial datum $u_0 \in L^2$, at least one solution satisfying the energy equality can be constructed.*

Remark 3. *Results in Hölder spaces have been recently extended to the boundary value problems for the Euler equations in Bardos and Titi [25], proving energy conservation for weak solutions such that*

$$v^E \in L^3(0,T;C^{0,\sigma}(\overline{\Omega})), \qquad \textit{with } \sigma > \frac{1}{3},$$

However, in order to consider the NSE (positive viscosity) or even inviscid limits, it seems necessary to restrict to the space–periodic case. In fact, there is limited knowledge about energy conservation for the Navier–Stokes equations (NSE) in the presence of boundaries under Hölder assumptions (see [21]). Additionally, the vanishing viscosity limit poses unsolved questions in the case of Dirichlet conditions. As a result, there is ongoing research to find further additional assumptions that allow extending known results near the boundary, such as in the work of Drivas and Nguyen [26,27].

Results collected in Sections 3 and 4 are useful to explain and interpret the new results proved in Section 5. The results proved here show that Leray–Hopf solutions of the NSE "quasi-singularities" are required in order to account for anomalous energy dissipation. In fact, such consequences follow even if the energy dissipation is vanishing in the limit of zero viscosity, as long as it goes to zero as slowly as certain positive powers of ν. In that case, we show that the solutions of the NSE cannot have Hölder norms, above a critical smoothness, which are bounded uniformly in viscosity. This observation is important (see Drivas and Eyink [1] (Remark 4)) because

> ...empirical studies cannot distinguish in principle between a dissipation rate which is independent of viscosity and one which is vanishing sufficiently slowly.

Our results thus considerably strengthen the conclusion that quasi-singularities are necessary to account for the enhanced energy dissipation rates observed in turbulent flow.

Plan of the paper: In Section 2, we introduce the necessary notation. In Section 3, we review recent results on the Euler equations, focusing on improving the conditions on the time-variable and extending the range of allowed Hölder exponents. The extension to the viscous case, namely the Navier–Stokes equations, is discussed in Section 4. Finally, in the last and more original Section 5, we combine these results to study the emergence of weak solutions to the Euler equations that conserve energy as the viscosity parameter $\nu > 0$ approaches zero. In addition, we prove a family of "Onsager singularity theorems" that show that sequences of "smooth enough" Leray–Hopf solutions cannot have a total dissipation that vanishes too slowly.

2. Notation and Preliminaries

In the sequel we will use, for $1 \leq p \leq \infty$, the Lebesgue $(L^p(\mathbb{T}^3), \|\,.\,\|_p)$ and Sobolev $(W^{1,p}(\mathbb{T}^3), \|\,.\,\|_{1,p})$ spaces; for simplicity, we denote by $(\,.\,,\,.\,)$ and $\|\,.\,\|$ the L^2 scalar product and norm (while the other norms are explicitly indicated). We say that a real function $L : \mathbb{R}^+ \to \mathbb{R}^+$ is slowly varying in the sense of Kumarata at $\nu = 0$ if

$$\lim_{\nu \to 0^+} \frac{L(\lambda\nu)}{L(\nu)} = 1 \qquad \text{for any } \lambda > 0.$$

Moreover, we will use the Banach space of (uniformly) Hölder continuous functions $C^\sigma(\mathbb{T}^3) = C^{0,\sigma}(\mathbb{T}^3)$, for $0 < \sigma \leq 1$, with the norm

$$\|u\|_{C^\sigma} = \max_{x \in \mathbb{T}^3} |u(x)| + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\sigma} = \|u\|_\infty + [u]_\sigma,$$

and by $\dot{C}^\sigma$, we denote the corresponding homogeneous space. We will focus on space–time properties of functions and we say that

$$u \in L^\beta(0, T; \dot{C}^\sigma(\mathbb{T}^3)),$$

if there exists $f_\sigma : [0, T] \to \mathbb{R}^+$ such that

$$\begin{aligned} &(a) \qquad |u(t,x) - u(t,y)| \leq f_\sigma(t)|x - y|^\sigma, \qquad \forall\, x, y \in \mathbb{T}^3, \text{ for a.e. } t \in [0, T], \\ &(b) \qquad \int_0^T f_\sigma^\beta(t)\,\mathrm{d}t < \infty, \end{aligned}$$

and note that $f_\sigma(t) = [u(t)]_\sigma$ for almost all $t \in [0, T]$. This space will be endowed with the following semi-norm

$$\|u\|_{L^\beta(0,T;\dot{C}^\sigma(\mathbb{T}^3))} := \Big(\int_0^T f_\sigma^\beta(t)\,\mathrm{d}t \Big)^{1/\beta}.$$

To define the notions of weak solution, we denote by H and V the closure of smooth, periodic, divergence-free, and zero mean-value vector fields in $L^2(\mathbb{T}^3)$ and $W^{1,2}(\mathbb{T}^3)$, respectively. As the space for test functions we use

$$\mathcal{D}_T := \Big\{ \varphi \in C_0^\infty([0, T[\times \mathbb{T}^3) : \ \mathrm{div}\,\varphi = 0 \Big\}.$$

We define the notion of weak solution in the inviscid case.

Definition 1 (Space periodic weak solutions to the Euler equations)**.** *Let $v_0 \in H$. A measurable function $v^E : (0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ is called a weak solution to the Euler Equation (1) if*

$$v^E \in L^\infty(0,T;H),$$

and if v^E solves the equations in the weak sense, that is

$$\int_0^\infty \int_{\mathbb{T}^3} \left[v^E \cdot \partial_t \varphi + (v^E \otimes v^E) : \nabla \varphi) \right] \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^3} v_0 \cdot \varphi(0) \, \mathrm{d}x, \tag{5}$$

for all $\varphi \in \mathcal{D}_T$.

We also recall the definition of a weak solution for the viscous problem.

