EQUILIBRIUM STATES OF ENDOMORPHISMS OF $\mathbb{P}^k$ I: EXISTENCE AND PROPERTIES

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Dedicated to the memory of Professor Nessim Sibony

Abstract. We develop a new method, based on pluripotential theory, to study the transfer (Perron-Frobenius) operator induced on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ by a holomorphic endomorphism and a suitable continuous weight. This method allows us to prove the existence and uniqueness of the equilibrium state and conformal measure for very general weights (due to Denker-Przytycki-Urbański in dimension 1 and Urbański-Zdunik in higher dimensions, both in the case of Hölder continuous weights). We establish a number of properties of the equilibrium states, including mixing, K-mixing, mixing of all orders, and an equidistribution of repelling periodic points. Our analytic method replaces all distortion estimates on inverse branches with a unique, global, estimate on dynamical currents, and allows us to reduce the dynamical questions to comparisons between currents and their potentials.

Contents

Notation. Throughout the paper, $\mathbb{P}^k$ denotes the complex projective space of dimension k endowed with the standard Fubini-Study form $\omega_{\rm FS}$. This is a Kähler $(1,1)$-form normalized so that $\omega_{\rm FS}^k$ is a probability measure. We will use the metric and distance $\mathrm{dist}(\cdot,\cdot)$ on $\mathbb{P}^k$ induced by $\omega_{\rm FS}$ and the standard ones on $\mathbb{C}^k$ when we work on open subsets of $\mathbb{C}^k$. We denote by $\mathbb{B}_{\mathbb{P}^k}(a,r)$ (resp. $\mathbb{B}^k_r, \mathbb{D}(a,r), \mathbb{D}_r$) the ball of center a and radius r in $\mathbb{P}^k$ (resp. the ball of center 0 and radius r in $\mathbb{C}^k$, the disc of center a and radius r in $\mathbb{C}$, and the disc of center 0 and radius r in $\mathbb{C}$). Leb denotes the standard Lebesgue measure on a Euclidean space or on a sphere. The oscillation $\Omega(\cdot)$, the modulus of continuity $m(\cdot,\cdot)$, and the semi-norms $\|\cdot\|_{\log^p}$ of a function are defined in Section 2.1. The currents ω_n and their dynamical potentials u_n are introduced in Section 2.4.

The pairing $\langle\cdot,\cdot\rangle$ is used for the integral of a function with respect to a measure or more generally the value of a current at a test form. If S and R are two $(1,1)$-currents, we will write $|R|\leq S$ when $\Re(\xi R)\leq S$ for every function $\xi\colon \mathbb{P}^k\to\mathbb{C}$ with $|\xi|\leq 1$, i.e., all currents $S-\Re(\xi R)$ with ξ as before are positive. Notice that this forces S to be real and positive. We also write other inequalities such as $|R|\leq|R_1|+|R_2|$ if $|R|\leq S_1+S_2$ whenever $|R_1|\leq S_1$ and $|R_2|\leq S_2$. Recall that $d^c=\frac{i}{2\pi}(\overline{\partial}-\partial)$ and $dd^c=\frac{i}{\pi}\partial\overline{\partial}$. The notations $\lesssim$ and $\gtrsim$ stand for inequalities up to a multiplicative constant. The function identically equal to 1 is denoted by $\mathbb{1}$. We also use the function $\log^\star(\cdot):=1+|\log(\cdot)|$.

Consider a holomorphic endomorphism $f\colon\mathbb{P}^k\to\mathbb{P}^k$ of algebraic degree $d\geq 2$ satisfying the Assumption **(A)** in the Introduction. Denote respectively by T, $\mu=T^k$, $\mathrm{supp}(\mu)$ the Green

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$(1,1)$-current, the measure of maximal entropy (also called the Green measure or the equilibrium measure), and the small Julia set of f. If S is a positive closed $(1,1)$-current on $\mathbb{P}^k$, its dynamical potential is denoted by u_S and is defined in Section 2.2. If ν is an invariant probability measure, we denote by $\mathrm{Ent}_f(\nu)$ the metric entropy of ν with respect to f.

We also consider a weight ϕ which is a real-valued continuous function on $\mathbb{P}^k$. The transfer operator (Perron-Frobenius operator) $\mathcal{L} = \mathcal{L}_\phi$ is introduced in the Introduction together with the scaling ratio $\lambda = \lambda_\phi$, the conformal measure m_ϕ, the density function $\rho = \rho_\phi$, the equilibrium state $\mu_\phi = \rho m_\phi$, the pressure $P(\phi)$, see also Section 3. The measures m_ϕ and μ_ϕ are probability measures. The operator L is a suitable modification of $\mathcal{L}$ and is introduced in Section 4.1.

1. Introduction and results

Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of the complex projective space $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$, with $k \geq 1$, of algebraic degree $d \geq 2$. Denote by μ the unique measure of maximal entropy for the dynamical system $(\mathbb{P}^k, f)$ [Lyu83; BD09; DS10a; BM01]. The support $\mathrm{supp}(\mu)$ of μ is called *the small Julia* set of f. The measure μ corresponds to the equilibrium state of the system in the case without weight, i.e., when the weight is zero. In this paper, we will consider the case where the weight, denoted by ϕ, is not necessarily equal to zero. This problem has been studied for Hölder continuous weights using a geometric approach, in dimension 1, see, e.g., Denker-Przytycki-Urbański [Prz90; DU91a; DU91b; DPU96] and Haydn [Hay99] just to name a few, and in higher dimensions, see Szostakiewicz-Urbański-Zdunik [UZ13; SUZ14]. We will develop here an analytic method which will allow us to obtain more general and more quantitative results. Many results are new even when for $k = 1$.

Throughout this paper, we make use of the following technical assumption for f:

(A) the local degree of the iterate $f^n := f \circ \cdots \circ f$ (n times) satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log \max_{a\in\mathbb{P}^k} \deg(f^n, a) = 0.$$

Here, $\deg(f^n, a)$ is the multiplicity of a as a solution of the equation $f^n(z) = f^n(a)$. Note that generic endomorphisms of $\mathbb{P}^k$ satisfy this condition, see [DS10b]. Our study still holds under a weaker condition that the exceptional set of f (i.e., the maximal proper analytic subset of $\mathbb{P}^k$ invariant by f^{-1}) is empty or more generally has no intersection with $\mathrm{supp}(\mu)$ (in particular, this condition is superfluous in dimension 1). However, this situation requires more technical conditions on the weight ϕ. We choose not to present this case here in order to simplify the notation and focus on the main new ideas introduced in this topic. Our main goal in this paper is to prove the following theorem (see Theorem 3.1 and Section 4 for more precise statements).

Theorem 1.1. *Let f be an endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying the Assumption* **(A)** *above. Let ϕ be a real-valued $\log^q$-continuous function on $\mathbb{P}^k$, for some $q > 2$, such that $\Omega(\phi) := \max\phi - \min\phi < \log d$. Then ϕ admits a unique equilibrium state μ_ϕ, whose support is equal to the small Julia set of f. This measure μ_ϕ is K-mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_ϕ as n goes to infinity. Moreover, there is a unique conformal measure m_ϕ associated to ϕ. We have $\mu_\phi = \rho m_\phi$ for some strictly positive continuous function ρ on $\mathbb{P}^k$ and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_ϕ as n goes to infinity.*

We say that a function is $\log^q$-continuous if its oscillation on a ball of radius r is bounded by a constant times $(\log^\star r)^{-q}$, see Section 2.1 for details. See also Section 4.1 for the K-mixing and mixing of all orders.

An *equilibrium state* as in the statement above is defined as follows, see for instance [Rue72; Wal00; PU10]. Given a *weight*, i.e., a real-valued continuous function, ϕ as above, we define the *pressure* of ϕ as

$$P(\phi) := \sup\big\{ \mathrm{Ent}_f(\nu) + \langle \nu, \phi\rangle \big\},$$

where the supremum is taken over all Borel f-invariant probability measures ν and $\mathrm{Ent}_f(\nu)$ denotes the metric entropy of ν. An equilibrium state for ϕ is then an invariant probability measure μ_ϕ realizing a maximum in the above formula, that is,

$$P(\phi) = \mathrm{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi \rangle.$$

On the other hand, a *conformal measure* is defined as follows. Define the *Perron-Frobenius* (or *transfer*) operator $\mathcal{L}$ with weight ϕ as (we often drop the index ϕ for simplicity)

$$\mathcal{L}g(y) := \mathcal{L}_\phi g(y) := \sum_{x \in f^{-1}(y)} e^{\phi(x)} g(x), \tag{1.1}$$

where $g\colon \mathbb{P}^k \to \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A conformal measure is an eigenvector for the dual operator $\mathcal{L}^*$ acting on positive measures.

Notice that, in the case where ϕ is Hölder continuous, a part of Theorem 1.1 was established by Urbański-Zdunik [UZ13] (also under a genericity assumption for f), see also [Prz90; DU91a; DU91b; DPU96] for previous results in dimension $k = 1$. When ϕ is constant, the operator $\mathcal{L}$ reduces to a constant times the push-forward operator f_* and we get $\mu_\phi = \mu$. For an account of the known results in this case, see for instance [DS10a].

A reformulation of Theorem 1.1 is the following: given ϕ as in the statement, there exist a number $\lambda > 0$ and a continuous function $\rho = \rho_\phi \colon \mathbb{P}^k \to \mathbb{R}$ such that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the following uniform convergence holds:

$$\lambda^{-n}\mathcal{L}^n g(y) \to c_g \rho \tag{1.2}$$

for some constant c_g depending on g. By duality, this is equivalent to the convergence, uniform on probability measures ν,

$$\lambda^{-n}(\mathcal{L}^*)^n \nu \to m_\phi, \tag{1.3}$$

where m_ϕ is a conformal measure associated to the weight ϕ. The equilibrium state μ_ϕ is then given by $\mu_\phi = \rho m_\phi$, and we have $c_g = \langle m_\phi, g \rangle$.

To prove Theorem 1.1, in Section 3 we develop a new and completely different approach with respect to [UZ13] and to the previous studies in dimension 1. As we will see in the second part of this work [BD20], the flexibility of this method will allow for a more quantitative understanding of the convergences (1.2) and (1.3), and for the direct establishment of several statistical properties of the equilibrium states.

The main idea of our method is the following. Let us just consider for now the case where both of the functions g and ϕ are of class $\mathcal{C}^2$ (the general case is technically quite involved and requires suitable approximations of g and ϕ by $\mathcal{C}^2$ functions). Given such a function g, first we want to prove that the ratio between the maximum and the minimum of $\mathcal{L}^n g$ stays bounded with n. This allows us to define the good scaling ratio λ and to get that the sequence $\lambda^{-n}\mathcal{L}^n g$ is uniformly bounded. Next, we would like to prove that this sequence is actually equicontinuous. This, together with other technical arguments, would imply the existence and uniqueness of the limit function ρ.

In order to establish the above controls, we study the sequence of $(1,1)$-currents given by $dd^c\mathcal{L}^n g$. First we prove that suitably normalized versions of these currents are uniformly bounded by a common positive closed $(1,1)$-current R. This is the core of our method which replaces all controls on the distortion of inverse branches of f^n in the geometric method of [UZ13] by a unique, global, and flexible estimate. Namely, for every $n \in \mathbb{N}$ we can get an estimate of the form

$$\Big| dd^c \frac{\mathcal{L}^n g}{c_n} \Big| \lesssim \sum_{j=0}^{\infty} \Big(\frac{e^{\Omega(\phi)}}{d}\Big)^j \frac{(f_*)^j \omega_{\mathrm{FS}}}{d^{(k-1)j}} \quad \text{with} \quad c_n := \|g\|_{\mathcal{C}^2} \langle \omega_{\mathrm{FS}}^k, \mathcal{L}^n \mathbb{1} \rangle. \tag{1.4}$$

Here, $\omega_{\rm FS}$ denotes the usual Fubini-Study form on $\mathbb{P}^k$ normalized so that $\omega_{\rm FS}^k$ is a probability measure. Notice that the last infinite sum gives a key reason for the assumption $\Omega(\phi) < \log d$ made on the weight ϕ as the mass of the current $(f_*)^j\omega_{\rm FS}$ is equal to $d^{(k-1)j}$.

We will establish in Section 2 some general criteria, interesting in themselves, which allow one to bound the oscillation of $c_n^{-1}\mathcal{L}^n g$ in terms of the oscillation of the potentials of the current in the RHS of (1.4). This latter oscillation is actually controllable. Assumption **(A)** allows us to have a simple control which makes the estimates less technical but such a control exists without Assumption **(A)**.

Combining all these ingredients, the existence and uniqueness of the equilibrium state and conformal measure, as well as the equidistribution of preimages and the equality $P(\phi) = \log\lambda$, follow from standard arguments that we recall in Sections 4.1 and 4.2 for completeness. We also prove that the entropy of μ_ϕ is larger than $k\log d - \Omega(\phi) > (k-1)\log d$, and that all the Lyapunov exponents of μ_ϕ are strictly positive, see Proposition 4.9. This also leads to a lower bound for the Hausdorff dimension of μ_ϕ. In Section 4.3 we establish the equidistribution of repelling periodic points with respect to μ_ϕ, see Theorem 4.10, which completes the proof of Theorem 1.1. This result is due to Lyubich [Lyu83] (for $k = 1$) and Briend-Duval [BD99] (for any $k \geq 1$) when $\phi = 0$, and is new even for $k = 1$ otherwise.

In the second part of our study [BD20], we will prove that the Perron-Frobenius operator and its complex perturbations admit spectral gaps, and deduce several statistical properties of the equilibrium states through a unified method.

Outline of the organization of the paper. In Section 2, we introduce some useful notions and establish comparison principles for currents and potentials that will be the technical key to prove Theorem 1.1. We also present the estimates on the sequence $f_*^n\omega_{\rm FS}$ (and on their potentials) that we will need in the sequel. Section 3 is dedicated to the proof of Theorem 3.1. For this purpose, we develop our method to get the uniform boundedness and equicontinuity for the sequence $\mathcal{L}^n g$, properly normalized, that lead to the good definition of the scaling ratio λ. Once this is done, we will complete the proof of Theorem 1.1 in Section 4.

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2. Dynamical potentials and some comparison principles

2.1. $\log^p$-continuous functions.

We will use the following notations throughout the paper.

Definition 2.1. Given a subset U of $\mathbb{P}^k$ or $\mathbb{C}^k$ and a real-valued function $g\colon U \to \mathbb{R}$, define the *oscillation* $\Omega_U(g)$ of g as

$$\Omega_U(g) := \sup g - \inf g$$

and its continuity modulus $m_U(g,r)$ at distance r as

$$m_U(g,r) := \sup_{x,y\in U\colon\ {\rm dist}(x,y)\leq r} |g(x)-g(y)|.$$

We may drop the index U when there is no possible confusion.

Definition 2.2. The semi-norm $\|\cdot\|_{\log^p}$ is defined for every $p>0$ and $g\colon \mathbb{P}^k\to\mathbb{R}$ as

$$\|g\|_{\log^p} := \sup_{a,b\in\mathbb{P}^k} |g(a)-g(b)|\cdot(\log^\star {\rm dist}(a,b))^p = \sup_{r>0,a\in\mathbb{P}^k} \Omega_{\mathbb{B}_{\mathbb{P}^k}(a,r)}(g)\cdot(1+|\log r|)^p,$$

where $\mathbb{B}_{\mathbb{P}^k}(a,r)$ denotes the ball of center a and radius r in $\mathbb{P}^k$.

The following technical lemma will be used in Section 3.

Lemma 2.3. *For every* $\log^p$*-continuous function* $g\colon \mathbb{P}^k \to \mathbb{R}$*,* $p > 0$*,* $s \geq 1$*, and* $0 < \epsilon \leq 1$*, there exist continuous functions* $g_\epsilon^{(1)}$ *and* $g_\epsilon^{(2)}$ *such that*

$$g = g_\epsilon^{(1)} + g_\epsilon^{(2)}, \qquad \|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \leq c\, \|g\|_\infty\, e^{(1/\epsilon)^{1/p}}, \qquad \text{and} \qquad \|g_\epsilon^{(2)}\|_\infty \leq c\, \|g\|_{\log^p} \epsilon,$$

where $c = c(p, s)$ *is a positive constant independent of* g *and* ϵ*. In particular, for every* $n \geq 1$ *there exist* $g_n^{(1)}$ *of class* $\mathcal{C}^2$ *and* $g_n^{(2)}$ *continuous such that*

$$g = g_n^{(1)} + g_n^{(2)}, \qquad \|g_n^{(1)}\|_{\mathcal{C}^2} \leq c\, \|g\|_\infty\, e^{\frac{1}{2} n^{2/p}}, \qquad \text{and} \qquad \|g_n^{(2)}\|_\infty \leq c\, \|g\|_{\log^p} n^{-2}.$$

Proof. Clearly, the second assertion is a consequence of the first one by taking $\epsilon = 2^p n^{-2}$ and replacing c by $2^p c$. We prove now the first assertion. Using a partition of unity, we can reduce the problem to the case where g is supported by the unit ball of an affine chart $\mathbb{C}^k \subset \mathbb{P}^k$.

Consider a smooth non-negative function χ with support in the unit ball of $\mathbb{C}^k$ whose integral with respect to the Lebesgue measure is 1. For $\nu > 0$, consider the function $\chi_\nu(z) := \nu^{-2k}\chi(z/\nu)$ which has integral 1 and tends to the Dirac mass at 0 when ν tends to 0. Define an approximation of g using the standard convolution operator $g_\nu := g * \chi_\nu$, and define $g_\epsilon^{(1)} := g_\nu$ and $g_\epsilon^{(2)} := g - g_\nu$. We consider $\nu := e^{-1/(M\epsilon)^{1/p}}$ for some constant $M > 0$ large enough. It remains to bound $\|g_\epsilon^{(1)}\|_{\mathcal{C}^s}$ and $\|g_\epsilon^{(2)}\|_\infty$.

By standard properties of the convolution we have, for some constant $\kappa > 0$,

$$\|g_\epsilon^{(2)}\|_\infty \lesssim m(g, \kappa\nu) \lesssim \|g\|_{\log^p} (\log^\star \nu)^{-p} \lesssim \|g\|_{\log^p} \epsilon$$

and, by definition of g_ν,

$$\|g_\epsilon^{(1)}\|_{\mathcal{C}^s} \lesssim \|g\|_\infty \|\chi_\nu\|_{\mathcal{C}^s} \operatorname{Leb}(\mathbb{B}_\nu^k) \lesssim \|g\|_\infty\, \nu^{-s} \lesssim \|g\|_\infty\, e^{(1/\epsilon)^{1/p}},$$

where we use the fact that M is large enough. This ends the proof of the lemma. □

2.2. **Dynamical potentials.** Let T denote the Green $(1,1)$-current of f. It is positive closed and of unit mass. Let S be any positive closed $(1,1)$-current of mass m on $\mathbb{P}^k$. There is a unique function $u_S\colon \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ which is p.s.h. modulo mT and such that

$$S = mT + dd^c u_S \qquad \text{and} \qquad \langle \mu, u_S \rangle = 0.$$

Locally, u_S is the difference between a potential of S and a potential of mT. We call it *the dynamical potential* of S. Observe that the dynamical potential of T is zero, i.e., $u_T = 0$.

