PARABOLIC IMPLOSION FOR ENDOMORPHISMS OF $\mathbb{C}^2$

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We give an estimate of the discontinuity of the large Julia set for a perturbation of a class of maps tangent to the identity, by means of a two-dimensional Lavaurs Theorem. We adapt to our situation a strategy due to Bedford, Smillie and Ueda in the semiattracting setting. We also prove the discontinuity of the filled Julia set for such perturbations of regular polynomials.

Notation. The symbol $O(x)$ will stand for some element in the ideal generated by x. More generally, given any f, $O(f)$ will stand for some element in the ideal generated by f. Analogously, $O(f_1, \ldots, f_k)$ will stand for some element in the ideal generated by $f_1, \ldots, f_k$.

The notation $O_2(x, y)$ will be a shortcut for $O(x^2, xy, y^2)$. Given a point $p \in \mathbb{C}^2$, we shall denote its coordinates by $x(p)$ and $y(p)$.

1. Introduction and results

Parabolic dynamics and the study of parabolic perturbations have been at the heart of holomorphic dynamics in the last couple of decades. Starting with the work by Lavaurs [14], the theory of parabolic implosion has provided useful tools for a very precise control of these perturbations and for the proof of some of the most striking recent results in the field, e.g., the construction of Julia sets of positive area [7, 5], significative steps toward the setting of the *hyperbolicity conjecture* for quadratic polynomials [8] and the construction of endomorphisms of $\mathbb{P}^2(\mathbb{C})$ with a wandering domain [4]. In particular, these techniques have proved extremely useful in the study of bifurcation loci (see, e.g., [15]).

In several complex variables, the study of parabolic perturbations and the theory of parabolic implosion are just at the start, with recent results only in the semiattracting setting [6, 11]. The goal of this paper is to provide a starting point for an analogous theory in the completely parabolic setting, by a precise study of perturbations of germs of endomorphisms of $\mathbb{C}^2$ tangent to the identity at the origin.

Let us briefly recall the foundational results of the one-dimensional theory. We refer to [10] for a more extended introduction to the subject, as well as to the original work by Lavaurs [14]. Consider an endomorphism of $\mathbb{C}$ tangent to the identity at the origin, given by $f(z) = z + z^2 + O(z^2)$. The origin is a *parabolic* fixed point for f. The dynamics is attracting near the negative real axis: there exists a parabolic basin $\mathcal{B}$ for 0, i.e., an open set of points converging to the origin after iteration. The origin is on the boundary of $\mathcal{B}$, and the convergence happens tangentially to the negative real axis. The iteration of f on $\mathcal{B}$ is semiconjugated to a translation by 1. More precisely, there exists an *incoming Fatou coordinate* (unique up to postcomposition with a translation) $\varphi^\iota : \mathcal{B} \to \mathbb{C}$ such that, for every $z \in \mathcal{B}$, we have $\varphi^\iota \circ f(z) = f(z) + 1$. One way to construct such a coordinate is to define it as the limit $\varphi^\iota(z) := \lim_n (w_0^\iota(f^n(z)) - n)$. Here $w_0^\iota(z) := -\frac{1}{z} - q\log(-z)$, where log is the principal branch of the logarithm and $q + 1$ is the coefficient of z^3 in the expression of f. We will consider this normalization in the sequel.

The same happens for the inverse iteration near the positive real axis: we have a repelling basin $\mathcal{R}$ of points converging to 0 under some inverse iteration, and the convergence happens tangentially to the positive real axis. We can construct in this case an *outgoing Fatou parametrization*, i.e., a map ψ^o from a subset of $\mathbb{C}$ to $\mathcal{R}$ (unique up to precomposition with a translation) such that $f \circ \psi^o(z) = \psi^o(z + 1)$. Here we will consider as ψ^o the inverse of the map φ^o arising as the limit $\varphi^0(z) := \lim_n (w_0^o(f^{-n}(z)) + n)$, where $w_0^o(z) := -\frac{1}{z} - q\log(z)$. It is worth noticing here that the

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union of $\mathcal{B}$ and $\mathcal{R}$ contains a full pointed neighbourhood of the origin. The map ψ^o can also be extended to $\mathbb{C}$, with image equal to $\mathbb{C}$.