Definition 2 (Space-periodic Leray–Hopf weak solution)**.** *Let $v_0^\nu \in H$. A measurable function $v^\nu : (0,T) \times \Omega \to \mathbb{T}^3$ is called a Leray–Hopf weak solution to the space-periodic NSE (2) if $v^\nu \in L^\infty(0,T;H) \cap L^2(0,T;V)$; and the following hold true:*
The function v^ν solves the equations in the weak sense:

$$\int_0^\infty \int_{\mathbb{T}^3} \left[v^\nu \cdot \partial_t \varphi - \nu \nabla v^\nu : \nabla \varphi - (v^\nu \cdot \nabla) \, v^\nu \cdot \varphi) \right] \mathrm{d}x \mathrm{d}t = - \int_{\mathbb{T}^3} v_0^\nu \cdot \varphi(0) \, \mathrm{d}x, \tag{6}$$

for all $\varphi \in \mathcal{D}_T$;
It holds the (global) energy inequality

$$\frac{1}{2} \|v^\nu(t)\|_2^2 + \nu \int_0^t \|\nabla v^\nu(\tau)\|_2^2 \, \mathrm{d}\tau \leq \frac{1}{2} \|v_0^\nu\|_2^2, \qquad \forall \, t \in [0,T]; \tag{7}$$

The initial datum is strongly attained

$$\lim_{t \to 0^+} \|v^\nu(t) - v_0^\nu\| = 0. \tag{8}$$

It is well-known that for all $v_0^\nu \in H$ there exists at least a Leray–Hopf weak solution in any time interval $(0,T)$.

The energy inequality (7) can be rewritten as an equality

$$\frac{1}{2} \|v^\nu(t)\|^2 + \nu \int_0^t \int_{\mathbb{T}^3} \epsilon[v^\nu(\tau,x)] \, \mathrm{d}x \mathrm{d}\tau = \frac{1}{2} \|v_0^\nu\|^2 \qquad \forall \, t \in [0,T], \tag{9}$$

where the total energy dissipation rate is defined as follows

$$\epsilon[v] := \nu |\nabla v^\nu|^2 + D(v^\nu),$$

with $D(v^\nu)$, a non-negative distribution (Radon measure). Note that if $D(v^\nu) = 0$, then energy dissipation arises entirely from viscosity, and energy equality holds.

For the Euler equations, we will essentially apply the same classical strategy as in [3,6,8], based on mollifications, to justify the calculations. Despite the approach being the same, results can be greatly improved for the NSE since one can take advantage of the already known L^2-integrability for ∇v^ν; a different combination of estimates (which are ν-dependent) could then be used.

3. Energy Conservation for the Euler Equations

We first start recalling the known results about energy conservation for weak solutions to the Euler equations; we report in this section recent results from reference [21]. Theorem 1 below is the sharpest known in this setting and extends previous results to the full range of $\sigma \in]1/3, 1[$. Note that the results about non-conservation of energy below the exponent $1/3$ in [10,11] are proved with L^∞ or C^σ assumptions in the time variable. Even if the original conjecture detected the value $\sigma = 1/3$ as borderline, we identified in [21] the space $L^{1/\sigma}(0,T;\dot{C}^{\sigma})$ as the critical one (for all $\sigma > 1/3$).

On the other hand, the second result, namely Theorem 2, was already known for $\sigma = 1/3$ (see [28]) and again in [21] the proof is extended to the full range of exponents.

Theorem 1. *Let v^E be a weak solution to the Euler Equation (1) such that for some $\delta > 0$*

$$v^E \in L^{\frac{1}{\sigma}+\delta}(0,T;\dot{C}^{\sigma}(\mathbb{T}^3)), \quad \textit{with } \sigma \in \left]\frac{1}{3},1\right[. \tag{10}$$

Then, the weak solution v^E conserves the energy.

Proof. We just sketch the proof, which is obtained by a proper, even if standard, analysis of the commutation term after mollification; full details can be found in [21], see also the results recalled in Appendix A. A traditional way to take advantage of the additional Hölder continuity of the solution is to use properties of mollifiers.

We define $v^E_\varepsilon = \rho_\varepsilon * v^E$ and the following equality is valid (see for instance [25])

$$\frac{1}{2}\|v^E_\varepsilon(t)\|^2 - \frac{1}{2}\|v^E_\varepsilon(0)\|^2 = \int_0^t \int_{\mathbb{T}^3} (v^E \otimes v^E)_\varepsilon : \nabla v^E_\varepsilon \, \mathrm{d}x\mathrm{d}\tau. \tag{11}$$

The key observation is the Constantin-E-Titi commutator identity

$$(u \otimes u)_\varepsilon = u_\varepsilon \otimes u_\varepsilon + r_\varepsilon(u,u) - (u - u_\varepsilon) \otimes (u - u_\varepsilon), \tag{12}$$

with

$$r_\varepsilon(u,u) := \int_{\mathbb{T}^3} \rho_\varepsilon(y)(\delta_y u(x) \otimes \delta_y u(x)) \, \mathrm{d}y,$$

where $\delta_y u(x) := u(x-y) - u(x)$. We apply the decomposition (12) to (11) and since by integration by parts

$$\int_{\mathbb{T}^3} (v^E_\varepsilon \otimes v^E_\varepsilon) : \nabla v^E_\varepsilon \, \mathrm{d}x = 0,$$

we are reduced to study only the contribution of the remaining two terms in (12), in the decomposition of the right-hand-side of (11).