Recall that T has Hölder continuous potentials. So, u_S is locally the difference between a p.s.h. function and a Hölder continuous one. The dynamical potential of S behaves well under the push-forward and pull-back operators associated to f. Indeed, because of the invariance properties of T, we have

$$f^*S = md \cdot T + dd^c(u_S \circ f) \qquad \text{and} \qquad f_*S = md^{k-1} \cdot T + dd^c(f_* u_S),$$

which, together with the invariance properties of μ, imply

$$u_{f^*S} = u_S \circ f \qquad \text{and} \qquad u_{f_*S} = f_* u_S.$$

We refer the reader to [DS10a] for details. In this paper, we only need currents S such that u_S is continuous.

2.3. **Comparisons between currents and their potentials.** A technical key point in the proof of our main theorem will be based on the following general idea: if u and v are two functions on some domain in $\mathbb{C}^k$ such that $|dd^c u| \leq dd^c v$, then u inherits some of the regularity properties of v. This section is devoted to make this idea precise and quantitative for our purposes. We start with the simplest occurrence of this fact in the first case in terms of the sup-norm.

Lemma 2.4. *There exists a positive constant A such that, for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ of mass 1 and for every positive closed $(1,1)$-current S on $\mathbb{P}^k$ with $S \leq S_0$, we have $\Omega(u_S) \leq A + \Omega(u_{S_0})$, where u_{S_0} and u_S denote the dynamical potentials of S_0 and S, respectively.*

Proof. We assume that $\Omega(u_{S_0})$ is finite, since otherwise the assertion trivially holds. Observe that the mass m of S is at most equal to 1 because $S \leq S_0$. Recall that u_S and u_{S_0} satisfy

$$S = mT + dd^c u_S, \quad S_0 = T + dd^c u_{S_0}, \quad \langle \mu, u_S \rangle = 0, \quad \text{and} \quad \langle \mu, u_{S_0} \rangle = 0.$$

The last identity implies that $\sup u_{S_0}$ is non-negative.

We first prove that u_S is bounded above by a constant. As mentioned above, the correspondence between positive closed $(1,1)$-currents and their dynamical potentials is a bijection. Moreover, we know that quasi-p.s.h. functions (i.e., functions that are locally difference between a p.s.h. and a smooth function) are integrable with respect to μ [DS10a, Th. 1.35]. Since the set of positive closed $(1,1)$-currents of mass less than or equal to 1 is compact, u_S belongs to a compact family of p.s.h. functions modulo mT. We deduce that there is a constant $A > 0$ independent of S such that $u_S \leq A/2$ on $\mathbb{P}^k$, see [DS10a, App. A.2] for more details. It follows that $\sup u_S \leq \sup u_{S_0} + A/2$ because $\sup u_{S_0}$ is non-negative.

Consider the current $S' := S_0 - S$ which is positive closed and smaller than S_0. By the uniqueness of the dynamical potential, we have $u_{S'} = u_{S_0} - u_S$, which implies $u_S = u_{S_0} - u_{S'}$. Since $S' \leq S$, as above, we also have $\sup u_{S'} \leq A/2$. It follows that

$$\inf u_S \geq \inf u_{S_0} - \sup u_{S'} \geq \inf u_{S_0} - A/2.$$

This estimate and the above inequality $\sup u_S \leq \sup u_{S_0} + A/2$ imply the lemma. □

Corollary 2.5. *There exists a positive constant A such that for every positive closed $(1,1)$-current S_0 on $\mathbb{P}^k$ and for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$ with $|dd^c g| \leq S_0$ we have $\Omega(g) \leq A\,\|S_0\| + 3\Omega(u_{S_0})$.*

Proof. By linearity we can assume that S_0 is of mass $1/2$. Define $R := dd^c g$ and write it as a difference of positive closed currents, $R = (R + S_0) - S_0$. Since $R + S_0$ and S_0 belong to the same cohomology class, they have the same mass $1/2$. We denote as usual by u_{R+S_0} and u_{S_0} the dynamical potentials of $R + S_0$ and S_0 respectively.

A direct computation gives $dd^c(g - u_{R+S_0} + u_{S_0}) = 0$ which implies that $g - u_{R+S_0} + u_{S_0}$ is a constant function. Thus,

$$\Omega(g) = \Omega(u_{R+S_0} - u_{S_0}) \leq \Omega(u_{R+S_0}) + \Omega(u_{S_0}).$$

The assertion follows from Lemma 2.4 applied to $R + S_0, 2S_0$ instead of S, S_0. We use here the fact that $R + S_0 = dd^c g + S_0 \leq 2S_0$ and that $2S_0$ is of mass 1. We also use a constant A which is equal to twice the one in Lemma 2.4. □

The following result gives a quantitative control on the oscillation of u in terms of the oscillation of v. Notice in particular that it implies that, if v is Hölder or $\log^p$-continuous for some $p > 0$, then u enjoys the same property with possibly a loss in the Hölder exponent, but not in the $\log^p$-exponent.

Proposition 2.6. *Let u and v be two p.s.h. functions on $\mathbb{B}^k_3$ such that $dd^c u \leq dd^c v$ and v is continuous. Then u is continuous and for every $0 < s \leq 1$ there is a positive constant A (independent of u and v) such that, for every $0 < r \leq 1/2$, we have*

$$m_{\mathbb{B}^k_1}(u, r) \leq m_{\mathbb{B}^k_2}(v, r^s) + A m_{\mathbb{B}^k_2}(u, r^s) r^{1-s} \leq m_{\mathbb{B}^k_2}(v, r^s) + A\Omega_{\mathbb{B}^k_2}(u) r^{1-s}.$$

Proof. The continuity of u is a well-known property. Indeed, since $dd^c v - dd^c u$ is a positive closed $(1,1)$-current, there is a p.s.h. function u' such that $dd^c u' = dd^c v - dd^c u$. So, both $u + u'$ and v are potentials of $dd^c v$. We deduce that they differ by a pluriharmonic function. Hence $u + u'$ is continuous. We then easily deduce that both u and u' are continuous because both are p.s.h. (and hence u.s.c.).

We prove now the estimate in the lemma. Let $x, y \in \mathbb{B}_1^k$ be such that $\|x-y\| \leq r$. We need to bound $u(y)-u(x)$. Without loss of generality, we can reduce the problem to the case $k=1$ by restricting ourselves to the complex line through x and y. Moreover, by translating and adding constants to u and v, we can assume that $x=0$, $|y| \leq r$, $u(x)=v(x)=0$, and $u(y) \geq 0$. It is then enough to prove that

$$u(y) \leq m_{\mathbb{D}_1}(v, r^s) + A\Omega_{\mathbb{D}_{r^s}}(u) r^{1-s}$$

for some positive constant A and for u, v defined on $\mathbb{D}_2$. Note that $\Omega_{\mathbb{D}_{r^s}}(u) \leq 2m_{\mathbb{D}_1}(u, r^s)$.

Claim. *We have, for some positive constant A,*

$$u(y) \leq \frac{1}{\operatorname{Leb}(\partial \mathbb{D}_{r^s})} \int_{|z|=r^s} u(z) d \operatorname{Leb}(z) + A\Omega_{\mathbb{D}_{r^s}}(u) r^{1-s}.$$

Assuming the claim, we first complete the proof of the lemma. Let $\widetilde{u}$ (resp. $\widetilde{v}$) be the radial subharmonic function on $\mathbb{D}_2$ such that $\widetilde{u}(z)$ (resp. $\widetilde{v}(z)$) is equal to the mean value of u (resp. v) on the circle of center 0 and radius $|z|$. Using the Claim, in order to obtain the lemma, it is enough to show that $\widetilde{u} \leq \widetilde{v}$.

Recall that $v-u$ is a subharmonic function vanishing at 0. Therefore, $\widetilde{v}-\widetilde{u}$ is a radial subharmonic function vanishing at 0. Radial subharmonic functions are increasing in $|z|$. Thus, $\widetilde{v}-\widetilde{u}$ is a non-negative function and the lemma follows. □

Proof of the Claim. Define $u'(z) := u(zr^s)$ and $y' := y/r^s$. We need to show that, for $|y'| \leq r^{1-s}$,

$$u'(y') \leq \frac{1}{\operatorname{Leb}(\partial \mathbb{D}_1)} \int_{\partial \mathbb{D}_1} u'(z) d \operatorname{Leb}(z) + A\Omega_{\mathbb{D}_1}(u') r^{1-s}.$$

We can assume, without loss of generality, that $y' = \alpha \in \mathbb{R}^+$ and $\alpha \leq r^{1-s}$. Consider the automorphism Ψ of the unit disc given by $\Psi(z) = \frac{z+\alpha}{1+\alpha z}$. The map Ψ satisfies $\Psi(0)=y'$ and moreover Ψ extends smoothly to $\partial \mathbb{D}_1$ and tends to the identity in the $\mathcal{C}^1$ norm as $\alpha \to 0$. It follows that $\left\|\Psi^{\pm 1} - \operatorname{id}\right\|_{\mathcal{C}^1} \leq A'\alpha \leq A' r^{1-s}$ for some positive constant A'.

Define $u'' := u' \circ \Psi$ and denote by ν the normalized standard Lebesgue measure on the unit circle. We deduce from the last inequalities that $\Psi_* \nu - \nu$ is given by a smooth 1-form on $\partial \mathbb{D}_1$ and $\|\Psi_*\nu - \nu\|_\infty = O(r^{1-s})$. Applying the submean inequality to the subharmonic function u'' we get

$$u'(y') = u''(0) \leq \langle \nu, u'' \rangle = \langle \nu, u' \circ \Psi \rangle = \langle \Psi_* \nu, u' \rangle = \langle \nu, u' \rangle + \langle \Psi_* \nu - \nu, u' \rangle.$$

Since $\Psi_* \nu$ and ν are probability measures, the integral $\langle \Psi_*\nu - \nu, u' \rangle$ does not change if we add to u' a constant c. With the choice $c = -\inf_{\mathbb{D}_1} u'$ (observe that u' is continuous on $\overline{\mathbb{D}}_1$) we get

$$u'(y') \leq \int_{\partial \mathbb{D}_1} u' \, d\nu + \sup_{\mathbb{D}_1} |u'+c| \, O(r^{1-s}) \leq \int_{\partial \mathbb{D}_1} u' \, d\nu + A\Omega_{\mathbb{D}_1}(u') r^{1-s}$$

for some positive constant A. This implies the desired inequality. □

Corollary 2.7. *Let v be a continuous p.s.h. function on $\mathbb{B}_3^k$. Let u be a continuous real-valued function on $\mathbb{B}_3^k$ such that $|dd^c u| \leq dd^c v$. Then for every $0 < s \leq 1$ we have for $0 < r \leq 1/2$*

$$m_{\mathbb{B}_1^k}(u, r) \leq 3 m_{\mathbb{B}_2^k}(v, r^s) + A\left(\Omega_{\mathbb{B}_2^k}(u) + \Omega_{\mathbb{B}_2^k}(v)\right) r^{1-s},$$

where A is a positive constant independent of u and v.

Proof. Since $|dd^c u| \leq dd^c v$, we have $dd^c(u+v) = dd^c u + dd^c v \geq 0$. So the function $u+v$ is p.s.h.; observe also that $dd^c(u+v) = dd^c u + dd^c v \leq 2 dd^c v$. Therefore, we can apply Proposition 2.6 to $u+v, 2v$ instead of u, v. This gives

$$\begin{aligned} m_{\mathbb{B}_1^k}(u, r) &\leq m_{\mathbb{B}_1^k}(u+v, r) + m_{\mathbb{B}_1^k}(v, r) \leq m_{\mathbb{B}_2^k}(2v, r^s) + A\Omega_{\mathbb{B}_2^k}(u+v) r^{1-s} + m_{\mathbb{B}_2^k}(v, r) \\ &\leq 3 m_{\mathbb{B}_2^k}(v, r^s) + A\left(\Omega_{\mathbb{B}_2^k}(u) + \Omega_{\mathbb{B}_2^k}(v)\right) r^{1-s}, \end{aligned}$$

which is the desired estimate. □

Corollary 2.8. *Let S_0 be a positive closed $(1,1)$-current on $\mathbb{P}^k$ with continuous local potentials. Let $\mathcal{F}(S_0)$ denote the set of all continuous real-valued functions g on $\mathbb{P}^k$ such that $|dd^c g| \leq S_0$. Then $\mathcal{F}(S_0)$ is equicontinuous.*

Proof. Let g be as in the statement. We cover $\mathbb{P}^k$ with a finite family of open sets of the form $\Phi_j(\mathbb{B}^k_{1/2})$ where Φ_j is an injective holomorphic map from $\mathbb{B}^k_4$ to $\mathbb{P}^k$. Write $S_0 = dd^c v_j$ for some continuous p.s.h. function v_j on $\Phi_j(\mathbb{B}^k_4)$ and define $V_j := \Phi_j(\mathbb{B}^k_3)$.

We apply Corollary 2.7 to g, v_j restricted to V_j instead of u, v and to $s = 1/2$. Taking into account the distortion of the maps Φ_j, we see that for all r smaller than some constant $r_0 > 0$

$$m_{\mathbb{P}^k}(g,r) \leq 3 \max_j m_{V_j}(v_j, c\sqrt{r}) + A\Big(\Omega_{\mathbb{P}^k}(g) + \max_j \Omega_{V_j}(v_j)\Big)\sqrt{r},$$

where $c \geq 1$ is a constant. Since $\Omega_{\mathbb{P}^k}(g)$ is bounded by Corollary 2.5, the RHS of the last inequality is bounded by a constant ϵ_r which is independent of g and tends to 0 when r tends to 0. It is now clear that the family $\mathcal{F}(S_0)$ is equicontinuous. □

2.4. **Dynamical potentials of $(f^n)_*\omega_{\rm FS}$.** In this section we consider the action of the operator $(f^n)_*$ on functions and currents. Some results and ideas here are of independent interest. Recall that we always assume that f satisfies the Assumption **(A)** in the Introduction.

We start by giving estimates on the potentials of the currents $(f^n)_*\omega_{\rm FS}$. As explained in the Introduction, these estimates will allow us to globally control the distortion of f^n. Define

$$\omega_n := d^{-(k-1)n}(f^n)_*\omega_{\rm FS}.$$

Recall that f_* multiplies the mass of a positive closed $(1,1)$-current by d^{k-1}. Therefore, all currents ω_n have unit mass. We denote by u_n the dynamical potential of ω_n. In particular, u_0 is the dynamical potential of $\omega_{\rm FS}$. It is known that u_0 is Hölder continuous, see [Kos97; DS10a].

Observe that $d^{-1}f^*\omega_{\rm FS}$ is a smooth positive closed $(1,1)$-form of mass 1. Therefore, there is a unique smooth function v such that

$$dd^c v = d^{-1}f^*\omega_{\rm FS} - \omega_{\rm FS} \qquad \text{and} \qquad \langle \mu, v\rangle = 0.$$

Lemma 2.9. *We have*

$$u_n = d^{-(k-1)n}(f^n)_* u_0 \qquad \textit{and} \qquad u_0 = -\sum_{n=0}^{\infty} d^{-n} v\circ f^n.$$

Proof. We prove the first identity. Denote by u'_n the RHS of this identity, which is a continuous function. By the definition of u_n and the invariance of T, we have

$$dd^c(u_n - u'_n) = (\omega_n - T) - d^{-(k-1)n}(f^n)_*(\omega_{\rm FS} - T) = (\omega_n - T) - (\omega_n - T) = 0.$$

Therefore, $u_n - u'_n$ is pluriharmonic and hence constant on $\mathbb{P}^k$. Moreover, the invariance of μ implies that

$$\langle \mu, u'_n\rangle = d^{-(k-1)n}\langle (f^n)^*\mu, u_0\rangle = d^n\langle \mu, u_0\rangle = 0.$$

By the definition of u_n, we also have $\langle \mu, u_n\rangle = 0$. We deduce that $u_n = u'_n$, which implies the first identity in the lemma.

It is clear that the sum in the RHS of the second identity in the lemma converges uniformly. Therefore, this RHS is a continuous function that we denote by u'_0. The invariance of μ also implies that $\langle \mu, u'_0\rangle = 0$. A direct computation gives

$$dd^c u'_0 = \lim_{N\to\infty}\Big(-\sum_{n=0}^{N-1} d^{-n} dd^c(v\circ f^n)\Big) = \lim_{N\to\infty} \omega_{\rm FS} - d^{-N}(f^N)^*\omega_{\rm FS} = \omega_{\rm FS} - T,$$

where the last identity is a consequence of the definition of T. Since $dd^c u_0$ is also equal to $\omega_{\rm FS} - T$, we obtain that $u_0 - u'_0$ is constant on $\mathbb{P}^k$. Finally, using that

$$\langle \mu, u_0\rangle = \langle \mu, u'_0\rangle = 0,$$

we conclude that $u_0 = u'_0$. This ends the proof of the lemma. □

In the sequel, we will need explicit bounds on the oscillation $\Omega(u_n)$ of u_n. These are provided in the next result.

Lemma 2.10. *For every constant $A > 1$, there exists a positive constant c independent of n such that $\|u_n\|_\infty \leq cA^n$ and $\Omega(u_n) \leq cA^n$ for all $n \geq 0$.*

Proof. Observe that the second assertion is deduced from the first one by replacing c with $2c$. We prove now the first assertion. By Lemma 2.9 we have, for any given $z \in \mathbb{P}^k$,

$$\begin{aligned} u_n(z) &= d^{-(k-1)n} \left((f^n)_* u_0 \right)(z) = \left\langle \delta_z, d^{-(k-1)n} (f^n)_* u_0 \right\rangle \\ &= d^n \left\langle d^{-kn} (f^n)^* \delta_z, u_0 \right\rangle = d^n \Big\langle d^{-kn} (f^n)^* \delta_z, - \sum_{m=0}^{\infty} d^{-m} v \circ f^m \Big\rangle \\ &= -d^n \Big\langle d^{-kn} (f^n)^* \delta_z, \sum_{m=0}^{n} d^{-m} v \circ f^m \Big\rangle - \Big\langle d^{-kn} (f^n)^* \delta_z, \sum_{m=n+1}^{\infty} d^{-m+n} v \circ f^m \Big\rangle. \end{aligned}$$

The absolute value of the second term in the last line is bounded by $\|v\|_\infty$ because $d^{-kn}(f^n)^* \delta_z$ is a probability measure. Observe that $(f^n)_*(v \circ f^m) = d^{km}(f^{n-m})_* v$ for all $n \geq m$. Hence, the absolute value of the first term is equal to

$$\Big| \sum_{m=0}^{n} d^{n-m} \Big\langle \delta_z, d^{-k(n-m)} (f^{n-m})_* v \Big\rangle \Big| \leq \sum_{j=0}^{n} d^j \|d^{-kj} (f^j)_* v\|_\infty. \tag{2.1}$$

Under the Assumption **(A)**, it is known that $\|d^{-kj}(f^j)_* v\|_\infty \lesssim \delta^{-j}$ for every $0 < \delta < d$. Indeed, the Assumption **(A)** implies the property **(A1)** below, see [DS10b, Cor. 1.2].

(A1) Let $g \colon \mathbb{P}^k \to \mathbb{R}$ be $\mathcal{C}^2$ and such that $\langle \mu, g \rangle = 0$. For every constant $1 < \delta < d$, there is a positive constant c independent of g and n such that

$$\|d^{-kn}(f^n)_* g\|_\infty \leq c \|g\|_{\mathcal{C}^2} \delta^{-n}.$$

By choosing $\delta > d/A$, we can bound the RHS of (2.1) by a constant times A^n. This ends the proof of the lemma. □

As an application of the previous estimates, we have the following lemma that can be used to study the regularity of functions $g \colon \mathbb{P}^k \to \mathbb{R}$.

Lemma 2.11. *Let $g \colon \mathbb{P}^k \to \mathbb{R}$ be a continuous function and $0 < \beta < 1$ a constant such that*

$$|dd^c g| \leq \sum_{n=0}^{\infty} \beta^n \omega_n. \tag{2.2}$$

Then, for every $q > 0$, there is a positive constant $c = c(q, \beta)$ independent of g such that

$$\|g\|_{\log^q} \leq c.$$

Proof. We bound the continuity modulus $m(g, r)$ of g by means of Corollary 2.7. We only need to consider $0 < r \leq 1/2$. For this purpose, since T has Hölder continuous local potentials, it suffices to bound the continuity modulus of the dynamical potential of the RHS of (2.2). This dynamical potential is equal to

$$u := \sum_{n=0}^{\infty} \beta^n u_n.$$

Fix a constant $1 < A < 1/\beta$. By Lemma 2.10, we have $\|u_n\|_\infty \lesssim A^n$. Hence, for every N, we have

$$m(u, r) \lesssim \sum_{n \leq N} \beta^n m(u_n, r) + \sum_{n > N} (A\beta)^n \lesssim \sum_{n \leq N} \beta^n m(u_n, r) + (A\beta)^N.$$

Applying [DS10b, Cor. 4.4] inductively to some iterate of f, we see that the Assumption **(A)** implies:

(A2) for every constant $\kappa > 1$, there are an integer $n_\kappa \geq 0$ and a constant $c_\kappa > 0$ independent of n such that for all $x, y \in \mathbb{P}^k$ and $n \geq n_\kappa$ we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_\kappa \operatorname{dist}(x, y)^{1/\kappa^n} \quad \text{for} \quad j = 1, \ldots, d^{kn}.$$

By definition, the function u_0 is γ-Hölder continuous for some Hölder exponent γ because T has Hölder continuous local potentials. The above property **(A2)** implies that $(f^n)_* u_0$ is $\gamma\kappa^{-n}$-Hölder continuous for all $n \geq n_\kappa$. More precisely, we have

$$m(d^{-kn}(f^n)_* u_0, r) \leq c' r^{\gamma\kappa^{-n}} \qquad \text{and hence} \qquad m(u_n, r) \leq c' d^n r^{\gamma\kappa^{-n}}$$

for some positive constant c' independent of $n \geq n_\kappa$ and r. Observe also that for $0 \leq n \leq n_\kappa$ all the u_n are α_κ-Hölder continuous for some $\alpha_\kappa > 0$. Indeed, as the multiplicity of f^n at a point is at most d^{kn}, we have (see again [DS10b, Cor. 4.4]):

(A2') there is a constant $c_0 > 0$ such that for every $n \geq 0$, for all $x, y \in \mathbb{P}^k$, we can write $f^{-n}(x) = \{x_1, \ldots, x_{d^{kn}}\}$ and $f^{-n}(y) = \{y_1, \ldots, y_{d^{kn}}\}$ (counting multiplicity) with the property that

$$\operatorname{dist}(x_j, y_j) \leq c_0 \operatorname{dist}(x, y)^{1/d^{kn}} \quad \text{for} \quad j = 1, \ldots, d^{kn}.$$

Therefore, we have

$$m(u, r) \lesssim r^{\alpha_\kappa} + \sum_{n_\kappa \leq n \leq N} (\beta d)^n \, r^{\gamma\kappa^{-n}} + (A\beta)^N. \tag{2.3}$$

Choose κ close enough to 1 so that $2q \log \kappa < |\log(A\beta)|$ and take

$$N = \frac{1}{2 \log \kappa} \log |\log r|$$

(recall that we only need to consider $r \leq 1/2$). Then, the last term in (2.3) satisfies

$$(A\beta)^N = e^{N \log(A\beta)} < e^{-2Nq \log \kappa} = |\log r|^{-q}.$$

It remains to prove that the sum in (2.3) satisfies a similar estimate. We have

$$\sum_{n \leq N} (\beta d)^n \, r^{\gamma\kappa^{-n}} \leq \sum_{n \leq N} \beta^n d^N r^{\gamma\kappa^{-N}} \lesssim d^N r^{\gamma\kappa^{-N}} = e^{\frac{\log d}{2 \log \kappa} \log |\log r|} e^{\gamma (\log r) e^{-\frac{1}{2} \log |\log r|}} = \frac{|\log r|^{\frac{\log d}{2 \log \kappa}}}{e^{\gamma \sqrt{|\log r|}}}.$$

The last expression is smaller than a constant times $|\log r|^{-q}$ because $e^t \gg t^M$ when $t \to \infty$ for every $M \geq 0$. This, together with the above estimates, gives $m(u, r) \lesssim |\log r|^{-q}$ and ends the proof of the lemma. $\square$

3. Existence of the scaling ratio and equilibrium state

In this section we prove the existence of a good scaling ratio λ, see Theorem 3.1 below.

3.1. Main statement and first step of the proof.

Recall that the Perron-Frobenius operator $\mathcal{L}$ is defined as in (1.1). A direct computation gives

$$\mathcal{L}^n(g)(y) = \sum_{f^n(x) = y} e^{\phi(x) + \phi(f(x)) + \cdots + \phi(f^{n-1}(x))} g(x).$$

Theorem 3.1. *Let f and ϕ be as in Theorem 1.1. There exist a number $\lambda > 0$ and a continuous function $\rho > 0$ on $\mathbb{P}^k$ such that for every continuous function $g \colon \mathbb{P}^k \to \mathbb{R}$ the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is equicontinuous and converges uniformly to $c_g \rho$, where c_g is a constant depending linearly on g. Moreover, if g is strictly positive, then c_g is strictly positive and the sequence $\mathcal{L}^n(g)^{1/n}$ converges uniformly to λ as n tends to infinity.*

We will first study the case where g is equal to $\mathbb{1}$. The general case will be deduced from this particular case. Define $\mathbb{1}_n := \mathcal{L}^n(\mathbb{1})$. Denote by ρ_n^+ and ρ_n^- the maximum and the minimum of $\mathbb{1}_n$, respectively. Consider also the ratio $\theta_n := \rho_n^+/\rho_n^-$ and the function $\mathbb{1}_n^* := (\rho_n^-)^{-1}\mathbb{1}_n$. Observe that the last function satisfies $\min \mathbb{1}_n^* = 1$. The following result will be crucial for us.

Proposition 3.2. *Under the hypotheses of Theorem 3.1, the sequence $\{\theta_n\}$ is bounded and the sequence of functions $\{\mathbb{1}_n^*\}$ is uniformly bounded and equicontinuous.*

The proof of this result will be given in Section 3.3 and uses the technical tools that were presented in Section 2. Before giving it, we need to first introduce some auxiliary objects.

By Lemma 2.3 applied to ϕ instead of g, we can find functions ϕ_n and ψ_n such that

$$\phi = \phi_n + \psi_n, \qquad \|\phi_n\|_{\mathcal{C}^2} \leq c\,\|\phi\|_\infty\, e^{\frac{1}{2}n^{2/q}}, \qquad \text{and} \qquad \|\psi_n\|_\infty \leq c\,\|\phi\|_{\log^q} n^{-2}. \tag{3.1}$$

Consider two integers $J \geq 0$ and $N \geq 0$, whose values will be specialised later. Define for $n \geq N+1$

$$\hat{\mathcal{L}}_n(g)(x) := \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\phi_{n+J-1}(f(x))+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))} g(x). \tag{3.2}$$

This operator will be used to approximate $\mathcal{L}^n$. The gain here is the fact that the involved functions ϕ_m have controlled $\mathcal{C}^2$ norms. As above, we define

$$\hat{\mathbb{1}}_n := \hat{\mathcal{L}}_n\mathbb{1}, \quad \hat{\rho}_n^+ := \max\hat{\mathbb{1}}_n, \quad \hat{\rho}_n^- := \min\hat{\mathbb{1}}_n, \quad \hat{\theta}_n := \hat{\rho}_n^+/\hat{\rho}_n^-, \quad \text{and} \quad \hat{\mathbb{1}}_n^* := (\hat{\rho}_n^-)^{-1}\hat{\mathbb{1}}_n.$$

The following lemma allows us to reduce our problem to the study of the functions $\hat{\mathbb{1}}_n$.

Lemma 3.3. *There exists a positive constant $c = c(N)$ such that, for all $n > N \geq 0$ and J,*

$$c^{-1} \leq \rho_n^+/\hat{\rho}_n^+ \leq c \quad \textit{and} \quad c^{-1} \leq \rho_n^-/\hat{\rho}_n^- \leq c.$$

In particular, the sequence $\{\hat{\theta}_n\}$ is bounded if and only if the sequence $\{\theta_n\}$ is bounded.

Proof. We have

$$\begin{aligned}\rho_n^+ &= \max\mathbb{1}_n = \max_y \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))}\\ &= \max_y \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))}\cdot e^{\psi_{n+J}(x)+\cdots+\psi_{J+N+1}(f^{n-N-1}(x))}\\ &\qquad\qquad \cdot e^{\phi(f^{n-N}x))+\cdots+\phi(f^{n-1}(x))}\end{aligned}$$

and similarly for ρ_n^-. So, both $\rho_n^+/\hat{\rho}_n^+$ and $\rho_n^-/\hat{\rho}_n^-$ are bounded from above and below by $e^{N\max\phi}C_{n,N,J}$ and $e^{N\min\phi}/C_{n,N,J}$ respectively, where

$$C_{n,N,J} := e^{\|\psi_{n+J}\|_\infty+\|\psi_{n+J-1}\|_\infty+\cdots+\|\psi_{J+N+1}\|_\infty}.$$

It follows from the estimate on ψ_n given above that $C_{n,N,J}$ is bounded from above by a positive constant which does not depend on n, J and N. Therefore, both $\rho_n^+/\hat{\rho}_n^+$ and $\rho_n^-/\hat{\rho}_n^-$ are bounded from below and above by positive constants as in the statement. The lemma follows. □

3.2. **An estimate for $dd^c\hat{\mathbb{1}}_n$.** Proposition 3.2 will be obtained using the following crucial estimate for $dd^c\hat{\mathbb{1}}_n$. We will see here the role of the estimate of the $\mathcal{C}^2$ norm of ϕ_n. Recall that $q > 2$, see Theorem 1.1. We also refer to Section 2.4 for notation.

Proposition 3.4. *There exists a sub-exponential function $\eta(t) = ct^3e^{(t+J)^{2/q}}$ with a positive constant $c = c(\|\phi\|_{\log^q}, \|\phi\|_\infty)$ independent of n, J and N such that for all $n > N \geq 0$ we have*

$$\left|dd^c\hat{\mathbb{1}}_n\right| \leq \sum_{m=N+1}^{n-N} \eta(m)e^{m\max\phi}\hat{\rho}_{n-m}^+ d^{(k-1)m}\omega_m + \sum_{m=n_0}^{n} d^{kN}\eta(m)e^{(n-N)\max\phi}d^{(k-1)m}\omega_m,$$

where $n_0 := \max(n-N+1, N+1)$.

Recall that the function $\hat{\mathbb{1}}_n$ is given by

$$\hat{\mathbb{1}}_n(y) = \sum_{f^n(x)=y} e^{\phi_{n+J}(x)+\phi_{n+J-1}(f(x))+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))}.$$

In order to estimate $dd^c\hat{\mathbb{1}}_n$, we will use a now classical construction due to Gromov [Gro03]. Define the manifold $\Gamma_n \subset (\mathbb{P}^k)^{n+1}$ by

$$\Gamma_n := \big\{(x, f(x), \ldots, f^n(x)) : x \in \mathbb{P}^k\big\},$$

which can also be seen as the graph of the map $(f, f^2, \ldots, f^n)$ in the product space $(\mathbb{P}^k)^{n+1}$. Consider the function $\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ given by

$$\mathbb{h}(x_0, \ldots, x_n) := e^{\phi_{n+J}(x_0)+\phi_{n+J-1}(x_1)+\cdots+\phi_{J+N+1}(x_{n-N-1})}.$$

The function $\hat{\mathbb{1}}_n$ on $\mathbb{P}^k$ is equal to the push-forward of the function $\mathbb{h}_{|\Gamma_n}$ to the last factor $\mathbb{P}^k$ of $(\mathbb{P}^k)^{n+1}$. Indeed, denoting by π_n the restriction of the projection $x \mapsto x_n$ to Γ_n, we have

$$(\pi_n)_* (\mathbb{h})(y) = \sum_{(x_0,\ldots,x_n)\in\Gamma_n:\, x_n=y} \mathbb{h}(x) = \sum_{x\in f^{-n}(y)} e^{\phi_{n+J}(x)+\cdots+\phi_{J+N+1}(f^{n-N-1}(x))} = \hat{\mathbb{1}}_n(y).$$

Recall that, since $dd^c\hat{\mathbb{1}}_n$ is real, estimating $|dd^c\hat{\mathbb{1}}_n|$ means finding a good positive closed $(1,1)$-current S on $\mathbb{P}^k$ such that both $S \pm dd^c\hat{\mathbb{1}}_n$ are positive. According to the identities above, we have

$$dd^c\hat{\mathbb{1}}_n = (\pi_n)_* \big(dd^c\mathbb{h}\big).$$

Thus, we need to estimate $dd^c\mathbb{h}$ on $(\mathbb{P}^k)^{n+1}$ and Γ_n. We define $\omega^{(m)}$ as the pullback of the Fubini-Study form ω_{FS} to $(\mathbb{P}^k)^{n+1}$ by the projection $x \mapsto x_m$. Equivalently, $\omega^{(m)}$ is a $(1,1)$-form on $(\mathbb{P}^k)^{n+1}$ such that $\omega^{(m)}(x) = \omega_{\text{FS}}(x_m)$.

Lemma 3.5. *There exists a sub-exponential function $\eta(t) = ct^3 e^{(t+J)^{2/q}}$ with a positive constant $c = c(\|\phi\|_{\log^q}, \|\phi\|_\infty)$ independent of $n, J,$ and N such that*

$$|dd^c\mathbb{h}| \leq \mathbb{h} \sum_{m=0}^{n-N-1} \eta(n-m)\omega^{(m)}.$$

Proof. A direct computation gives

$$i\partial\bar{\partial}\mathbb{h} = \mathbb{h}\Big(\sum_{m=0}^{n-N-1} i\partial\overline{\partial}\phi_{n+J-m}(x_m) + \sum_{m,m'=0}^{n-N-1} i\partial\phi_{n+J-m}(x_m) \wedge \overline{\partial}\phi_{n+J-m'}(x_{m'})\Big).$$

For the first sum, observe that

$$|i\partial\overline{\partial}\phi_{n+J-m}(x_m)| \lesssim \|\phi_{n+J-m}\|_{\mathcal{C}^2}\omega^{(m)}(x) \lesssim e^{(n+J-m)^{2/q}}\omega^{(m)}(x).$$

For the second sum, consider $m' \leq m \leq n-N-1$. By using Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
(3.3) \quad & |i\partial\phi_{n+J-m}(x_m) \wedge \overline{\partial}\phi_{n+J-m'}(x_{m'})| \\
\leq\ & (m-m'+1)^{-2} i\partial\phi_{n+J-m}(x_m) \wedge \overline{\partial}\phi_{n+J-m}(x_m) \\
& + (m-m'+1)^2 i\partial\phi_{n+J-m'}(x_{m'}) \wedge \overline{\partial}\phi_{n+J-m'}(x_{m'}) \\
\lesssim\ & (m-m'+1)^{-2}\|\phi_{n+J-m}\|^2_{\mathcal{C}^1}\omega^{(m)}(x) + (m-m'+1)^2\|\phi_{n+J-m'}\|^2_{\mathcal{C}^1}\omega^{(m')}(x) \\
\lesssim\ & (m-m'+1)^{-2}e^{(n+J-m)^{2/q}}\omega^{(m)}(x) + (n-m'+1)^2 e^{(n+J-m')^{2/q}}\omega^{(m')}(x).
\end{aligned}$$

This and the fact that $\sum_{j=1}^{\infty} j^{-2}$ is finite imply that

$$
\begin{aligned}
(3.4)\qquad & \Big|\sum_{0\le m'\le m\le n-N-1} i\partial\phi_{n+J-m}(x_m)\wedge\overline{\partial}\phi_{n+J-m'}(x_{m'})\Big| \\
&\lesssim \sum_{m=0}^{n-N-1} e^{(n+J-m)^{2/q}}\omega^{(m)}(x) + \sum_{m'=0}^{n-N-1}(n-m'+1)^3 e^{(n+J-m')^{2/q}}\omega^{(m')}(x) \\
&\lesssim \sum_{m=0}^{n-N-1}(n-m)^3 e^{(n+J-m)^{2/q}}\omega^{(m)}(x).
\end{aligned}
$$

We obtain by symmetry a similar estimate for the case where $m < m' \le n-N-1$.

Finally, combining all the above identities and estimates we get

$$
|i\partial\bar\partial \mathbb{h}| \lesssim \mathbb{h}\sum_{m=0}^{n-N-1}(n-m)^3 e^{(n+J-m)^{2/q}}\omega^{(m)}.
$$

The lemma follows. □

Proof of Proposition 3.4. We are only interested in the restriction of $\mathbb{h}$ to the graph Γ_n. We deduce from Lemma 3.5 that

$$
(3.5)\qquad \big|dd^c\hat{\mathbb{1}}_n\big| = |(\pi_n)_* dd^c\mathbb{h}| \le \sum_{m=0}^{n-N-1}\eta(n-m)(\pi_n)_*\big(\mathbb{h}\omega^{(m)}\big).
$$

We split the last sum into the two sums corresponding to $m < N$ and $m \ge N$. Note that when $n \le 2N$, in the sum in (3.5) we always have $m < N$ and the first sum in the statement of the proposition vanishes. So, for simplicity, we assume that $n > 2N$ and we will see in the proof below that the arguments also work when $n \le 2N$.

For $m < N$, using the definition of ϕ_m we have $\|\phi-\phi_m\|_\infty = \|\psi_m\|_\infty \le c'm^{-2}$ and hence $\max\phi_m \le \max\phi + c'm^{-2}$ for some positive constant c' which may depend on $\|\phi\|_{\log^q}$. It follows that $\mathbb{h} \lesssim e^{(n-N)\max\phi}$. Then, using the definition of Γ_n, we have for $m < N$

$$
\begin{aligned}
(\pi_n)_*\big(\mathbb{h}\omega^{(m)}\big) &\lesssim e^{(n-N)\max\phi}(\pi_n)_*\big(\omega^{(m)}\big) = e^{(n-N)\max\phi}d^{km}(f^{n-m})_*(\omega_{\rm FS}) \\
&= e^{(n-N)\max\phi}d^{km}d^{(k-1)(n-m)}\omega_{n-m}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\sum_{m=0}^{N-1}\eta(n-m)(\pi_n)_*\big(\mathbb{h}\omega^{(m)}\big) &\lesssim \sum_{m=0}^{N-1}\eta(n-m)e^{(n-N)\max\phi}d^{km}d^{(k-1)(n-m)}\omega_{n-m} \\
(3.6)\qquad &\le \sum_{m=n-N+1}^{n} d^{kN}\eta(m)e^{(n-N)\max\phi}d^{(k-1)m}\omega_m.
\end{aligned}
$$

The last expression is the second sum in the statement of the present proposition (this step also works for $n \le 2N$ but in this case the above sums $\sum_0^{N-1}$ and $\sum_{n-N+1}^{n}$ are replaced by $\sum_0^{n-N-1}$ and $\sum_{N+1}^{n}$ respectively). In order to finish the proof, it is enough to have a similar estimate for $m \ge N$ (this step is superfluous when $n \le 2N$, see (3.5)).