We start from the last one and split the absolute value of the integrand as follows

$$|v^E - v^E_\varepsilon|^2 |\nabla v^E_\varepsilon| = |v^E - v^E_\varepsilon|^\eta |v^E - v^E_\varepsilon|^{2-\eta} |\nabla v^E_\varepsilon|, \qquad \text{for some } 0 \leq \eta \leq 2.$$

Hence, by using (10) and Lemma A1, we get

$$\int_{\mathbb{T}^3} |v^E - v^E_\varepsilon|^2 |\nabla v^E_\varepsilon| \, \mathrm{d}x \leq f_\sigma^{1+\eta}(t) \varepsilon^{\sigma\eta+\sigma-1} \int_{\mathbb{T}^3} |v^E - v^E_\varepsilon|^{2-\eta} \, \mathrm{d}x,$$

and, by Hölder inequality, by recalling that from Definition 1 $\|v^E_\varepsilon(t)\| \leq \|v^E(t)\| \leq C$, we get

$$\int_0^T \int_{\mathbb{T}^3} |v^E - v^E_\varepsilon|^2 |\nabla v^E_\varepsilon| \, \mathrm{d}x\mathrm{d}t \leq C\, \varepsilon^{\sigma\eta+\sigma-1} \int_0^T f_\sigma^{1+\eta}(t) \, \mathrm{d}t.$$

To ensure that this term vanishes as $\varepsilon \to 0$ it is sufficient to fix $\eta \in [0,2]$ such that

$$\frac{1-\sigma}{\sigma} < \eta \leq 2, \tag{13}$$

and such a choice is possible only if $\sigma > 1/3$; furthermore, one has to assume $f_\sigma \in L^{1+\eta}(0,T)$, which follows if $1 + \eta \leq 1/\sigma + \delta$. We find that we have to choose η such that $1/\sigma - 1 < \eta \leq 1/\sigma - 1 + \delta$.

With the same fixed η, the other term arising from the commutator is estimated as follows:

$$|r_\varepsilon(v^E, v^E)| \leq \int_{B(0,\varepsilon)} \rho_\varepsilon(y)|v^E(x-y) - v^E(x)|^\eta |v^E(x-y) - v^E(x)|^{2-\eta}\,\mathrm{d}y$$
$$\leq f_\sigma^\eta(t) \int_{B(0,\varepsilon)} \rho_\varepsilon(y)|y|^{\sigma\eta}|v^E(x-y) - v^E(x)|^{2-\eta}\,\mathrm{d}y,$$
$$\leq C(\eta) f_\sigma(t)^\eta \varepsilon^{\sigma\eta} \int_{\mathbb{T}^3} \rho_\varepsilon(y)(|v^E(x-y)|^{2-\eta} + |v^E(x)|^{2-\eta})\,\mathrm{d}y.$$

Next, by properties of convolution and Hölder inequality we obtain

$$\left| \int_{\mathbb{T}^3} r_\varepsilon(v^E, v^E) : \nabla v_\varepsilon^E\,\mathrm{d}x \right| \leq C\, f_\sigma(t)^{1+\eta} \varepsilon^{\sigma\eta+\sigma-1},$$

showing that this term can be treated as the previous one. Hence, we obtain that

$$\left| \int_0^t \int_{\mathbb{T}^3} (v^E \otimes v^E)_\varepsilon : \nabla v_\varepsilon^E\,\mathrm{d}x\mathrm{d}\tau \right| \leq C\,\varepsilon^\gamma \int_0^T f_\sigma^{1/\sigma+\delta}(t)\,\mathrm{d}t \overset{\varepsilon\to 0^+}{\to} 0, \tag{14}$$

since by (10), the time-integral is finite and $\gamma := \sigma\eta + \sigma - 1 > 0$. □

Remark 4. *Apart from some "δ-technical conditions", the critical space for energy conservation of weak solutions of the Euler equations is*

$$v^E \in L^{1/\sigma}(0, T; \dot{C}^\sigma) \quad \text{with } \sigma > 1/3.$$

Observe also that if $\sigma > 1/3$, then $1/\sigma < 3$, and besides having a full-range of scaled space–time results, there is a technical-improvement in the time variable with respect to the previous results: we can have $v^E \in L^q(0, T; \dot{C}^\sigma(\mathbb{T}^3))$ for some $q < 3$, even for $\sigma > 1/3$ arbitrarily close to $1/3$. Related results in Besov spaces can be proved using Lemma A3 instead of that for Hölder functions.

Some improvements can be obtained considering the following spaces $C_\omega^\sigma(\mathbb{T}^3) \subset C^\sigma(\mathbb{T}^3)$ defined through the norm

$$\|u\|_{C_\omega^\sigma} = \max_{x\in\mathbb{T}^3} |u(x)| + \sup_{x\neq y} \frac{|u(x) - u(y)|}{\omega(|x-y|)|x-y|^\sigma} = \|u\|_\infty + [u]_{\omega,\sigma},$$

with $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, a non-decreasing function such that $\lim_{s\to 0^+} \omega(s) = 0$. These spaces have already been considered by Duchon and Robert [3] and, more recently in [28], for $\sigma = 1/3$. By using this slightly smaller space $C_\omega^\sigma(\mathbb{T}^3)$ (which is a counterpart of the space $B^\sigma_{p,c(\mathbb{N})}$ as in Cheskidov et al. [29] and in the same spirit of the results by Duchon and Robert [3]), it is also possible to show the following result.

Theorem 2. *Let v^E be a weak solution to the Euler Equation (1) such that*

$$v^E \in L^{1/\sigma}(0, T; \dot{C}_\omega^\sigma(\mathbb{T}^3)) \quad \text{with } \sigma \in \left[\frac{1}{3}, 1\right[. \tag{15}$$

Then v^E conserves the energy.

The proof given in [21] follows the same path as in the previous theorem, by employing Lemma A2.

4. Energy Conservation for the Navier–Stokes Equations

In this section, we focus on the viscous case and recall some recent results, proved in the setting of Hölder solutions, see [13,21].

Theorem 3. *Let v^ν be a Leray–Hopf weak solution to the NSE in $(0,T)\times\mathbb{T}^3$ such that*

$$v^\nu \in L^{\frac{2}{1+\sigma}}_{loc}(]0,T];\dot{C}^\sigma(\mathbb{T}^3)) \qquad \text{with } \sigma \in]0,1[. \tag{16}$$

Then, v^ν satisfies the energy equality in $[0,T]$.

Recall that condition (16) means that that there exists $f_\sigma :]0,T] \to \mathbb{R}^+$ such that $|v^\nu(t,x)-v^\nu(t,y)| \le f_\sigma(t)|x-y|^\sigma$ for almost all $t \in]0,T]$ and $f_\sigma \in L^{\frac{2}{1+\sigma}}(\lambda,T)$ for all $\lambda \in]0,T[$, but it is not excluded that

$$\int_0^T f_\sigma^{\frac{2}{1+\sigma}}(t)\,\mathrm{d}t = +\infty.$$

Remark 5. *The conditions derived from Theorem 3 are less restrictive compared to those previously established for the Euler equations. This is because for all $\sigma \in]0,1[$, we have $2/(1+\sigma) < 1/\sigma$.*

Additionally, the conditions in Theorem 3 have a similar scaling as the condition (3), from [22] in the case $p=\infty$. However, it is important to note that the two results are not directly comparable, as the space $L^\beta_w(0,T)$ is larger than $L^\beta(0,T)$, but our results allow for less regularity near $t=0$, through the use of Leray–Hopf solutions, an observation that was also utilized in [30].

We do not prove the result here since the techniques are similar to those used in the previous section, but see some sketch of the proof in Appendix A. The proof uses additional estimates valid for Leray–Hopf solutions, which are not uniform in $\nu > 0$. In particular, the additional regularity, $\nabla v^\nu \in L^2(0,T;L^2(\mathbb{T}^3))$, being v^ν a Leray–Hopf weak solution. Hence, these results are not applicable to the vanishing viscosity limit. Nevertheless, we reported them to highlight the possible differences between a) assumptions producing Leray–Hopf weak solutions conserving energy and b) assumptions on Leray–Hopf weak solutions that allow for the identification of the inviscid limit and convergence to energy-conserving weak solutions of the Euler equations. See the next section.

5. Inviscid Limits Conserving Energy

We prove now an original theorem concerning weak solutions to Euler equations as limits of the NSE. Results in this section are an extension of those by Drivas and Eyink [1] to a wider range of parameters, but considering Hölder functions: this allows us to identify some critical exponent. Note that for the smaller values of σ, namely $1/3 < \sigma \le 1/2$, the result represents a direct extension (with improvement in the time variable requirements), while for $\sigma > 1/2$, the result is rather different, showing a possible criticality for $\sigma = 1/2$. This also shows that there could be a difference if the solution of the Euler equations is obtained as a inviscid limit, in comparison with general weak solutions.

Remark 6. *For technical reasons related to the Fourier characterization, the critical role of the exponent $1/2$ was also present in the early work of Eyink [6].*

Remark 7. *No assumptions are required regarding the existence of limiting Euler solutions. However, based on our hypotheses, weak Euler solutions v^E can be obtained as the limit of Leray–Hopf solutions v^ν as $\nu \to 0$.*

The following is the first original result of this paper. In particular, it represents an improvement of [1] (Thm. 2), where only the case corresponding to $\beta = 3$ has been considered (even if in a slightly larger Besov space in the space variables). Moreover, under our assumptions, the convergence of v^ν in $L^3((0,T)\times\mathbb{T}^3)$ is a consequence of the assumptions and has not to be additionally added, as it was also in [3].

Theorem 4. *Let us assume that $\{v^\nu\}$, $\nu \in]0,1]$, is a family of weak solutions of the NSE with the same initial datum $v_0 \in H \cap C^{0,\sigma}(\mathbb{T}^3)$. Let us assume that for some $\sigma \in]1/3, 1[$, there exists a constant $C_{\sigma,\beta} > 0$, independent of $\nu > 0$, such that*

$$\|v^\nu\|_{L^\beta(0,T;C^{0,\sigma}(\mathbb{T}^3))} \leq C_{\sigma,\beta} \qquad \forall\, \nu \in]0,1], \tag{17}$$

with

$$\begin{aligned} &\beta > \frac{1}{\sigma} && \text{if} \quad 1/3 < \sigma \leq 1/2, \\ &\beta = 2 && \text{if} \quad \sigma > 1/2. \end{aligned}$$

Then, in the limit $\nu \to 0$, the family v^ν converges to a weak solution v^E in $[0,T]$ of the Euler equations satisfying the energy equality.

Proof of Theorem 4. Let v^ν be a Leray–Hopf solution with initial datum v_0 and viscosity $\nu > 0$. Then, by the energy inequality (7) it holds, uniformly in $\nu > 0$, the estimate

$$2^{-1/2}\|v^\nu\|_{L^\infty(0,T;L^2(\mathbb{T}^3))} + \|\sqrt{\nu}\nabla v^\nu\|_{L^2(0,T;L^2(\mathbb{T}^3))} \leq 2^{-1/2}\|v_0\|.$$

Moreover, for each fixed $\nu > 0$, it holds that $\partial_t v^\nu \in L^{4/3}(0,T;V')$, hence the duality $\langle \cdot\,,\cdot \rangle$ between V and its topological dual is justified. By comparison, for any smooth, periodic, and divergence-free vector ϕ, it holds

$$\int_0^T \langle \partial_t v^\nu, \phi \rangle, \mathrm{d}t = -\nu \int_0^T \int_{\mathbb{T}^3} \nabla v^\nu : \nabla\phi \,\mathrm{d}x\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} v^\nu \otimes v^\nu : \nabla\phi \,\mathrm{d}x\mathrm{d}t,$$

and consequently

$$\begin{aligned} \int_0^T \langle \partial_t v^\nu, \phi \rangle \,\mathrm{d}t \leq{}& \Big(\int_0^T \nu \|\nabla v^\nu\|^2 \,\mathrm{d}t \Big)^{1/2} \Big(\int_0^T \nu \|\nabla \phi\|^2 \,\mathrm{d}t \Big)^{1/2} \\ &+ \sup_{t\in[0,T]} \|v^\nu(t)\| \Big(\int_0^T \|v^\nu\|_\infty^2 \mathrm{d}t \Big)^{1/2} \Big(\int_0^T \|\nabla\phi\|_2^2 \,\mathrm{d}t \Big)^{1/2} \\ \leq{}& 2^{-1/2}\sqrt{\nu}\|v_0\|\|\nabla\phi\|_{L^2(0,T;L^2(\mathbb{T}^3))} + C_{\sigma,\beta}\|v_0\|\|\nabla\phi\|_{L^2(0,T;L^2(\mathbb{T}^3))}, \end{aligned}$$

showing by duality that

$$\|\partial_t v^\nu\|_{L^2(0,T;V')} \leq (2^{-1/2} + C_{\sigma,\beta})\|v_0\| \qquad \forall\, \nu \in]0,1].$$

Hence, we can apply the Aubin–Lions compactness theorem (in the version of Simon [31], for non-reflexive Banach spaces) with the triple of Banach spaces

$$C^{0,\sigma}(\mathbb{T}^3) \cap H \hookrightarrow\hookrightarrow H \hookrightarrow V',$$

where the first inclusion is compact by Arzelà–Ascoli theorem and the second is continuous.