As above, using the definition of $\mathbb{h}$ and the estimates on $\max\phi_m$ and $\|\phi-\phi_m\|_\infty$, we have

$$
\mathbb{h} \lesssim e^{(n-m)\max\phi}e^{\phi(x_0)+\phi(x_1)+\cdots+\phi(x_{m-N-1})} \lesssim e^{(n-m)\max\phi}\mathbb{h}'
$$

with

$$
\mathbb{h}' := e^{\phi_{m+J}(x_0)+\phi_{m+J-1}(x_1)+\cdots+\phi_{J+N+1}(x_{m-N-1})}.
$$

Note that the sum in the definition of $\mathbb{h}'$ contains $m-N$ terms while the one of $\mathbb{h}$ contains $n-N$ terms. The specific choice of $\mathbb{h}'$ is convenient for our next computation as it is related to the function $\hat{\mathbb{1}}_m$.

Consider the map $\pi' : \Gamma_n \to (\mathbb{P}^k)^{n-m+1}$ defined by $\pi'(x) := x' := (x_m, \ldots, x_n)$. Denote by Γ' the image of Γ_n by π'. It is the graph of the map $(f, \ldots, f^{n-m})$ from $\mathbb{P}^k$ to $(\mathbb{P}^k)^{n-m}$. We also have for $x' \in \Gamma'$

$$\pi'^{-1}(x') = \big\{ \big(y, f(y), \ldots, f^{m-1}(y), x'\big) \quad \text{with} \quad y \in f^{-m}(x_m) \big\}.$$

So $\pi' : \Gamma_n \to \Gamma'$ is a ramified covering of degree d^{km}.

Consider the map $\pi'' : \Gamma' \to \mathbb{P}^k$ defined by $\pi''(x') := x_n$. We have, for $x_n \in \mathbb{P}^k$,

$$\pi''^{-1}(x_n) = \big\{ \big(z, f(z), \ldots, f^{n-m}(z)\big) \quad \text{with} \quad z \in f^{-n+m}(x_n) \big\}.$$

So $\pi'' : \Gamma' \to \mathbb{P}^k$ is a ramified covering of degree $d^{k(n-m)}$. We have $\pi_n = \pi'' \circ \pi'$. Observe that $\pi'_*(\mathbb{h}'\omega^{(m)})$ is a $(1,1)$-form on Γ' such that

$$\begin{aligned} \pi'_*(\mathbb{h}'\omega^{(m)})(x') &= \Big(\sum_{y \in f^{-m}(x_m)} e^{\phi_{m+J}(y) + \cdots + \phi_{J+N+1}(f^{m-N-1}(y))} \Big) \omega_{\mathrm{FS}}(x_m) \\ &\leq \hat\rho_m^+ \omega_{\mathrm{FS}}(x_m) =: \hat\rho_m^+ \omega'(x'), \end{aligned}$$

where we define ω' as the pull-back of ω_{FS} to Γ' by the map $x' \mapsto x_m$. We also have

$$\pi''_*(\omega')(x_n) = \sum_{x_m \in f^{-n+m}(x_n)} \omega_{\mathrm{FS}}(x_m) = (f^{n-m})_*(\omega_{\mathrm{FS}})(x_n) = d^{(k-1)(n-m)} \omega_{n-m}(x_n).$$

Thus,

$$(\pi_n)_*(\mathbb{h}\omega^{(m)}) \lesssim e^{(n-m)\max\phi} \pi''_* \pi'_*(\mathbb{h}'\omega^{(m)}) \leq e^{(n-m)\max\phi} \hat\rho_m^+ d^{(k-1)(n-m)} \omega_{n-m}$$

and

$$\begin{aligned} \sum_{m=N}^{n-N-1} \eta(n-m)(\pi_n)_* \big(\mathbb{h}\omega^{(m)}\big) &\lesssim \sum_{m=N}^{n-N-1} \eta(n-m) e^{(n-m)\max\phi} \hat\rho_m^+ d^{(k-1)(n-m)} \omega_{n-m} \\ &= \sum_{m=N+1}^{n-N} \eta(m) e^{m\max\phi} \hat\rho_{n-m}^+ d^{(k-1)m} \omega_m. \end{aligned} \tag{3.7}$$

Finally, we deduce the proposition from (3.5), (3.6), and (3.7) by multiplying η with a large enough constant. □

3.3. **Proof of Proposition 3.2.** We are working under the hypotheses of Theorem 3.1. We will obtain Proposition 3.2 using Lemmas 3.6 and 3.7 below.

Lemma 3.6. *Under the hypotheses of Theorem 3.1, given an integer $J \geq 0$, we have $\hat\theta_n \leq d^{kN}$ for all $n > N$, with N large enough. In particular, the sequences (θ_n) and $(\hat\theta_n)$ are bounded for all $J \geq 0$ and $N \geq 0$.*

Proof. Observe that the last assertion is a consequence of the first one. Indeed, we can first fix J and N satisfying the first assertion of the lemma. Then, by Lemma 3.3, the sequence (θ_n) is bounded. Applying again Lemma 3.3 for arbitrary J and N gives that the sequence $(\hat\theta_n)$ is also bounded. We prove now the first assertion in the lemma with J fixed and N large enough.

Observe that, by the definition of $\hat\rho_n^\pm, \hat\theta_n$, and $\Omega(\cdot)$, for every $K \geq 1$ the two inequalities $\hat\theta_n \leq K$ and $\Omega(\hat{\mathbb{1}}_n) \leq (K-1)\hat\rho_n^-$ are equivalent. Hence, in order to get the first assertion in the lemma, it is enough to show that $\Omega(\hat{\mathbb{1}}_n)/\hat\rho_n^- \leq d^{kN}/2$. The constants that we use below are independent of N and n. Fix a constant δ such that $e^{\Omega(\phi)} < \delta < d$. By the estimate on $\|\psi_n\|_\infty$ in (3.1), for every j sufficiently large, we have $\Omega(\phi_j) \leq \Omega(\phi) + \Omega(\psi_j) < \log\delta$. Since we assume that N is large enough, the last inequality holds for all $j \geq N$.

We use Proposition 3.4 and Corollary 2.5 in order to estimate $\Omega(\hat{\mathbb{1}}_n)$ in terms of $\Omega(u_m)$. Recall that u_m is the dynamical potential of ω_m. We also use Lemma 2.10, which gives $\Omega(u_m) \lesssim d^m \delta'^{-m}$

for any δ' such that $\delta < \delta' < d$. More precisely, we obtain from those results that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim \sum_{m=N+1}^{n-N} \eta(m) e^{m \max\phi} d^{km} \delta'^{-m} \hat{\rho}^+_{n-m} + \sum_{m=\max(n-N+1,N+1)}^{n} d^{kN} \eta(m) e^{(n-N)\max\phi} d^{km} \delta'^{-m}.$$

Since $\delta < \delta'$ and N is large, the fact that η is sub-exponential and independent of n and N implies that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim \sum_{m=N+1}^{n-N} e^{m \max\phi} d^{km} \delta^{-m} \hat{\rho}^+_{n-m} + \sum_{m=\max(n-N+1,N+1)}^{n} d^{kN} e^{(n-N)\max\phi} d^{km} \delta^{-m}. \tag{3.8}$$

We now distinguish two cases.

Case 1. Assume that $N < n \leq 2N$. In this case, the first sum in (3.8) is empty. We thus deduce from (3.8) that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim d^{kN} e^{(n-N)\max\phi} \Big(\frac{d^{k(N+1)}}{\delta^{N+1}} + \cdots + \frac{d^{kn}}{\delta^n} \Big) \lesssim d^{kN} e^{(n-N)\max\phi} \frac{d^{kn}}{\delta^n}.$$

On the other hand, by the definitions of $\hat{\rho}^-_n$ we have the following general estimates (with $n \geq N$ in the first inequality and $n - m \geq N$ in the second one)

$$\hat{\rho}^-_n \gtrsim d^{kn} e^{(n-N)\min\phi} \qquad \text{and} \qquad \hat{\rho}^-_n \gtrsim d^{km} e^{m \min\phi} \hat{\rho}^-_{n-m}. \tag{3.9}$$

The first inequality and the above estimate of $\Omega(\hat{\mathbb{1}}_n)$ imply that

$$\frac{\Omega(\hat{\mathbb{1}}_n)}{\hat{\rho}^-_n} \lesssim d^{kN} \frac{e^{(n-N)\Omega(\phi)}}{\delta^n} \leq d^{kN} \frac{e^{n\Omega(\phi)}}{\delta^n}.$$

Hence, $\Omega(\hat{\mathbb{1}}_n)/\hat{\rho}^-_n \leq d^{kN}/2$ because N is chosen large enough and $\delta > e^{\Omega(\phi)}$. The lemma in this case follows.

Case 2. Assume now that $n > 2N$. By induction on n and the previous case, we can assume that $\Omega(\hat{\mathbb{1}}_m)/\hat{\rho}^-_m \leq d^{kN}/2$, which implies $\hat{\rho}^+_m \leq d^{kN}\hat{\rho}^-_m$, for all $m < n$. We need to prove the same inequality for $m = n$. From (3.8) and the induction hypothesis, we have

$$\begin{aligned}
\Omega(\hat{\mathbb{1}}_n) &\lesssim d^{kN} \sum_{m=N+1}^{n-N} e^{m\max\phi} d^{km} \delta^{-m} \hat{\rho}^-_{n-m} + d^{kN} \sum_{m=n-N+1}^{n} e^{(n-N)\max\phi} d^{km} \delta^{-m} \\
&\lesssim d^{kN} \sum_{m=N+1}^{n-N} e^{m\max\phi} d^{km} \delta^{-m} \hat{\rho}^-_{n-m} + d^{kN} e^{(n-N)\max\phi} d^{kn} \delta^{-n}.
\end{aligned}$$

This and the second inequality in (3.9) imply that

$$\Omega(\hat{\mathbb{1}}_n) \lesssim d^{kN} \sum_{m=N+1}^{n-N} e^{m\Omega(\phi)} \delta^{-m} \hat{\rho}^-_n + d^{kN} e^{(n-N)\max\phi} d^{kn} \delta^{-n}.$$

Then, by the first inequality in (3.9) and using that $\delta > e^{\Omega(\phi)}$ and $n > 2N$, we obtain

$$\frac{\Omega(\hat{\mathbb{1}}_n)}{\hat{\rho}^-_n} \lesssim d^{kN} \sum_{m=N+1}^{n-N} e^{m\Omega(\phi)} \delta^{-m} + d^{kN} e^{(n-N)\Omega(\phi)} \delta^{-n} \lesssim d^{kN} e^{N\Omega(\phi)} \delta^{-N}.$$

Recall that all the constants involved in our computations do not depend on n and N. Since N is chosen large enough, we obtain that $\Omega(\hat{\mathbb{1}}_n)/\hat{\rho}^-_n \leq d^{kN}/2$. This ends the proof of the lemma. □

Lemma 3.7. *Under the hypotheses of Theorem 3.1, for all $J \geq 0$, $N \geq 0$, and $p > 0$, the sequence $\|\hat{\mathbb{1}}^*_n\|_{\log^p}$ is bounded. In particular, the sequence of functions $\hat{\mathbb{1}}^*_n$ is equicontinuous.*

Proof. We only need to consider $n > 2N$, and the implicit constants below may depend on N. We will use Lemma 2.11 and need to estimate $dd^c\hat{\mathbb{1}}_n^*$. By Lemma 3.6 the sequence $(\hat{\theta}_n)$ is bounded. This and Proposition 3.4 imply that

$$\left|dd^c\hat{\mathbb{1}}_n^*\right| \lesssim \frac{1}{\hat{\rho}_n^-}\Big(\sum_{m=N+1}^{n-N}\eta(m)e^{m\max\phi}\hat{\rho}_{n-m}^-d^{(k-1)m}\omega_m + \sum_{m=n-N+1}^{n}\eta(m)e^{(n-N)\max\phi}d^{(k-1)m}\omega_m\Big).$$

Then, using the two inequalities in (3.9), we obtain

$$\begin{aligned}\left|dd^c\hat{\mathbb{1}}_n^*\right| &\lesssim \sum_{m=N+1}^{n-N}\eta(m)e^{m\Omega(\phi)}d^{-m}\omega_m + \sum_{m=n-N+1}^{n}\eta(m)e^{(n-N)\Omega(\phi)}d^{(k-1)m-kn}\omega_m\\ &\lesssim \sum_{m=0}^{\infty}\eta(m)e^{m\Omega(\phi)}d^{-m}\omega_m.\end{aligned}$$

Finally, since η is sub-exponential and $e^{\Omega(\phi)} < d$, Lemma 2.11 implies the result. □

End of the proof of Proposition 3.2. By Lemma 3.6, we already know that the sequence (θ_n) is bounded. Since $\min\mathbb{1}_n^* = 1$, we have $\max\mathbb{1}_n^* = \theta_n$, hence the sequence $(\mathbb{1}_n^*)$ is uniformly bounded. In order to show that this sequence is equicontinuous, it is enough to approximate it uniformly by an equicontinuous sequence.

Take $N = 0$. Fix an arbitrary constant $0 < \epsilon < 1$. Since $\|\phi - \phi_m\|_\infty \lesssim m^{-2}$ by (3.1), we can choose an integer J large enough so that for every $n \geq 0$ we have

$$(1-\epsilon)\hat{\mathbb{1}}_n \leq \mathbb{1}_n \leq (1+\epsilon)\hat{\mathbb{1}}_n.$$

This implies

$$\frac{1-\epsilon}{1+\epsilon}\,\hat{\mathbb{1}}_n^* \leq \mathbb{1}_n^* \leq \frac{1+\epsilon}{1-\epsilon}\,\hat{\mathbb{1}}_n^*.$$

Therefore, $|\mathbb{1}_n^* - \hat{\mathbb{1}}_n^*|$ is bounded uniformly by a constant times ϵ. By Lemma 3.7, the sequence $(\hat{\mathbb{1}}_n^*)$ is equicontinuous. We easily deduce that the sequence $(\mathbb{1}_n^*)$ is equicontinuous as well. □

3.4. **Proof of Theorem 3.1.** We first define the scaling ratio λ. By definition of ρ_n^+, we easily see that the sequence (ρ_n^+) is sub-multiplicative, that is, $\rho_{n+m}^+ \leq \rho_m^+\rho_n^+$ for all $m, n \geq 0$. It follows that the first limit in the following line exists

$$\lambda := \lim_{n\to\infty}\left(\rho_n^+\right)^{1/n} = \lim_{n\to\infty}\left(\rho_n^-\right)^{1/n},$$

where the last identity is due to the fact that (θ_n) is bounded, see Lemma 3.6. We have the following lemma.

Lemma 3.8. *The sequences $(\lambda^{-n}\rho_n^+)$ and $(\lambda^{-n}\rho_n^-)$ are both bounded above and below by positive constants. In particular, the sequence $\left(\lambda^{-n}\mathbb{1}_n\right)$ is uniformly bounded and equicontinuous.*

Proof. It is clear that the second assertion is a consequence of the first one and Proposition 3.2. We prove now the first assertion. Since the sequence ρ_n^+ is sub-multiplicative, it is well-known that $\inf_n(\rho_n^+)^{1/n}$ is equal to λ. Hence, we have $\lambda^{-n}\rho_n^+ \geq 1$. Since θ_n is bounded, we have $\rho_n^+ \lesssim \rho_n^-$. It follows that both $\lambda^{-n}\rho_n^\pm$ are bounded from below by positive constants. Similarly, the sequence ρ_n^- is super-multiplicative, i.e., $\rho_{n+m}^- \geq \rho_m^-\rho_n^-$ for all $m, n \geq 0$, and we deduce that that both $\lambda^{-n}\rho_n^\pm$ are bounded from above by positive constants. The lemma follows. □

We can extend the above result to all continuous test functions.

Lemma 3.9. *Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\mathcal{F}_{\mathbb{N}} := \{\lambda^{-n}\mathcal{L}^n(g) \;:\; g \in \mathcal{F}, n \geq 0\}$$

is also uniformly bounded and equicontinuous.

Proof. By Lemma 3.8, the family $\mathcal{F}_{\mathbb{N}}$ is uniformly bounded. We prove now that it is equicontinuous. Given any constant $\epsilon > 0$, using a convolution, we can find for every $g \in \mathcal{F}$ a smooth function g' such that $\|g - g'\|_\infty \leq \epsilon$ and $\|g'\|_{\mathcal{C}^2}$ is bounded by a constant depending on ϵ. Denote by $\mathcal{F}'$ the family of these g'. Observe that

$$|\lambda^{-n}\mathcal{L}^n(g) - \lambda^{-n}\mathcal{L}^n(g')| = |\lambda^{-n}\mathcal{L}^n(g - g')| \leq \epsilon\lambda^{-n}\mathbb{1}_n \leq \epsilon\lambda^{-n}\rho_n^+$$

and the last expression is bounded by a constant times ϵ. Therefore, in order to prove the lemma, it is enough to show that the family $\mathcal{F}'_{\mathbb{N}}$, defined in a similar way as for $\mathcal{F}_{\mathbb{N}}$, is equicontinuous. For simplicity, we replace $\mathcal{F}$ by $\mathcal{F}'$ and assume that $\|g\|_{\mathcal{C}^2}$ is bounded by a constant for $g \in \mathcal{F}$. The constants involved in the computation below do not depend on $g \in \mathcal{F}$.

We continue to use the notation introduced above. Consider an arbitrary constant $\epsilon > 0$. Take $N = 0$ and choose J large enough depending on ϵ. From the definitions of $\mathcal{L}$ and $\hat{\mathcal{L}}_n$ (see (3.2)) and the fact that $\|\phi - \phi_m\|_\infty \lesssim m^{-2}$ we obtain that

$$|\lambda^{-n}\mathcal{L}^n(g)(x) - \lambda^{-n}\hat{\mathcal{L}}_n(g)(x)| \leq \epsilon\lambda^{-n} \sum_{f^n(x)=y} e^{\phi(x)+\phi(f(x))+\cdots+\phi(f^{n-1}(x))}|g(x)|.$$

This and Lemma 3.8 imply that

$$\|\lambda^{-n}\mathcal{L}^n(g) - \lambda^{-n}\hat{\mathcal{L}}_n(g)\|_\infty \leq \epsilon\lambda^{-n}\rho_n^+\|g\|_\infty \lesssim \epsilon.$$

So, in order to prove that the family $\lambda^{-n}\mathcal{L}^n(g)$ is equicontinuous, it is enough to show the same property for the family $\lambda^{-n}\hat{\mathcal{L}}_n(g)$.

We will use the same idea as in Proposition 3.4 and Lemma 3.5. Instead of the function $\mathbb{h}$, we need to consider the following slightly different function (recall that $N = 0$)

$$\mathbb{H}(x_0, \ldots, x_n) := e^{\phi_{n+J}(x_0)+\phi_{n+J-1}(x_1)+\cdots+\phi_{J+1}(x_{n-1})}g(x_0) = \mathbb{h}(x_0, \ldots, x_n)g(x_0).$$

We have

$$i\partial\overline{\partial}\mathbb{H} = (i\partial\overline{\partial}\mathbb{h})g(x_0) + \mathbb{h}(i\partial\overline{\partial}g(x_0)) + i\partial\mathbb{h} \wedge \overline{\partial}g(x_0) - i\overline{\partial}\mathbb{h} \wedge \partial g(x_0).$$

Applying Cauchy-Schwarz's inequality to the last two terms, and since g has a bounded $\mathcal{C}^2$ norm, we obtain

$$\begin{aligned}
|i\partial\overline{\partial}\mathbb{H}| &\leq |(i\partial\overline{\partial}\mathbb{h})g(x_0)| + |\mathbb{h}(i\partial\overline{\partial}g(x_0))| + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} + i\mathbb{h}\partial g(x_0) \wedge \overline{\partial}g(x_0) \\
&\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} + \mathbb{h}\omega_{\mathrm{FS}}(x_0) \\
&\lesssim |i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h}.
\end{aligned}$$

We claim that the last sum satisfies

$$|i\partial\overline{\partial}\mathbb{h}| + \mathbb{h}\omega_{\mathrm{FS}}(x_0) + i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} \lesssim \mathbb{h}\sum_{m=0}^{n-1} \eta(n-m)\omega^{(m)}.$$

Lemma 3.5 shows that the first term $|i\partial\overline{\partial}\mathbb{h}|$ of the LHS is bounded by the RHS. The second term clearly satisfies the same property (consider $m = 0$ in the above sum). For the last term, by Cauchy-Schwarz's inequality and using a computation as in the proof of Lemma 3.5, we have (recall that $N = 0$)

$$i\mathbb{h}^{-1}\partial\mathbb{h} \wedge \overline{\partial}\mathbb{h} = \mathbb{h} \sum_{m,m'=0}^{n-1} i\partial\phi_{n+J-m}(x_m) \wedge \overline{\partial}\phi_{n+J-m'}(x_{m'}) \lesssim \mathbb{h}\sum_{m=0}^{n-1} \eta(n-m)\omega^{(m)}.$$

This implies the claim and gives a bound for $|i\partial\overline{\partial}\mathbb{H}|$.

Since $\hat{\mathcal{L}}_n(g) = (\pi_n)_*(\mathbb{H})$, we obtain as in the proof of Proposition 3.4 that

$$|dd^c\lambda^{-n}\hat{\mathcal{L}}_n(g)| \lesssim \lambda^{-n}\sum_{m=1}^{n} \eta(m)e^{m\max\phi}\hat{\rho}_{n-m}^+ d^{(k-1)m}\omega_m.$$

By Lemmas 3.3 and 3.8 we have $\hat{\rho}^{\pm}_{n-m} \lesssim \rho^{\pm}_{n-m} \lesssim \lambda^{n-m}$. Therefore, we obtain

$$|dd^c \lambda^{-n} \hat{\mathcal{L}}_n(g)| \lesssim \sum_{m=1}^{n} \eta(m) e^{m \max \phi} \lambda^{-m} d^{(k-1)m} \omega_m.$$

Finally, since $\lambda \geq d^k e^{\min \phi}$ by definition of λ, the last estimate implies that

$$|dd^c \lambda^{-n} \hat{\mathcal{L}}_n(g)| \lesssim \sum_{m=1}^{n} \eta(m) e^{m\Omega(\phi)} d^{-m} \omega_m.$$

Lemma 2.11 and the fact that $d > e^{\Omega(\phi)}$ imply the result. □

We now construct the density function ρ on $\mathbb{P}^k$. Recall that the sequence $\lambda^{-n}\mathbb{1}_n$ is uniformly bounded and equicontinuous. Therefore, the Cesaro sums

$$\widetilde{\mathbb{1}}_n := \frac{1}{n} \sum_{j=0}^{n-1} \lambda^{-j} \mathbb{1}_j$$

also form a uniformly bounded and equicontinuous sequence of functions. It follows that there is a subsequence of $\widetilde{\mathbb{1}}_n$ which converges uniformly to a continuous function ρ. Observe that $\rho \geq \inf_n \lambda^{-n} \rho^-_n$. Hence, by Lemma 3.8, the function ρ is strictly positive. A direct computation gives

$$\lambda^{-1}\mathcal{L}(\widetilde{\mathbb{1}}_n) - \widetilde{\mathbb{1}}_n = \frac{1}{n}(\lambda^{-n}\mathbb{1}_n - \mathbb{1}_0).$$

Since $\lambda^{-n}\mathbb{1}_n$ is bounded uniformly in n, the last expression tends uniformly to 0 when n tends to infinity. We then deduce from the definition of ρ that $\lambda^{-1}\mathcal{L}(\rho) = \rho$.

End of the proof of Theorem 3.1. Observe that we only need to show that $\lambda^{-n}\mathcal{L}^n(g)$ converges to $c_g\rho$ for some constant c_g. The remaining part of the theorem is then clear. Let $\mathcal{G}$ denote the family of all limit functions of subsequences of $\lambda^{-n}\mathcal{L}^n(g)$. By Lemma 3.9, the sequence $\lambda^{-n}\mathcal{L}^n(g)$ is uniformly bounded and equicontinuous. Therefore, by Arzelà-Ascoli theorem, $\mathcal{G}$ is a uniformly bounded and equicontinuous family of functions which is compact for the uniform topology. Observe also that $\mathcal{G}$ is invariant under the action of $\lambda^{-1}\mathcal{L}$. Define

$$M := \max\{l(a)/\rho(a) \;:\; l \in \mathcal{G}, a \in \mathbb{P}^k\}.$$

We first prove the following properties.

Claim 1. We have $\max_{\mathbb{P}^k}(l/\rho) = M$ for every $l \in \mathcal{G}$.

Assume by contradiction that there is a sequence $\lambda^{-n_j}\mathcal{L}^{n_j}(g)$ which converges uniformly to a function $l \in \mathcal{G}$ such that $l \leq (M - 2\epsilon)\rho$ for some constant $\epsilon > 0$. Then, for j large enough, we have $\lambda^{-n_j}\mathcal{L}^{n_j}(g) \leq (M - \epsilon)\rho$. Fix such an index j. For $n > n_j$ we have

$$\lambda^{-n}\mathcal{L}^n(g) = \lambda^{-n+n_j}\mathcal{L}^{n-n_j}(\lambda^{-n_j}\mathcal{L}^{n_j}(g)) \leq (M-\epsilon)\lambda^{-n+n_j}\mathcal{L}^{n-n_j}(\rho) = (M-\epsilon)\rho.$$

Since this is true for every $n > n_j$, we get a contradiction with the definition of M. This ends the proof of Claim 1.

Claim 2. We have $l/\rho = M$ on the small Julia set $\mathrm{supp}(\mu)$ for every $l \in \mathcal{G}$.

Consider an arbitrary function $l \in \mathcal{G}$ and define $l_n := \lambda^{-n}\mathcal{L}^n(l) \in \mathcal{G}$. By Claim 1, there is a point $a_n \in \mathbb{P}^k$ such that $l_n(a_n) = M\rho(a_n)$. By definition of M, we have $l \leq M\rho$ and hence

$$M\rho(a_n) = l_n(a_n) = \lambda^{-n}\mathcal{L}^n(l)(a_n) \leq \lambda^{-n}\mathcal{L}^n(M\rho)(a_n) = M\rho(a_n).$$

So the inequality in the last line is actually an equality. This and the definition of $\mathcal{L}$ imply that $l/\rho = M$ on $f^{-n}(a_n)$. Observe that when n tends to infinity, the limit of $f^{-n}(a_n)$ contains $\mathrm{supp}(\mu)$, see, e.g., [DS10b, Cor. 1.4]. By continuity, we obtain $l/\rho = M$ on $\mathrm{supp}(\mu)$. This ends the proof of Claim 2.

Applying the above claims to the function $-g$ instead of g, we obtain that l/ρ is equal on $\mathrm{supp}(\mu)$ to $\min_{\mathbb{P}^k}(l/\rho)$. We can now conclude that $l = M\rho$ on $\mathbb{P}^k$ for every $l \in \mathcal{G}$. Define

$c_g := M$. We obtain that $\lambda^{-n}\mathcal{L}^n(g)$ converges uniformly to $c_g\rho$. This completes the proof of the theorem. □

4. Properties of equilibrium states

In this section we conclude the proof of Theorem 1.1. In Sections 4.1 and 4.2, we deduce the main properties of the equilibrium states in Theorem 1.1 from Theorem 3.1. In Section 4.3 we prove the equidistribution of repelling periodic points, which concludes the proof of Theorem 1.1.

4.1. **Equidistribution of preimages and mixing properties.** We have seen that the operator $\mathcal{L}$ acts on the space of continuous functions $g\colon \mathbb{P}^k \to \mathbb{R}$. It is also positive, i.e., $\mathcal{L}(g) \geq 0$ when $g \geq 0$. Therefore, $\mathcal{L}$ induces by duality a linear operator $\mathcal{L}^*$ acting on the space of measures and preserving the cone of positive measures.

Proposition 4.1. *Under the assumptions of Theorem 1.1, there exists a unique conformal measure associated with ϕ, that is, there exists a unique probability measure m_ϕ which is an eigenvector of $\mathcal{L}^*$. We also have $\mathcal{L}^*(m_\phi) = \lambda m_\phi$, $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$, and if ν is a positive measure, $\lambda^{-n}(\mathcal{L}^n)^*(\nu)$ converges to $\langle \nu, \rho\rangle m_\phi$ when n tends to infinity. Moreover, if $\mathcal{F}$ is a uniformly bounded and equicontinuous family of functions on $\mathbb{P}^k$, then $\lambda^{-n}\mathcal{L}^n(g) - c_g\rho$ converges to 0 when n goes to infinity, uniformly on $g \in \mathcal{F}$, where $c_g := \langle m_\phi, g\rangle$.*

Proof. For any probability measure m_ϕ as in the first assertion, there is a constant $\lambda' > 0$ such that $\mathcal{L}^*(m_\phi) = \lambda' m_\phi$. It follows that, for every continuous function g,

$$\langle m_\phi, g\rangle = \lim_{n\to\infty} \langle \lambda'^{-n}(\mathcal{L}^n)^*(m_\phi), g\rangle = \lim_{n\to\infty} \langle m_\phi, \lambda'^{-n}\mathcal{L}^n(g)\rangle.$$

We necessarily have $\lambda' = \lambda$ because we know from the end of the proof of Theorem 3.1 that $\lambda^{-n}\mathcal{L}^n(g)$ converges uniformly to $c_g\rho$ and c_g is not always 0. We conclude that $\langle m_\phi, g\rangle = c_g\langle m_\phi, \rho\rangle$. Since $\langle m_\phi, g\rangle = c_g = 1$ when $g = \mathbb{1}$ (because m_ϕ is a probability measure) we deduce that $\langle m_\phi, \rho\rangle = 1$ and hence $\langle m_\phi, g\rangle = c_g$ for every continuous function g. This gives the uniqueness of m_ϕ.

Consider now an arbitrary probability measure ν on $\mathbb{P}^k$. We have

$$\langle \lambda^{-n}(\mathcal{L}^n)^*(\nu), g\rangle = \langle \nu, \lambda^{-n}\mathcal{L}^n(g)\rangle \to \langle \nu, c_g\rho\rangle = \langle \nu, \rho\rangle\langle m_\phi, g\rangle.$$

It follows that $\lambda^{-n}(\mathcal{L}^n)^*(\nu)$ converges to $\langle \nu, \rho\rangle m_\phi$. If ν is supported by $\mathrm{supp}(\mu)$ and g vanishes on $\mathrm{supp}(\mu)$, by definition of $\mathcal{L}$, the function $\mathcal{L}^n(g)$ also vanishes on $\mathrm{supp}(\mu)$ and the last computation implies that $\langle m_\phi, g\rangle = 0$. Equivalently, the measure m_ϕ is supported by $\mathrm{supp}(\mu)$.

In order to show that $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$, we assume by contradiction that there is a continuous function $g \geq 0$ on $\mathbb{P}^k$ such that $g > 0$ on some open subset U of $\mathrm{supp}(\mu)$ and $\langle m_\phi, g\rangle = 0$. The $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ implies that $\langle m_\phi, \mathcal{L}^n g\rangle = \lambda^n\langle m_\phi, g\rangle = 0$ and the definition of $\mathcal{L}$ implies that $\mathcal{L}^n(g) > 0$ on $f^n(U)$. It follows that m_ϕ has no mass on $f^n(U)$ and hence on $\cup_{n\geq 0} f^n(U)$. On the other hand, we have for every $x \in \mathbb{P}^k$ that $d^{-kn}(f^n)^*(\delta_x)$ converges to μ, see, e.g., [DS10b, Cor. 1.4]. Therefore, $f^{-n}(\delta_x) \cap U \neq \varnothing$ for some n or equivalently $x \in \cup_{n\geq 0} f^n(U)$. So we have $\cup_{n\geq 0} f^n(U) = \mathbb{P}^k$. This contradicts the fact that m_ϕ has no mass on this union. So we have $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$ as desired.

For the last assertion of the proposition, we can replace g with $g - c_g\rho$ in order to assume that $c_g = 0$ for $g \in \mathcal{F}$. By Lemma 3.9, the family $\mathcal{F}_\mathbb{N}$ is uniformly bounded and equicontinuous. So the limit of the sequence of sets $\lambda^{-n}\mathcal{L}^n(\mathcal{F})$ is a compact, uniformly bounded and equicontinuous family of functions that we denote by $\mathcal{F}_\infty$. This family is invariant by $\lambda^{-1}\mathcal{L}$ and we also have $c_g = 0$ for $g \in \mathcal{F}_\infty$. We want to show that it contains only the function 0.

Define

$$M := \max\{l(a)/\rho(a) \ : \ l \in \mathcal{F}_\infty, a \in \mathbb{P}^k\}.$$

Choose a function $l \in \mathcal{F}_\infty$ and a point a such that $l(a)/\rho(a) = M$. There are an increasing sequence of integers (n_j) and a sequence $(g_j) \subset \mathcal{F}$ such that $\lambda^{-n_j}\mathcal{L}^{n_j}(g_j)$ converges uniformly to l. For every $n \geq 0$, choose a limit function l_{-n} of the sequence $\lambda^{-n_j+n}\mathcal{L}^{n_j-n}(g_j)$. We have $l = \lambda^{-n}\mathcal{L}^n(l_{-n})$ and $l_{-n} \in \mathcal{F}_\infty$.

As in the end of the proof of Theorem 3.1, we obtain that $l_{-n}/\rho = M$ on the set $f^{-n}(a)$ and if $l_{-\infty}$ is a limit of the sequence l_{-n} then $l_{-\infty}$ belongs to $\mathcal{F}_\infty$ and $l_{-\infty}/\rho = M$ on the small Julia set $\mathrm{supp}(\mu)$. Since m_ϕ is supported by the small Julia set and $\langle m_\phi, g\rangle = c_g = 0$ for $g \in \mathcal{F}_\infty$, we conclude that $M = 0$. Using the same argument for $-g$ with $g \in \mathcal{F}$, we obtain that the minimal value of the functions in $\mathcal{F}_\infty$ is also 0. So $\mathcal{F}_\infty$ contains only the function 0. This ends the proof of the proposition. □

Proposition 4.1 in particular gives the following equidistribution result for the (weighted) preimages of a given point.

Corollary 4.2. *Under the assumptions of Theorem 1.1, for every $x \in \mathbb{P}^k$ the points in $f^{-n}(x)$, with suitable weights, are equidistributed with respect to the conformal measure m_ϕ when n tends to infinity. More precisely, if δ_a denotes the Dirac mass at a, then*

$$\lim_{n\to\infty} \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))}\delta_a = \rho(x) m_\phi$$

for every $x \in \mathbb{P}^k$.

Proof. Denote by μ_n the measure in the LHS of the last identity. Let g be any continuous function on $\mathbb{P}^k$. We have

$$\langle \mu_n, g\rangle = \lambda^{-n} \sum_{f^n(a)=x} e^{\phi(a)+\cdots+\phi(f^{n-1}(a))} g(a) = \lambda^{-n}(\mathcal{L}^n g)(x).$$

The last expression converges to $c_g\rho(x) = \rho(x)\langle m_\phi, g\rangle$. The result follows. □

For our convenience, define the operator L by $L(g) := (\lambda\rho)^{-1}\mathcal{L}(\rho g)$. Define also the positive measure μ_ϕ by $\mu_\phi := \rho m_\phi$. We have the following lemma.

Lemma 4.3. *For any continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the sequence $L^n(g)$ converges uniformly to the constant $c_{\rho g} = \langle \mu_\phi, g\rangle = \langle m_\phi, \rho g\rangle$. We also have that μ_ϕ is an f-invariant probability measure such that* $\mathrm{supp}(\mu_\phi) = \mathrm{supp}(\mu)$.

Proof. Define $g' := \rho g$. We have $c_{g'} = \langle m_\phi, \rho g\rangle = \langle \mu_\phi, g\rangle$. The first assertion is a direct consequence of the fact that $\lambda^{-n}\mathcal{L}^n(g')$ converges uniformly to $c_{g'}\rho$.

For the second assertion, we have seen in the proof of Proposition 4.1 that $\langle m_\phi, \rho\rangle = 1$. It follows that μ_ϕ is a probability measure. Moreover, we obtain from the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ that

$$\langle \mu_\phi, g\circ f\rangle = \langle m_\phi, \rho(g\circ f)\rangle = \langle \lambda^{-1}\mathcal{L}^*(m_\phi), \rho(g\circ f)\rangle = \langle m_\phi, \lambda^{-1}\mathcal{L}(\rho(g\circ f))\rangle = \langle \mu_\phi, L(g\circ f)\rangle.$$

Using that $\lambda^{-1}\mathcal{L}(\rho) = \rho$ and the definition of $\mathcal{L}$, we can easily check that $L(g\circ f) = g$. So the previous identities imply that $\langle \mu_\phi, g\circ f\rangle = \langle \mu_\phi, g\rangle$. Hence, μ_ϕ is an invariant measure. The assertion on the support of μ_ϕ is clear because $\mathrm{supp}(m_\phi) = \mathrm{supp}(\mu)$ by Proposition 4.1 and ρ is strictly positive. □

The operator $\mathcal{L}$ can also be extended to a continuous operator on $L^2(\mu_\phi)$ and $L^2(m_\phi)$. Since $\mu_\phi = \rho m_\phi$ and ρ is positive and continuous, these two spaces are actually the same and the corresponding norms are equivalent.

Lemma 4.4. *Under the assumptions of Theorem 1.1, the operator $\mathcal{L}$ extends to a linear continuous operator on $L^2(m_\phi)$ whose norm is bounded by $\lambda e^{\frac{1}{2}\Omega(\phi)}$. Moreover, there exists a positive constant c such that $\|\lambda^{-n}\mathcal{L}^n\|_{L^2(m_\phi)} \leq c$ for all $n \geq 0$.*

Proof. By Cauchy-Schwarz's inequality and using the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ, we have

$$\big\langle m_\phi, |\mathcal{L}^n g|^2\big\rangle \leq \big\langle m_\phi, (\mathcal{L}^n \mathbb{1})\cdot(\mathcal{L}^n |g|^2)\big\rangle \leq \rho_n^+ \big\langle m_\phi, \mathcal{L}^n |g|^2\big\rangle = \rho_n^+\lambda^n\big\langle m_\phi, |g|^2\big\rangle$$

for every $g \in L^2(m_\phi)$ and $n \geq 0$. The second assertion of the lemma follows because $\rho_n^+ \lesssim \lambda^n$.