By using the uniform boundedness of the sequence $\{v^\nu\}$ in the space

$$\mathcal{W} := \Big\{ u \in L^\beta(0,T;C^{0,\sigma}(\mathbb{T}^3) \cap H), \quad \text{s.t.} \quad \partial_t u \in L^2(0,T;V') \Big\},$$

we infer, by using the uniform bound in $L^\infty(0,T;H)$ (cf. again [31]), that there exists v^E such that, along some sequence $\nu \to 0$,

$$\lim_{\nu\to 0} v^\nu = v^E \qquad \text{in } Ł^q(0,T;H) \quad \forall\, q < \infty.$$

Next, by using the bound $v^\nu \in L^\infty(0,T;H)$, by convex interpolation, one also gets that $v^\nu \to v^E$ in $L^{3\beta'}(0,T;L^3(\mathbb{T}^3))$, for any $\beta' < \beta$, hence it follows that

$$\lim_{\nu\to 0} v^\nu = v^E \qquad \text{in } L^3(0,T;L^3(\mathbb{T}^3)),$$

and, passing to the limit as $\nu \to 0$ in (6), we get

$$\int_0^\infty \int_{\mathbb{T}^3} \Big[v^E \cdot \partial_t \varphi + (v^E \otimes v^E) : \nabla\phi \Big]\, \mathrm{d}x\mathrm{d}t = -\int_{\mathbb{T}^3} v_0 \cdot \varphi(0)\, \mathrm{d}x,$$

for all $\varphi \in \mathcal{D}_T$ since $\int_0^T (\sqrt{\nu}\nabla v^\nu, \sqrt{\nu}\nabla\varphi)\, \mathrm{d}t \to 0$. This shows that v^E is a weak solution to the Euler equations. Moreover, by using well-known results by Duchon and Robert (see [3] (Proposition 4)) from the combination of vanishing viscosity and strong limit in $L^3(0,T;L^3(\mathbb{T}^3))$, it follows that v^E is a dissipative solution of the Euler equations in $(0,T) \times \mathbb{T}^3$, that is

$$\frac{1}{2}\partial_t(|v^E|^2) + \nabla \cdot (v^E \frac{1}{2}(|v^E|^2 + p^E)) + D(v^E) = 0,$$

in the sense of space–time distributions, where the pressure p^E associated to v^E is defined as follows $p^E := -\sum_{i,j=1}^3 \Delta^{-1}\partial_i\partial_j(v_i^E v_j^E)$.

The dissipative term is then estimated in the following way

$$\int_0^T \nu \int_{\mathbb{T}^3} |\nabla v_\varepsilon^\nu|^2\, \mathrm{d}x\, \mathrm{d}t \leq 8\pi^3\, \nu\, \varepsilon^{2(\sigma-1)} \int_0^T f_\sigma^2(t)\, \mathrm{d}t, \tag{18}$$

and note that, since the limit v^E has not bounded gradients, contrary to the case at fixed viscosity, we have to estimate it without using properties of the space derivatives. This implies that the power of f_σ in the integral is exactly equal to 2. This gives a limitation on the range of possible results since any conditions derived should be using the space $L^\beta(0,T;C^{0,\sigma}(\mathbb{T}^3))$, with $\beta \geq 2$.

This implies that, to have vanishing dissipation, besides $\beta \geq 2$, also the following condition should be assumed:

$$\nu\, \varepsilon^{2(\sigma-1)} = o(1). \tag{19}$$

Next, we apply the same procedure as before using as test function $\rho_\varepsilon * (\rho_\varepsilon * v^\nu)$ and we estimate again the various commutator terms. We get, for any $0 < \eta \leq 2$

$$\begin{aligned} \left| \int_{\mathbb{T}^3} (v^\nu - v_\varepsilon^\nu) \otimes (v^\nu - v_\varepsilon^\nu) : \nabla v_\varepsilon^\nu\, \mathrm{d}x \right| &\leq \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^\eta |v - v_\varepsilon^\nu|^{2-\eta} |\nabla v_\varepsilon^\nu|\, \mathrm{d}x \\ &\leq f_\sigma^{1+\eta}(t) \varepsilon^{\sigma\eta+\sigma-1} \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^{2-\eta}\, \mathrm{d}x \\ &\leq C(\|v_0\|) f_\sigma^{1+\eta}(t)\, \varepsilon^{\sigma\eta+\sigma-1}. \end{aligned}$$

This shows that in order to have a vanishing limit one has to assume the same condition (13) as before. Next, having fixed ε and ν as in (19), it follows that as $\nu \to 0$, also $\varepsilon \to 0$, and the total dissipation vanishes. □

Remark 8. *Summarizing, to characterize the vanishing limit and its energy, we have to impose the uniform boundedness of*

$$L^\beta(0,T;\dot{C}^\sigma(\mathbb{T}^3)) \quad \textit{with } \beta = \max\left\{\frac{1}{\sigma} + \delta, 2\right\}.$$

In the case $\sigma \in]1/3, 1/2]$, the condition is the same valid for the Euler equations, while for $\sigma > 1/2$, the situation is different since the presence of the viscous term gives a further restriction on the integrability only in the time variable. The estimate on the viscous term seems not easily improvable, *since even employing the bound $\sqrt{\nu}\nabla v^\nu \in L^2((0,T)\times\mathbb{T}^3)$ (the only additional one we have at disposal and using interpolation techniques to interpolate between $\nabla v \in L^2(0,T;L^2)$ and $v^\nu \in L^{2/(1+\sigma)}(0,T;L^\infty)$) the estimations seem not to become sharper, especially for what concerns the time-integrability.*

As a consequence of the estimates we proved, we can easily deduce the following *Onsager singularity theorem* for Leray–Hopf solutions, in the same spirit of the results in [1]. Furthermore, this result represents a slight improvement of the criterion in [1] (Thm. 1), which was limited to the case $\beta = 3$ and $\sigma > 1/3$ (but for some Besov space).

Theorem 5 (An Onsager singularity theorem)**.** *Let $\{v^\nu\}$, $\nu \in]0,1]$, be a sequence of Leray solutions of incompressible NSE in $[0,T]\times\mathbb{T}^3$ corresponding to the viscosity $\nu > 0$ and to the same datum v_0. Suppose that*

$$\int_0^T \int_{\mathbb{T}^3} \epsilon[v^\nu]\,\mathrm{d}x\mathrm{d}t \geq \nu^\alpha L(\nu) \qquad \textit{for some } \alpha \in [0,1), \tag{20}$$

where L is a function slowly-varying at $\nu = 0$. Let $\beta \geq 2$. Then, for any $\gamma > 0$, the family $\{v^\nu\}$ of Leray solutions cannot have norms

$$\|v^\nu\|_{L^\beta(0,T;C^{\sigma_\alpha+\gamma}(\mathbb{T}^3))} \qquad \textit{for} \quad \sigma_\alpha := \frac{1+\alpha}{\beta-\alpha(\beta-2)},$$

which are bounded uniformly in $\nu > 0$

Remark 9. *The case $\beta = 3$ and $\sigma_\alpha = \frac{1+\alpha}{3-\alpha} \in [1/3,1)$ corresponds to the result proved in [1] (Theorem 1) in the context of Besov spaces. In fact, the condition proved in that paper identifies the critical space $L^3(0,T;B_3^{\sigma_\alpha,\infty})$. Moreover, in [1] (Remark 1), it is pointed out that the same method applies also to prove a condition with the critical space $L^p(0,T;B_p^{\sigma_\alpha,\infty})$ for all $p \geq 3$. Our spaces of classical functions are such that $C^{\sigma_\alpha}(\mathbb{T}^3) = B_\infty^{\sigma_\alpha,\infty}$, and this shows the improvement of our result concerning the time-variable, having the same range of Hölder/Besov exponents in time. Again, the factor $\gamma > 0$, can be removed by considering the space $C_\omega^{\sigma_\alpha}(\mathbb{T}^3)$.*

Proof of Theorem 5. The proof is an application of the estimates we already deduced for the dissipation terms in the mollified equations. In order to balance the two contributions to $\epsilon[v^\nu]$ coming from the dissipative term and from the commutation term, in (18) and (14) (where $\beta = 1+\eta$), respectively, we assume that

$$\nu\varepsilon^{2(\sigma-1)} \sim \varepsilon^{\sigma\beta-1},$$

implying that $\varepsilon = \nu^{1/(2(\sigma-1))}$, which produces

$$\nu^\alpha L(\nu) \leq \int_0^T \int_{\mathbb{T}^3} \epsilon[v^\nu]\,\mathrm{d}x\mathrm{d}t \leq C\nu^{\frac{\beta\sigma-1}{(\beta-2)\sigma+1}},$$

for some C independent of ν.

With $L(\nu)$ being slow varying we get an absurd if $\frac{\beta\sigma-1}{(\beta-2)\sigma+1} - \alpha > 0$, which turns out to be the case if

$$\sigma > \sigma_\alpha := \frac{1+\alpha}{\beta-\alpha(\beta-2)}.$$

Note that the denominator $\beta-\alpha(\beta-2)$ is always positive for $\alpha \in [0,1)$, being $\beta \geq 2$ and moreover, $\sigma_\alpha \in [1/\beta,1)$ for all $\alpha \in [0,1)$. $\square$

6. Conclusions

Our results demonstrate that the conservation of energy in solutions to the Euler equations can depend on the method of construction. Specifically, for $\sigma > 1/2$, the conditions for both the existence of the inviscid limit and the conservation of energy are more stringent compared to the conditions for just the conservation of energy in the limit problem.

Furthermore, we have shown that if energy dissipation does not decrease rapidly enough, it may result in the formation of singularities and cause some "higher norm" of the Leray–Hopf weak solutions to become unbounded. This supports the idea that "quasi-singularities" are needed to explain the enhanced energy dissipation rates observed in turbulent flows.

Our results provide a clear understanding of the presence of "quasi-singularities" in the sequence of Leray solutions, which are consistent with the slow decrease of energy dissipation as ν approaches zero. Solutions to the NSE (excluding possible true Leray-type singularities) are smooth in space for any $\nu > 0$, but this smoothness cannot be uniform in viscosity. Our findings are motivated by turbulence in three dimensions, where a forward energy cascade is expected.

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Appendix A

To fully use the features of the hypothesis on the Hölder continuity, the approximation should be made by means of a mollification argument. To this end, we fix a "symmetric" $\rho \in C_0^\infty(\mathbb{R}^3)$, such that

$$\rho \geq 0, \qquad \operatorname{supp} \rho \subset B(0,1) \subset \mathbb{R}^3, \qquad \int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = 1,$$

and we define, for $\varepsilon \in (0,1]$, the Friedrichs family $\rho_\varepsilon(x) := \varepsilon^{-3}\rho(\varepsilon^{-1}x)$. Then, for any function $f \in L^1_{loc}(\mathbb{R}^3)$, we define by the usual convolution

$$f_\varepsilon(x) := \int_{\mathbb{R}^3} \rho_\varepsilon(x-y) f(y)\,\mathrm{d}y = \int_{\mathbb{R}^3} \rho_\varepsilon(y) f(x-y)\,\mathrm{d}y.$$

It turns out that the last integral is evaluated on $B(0,\varepsilon) = \{y : |y| < \varepsilon\}$, which is contained in $]-\pi,\pi[^3$, for small $\varepsilon > 0$. If needed, we can restate the definition as

$$f_\varepsilon(x) := \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x-y)\,\mathrm{d}y.$$

If $f \in L^1(\mathbb{T}^3)$, then $f \in L^1_{loc}(\mathbb{R}^3)$, and it turns out that f_ε is 2π-periodic along the x_j-direction, for $j = 1,2,3$ since

$$f_\varepsilon(x + 2\pi e_j) = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x + 2\pi e_j - y)\,\mathrm{d}y = \int_{\mathbb{T}^3} \rho_\varepsilon(y) f(x-y)\,\mathrm{d}y,$$

by the periodicity of f, where e_j, $j = 1,2,3$, is the unit vector in the x_j direction and—in addition—it turns out that $f_\varepsilon \in C^\infty(\mathbb{T}^3)$.