For the first assertion, take $n=1$. From the definition of ρ_n^+ and λ, we have $\rho_1^+ \leq d^k e^{\max\phi}$ and $\lambda \geq d^k e^{\min\phi}$. The above inequality implies that

$$\big\langle m_\phi, |\mathcal{L}g|^2\big\rangle \leq e^{\Omega(\phi)}\lambda^2 \big\langle m_\phi, |g|^2\big\rangle.$$

The first assertion in the lemma follows. □

Proposition 4.5. *Under the assumptions of Theorem 1.1, the measure $\mu_\phi = \rho m_\phi$ is K-mixing and mixing of all orders.*

Proof. We start with the second property. Let $\{g_0, \ldots, g_r\}$ be any finite family of continuous test functions on $\mathbb{P}^k$. We need to show that, for $0 = n_0 \leq n_1 \leq \cdots \leq n_r$,

$$\big\langle \mu_\phi, g_0(g_1\circ f^{n_1})\ldots(g_r\circ f^{n_r})\big\rangle - \prod_{j=0}^r \langle \mu_\phi, g_j\rangle \to 0$$

when $n := \inf_{0\leq j<r}(n_{j+1}-n_j)$ tends to infinity. This property is clearly true for $r=0$. Take $r\geq 1$. By induction, we can assume that the above convergence holds for the case of $r-1$ test functions. We prove now the same property for r test functions.

By the f_*-invariance of μ_ϕ and the induction hypothesis, we have

$$\big\langle \mu_\phi, (g_1\circ f^{n_1})\ldots(g_r\circ f^{n_r})\big\rangle = \big\langle \mu_\phi, g_1(g_2\circ f^{n_2-n_1})\ldots(g_r\circ f^{n_r-n_1})\big\rangle \to \prod_{j=1}^r \langle \mu_\phi, g_j\rangle.$$

So the desired property holds when g_0 is a constant function. Therefore, we can subtract from g_0 a constant and assume that $\langle\mu_\phi, g_0\rangle = 0$, which implies that the product $\prod_{j=0}^n \langle \mu_\phi, g_j\rangle$ vanishes. Using that $\lambda^{-1}\mathcal{L}(\rho) = \rho$, the $\lambda^{-1}\mathcal{L}^*$-invariance of m_ϕ, and the definition of $\mathcal{L}$, we can easily check by induction that for all functions g, l we have

$$\langle \mu_\phi, g\rangle = \langle \mu_\phi, L^n(g)\rangle \qquad \text{and} \qquad L^n(g(l\circ f^n)) = L^n(g)l.$$

We then deduce that

$$\begin{aligned}\big\langle \mu_\phi, g_0(g_1\circ f^{n_1})\ldots(g_r\circ f^{n_r})\big\rangle &= \big\langle \mu_\phi, L^{n_1}\big(g_0(g_1\circ f^{n_1})\ldots(g_r\circ f^{n_r})\big)\big\rangle \\ &= \big\langle \mu_\phi, L^{n_1}(g_0)g_1\ldots(g_r\circ f^{n_r-n_1})\big\rangle.\end{aligned}$$

By Lemma 4.3, the sequence $L^{n_1}(g_0)$ converges uniformly to 0 as n_1 tends to ∞. So the last integral tends to 0 because the function $g_1\ldots(g_r\circ f^{n_r-n_1})$ is bounded. We then conclude that μ_ϕ is mixing of all orders.

We prove now that μ_ϕ is K-mixing, that is, that given $g\in L^2(\mu_\phi)$, when n tends to infinity the difference

$$\big\langle \mu_\phi, g(l\circ f^n)\big\rangle - \langle\mu_\phi, g\rangle\langle \mu_\phi, l\rangle$$

tends to 0 uniformly on test functions l whose $L^2(\mu_\phi)$-norm is bounded by a constant. As above, we can assume that $\langle \mu_\phi, g\rangle = 0$. We can also assume that the $L^2(\mu_\phi)$-norms of g and l are bounded by 1. Fix an arbitrary constant $\epsilon > 0$. It is enough to show the existence of an integer $N = N(\epsilon)$ independent of l such that $|\langle \mu_\phi, g(l\circ f^n)\rangle| \leq 2\epsilon$ for $n\geq N$.

Choose a continuous function g' such that $\langle \mu_\phi, g'\rangle = 0$ and $\|g-g'\|_{L^2(\mu_\phi)} \leq \epsilon$. Using the invariance of μ_ϕ we have

$$\begin{aligned}|\langle \mu_\phi, g(l\circ f^n)\rangle - \langle \mu_\phi, g'(l\circ f^n)\rangle| &= |\langle \mu_\phi, (g-g')(l\circ f^n)\rangle| \leq \|g-g'\|_{L^2(\mu_\phi)}\|l\circ f^n\|_{L^2(\mu_\phi)} \\ &= \|g-g'\|_{L^2(\mu_\phi)}\|l\|_{L^2(\mu_\phi)} \leq \epsilon.\end{aligned}$$

It remains to show that $|\langle \mu_\phi, g'(l\circ f^n)\rangle| \leq \epsilon$ when $n\geq N$ for some N large enough. As above, we have

$$|\langle \mu_\phi, g'(l\circ f^n)\rangle| = |\langle \mu_\phi, L^n(g')l\rangle| \leq \|L^n(g')\|_\infty.$$

Lemma 4.3 and the identity $\langle \mu_\phi, g'\rangle = 0$ imply the result. □

For positive real numbers q, M, and Ω with $q > 2$ and $\Omega < \log d$, consider the following set of weights

$$\mathcal{P}(q, M, \Omega) := \big\{\phi\colon \mathbb{P}^k \to \mathbb{R} \ : \ \|\phi\|_{\log^q} \le M, \ \Omega(\phi) \le \Omega\big\}$$

and the uniform topology induced by the sup norm. Observe that this family is equicontinuous. In the two lemmas below, we study the dependence on $\phi \in \mathcal{P}(q, M, \Omega)$ of the objects introduced in this section. Therefore, we will use the index ϕ or parameter ϕ for objects which depend on ϕ, e.g., we will write $\lambda_\phi, \mathcal{L}_\phi, \rho_\phi, \mathbb{1}_n(\phi)$ instead of $\lambda, \mathcal{L}, \rho$ and $\mathbb{1}_n$.

Lemma 4.6. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. The maps $\phi \mapsto \lambda_\phi$, $\phi \mapsto m_\phi$, $\phi \mapsto \mu_\phi$, and $\phi \mapsto \rho_\phi$ are continuous on $\phi \in \mathcal{P}(q, M, \Omega)$ with respect to the standard topology on $\mathbb{R}$, the weak topology on measures, and the uniform topology on functions. In particular, ρ_ϕ is bounded from above and below by positive constants which are independent of $\phi \in \mathcal{P}(q, M, \Omega)$. Moreover, $\|\lambda_\phi^{-n}\mathcal{L}_\phi^n\|_\infty$ is bounded by a constant which is independent of n and of $\phi \in \mathcal{P}(q, M, \Omega)$.*

Proof. Fix q, M, and Ω as above. Observe that when we add to ϕ a constant c the scaling ratio λ_ϕ and the operator $\mathcal{L}_\phi$ are both changed by a factor e^c. It follows that the operator $\lambda_\phi^{-1}\mathcal{L}_\phi$, the measures m_ϕ, μ_ϕ, and the density function ρ_ϕ do not change. So, for simplicity, it is enough to prove the lemma for ϕ in the family

$$\mathcal{P}_0(q, M, \Omega) := \big\{\phi\colon \mathbb{P}^k \to \mathbb{R} \ : \ \min \phi = 0, \ \|\phi\|_{\log^q} \le M, \ \Omega(\phi) \le \Omega\big\}.$$

Notice that this family is compact for the uniform topology.

Consider two weights ϕ and ϕ' in this space. From the definition of λ_ϕ and $\lambda_{\phi'}$, we have $e^{-\|\phi-\phi'\|_\infty} \le \lambda_\phi/\lambda_{\phi'} \le e^{\|\phi-\phi'\|_\infty}$. It follows that $\phi \mapsto \lambda_\phi$ is continuous. When $\phi' \to \phi$, any limit value of $m_{\phi'}$ is a probability measure invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi^*$ thanks to the invariance of $m_{\phi'}$ by $\lambda_{\phi'}^{-1}\mathcal{L}_{\phi'}^*$. Since m_ϕ is the only probability measure which is invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi^*$, this limit value must be m_ϕ. Thus, $\phi \mapsto m_{\phi'}$ is continuous.

We deduce from the proof of Proposition 3.2 that $\theta_n(\phi) = \rho_n^+(\phi)/\rho_n^-(\phi)$ is bounded by a constant independent of n and ϕ. Moreover, the family of functions

$$\big\{\mathbb{1}_n^*(\phi) \quad \text{with} \quad n \ge 0 \quad \text{and} \quad \phi \in \mathcal{P}_0(q, M, \Omega)\big\}$$

is uniformly bounded and equicontinuous. Recall that $\mathbb{1}_n^*(\phi) = (\rho_n^-(\phi))^{-1}\mathbb{1}_n(\phi)$ and $\rho_n^-(\phi) \le \lambda_\phi^n \le \rho_n^+(\phi)$, see the proof of Lemma 3.8. It follows that $\lambda_\phi^{-n}\mathbb{1}_n(\phi)$ belongs to a uniformly bounded and equicontinuous family of functions.

From the definition of ρ_ϕ and $\rho_{\phi'}$, we also see that these functions belong to a uniformly bounded and equicontinuous family of functions. When $\phi' \to \phi$, if ρ' is any limit of $\rho_{\phi'}$, then ρ' is continuous and invariant by $\lambda_\phi^{-1}\mathcal{L}_\phi$ because $\rho_{\phi'}$ satisfies a similar property. It follows from Theorem 3.1 that $\rho' = c\rho_\phi$ for some constant c. On the other hand, since $\mu_{\phi'} = \rho_{\phi'}m_{\phi'}$ is a probability measure, any limit of $\rho_{\phi'}m_{\phi'}$ is a probability measure. Thus, $\rho' m_\phi = c\mu_\phi$ is a probability measure and hence $c = 1$. We conclude that $\rho_{\phi'} \to \rho_\phi$ and also $\mu_{\phi'} \to \mu_\phi$. In other words, the maps $\phi \mapsto \mu_\phi$ and $\phi \mapsto \rho_\phi$ are continuous. Since ρ_ϕ is strictly positive and the family $\mathcal{P}_0(q, M, \Omega)$ is compact, we deduce that ρ_ϕ is bounded from above and below by positive constants independent of ϕ.

The last assertion in the lemma is also clear because $\|\lambda_\phi^{-n}\mathcal{L}_\phi^n\|_\infty = \lambda_\phi^{-n}\|\mathbb{1}_n(\phi)\|_\infty \le \theta_n(\phi)$. This ends the proof of the lemma. □

Lemma 4.7. *Let q, M, and Ω be positive real numbers such that $q > 2$ and $\Omega < \log d$. Let $\mathcal{F}$ be a uniformly bounded and equicontinuous family of real-valued functions on $\mathbb{P}^k$. Then the family*

$$\big\{\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) \ : \ n \ge 0, \ \phi \in \mathcal{P}(p, M, \Omega), \ g \in \mathcal{F}\big\}$$

is equicontinuous. Moreover, $\big\|\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) - \langle m_\phi, g\rangle\big\|_\infty$ tends to 0 uniformly on $\phi \in \mathcal{P}(p, M, \Omega)$ and $g \in \mathcal{F}$ when n goes to infinity.

Proof. As in Lemma 4.6, we can assume that $\phi \in \mathcal{P}_0(p, M, \Omega)$. The first assertion is clear from the proof of Lemma 3.9. We prove now the second assertion. By Lemma 4.6, m_ϕ belongs to a compact family of probability measures. It follows that $|\langle m_\phi, g\rangle|$ is bounded by a constant independent of ϕ and g. It follows that the family

$$\mathcal{F}'_{\mathbb{N}} := \left\{\lambda_\phi^{-n}\mathcal{L}_\phi^n(g) - \langle m_\phi, g\rangle \ : \ n \geq 0,\ \phi \in \mathcal{P}_0(p, M, \Omega),\ g \in \mathcal{F}\right\}$$

is uniformly bounded and equicontinuous. Denote by $\mathcal{F}'_\infty$ the set of all functions l' obtained as the limit of a sequence

$$h_j := \lambda_{\phi_j}^{-n_j}\mathcal{L}_{\phi_j}^{n_j}(g_j) - \langle m_{\phi_j}, g_j\rangle$$

in $\mathcal{F}'_{\mathbb{N}}$ with $n_j \to \infty$. By taking a subsequence, we can assume that ϕ_j converges uniformly to some function $\phi \in \mathcal{P}_0(p, M, \Omega)$. Since $\langle m_{\phi_j}, h_j\rangle = 0$, we also obtain that $\langle m_\phi, l'\rangle = 0$ by the continuity of $\phi \mapsto m_\phi$. Now, as in the end of the proof of Proposition 4.1, we obtain that $l' = \lambda_\phi^{-n}\mathcal{L}_\phi^n(l'_{-n})$ for some $l'_{-n} \in \mathcal{F}'_\infty$ and then deduce that $l' = 0$. The lemma follows. □

4.2. **Pressure and uniqueness of the equilibrium state.** Using the results in the previous section, to prove the next proposition we only need to follow the arguments in [UZ13, Sections 6 and 7] and [PU10, Section 5.6].

Proposition 4.8. *The probability measure μ_ϕ is a unique equilibrium state associate to ϕ. Moreover, the pressure $P(\phi)$ is equal to* $\log \lambda$.

Proof. We follow the approach in [PU10, Th. 5.6.5]. To simplify the notation, set $S_n(g) := \sum_{j=0}^{n-1} g \circ f^j$ for any function $g\colon \mathbb{P}^k \to \mathbb{R}$. Recall that, given $\phi'\colon \mathbb{P}^k \to \mathbb{R}$ with $\|\phi'\|_{\log^q} < \infty$ and $\Omega(\phi') < \log d$, we denote by $\lambda_{\phi'}, \rho_{\phi'}$ the objects associated to $\mathcal{L}_{\phi'}$.

Claim 1. We have $\mathrm{Ent}_f(\mu_{\phi'}) + \langle \mu_{\phi'}, \phi'\rangle = P(\phi') = \log \lambda_{\phi'}$ for all $\phi'\colon \mathbb{P}^k \to \mathbb{R}$ such that $\|\phi'\|_{\log^q} < \infty$ and $\Omega(\phi') < \log d$.

Proof of Claim 1. The proof of the inequality $P(\phi') \leq \log \lambda_{\phi'}$ is an adaptation of Gromov's proof of the fact that the topological entropy of f is bounded above by $k \log d$, see [Gro03]. We refer to [UZ13, Th. 6.1] for the complete details. To complete the proof, it is enough to show that $\mathrm{Ent}_f(\mu_{\phi'}) + \langle \mu_{\phi'}, \phi'\rangle \geq \log \lambda_{\phi'}$.

It follows from [Par69] that $\mathrm{Ent}_f(\mu_{\phi'}) \geq \langle \mu_{\phi'}, \log J_{\mu_{\phi'}}\rangle$, where $J_{\mu_{\phi'}}$ is defined as the Radon-Nikodym derivative of $f^*\mu_{\phi'}$ with respect to $\mu_{\phi'}$ (when this derivative exists). In our setting, it follows from a straightforward computation that $J_{\mu_{\phi'}}$ is well defined and given by

$$J_{\mu_{\phi'}} = \lambda_{\phi'}\rho_{\phi'}^{-1}e^{-\phi'}(\rho_{\phi'} \circ f).$$

Indeed, denoting by J' the RHS in the above expression, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have

$$\begin{aligned}\langle \mu_{\phi'}, J'g\rangle &= \langle \lambda m_{\phi'}, e^{-\phi'}(\rho_{\phi'} \circ f)g\rangle = \langle \mathcal{L}^*_{\phi'} m_{\phi'}, e^{-\phi'}(\rho_{\phi'} \circ f)g\rangle = \langle m_{\phi'}, \mathcal{L}_{\phi'}(e^{-\phi'}(\rho_{\phi'} \circ f)g)\rangle \\ &= \langle m_{\phi'}, \rho_{\phi'}\mathcal{L}_{\phi'}(e^{-\phi'}g)\rangle = \langle \mu_{\phi'}, f_*g\rangle = \langle f^*\mu_{\phi'}, g\rangle,\end{aligned}$$

which proves that $J' = J_{\mu_{\phi'}}$. We then have, using the f_*-invariance of $\mu_{\phi'}$,

$$\mathrm{Ent}_f(\mu_{\phi'}) + \langle \mu_{\phi'}, \phi'\rangle \geq \langle \mu_{\phi'}, \log J_{\mu_{\phi'}}\rangle + \langle \mu_{\phi'}, \phi'\rangle = \langle \mu_{\phi'}, \log(\rho_{\phi'} \circ f) - \log \rho_{\phi'}\rangle + \log \lambda_{\phi'} = \log \lambda_{\phi'}$$

and the proof is complete. □

Claim 2. Let M and Ω be positive real numbers such that $\Omega < \log d$, and $g\colon \mathbb{P}^k \to \mathbb{R}$ a continuous function. Then, for every $y \in \mathbb{P}^k$, we have

$$\frac{1}{n}\frac{\sum_{f^n(x)=y} S_n(g)(x)e^{S_n(\phi')(x)}}{\mathcal{L}^n_{\phi'}\mathbb{1}(y)} \to \langle \mu_{\phi'}, g\rangle \tag{4.1}$$

where the convergence is uniform on $\phi' \in \mathcal{P}(q, M, \Omega)$.

Proof of Claim 2. Observe that the LHS of (4.1) is equal to

$$\frac{1}{n}\frac{\lambda_{\phi'}^{-n}\sum_{f^n(x)=y}S_n(g)(x)e^{S_n(\phi')(x)}}{\lambda_{\phi'}^{-n}\mathcal{L}_{\phi'}^n\mathbb{1}(y)}.$$

The denominator of the last quotient converges to $\rho_{\phi'}(y)$ and the numerator satisfies

$$\lambda_{\phi'}^{-n}\sum_{f^n(x)=y}S_n(g)(x)e^{S_n(\phi')(x)}=\lambda_{\phi'}^{-n}\sum_{j=0}^{n-1}\mathcal{L}_{\phi'}^n(g\circ f^j)(y)=\lambda_{\phi'}^{-j}\sum_{j=0}^{n-1}\lambda_{\phi'}^{j-n}\mathcal{L}_{\phi'}^{n-j}(g\cdot\mathcal{L}_{\phi'}^j\mathbb{1})(y). \tag{4.2}$$

It follows from Lemma 4.7 that

$$\lambda_{\phi'}^{j-n}\mathcal{L}_{\phi'}^{n-j}(g\cdot\mathcal{L}_{\phi'}^j\mathbb{1})\to\langle m_{\phi'},g\cdot\mathcal{L}_{\phi'}^j\mathbb{1}\rangle\rho_{\phi'} \tag{4.3}$$

as $n-j\to\infty$, where the convergence is uniform on $\phi'\in\mathcal{P}(q',M,\Omega)$. We deduce from (4.2), (4.3), and the fact that $\lambda_{\phi'}^{-j}\mathcal{L}_{\phi'}^j\mathbb{1}\to\rho_{\phi'}$ as $j\to\infty$ that, as $n\to\infty$, the LHS in (4.1) tends to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\lambda_{\phi'}^{-j}\langle m_{\phi'},g\cdot\mathcal{L}_{\phi'}^j\mathbb{1}\rangle=\langle m_{\phi'},g\cdot\rho_{\phi'}\rangle=\langle\mu_{\phi'},g\rangle.$$

The proof is complete. □

Claim 3. For every $\psi\colon\mathbb{P}^k\to\mathbb{R}$ such that $\|\psi\|_{\log^q}<\infty$ the function $t\mapsto P(\phi+t\psi)$ is differentiable in a neighbourhood of 0.