We report now the basic calculus estimates used in the proof of the various results.

Lemma A1. *Let ρ be as above and let $u \in \dot{C}^\sigma(\mathbb{T}^3) \cap L^1_{loc}(\mathbb{T}^3)$, then it follows*

$$\max_{x\in\mathbb{T}^3} |u(x+y) - u(x)| \leq [u]_\sigma |y|^\sigma, \tag{A1}$$

$$\max_{x\in\mathbb{T}^3} |u(x) - u_\varepsilon(x)| \leq [u]_\sigma\, \varepsilon^\sigma, \tag{A2}$$

$$\max_{x\in\mathbb{T}^3} |\nabla u_\varepsilon(x)| \leq C[u]_\sigma\, \varepsilon^{\sigma-1}, \tag{A3}$$

where $C := \int_{\mathbb{R}^3} |\nabla\rho(x)|\,\mathrm{d}x$.

Proof. The first one is just the statement of Hölder continuity. Concerning the second one, we can write that

$$\begin{aligned}
|u(x) - u_\varepsilon(x)| &= \left| u(x) - \int_{B(0,\varepsilon)} \rho_\varepsilon(y) u(x-y)\,\mathrm{d}y \right| \\
&\leq [u]_\alpha \int_{B(0,\varepsilon)} \rho_\varepsilon(y)|y|^\alpha\,\mathrm{d}y \\
&\leq C\varepsilon^\alpha \int_{B(0,\varepsilon)} \rho_\varepsilon(y)\,\mathrm{d}y,
\end{aligned}$$

hence the thesis, where we used that $\rho \geq 0$.

The third estimate follows by observing that

$$\frac{\partial u_\varepsilon(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{1}{\varepsilon^3}\int_{\mathbb{R}^3} \rho\left(\frac{x-y}{\varepsilon}\right) u(y)\,\mathrm{d}y = \frac{1}{\varepsilon}\int_{\mathbb{R}^3} g^i_\varepsilon(x-y)u(y)\,\mathrm{d}y,$$

where $g^i_\varepsilon(x) := \frac{1}{\varepsilon^3}\frac{\partial\rho}{\partial x_i}\left(\frac{x}{\varepsilon}\right)$. Note that $\int_{\mathbb{R}^3} g^i_\varepsilon(x)\,dx = 0$, hence we can write

$$\begin{aligned}
\frac{\partial u_\varepsilon(x)}{\partial x_i} &= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} g^i_\varepsilon(x-y)u(y)\,\mathrm{d}y - \frac{u(x)}{\varepsilon}\int_{\mathbb{R}^3} g^i_\varepsilon(x-y)\,\mathrm{d}y, \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \Big(g^i_\varepsilon(x-y)u(y) - g^i_\varepsilon(x-y)u(x) \Big)\,\mathrm{d}y,
\end{aligned}$$

and then

$$\begin{aligned}
\left|\frac{\partial}{\partial x_i} u_\varepsilon(x)\right| &= \frac{1}{\varepsilon}\int_{|x-y|<\varepsilon} |g^i_\varepsilon(x-y)||u(y) - u(x)|\,\mathrm{d}y, \\
&\leq \frac{[u]_\alpha}{\varepsilon}\int_{|x-y|<\varepsilon} |g^i_\varepsilon(x-y)||y-x|^\alpha\,\mathrm{d}y \\
&\leq \frac{[u]_\alpha\varepsilon^\alpha}{\varepsilon}\int_{|x-y|<\varepsilon} |g^i_\varepsilon(x-y)|\,\mathrm{d}y,
\end{aligned}$$

hence the thesis. □

A similar result is also valid for Besov functions and for functions with general modulus of continuity, which is going to zero faster than a Hölder one. By similar arguments, one can prove also the following two lemma, which can be used to handle the cases just sketched in Section 3.

Lemma A2. *Let ρ be as above and let $u \in \dot{C}^\alpha_\omega(\mathbb{T}^3) \cap L^1_{loc}(\mathbb{T}^3)$, then it follows*

$$\max_{x\in\mathbb{T}^3} |u(x+y) - u(x)| \leq [u]_{\omega,\alpha}\, \omega(|y|)\, |y|^\alpha, \tag{A4}$$

$$\max_{x\in\mathbb{T}^3} |u(x) - u_\varepsilon(x)| \leq [u]_{\omega,\alpha}\, \omega(\varepsilon)\varepsilon^\alpha, \tag{A5}$$

$$\max_{x\in\mathbb{T}^3} |\nabla u_\varepsilon(x)| \leq C[u]_{\omega,\alpha}\, \omega(\varepsilon)\varepsilon^{\alpha-1}. \tag{A6}$$

Lemma A3. *Let u_ε be the usual mollification of $u \in B_3^{\sigma,\infty}(\mathbb{T}^3) \cap L^1_{loc}(\mathbb{T}^3)$. Then, it follows that*

$$\|u(\cdot + y) - u(\cdot)\|_3 \le \|u\|_{B_3^{\sigma,\infty}} |y|^\sigma, \tag{A7}$$

$$\|u(x) - u_\varepsilon\|_3 \le C\|u\|_{B_3^{\sigma,\infty}} \varepsilon^\sigma, \tag{A8}$$

$$\|\nabla u_\varepsilon(x)\|_3 \le C\|u\|_{B_3^{\sigma,\infty}} \varepsilon^{\sigma-1}. \tag{A9}$$

We now give some ideas on how to modify the proof of the Theorem valid for the Euler equations to the case of the NSE with given fixed viscosity $\nu > 0$.