Proof of Claim 3. Fix $y\in\mathbb{P}^k$ and set

$$P_n(t):=\frac{1}{n}\log\mathcal{L}_{\phi+t\psi}^n\mathbb{1}(y)\quad\text{and}\quad Q_n(t):=\frac{d}{dt}P_n(t)=\frac{1}{n}\frac{\sum_{f^n(x)=y}S_n(\psi)(x)e^{S_n(\phi+t\psi)(x)}}{\mathcal{L}_{\phi+t\psi}^n\mathbb{1}(y)}.$$

Notice that $\Omega(\phi+t\psi)<\log d$ for t sufficiently small. A direct computation and Claim 2 (applied with $\phi+t\psi,\psi$ instead of ϕ',g) imply that $Q_n(t)\to\langle\mu_{\phi+t\psi},\psi\rangle$ as $n\to\infty$, locally uniformly with respect to t. We also have $P_n(t)\to\log\lambda_{\phi+t\psi}=P(\phi+t\psi)$, where the convergence follows from Lemma 3.8 and the equality from Claim 1 applied with ϕ' instead of $\phi+t\psi$. We deduce that the pressure function P, in a neighbourhood of $t=0$, is the uniform limit of the $\mathcal{C}^1$ functions $P_n(t)$, whose derivatives $Q_n(t)$ are also uniformly convergent. Thus, the function P is differentiable in a neighbourhood of $t=0$, with derivative at t equal to $\langle\mu_{\phi+t\psi},\psi\rangle$. □

It follows from Claim 1 that μ_ϕ is an equilibrium state. By [PU10, Cor. 3.6.7], the fact that the pressure function $t\mapsto P(\phi+t\psi)$ is differentiable at $t=0$ with respect to a dense set of continuous functions ψ implies the uniqueness of the equilibrium state for the weight ϕ. Since this property holds by Claim 3 for all ψ such that $\|\psi\|_{\log^q}<\infty$, the proof is complete. □

In the second part of this work, we will prove that, when ϕ and ψ are Hölder continuous, the pressure function $P(t)$ defined above is actually analytic, see [BD20, Theorem 1.3].

We conclude this section with the following properties of the equilibrium state μ_ϕ that we will use in the next section.

Proposition 4.9. *Under the assumptions of Theorem 1.1, the metric entropy* $\mathrm{Ent}_f(\mu_\phi)$ *of* μ_ϕ *is strictly larger than* $(k-1)\log d$*. In particular,* μ_ϕ *has no mass on proper analytic subsets of* $\mathbb{P}^k$*, its Lyapunov exponents are strictly positive and at least equal to* $\frac{1}{2}(\mathrm{Ent}_f(\mu_\phi)-(k-1)\log d)$*, and the function* $\log|\mathrm{Jac}\,Df|$ *is integrable with respect to* μ_ϕ*. Moreover, the Hausdorff dimension of* μ_ϕ *satisfies*

$$\dim_H(\mu_\phi)\geq\frac{(k-1)\log d}{\lambda_1}+\frac{\mathrm{Ent}_f(\mu_\phi)-(k-1)\log d}{\lambda_k}.$$

Proof. Since μ_ϕ maximizes the pressure and $\mathrm{Ent}_f(\mu) = k\log d$, we have

$$\mathrm{Ent}_f(\mu_\phi) + \langle \mu_\phi, \phi\rangle \geq \mathrm{Ent}_f(\mu) + \langle \mu, \phi\rangle \geq k\log d + \min\phi.$$

Since by assumption we have $\Omega(\phi) < \log d$, it follows that

$$\mathrm{Ent}_f(\mu_\phi) \geq k\log d + \min\phi - \langle \mu_\phi, \phi\rangle \geq k\log d - \Omega(\phi) > (k-1)\log d.$$

The Lyapunov exponents of every ergodic invariant probability measure satisfying this property are bounded below as in the statement, and in particular the function $\log|\mathrm{Jac}|$ is integrable with respect to it, see de Thélin [De 08] and Dupont [Dup12]. The bound on the Hausdorff dimension of μ_ϕ is then a consequence of [Dup11], see also [DV15].

Let now X be a proper analytic subset of $\mathbb{P}^k$. Assume by contradiction that $m := \mu_\phi(X) > 0$. We choose such an X which is irreducible and of minimal dimension p. So, for all $n \geq 0$, $f^n(X)$ is also an irreducible analytic set of dimension p. We have

$$\mu_\phi(f^n(X)) = \mu_\phi(f^{-n}(f^n(X))) \geq \mu_\phi(X) = m.$$

It follows that $\mu_\phi(f^n(X) \cap f^{n'}(X)) > 0$ for some $n' > n \geq 0$. The minimality of the dimension p implies that $f^n(X) = f^{n'}(X)$.

Replacing X, f, and ϕ by $f^n(X), f^{n'-n}$, and $\phi + \cdots + \phi\circ f^{n'-n-1}$ we can assume that X is invariant and $\mu_\phi(X) > 0$. Since μ_ϕ is mixing, it is ergodic. We then deduce that $\mu_\phi(X) = 1$. Therefore, the metric entropy of μ_ϕ is smaller than the topological entropy of f on X. But this is a contradiction because the last one is at most equal to $p\log d$, see [DS10a, Th. 1.108 and Ex. 1.122]. The result follows. □

4.3. **Equidistribution of periodic points and end of the proof of Theorem 1.1.** Because of Proposition 4.1, Corollary 4.2, Lemma 4.3, and Propositions 4.5 and 4.8, to prove Theorem 1.1 it only remains to establish the equidistribution of (weighted) repelling periodic points of period n with respect to μ_ϕ, as $n \to \infty$.

Theorem 4.10. *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$ and satisfying Assumption* **(A)**. *Let $\phi\colon \mathbb{P}^k \to \mathbb{R}$ satisfy $\|\phi\|_{\log^q} < \infty$ for some $q > 2$ and $\Omega(\phi) < \log d$. Let μ_ϕ be the unique equilibrium state associated to ϕ, and λ the scaling ratio. Then for every $n \in \mathbb{N}$ there exists a set P'_n of repelling periodic points of period n in the small Julia set such that*

$$\lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))}\delta_y = \mu_\phi. \tag{4.4}$$

Note that a related equidistribution property for Hölder continuous weights was proved by Comman-River-Letelier [CR11] for (hyperbolic and) topologically Collect-Eckmann rational maps on $\mathbb{P}^1$.

To prove Theorem 4.10, we follow a now classical strategy due to Briend-Duval [BD99] for the measure of maximal entropy (which corresponds to the case $\phi \equiv 0$). We employ a trick due to X. Buff which simplifies the original proof. An extra difficulty with respect to the case $\phi \equiv 0$ is due to the fact that there is no a priori upper bound for the mass of the left hand side of (4.4) when P'_n is replaced by the set of all repelling periodic points of period n.

Given any point $x \in \mathbb{P}^k$ we denote by $\mu_{x,n}$ the measure

$$\mu_{x,n} := \lambda^{-n}\rho(x)^{-1} \sum_{f^n(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))}\rho(a)\delta_a.$$

It follows from Corollary 4.2 that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, we have

$$\langle \mu_{x,n}, g\rangle = \lambda^{-n}\rho(x)^{-1} \sum_{f^n(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))}\rho(a)g(a) \to \rho(x)^{-1}\langle \rho(x)m_\phi, \rho g\rangle = \langle \mu_\phi, g\rangle$$

as $n \to \infty$. This means that, for all $x \in \mathbb{P}^k$, we have $\mu_{x,n} \to \mu_\phi$ as $n \to \infty$.

We denote by $0 < L_1 \leq \cdots \leq L_k$ the Lyapunov exponents of μ_ϕ, see Proposition 4.9. We fix in what follows a constant $0 < L_0 < L_1$. Given $x \in X$, a ball B of center x, and $n \in \mathbb{N}$, we say that $g\colon B \to B'$ is an *m-good inverse branch of f of order n on B* if

$$g \circ f^n = \mathrm{id}_{|B'} \quad \text{and} \quad \operatorname{diam} f^l(B') \leq e^{-m-(n-l)L_0} \text{ for all } 0 \leq l \leq n.$$

Notice that the definition in particular implies that $\operatorname{diam}(B) \leq e^{-m}$. We denote by $\mu^{(m)}_{B,n}$ the measure

$$\mu^{(m)}_{B,n} := \lambda^{-n} \rho(x)^{-1} \sum_{a=g(x)} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{n-1}(a))} \rho(a) \delta_a,$$

where the sum is taken on the m-good inverse branches g of f of order n on B. Since we have $\mu^{(m)}_{B,n} \leq \mu_{x,n}$ for all $n \geq 0$, it follows that any limit value μ'_B of the sequence $\{\mu^{(m)}_{B,n}\}$ satisfies $\mu'_B \leq \mu_\phi$. In particular, we have $\|\mu'_B\| \leq 1$.

Given $m > 1$ we say that a ball B centred at x is *m-nice* if

(i) $\inf_B \rho > (1 - 1/m) \sup_B \rho$;

(ii) $\|\mu^{(m)}_{B,n}\| \geq 1 - 1/m$ for every n sufficiently large.

Observe that the second condition implies that $\operatorname{diam}(B) \leq e^{-m}$ for every m-nice ball B. Moreover, we have $\|\mu'_B\| \geq 1 - 1/m$ for every limit value μ'_B of the sequence $\mu^{(m)}_{B,n}$.

Lemma 4.11. *For μ_ϕ-almost every $x \in \mathbb{P}^k$, every sufficiently small ball centred at x is m-nice.*

The proof of Lemma 4.11 is elementary but makes uses of the natural extension of the system $(\mathbb{P}^k, f, \mu_\phi)$, see for instance [CFS12, Sec. 10.4]. We denote by X_0, C_f, PC_f the small Julia set, the critical set and the postcritical set $PC_f := \cup_{n \geq 0} f^n(C_f)$ of f, respectively. We also set $X := X_0 \setminus \cup_{m \in \mathbb{N}} f^{-m}(PC_f)$. By Proposition 4.9 we have $\mu_\phi(f^{-m}(PC_f)) = 0$ for every $m \in \mathbb{N}$, hence $\mu(X) = 1$. We denote by $\hat{X}$ the set

$$\hat{X} := \{\hat{x} := (x_n)_{n \in \mathbb{Z}} \colon x_n \in X, f(x_n) = x_{n+1}\},$$

by $\pi_n : \hat{x} \mapsto x_n$ the natural projection from $\hat{X}$ to X and by $\hat{f}\colon \hat{X} \to \hat{X}$ the map

$$\hat{f}(\ldots, x_{-1}, x_0, x_1, \ldots) := (\ldots, f(x_{-1}), f(x_0), f(x_1), \ldots) = (\ldots, x_0, x_1, x_2, \ldots).$$

Observe that $\pi_n \circ \hat{f} = f \circ \pi_n$ for all $n \in \mathbb{Z}$. Let us consider on $\hat{X}$ the σ-algebra $\hat{\mathcal{B}}$ generated by all *cylinders*, i.e., the sets of the form

$$A_{n,B} := \pi_n^{-1}(B) = \{\hat{x} \colon x_n \in B\} \text{ for } n \leq 0 \text{ and } B \subseteq \mathbb{P}^k \text{ a Borel set}$$

and set

$$\hat{\mu}_\phi(A_{n,B}) := \mu_\phi(B) \text{ for all } A_{n,B} \text{ as above.}$$

It follows from the invariance of μ_ϕ and the fact that $x_n \in B$ if and only if $x_{n-m} \in f^{-m}(B)$ (with $m \geq 0$) that $\hat{\mu}_\phi$ is well defined on the collection of the sets $A_{n,B}$ and

$$\hat{\mu}_\phi(A_{n,B}) = \hat{\mu}_\phi(A_{n-m,B}) \text{ for all } m \geq 0.$$

Similarly, for every $m > 0$ and Borel sets $B_0, B_{-1}, \ldots, B_{-m} \subseteq \mathbb{P}^k$ we then have

$$\begin{aligned}
\hat{\mu}_\phi(\{\hat{x} \colon x_0 \in B_0, x_{-1} \in B_{-1}, \ldots, x_{-m} \in B_{-m}\}) \\
&= \hat{\mu}_\phi(\{\hat{x} \colon x_{-m} \in f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}\}) \\
&= \mu_\phi(f^{-m}(B_0) \cap f^{-(m-1)}(B_{-1}) \cap \cdots \cap B_{-m}).
\end{aligned}$$

We then extend $\hat{\mu}_\phi$ to a probability measure, still denoted by $\hat{\mu}_\phi$, on $\hat{\mathcal{B}}$. Observe that $\hat{\mu}_\phi$ is $\hat{f}$-invariant by construction and satisfies $(\pi_0)_* \hat{\mu}_\phi = \mu_\phi$.

For $n > 0$ we denote by $f_{\hat{x}}^{-n}$ the inverse branch of f^n defined in a neighbourhood of x_0 and such that $f_{\hat{x}}^{-n}(x_0) = x_{-n}$. This branch exists for all $x_0 \in X$. We have the following lemma.

Lemma 4.12. *For every $0 < L < L_1$ there exist two measurable functions $\eta_L\colon \hat{X} \to (0,1]$ and $S_L\colon \hat{X} \to (1,+\infty)$ such that, for $\hat{\mu}_\phi$-almost every $\hat{x} \in \hat{X}$, the map $f_{\hat{x}}^{-n}$ is defined on $\mathbb{B}_{\mathbb{P}^k}(x_0, \eta_L(\hat{x}))$ with $\mathrm{Lip}(f_{\hat{x}}^{-n}) \le S_L(\hat{x})e^{-nL}$ for every $n \in \mathbb{N}$.*

Sketch of proof. The statement is a consequence of Proposition 4.9. A direct proof in the case $\phi = 0$ is given in [BD99, Sec. 2] and [BDM08, Thm. 1.4(3)]. The case $n = 1$ comes from a (quantitative) application of the inverse mapping theorem, which is then iterated to get functions η_L and S_L valid for all n. The main point in the proof is an application of the Birkhoff ergodic theorem to the function $\log|\mathrm{Jac}\, Df|$. This function is integrable with respect to the measure of maximal entropy μ_0, which has continuous potentials, because of the Chern-Levine-Nirenberg inequality [CLN69]. Since this function is integrable with respect to μ_ϕ by Proposition 4.9, the same proof applies in our setting. □

Proof of Lemma 4.11. Since ρ is continuous and strictly positive, we only need to check that, for μ_ϕ-almost every $x \in \mathbb{P}^k$, every sufficiently small ball B centred at x satisfies $\|\mu_{B,n}^{(m)}\| \ge 1 - 1/m$ for every n sufficiently large.

Let us consider the disintegration of the measure $\hat{\mu}_\phi$ with respect to μ_ϕ and the projection π_0. We denote by $\hat{\mu}_\phi^x$ the conditional measure on $\{x_0 = x\}$. The measure $\hat{\mu}_\phi^x$ is uniquely defined for μ_ϕ-almost all $x \in X$ and characterized by the identity

$$\langle \hat{\mu}_\phi, g\rangle = \langle \mu_\phi, u(x)\rangle\,, \text{ where } u(x) := \langle \hat{\mu}_\phi^x, g\rangle$$

for all bounded measurable functions $g\colon \hat{X} \to \mathbb{R}$. Since $(\pi_0)_*\hat{\mu}_\phi = \mu_\phi$, $\hat{\mu}_\phi^x$ is a probability measure for μ_ϕ-almost every x.

We will need a more explicit description of the conditional measures $\hat{\mu}_\phi^x$. For $n > 0$ and $x \in X$ we consider the measure $\hat{\mu}_n^x$ on $\hat{X}$ defined as follows. First, let us consider the projection $\hat{X} \to X^{n+1}$ given by

$$\hat{\pi}^n := (\pi_{-n}, \dots, \pi_{-1}, \pi_0).$$

For every element $(y_{-n}, \dots, y_0) \in X^{n+1}$ we choose a representative $\hat{z} \in \hat{X}$ such that $z_j = y_j$ for all $-n \le j \le 0$. For any given y_0 and any $n > 0$ we then have d^{kn} distinct such representatives, and we denote by $\hat{Z}_n$ their collection. We then set

$$\hat{\mu}_n^x := \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z}\in\hat{Z}_n\colon z_0 = x} e^{\phi(z_{-n})+\phi(z_{-n+1})+\cdots+\phi(z_{-1})}\rho(z_{-n})\delta_{\hat{z}}.$$

Since this is a finite sum, the measures $\hat{\mu}_n^x$ are well defined on $\hat{X}$.

Claim. We have $\lim_{n\to\infty} \hat{\mu}_n^x = \hat{\mu}_\phi^x$ for μ_ϕ-almost every $x \in X$.