Sketch of the Proof of Theorem 3. The testing is done with the double regularized velocity $\rho_\varepsilon * (\rho_\varepsilon * v^\nu)$ and also the time derivative is treated in the same way. Next, the properties of the convolution show also that

$$\int_s^t \int_{\mathbb{T}^3} \nabla v_\varepsilon^\nu : \nabla v_\varepsilon^\nu \, \mathrm{d}x\mathrm{d}\tau \overset{\varepsilon \to 0}{\to} \int_s^t \int_{\mathbb{T}^3} |\nabla v^\nu|^2 \, \mathrm{d}x\mathrm{d}\tau.$$

The convective term is treated with the decomposition (12) into three terms, with the first one vanishing when tested by v_ε^ν by integration by parts. By using the properties (A2) and (A3) of the convolution we get

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} (v^\nu - v_\varepsilon^\nu) \otimes (v^\nu - v_\varepsilon^\nu) : \nabla v_\varepsilon^\nu \, \mathrm{d}x \right| &\le \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^2 |\nabla v_\varepsilon^\nu| \, \mathrm{d}x \\
&\le \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^{1+\alpha} |v^\nu - v_\varepsilon^\nu|^{1-\alpha} |\nabla v_\varepsilon^\nu|^\alpha |\nabla v_\varepsilon^\nu|^{1-\alpha} \, \mathrm{d}x \\
&\le \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^{1+\alpha} f_\alpha(t)^{1-\alpha} \varepsilon^{\alpha(1-\alpha)} f_\alpha(t)^\alpha \varepsilon^{\alpha(\alpha-1)} |\nabla v_\varepsilon^\nu|^{1-\alpha} \, \mathrm{d}x \\
&\le f_\alpha(t) \int_{\mathbb{T}^3} |v^\nu - v_\varepsilon^\nu|^{1+\alpha} |\nabla v_\varepsilon^\nu|^{1-\alpha} \, \mathrm{d}x.
\end{aligned}$$

By using the Hölder inequality with exponents $r = \frac{2}{1+\alpha}$ and $r' = \frac{2}{1-\alpha}$ we obtain

$$\begin{aligned}
\left| \int_{\mathbb{T}^3} (v^\nu - v_\varepsilon^\nu) \otimes (v^\nu - v_\varepsilon^\nu) : \nabla v_\varepsilon^\nu \, \mathrm{d}x \right| &\le f_\alpha(t) \|v^\nu - v_\varepsilon^\nu\|^{1+\alpha} \|\nabla v_\varepsilon^\nu\|^{1-\alpha} \\
&\le (2\|v_0\|)^{1+\alpha} f_\alpha(t) \|\nabla v_\varepsilon^\nu\|^{1-\alpha},
\end{aligned}$$

Concerning the *remainder* term $r_\varepsilon(v^\nu, v^\nu)$, we can use a similar argument based on the Young theorem on convolutions, to prove the same estimate.

The above calculations prove that the family of functions $F_\varepsilon : (0,T] \to \mathbb{R}^+$, indexed by $\varepsilon > 0$, and defined by

$$F_\varepsilon(\tau) := \int_{\mathbb{T}^3} (v^\nu(\tau,x) \otimes v^\nu(\tau,x))_\varepsilon : \nabla v_\varepsilon^\nu(\tau,x) \, \mathrm{d}x \qquad \forall \tau \in]0,T],$$

is such that for any fixed $\nu > 0$ and for any $s \in (0,T)$

1. $|F_\varepsilon(\tau)|$ is uniformly bounded in (s,T) by the function $G(\tau) := C_2 f_\alpha(\tau) \|\nabla v(\tau)\|_2^{1-\alpha}$, which belongs to $L^1(s,T)$ by hypothesis;
2. Due to the a.e. in $(0,T)$ convergence of $v_\varepsilon^\nu(\tau)$ towards $v^\nu(\tau) \in V$ (valid being $v \in L^2(0,T;V)$) it follows that

$$\lim_{\varepsilon \to 0} F_\varepsilon(\tau) = \int_{\mathbb{T}^3} (v^\nu(\tau,x) \otimes v^\nu(\tau,x)) : \nabla v^\nu(\tau,x) \, \mathrm{d}x = 0, \quad \text{for a.e. } \tau \in (0,T).$$

By using the Lebesgue dominated convergence theorem, the two properties imply that

$$\lim_{\varepsilon \to 0} \int_s^t F_\varepsilon(\tau) \, \mathrm{d}\tau = \lim_{\varepsilon \to 0} \int_s^t \int_{\mathbb{T}^3} (v^\nu(\tau,x) \otimes v^\nu(\tau,x))_\varepsilon : \nabla v_\varepsilon^\nu(\tau,x) \, \mathrm{d}x\mathrm{d}\tau = 0,$$

for all $t \in]s, T]$, hence finally proving the energy conservation in $[s, t] \subseteq]0, T]$, that is

$$\frac{1}{2}\|v^\nu(t)\|^2 + \nu \int_s^t \|\nabla v^\nu(\tau)\|^2 \, \mathrm{d}\tau = \frac{1}{2}\|v^\nu(s)\|^2, \quad \forall\, s, t \text{ s.t. } 0 < s < t \leq T.$$

Next, we take a sequence $\{s_m\}$ of strictly positive times such that $s_m \to 0$. By the definition of a weak solution it holds that $\lim_{m \to +\infty} \|v(s_m)\| = \|v_0\|$, being the initial datum strongly attained, and by the absolute continuity of the time-integral of $\|\nabla v(t)\|^2$ taking the limit as $m \to +\infty$ in the energy equality over $[s_m, t]$ is justified, and we get that

$$\frac{1}{2}\|v^\nu(t)\|^2 + \nu \int_0^t \|\nabla v^\nu(\tau)\|^2 \, \mathrm{d}\tau = \frac{1}{2}\|v_0^\nu\|^2, \qquad \forall\, t \in [0, T],$$

ending the proof. □

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