Proof. It is enough to check the assertion on the cylinders $A_{-i,B}$ for $i \ge 0$ and $B \subseteq \mathbb{P}^k$ a Borel set. It is clear that, for all $n > 0$, we have $\hat{\mu}_n^x(A_{0,B}) = \delta_x(B)$, which implies that

$$\int \hat{\mu}_n^x(A_{0,B})\mu_\phi(x) = \int \delta_x(B)\mu_\phi(x) = \mu_\phi(B).$$

Moreover, for all $n > i$, using the invariance of ρ by $\lambda^{-1}\mathcal{L}$ we have

$$\begin{aligned}
\hat{\mu}_n^x(A_{-i,B}) &= \hat{\mu}_n^x(A_{-i,B} \cap \pi_0^{-1}(x)) \\
&= \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z}\in\hat{Z}_n\colon z_0=x} e^{\phi(z_{-n})+\phi(z_{-n+1})+\cdots+\phi(z_{-1})}\rho(z_{-n})\delta_{\hat{z}}(A_{-i,B}) \\
&= \lambda^{-n}\rho(x)^{-1} \sum_{\hat{z}\in\hat{Z}_i\colon z_0=x} (\mathcal{L}^{n-i}\rho)(z_{-i})e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\delta_{\hat{z}}(A_{-i,B}) \\
&= \lambda^{-i}\rho(x)^{-1} \sum_{\hat{z}\in\hat{Z}_i\colon z_0=x} \rho(z_{-i})e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\delta_{\hat{z}}(A_{-i,B}) \\
&= \hat{\mu}_i^x(A_{-i,B}).
\end{aligned}$$

In order to conclude it is enough to prove that

$$\int \hat{\mu}_i^x(A_{-i,B})\mu_\phi(x) = \mu_\phi(B) \text{ for all } i > 0.$$

We have

$$\begin{aligned}
\int \hat{\mu}_i^x(A_{-i,B})\mu_\phi(x) &= \int \Big(\lambda^{-i}\rho(x)^{-1} \sum_{\hat{z}\in\hat{Z}_i :\, z_0=x} e^{\phi(z_{-i})+\phi(z_{-i+1})+\cdots+\phi(z_{-1})}\rho(z_{-i})\delta_{\hat{z}}(A_{-i,B})\Big)\mu_\phi(x) \\
&= \int \Big(\lambda^{-i}\rho(x)^{-1} \sum_{f^i(a)=x} e^{\phi(a)+\phi(f(a))+\cdots+\phi(f^{i-1}(a))}\rho(a)\mathbb{1}_B(a)\Big)\mu_\phi(x) \\
&= \Big\langle \mu_\phi, \lambda^{-i}\rho^{-1}f_*^i(e^{\phi+\phi\circ f+\cdots+\phi\circ f^{i-1}}\rho\mathbb{1}_B)\Big\rangle \\
&= \Big\langle \frac{\rho e^{\phi+\phi\circ f+\cdots+\phi\circ f^{i-1}}}{\lambda^i(\rho\circ f^i)}(f^i)^*\mu_\phi, \mathbb{1}_B\Big\rangle = \mu_\phi(B),
\end{aligned}$$

where in the last step we used the fact that the Jacobian of μ_ϕ (i.e., the Radon-Nidokym derivative $\frac{f^*\mu_\phi}{\mu_\phi}$) is given by $\lambda\rho^{-1}e^{-\phi}(\rho\circ f)$, which implies that

$$(f^i)^*\mu_\phi = \lambda^i\rho^{-1}e^{-\sum_{j=0}^{i-1}\phi\circ f^j}(\rho\circ f^i)\mu_\phi.$$

This completes the proof of the Claim. □

Let us now fix an integer $m > 0$, a constant $L_0 < L < L_1$, and a second positive integer γ. For every integer $N > 0$ we set

$$\hat{X}_N := \big\{\hat{x}\in\hat{X} : \eta_L(\hat{x}) \geq N^{-1} \text{ and } S_L(\hat{x}) \leq N\big\}.$$

Observe that $\hat{\mu}_\phi(\hat{X}_N)\to 1$ as $N\to\infty$. In particular, there exists $N_0 = N_0(m,\gamma)$ such that, for every $N > N_0$, we have $\hat{\mu}_\phi(\hat{X}_N) > 1 - 1/(2m^{\gamma+1})$. It follows by Markov inequality that there exists a subset $X_\gamma\subset X$ with $\mu_\phi(X_\gamma) > 1 - 1/m^\gamma$ such that, for all $N > N_0$,

$$\hat{\mu}_\phi^x(\hat{X}_N\cap\{x_0 = x\}) > 1 - 1/(2m) \text{ for all } x\in X_\gamma.$$

It is enough to prove the property in the lemma for all $x\in X_\gamma$. Let us fix one such x. By Lemma 4.12 and the definition of $\hat{X}_N$, for every $\hat{x}\in\hat{X}_N$ and $n\geq 0$ the inverse branch $f_{\hat{x}}^{-n}$ is defined on the ball $B_{\mathbb{P}^k}(x_0, N^{-1})$ with $\mathrm{Lip}(f_{\hat{x}}^{-n}) \leq Ne^{-nL}$. In particular, $\mathrm{diam}(f_{\hat{x}}^{-n}(B_{\mathbb{P}^k}(x_0, e^{-m}/(2N)))) \leq e^{-m-nL_0}$ for all $n\geq 0$. It follows that all inverse branches on $B_{\mathbb{P}^k}(x, e^{-m}/(2N))$ corresponding to elements $\hat{x}\in\hat{X}_N\cap\{x_0 = x\}$ are m-good for all n.

The Claim above implies that

$$\hat{\mu}_n^x(\hat{X}_N\cap\{x_0 = x\}) > 1 - 1/m \text{ for all } n \text{ large enough.}$$

This precisely means that, for all n sufficiently large, we have $\|\mu_{B,n}^{(m)}\| > 1 - 1/m$, where $B = B_{\mathbb{P}^k}(x, e^{-m}/(2N))$. This implies that such a ball B is m-nice. The proof is complete. □

Lemma 4.13. *There exists a positive constant $C = C(L_0, q)$ such that, for all $n\in\mathbb{N}, m > 0$, and every m-good inverse branch $g\colon B\to B'$ of f of order n on a ball B, and for all sequences of points $\{x_l\},\{y_l\}$ with $0\leq l\leq n-1$ and $x_l, y_l\in f^l(B')$ we have*

$$\sum_{l=0}^{n-1}|\phi(x_l) - \phi(y_l)| \leq Cm^{-(q-1)}.$$

Proof. Since g is m-good, we have $\mathrm{dist}(x_l, y_l) \leq e^{-m-(n-l)L_0}$ for all $0\leq l\leq n-1$. Hence,

$$\sum_{l=0}^{n-1}|\phi(x_l) - \phi(y_l)| \leq \sum_{l=0}^{n-1}\|\phi\|_{\log^q}|\log^\star\mathrm{dist}(x_l, y_l)|^{-q} \leq \|\phi\|_{\log^q}\sum_{l=1}^{\infty}|1 + m + lL_0|^{-q} \lesssim m^{-(q-1)},$$

where the implicit constant depends on L_0, q and we used the assumption that $q > 2$. □

Lemma 4.14. *Let $\mathcal{U}$ be a finite collection of disjoint open subsets of $\mathbb{P}^k$. For every $m > 0$ there exists $n(m,\mathcal{U}) > m$ and, for every $n \geq n(m,\mathcal{U})$, a set $Q_{m,n}$ of repelling periodic points of period n in the intersection of the union of the sets in $\mathcal{U}$ with the small Julia set such that, for all $U \in \mathcal{U}$,*

$$(1-1/m)\mu_\phi(U) \leq \lambda^{-n} \sum_{y \in Q_{m,n} \cap U} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \leq (1+1/m)\mu_\phi(U).$$

Proof. We can assume that $\mathcal{U}$ consists of a single open set U, the general case follows by taking $n(m,\mathcal{U})$ to be the maximum of the $n(m,U)$, for $U \in \mathcal{U}$. We can also assume that $\mu_\phi(U) > 0$ because otherwise we can choose $n(m,U) = m+1$ and $Q_{m,n} = \varnothing$. Fix integers $m_2 \gg m_1 \gg m$. By Lemma 4.11, for μ_ϕ-almost every point a, every ball of sufficiently small radius centred at a is m_2-nice. Hence, we can find a finite family of disjoint m_2-nice balls $B_i \Subset U$, such that $\mu_\phi(U \setminus \cup B_i) < \mu_\phi(U)/m_2$. It is then enough to prove the lemma for each B_i instead of U. More precisely, let $B = B_{\mathbb{P}^k}(a,r)$ be an m_2-nice ball. It is enough to find an $n(m_2) > m_2$ and, for all $n \geq n(m_2)$, a set Q of repelling periodic points of period n in $B \cap \mathrm{supp}(\mu_\phi)$ such that

$$(1-1/m_1)\mu_\phi(B) \leq \lambda^{-n} \sum_{y \in Q} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \leq (1+1/m_1)\mu_\phi(B). \tag{4.5}$$

We fix in what follows an integer $m_3 \gg m_2/\mu_\phi(B)$ and a second ball $B^\star = B_{\mathbb{P}^k}(a, r^\star)$, with $r^\star < r$, such that $\mu_\phi(B^\star) > (1-1/m_2)\mu_\phi(B)$. Choose a finite family of disjoint m_3-nice balls D_i with the property that $\mu_\phi(\cup D_i) > 1-1/m_3$. We set $D := \cup D_i$ and let b_i be the center of D_i. We also fix balls $D_i^\star \Subset D_i$ centred at b_i and such that $\mu_\phi(\cup D_i^\star) > 1-1/m_3$ and set $D^\star := \cup D_i^\star$.

Claim 1. There is an integer $M_1 = M_1(m_2, B, B^\star, D_i)$ such that, for all $N \geq M_1$, we have

$$(1-4/m_2)\mu_\phi(B) \leq \mu^{(m_3)}_{D_i,N}(B^\star) \leq (1+4/m_2)\mu_\phi(B) \quad \text{for all } i. \tag{4.6}$$

Proof. Since the balls D_i are m_3-nice and $m_3 \gg m_2/\mu_\phi(B)$, for every i we have

$$\big\|\mu^{(m_3)}_{D_i,N}\big\| \geq (1-\mu_\phi(B)/m_2) \text{ for all } N \text{ large enough.}$$

Hence, since $\mu^{(m_3)}_{D_i,N} \leq \mu_{b_i,N}$ and $\|\mu_{b_i,N}\| \leq 1+o(1)$, we have $\big\|\mu_{b_i,N} - \mu^{(m_3)}_{D_i,N}\big\| \leq \mu_\phi(B)/m_2 + o(1)$. Therefore, in order to prove the claim it is enough to show that

$$(1-2/m_2)\mu_\phi(B) \leq \mu_{b_i,N}(B^\star) \leq (1+2/m_2)\mu_\phi(B)$$

for all i and all N large enough. This is a consequence of Corollary 4.2 and of the inequality $\mu_\phi(B^\star) > (1-1/m_2)\mu_\phi(B)$. □

Similarly, we also have the following.

Claim 2. There is an integer $M_2 = M_2(m_2, B, D^\star)$ such that, for all $N \geq M_2$, we have

$$1-4/m_2 \leq \mu^{(m_2)}_{B,N}(D^\star) \leq 1+4/m_2. \tag{4.7}$$

Proof. Since the ball B is m_2-nice, we have

$$\big\|\mu^{(m_2)}_{B,N}\big\| \geq (1-1/m_2) \text{ for all } N \text{ large enough.}$$

Hence, by the fact that $\mu^{(m_2)}_{B,N} \leq \mu_{a,N}$ and $\|\mu_{a,N}\| \leq 1+o(1)$, in order to prove the claim it is enough to show that

$$1-2/m_2 \leq \mu_{a,N}(D^\star) \leq 1+2/m_2$$

for all N large enough. This is again a consequence of Corollary 4.2 and of the inequality $\mu_\phi(D^\star) > (1-1/m_3)$. □

For every N_1 sufficiently large, every point in the support of $\mathbb{1}_{B^\star}\mu^{(m_3)}_{D_i,N_1}$ corresponds to an m_3-good inverse branch of f of order N_1 mapping D_i to a relatively compact subset of B. Similarly, for every N_2 sufficiently large every point in the support of $\mathbb{1}_{D^\star}\mu^{(m_2)}_{B,N_2}$ corresponds to an m_2-good inverse branch of f of order N_2 mapping B to a relatively compact subset of

D. Composing such inverse branches we get inverse branches g_j of $f^{N_1+N_2}$ defined on B whose images are relatively compact in B. In what follows, we only consider these inverse branches g_j. We also write g_j as $g_j^{(1)} \circ g_j^{(2)}$, where $g_j^{(2)}$ is the corresponding inverse branch of f^{N_2} on B (whose image is then in D) and $g_j^{(1)}$ is the corresponding inverse branch of f^{N_1} on $g_j^{(2)}(B)$. We also set $i = i(j)$, where $g_j^{(2)}(B) \subset D_i$.

Each inverse branch g_j as above contracts the Kobayashi metric of B, and thus admits a unique fixed point y_j, which is attracting for g_j and hence repelling for $f^{N_1+N_2}$. Up to possibly increasing the integers M_1 and M_2 given by the Claims above, we can assume that the above properties hold for $N_1 = M_1$ and $N_2 = M_2$. We set $n(m) := M_1(m_2) + M_2(m_2)$ for a fixed choice of sufficiently large m_1, m_2, m_3 and, for all $n \geq n(m)$, we define the set Q as the union of all such fixed points constructed as above with $N_1 = M_1(m_2)$ and $N_2 = n - N_1 \geq M_2(m_2)$. The points in Q are then repelling periodic points of period $n = N_1 + N_2$ for f. Observe that, for all j and all $z \in B$, since $g_j(B) \Subset B$ we have $g_j^l(z) \to y_j$ as $l \to \infty$. Since B intersects the small Julia set, by taking z in the small Julia set we see that y_j belongs to the small Julia set. To conclude, we need to prove (4.5) for this choice of Q. We set

$$\mu_n := \lambda^{-n} \sum_{y \in Q} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \delta_y = \sum_j e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \delta_{y_j}$$

and

$$\tilde{\mu}_n := \lambda^{-n} \sum_j \Big(e^{\phi(g_j^{(1)}(b_{i(j)}))+\phi(f\circ g_j^{(1)}(b_{i(j)}))+\cdots+\phi(f^{N_1-1}\circ g_j^{(1)}(b_{i(j)}))} \frac{\rho(g_j^{(1)}(b_{i(j)}))}{\rho(b_{i(j)})} \cdot$$

$$\cdot\, e^{\phi(g_j^{(2)}(a))+\phi(f\circ g_j^{(2)}(a))+\cdots+\phi(f^{N_2-1}\circ g_j^{(2)}(a))} \frac{\rho(g_j^{(2)}(a))}{\rho(a)} \delta_{g_j(a)} \Big).$$

Observe that there is a correspondence between the terms in μ_n and those in $\tilde{\mu}_n$. Moreover, since all the balls B and D_i are m_2-nice, we have

$$|\rho(g_j^{(1)}(b_{i(j)}))/\rho(a) - 1| \lesssim m_2^{-1} \text{ and } |\rho(g_j^{(2)}(a))/\rho(b_{i(j)}) - 1| \lesssim m_2^{-1} \text{ for all } i \text{ and } j.$$

It follows from these inequalities and Lemma 4.13 that $|\mu_n(B) - \tilde{\mu}_n(B)| \lesssim \tilde{\mu}_n(B) m_2^{-1}$. Hence, in order to conclude it is enough to prove that

$$(1 - 1/(2m_1))\mu_\phi(B) \leq \tilde{\mu}_n(B) \leq (1 + 1/(2m_1))\mu_\phi(B)$$

because m_2 is chosen large enough. By construction, we have

$$\tilde{\mu}_n(B) = \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) \cdot \mu_{D_i,N_1}^{(m_3)}(B^\star).$$

By Claim 1, this implies that

$$(1 - 4/m_2)\mu_\phi(B) \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) \leq \tilde{\mu}_n(B) \leq (1 + 4/m_2)\mu_\phi(B) \sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star).$$

The assertion then follows from Claim 2 and the fact that $\sum_i \mu_{B,N_2}^{(m_2)}(D_i^\star) = \mu_{B,N_2}^{(m_2)}(D^\star)$, by taking m_2 large enough. □

We can now conclude the proof of Theorem 4.10. As mentioned at the beginning of the section, this also completes the proof of Theorem 1.1.

End of the proof of Theorem 4.10. For every $i \in \mathbb{N}$ we construct a finite family of disjoint open sets $\mathcal{U}_i := \{U_{i,j}\}_{1 \leq j \leq J_i}$ with the following properties:

(i) $\mu_\phi(\cup_{1 \leq j \leq J_i} U_{i,j}) = 1$;
(ii) for all $1 \leq j \leq J_i$ we have $\operatorname{diam}(U_{i,j}) < 1/i$;
(iii) for all $i \geq 2$ and $1 \leq j \leq J_i$ there exists $1 \leq j' \leq J_{i-1}$ such that $U_{i,j} \subset U_{i-1,j'}$.

We can construct these sets using local coordinates and generic real hyperplanes which are parallel to the coordinate hyperplanes. Observe also that, by the first condition, we have $\mu_\phi(\partial U_{i,j}) = 0$ for all i and $1 \le j \le J_i$.

For every n, we define $i_n := \max\{m \le n \colon n \ge n(m, \mathcal{U}_m)\}$, where $n(m,\mathcal{U}_m)$ is given by Lemma 4.14. Observe that $i_n \to \infty$ as $n \to \infty$. We define $P'_n \subset \cup_j U_{i_n,j}$ as the union of the sets of repelling periodic points of period n in the small Julia set obtained by applying Lemma 4.14 to the collection $\mathcal{U}_{i_n}$ instead of $\mathcal{U}$, and set

$$\mu'_n := \lambda^{-n} \sum_{y \in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \delta_y.$$

By Properties (i) and (ii) of the open sets $U_{i,j}$ and Lemma 4.14, any limit μ' of the sequence $\{\mu'_n\}$ has mass 1. So, since $\mu_\phi(\cup_j U_{i_n,j}) = 1$ for all n and $\operatorname{diam}(U_{i,j}) < 1/i$ for all i, it is enough to prove that

$$\liminf_{n\to\infty} \mu'_n(U_{i^\star,j^\star}) \ge \mu_\phi(U_{i^\star,j^\star}) \text{ for all } i^\star \in \mathbb{N} \text{ and } 1 \le j^\star \le J_{i^\star}. \tag{4.8}$$

Indeed, given any open set $A \subseteq \mathbb{P}^k$, we can write A as a countable union of compact sets of the form $\bar U_{i,j} \Subset A$, overlapping only on their boundaries. We then see that (4.8) implies that $\mu_\phi(A) \le \mu'(A)$ for every open set A, and the facts that $\|\mu_\phi\| = \|\mu'\|$ and $\mu_\phi(\partial U_{i,j}) = 0$ for all i,j imply that $\mu_\phi = \mu'$.

We can then fix $i^\star, j^\star$ as in (4.8) and a positive number ϵ, and it is enough to prove that

$$\mu'_n(U_{i^\star,j^\star}) \ge \mu_\phi(U_{i^\star,j^\star}) - \epsilon \quad \text{for all } n \text{ sufficiently large.}$$

We only consider in what follows integers n such that $i_n > i^\star$ and the sets $U_{i_n,j}$ which are contained in $U_{i^\star,j^\star}$. For all such n, we have $\mu_\phi(U_{i^\star,j^\star}) = \sum_j \mu_\phi(U_{i_n,j})$ and $\mu'_n(U_{i^\star,j^\star}) = \sum_j \mu'_n(U_{i_n,j})$. It follows by the definition of μ'_n and Lemma 4.14 that

$$\left|\mu'_n(U_{i^\star,j^\star}) - \mu_\phi(U_{i^\star,j^\star})\right| \le \sum_j \left|\mu'_n(U_{i_n,j}) - \mu_\phi(U_{i_n,j})\right| \le i_n^{-1} \sum_j \mu_\phi(U_{i_n,j}) = i_n^{-1}\mu_\phi(U).$$

The assertion follows. □

Remark 4.15. One could improve the argument in the proof of Lemma 4.14 to obtain that P'_n can be taken to be a subset of the repelling periodic points with a good control of the eigenvalues, see for instance [BDM08; BD19]. This implies that, setting $\Sigma_j := L_{k-j+1} + \cdots + L_k$, we have

$$\Sigma_j = \lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} \frac{1}{n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \log \left\| \bigwedge^j Df^n_y \right\|$$

and, in particular,

$$\Sigma_k = \sum_{j=1}^k L_j = \lim_{n\to\infty} \lambda^{-n} \sum_{y\in P'_n} e^{\phi(y)+\phi(f(y))+\cdots+\phi(f^{n-1}(y))} \log|\operatorname{Jac} Df_y|.$$

Here, $Df^n_x \colon T_x\mathbb{P}^k \to T_{f^n(x)}\mathbb{P}^k$ denotes the differential of f^n at x. This is a linear map from the complex tangent space of $\mathbb{P}^k$ at x to the one at $f^n(x)$. It induces the natural linear map $\bigwedge^j Df^n_x$ from the exterior power $\bigwedge^j T_x\mathbb{P}^k$ to $\bigwedge^j T_{f^n(x)}\mathbb{P}^k$.

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