Notice that the incoming Fatou coordinate is a map from the dynamical plane to $\mathbb{C}$, while the outgoing Fatou parametrization is a map from $\mathbb{C}$ to the dynamical plane. In particular, given any $\alpha \in \mathbb{C}$ and denoting by t_α the translation by α on $\mathbb{C}$, the composition $L_\alpha := \psi^o \circ t_\alpha \circ \varphi^\iota$ is well defined as a function from $\mathcal{B}$ to $\mathcal{R}$. Such a map is usually called a *Lavaurs map*, or a *transfer map*.

We consider now the perturbation $f_\varepsilon(z) = z + (z^2 + \varepsilon^2)(1 + O(z) + O(\varepsilon^2))$ of the system f, for ε real and positive. As $\varepsilon \neq 0$, the dynamics abruptly changes: the parabolic point splits in two (repelling) points $\pm i\varepsilon$, and the orbits of points in $\mathcal{B}$ can now pass through the "gate" between these two points. Using the Lavaurs map it is possible to give a very precise description of this phenomenon, by studying the dynamics of high iterates of the perturbed maps f_ε, as $\varepsilon \to 0$. The following definition plays a central role in this study.

Definition 1.1. Given $\alpha \in \mathbb{C}$, an *α-sequence* is a sequence $(\varepsilon_\nu, n_\nu)_{\nu \in \mathbb{N}} \in (\mathbb{C} \times \mathbb{N})^{\mathbb{N}}$ such that $n_\nu \to \infty$ and $n_\nu - \frac{\pi}{\varepsilon_\nu} \to \alpha$ as $\nu \to +\infty$.

Notice in particular that the definition of α-sequence implies that ε_ν tends to the origin tangentially to the positive real axis. More precisely, there exists a constant c such that, for every ν sufficiently large, we have $|\mathrm{Im}\, \varepsilon_\nu| \le c\, |\varepsilon_\nu|^2$. The following result gives the limit description of suitable high iterates of f_ε.

Theorem 1.2 (Lavaurs [14])**.** *Let $f_\varepsilon(z) = z + (z^2 + \varepsilon^2)(1 + O(z) + O(\varepsilon^2))$ and (ε_ν, n_ν) be an α-sequence. Then $f_{\varepsilon_\nu}^{n_\nu} \to L_\alpha$, locally uniformly on $\mathcal{B}$, as $\nu \to +\infty$.*

One of the most direct consequences of Lavaurs theorem is the fact that the set-valued functions $\varepsilon \mapsto J(f_\varepsilon)$ and $\varepsilon \mapsto K(f_\varepsilon)$ are discontinuous at $\varepsilon = 0$ for the Hausdorff topology. Here $J(f_\varepsilon)$ and $K(f_\varepsilon)$ denote the Julia set and the filled Julia set of f_ε, respectively (recall – see e.g. [10] – that $\varepsilon \mapsto J(f_\varepsilon)$ is always lower semicontinuous, while $\varepsilon \mapsto K(f_\varepsilon)$ is always upper semicontinuous). More precisely, define the *Lavaurs-Julia set* $J(f_0, L_\alpha)$ and the *filled Lavaurs-Julia set* $K(f_0, L_\alpha)$ by

$$J(f_0, L_\alpha) := \overline{\{\, z \in \mathbb{C} \colon \exists m \in \mathbb{N}, L_\alpha^m(z) \in J(f_0)\,\}}$$
$$K(f_0, L_\alpha) := \{\, z \in \mathbb{C} \colon \exists m \in \mathbb{N}, L_\alpha^m(z) \notin K(f_0)\,\}^c$$

Notice that the Lavaurs-Julia set $J(f_0, L_\alpha)$ is in general larger than the Julia set of f_0. On the other hand, the set $K(f_0, L_\alpha)$ is in general smaller than $K(f_0)$. The following Theorem then gives an estimate of the discontinuity of the maps $\varepsilon \mapsto J(f_\varepsilon)$ at $\varepsilon = 0$.

Theorem 1.3 (Lavaurs [14])**.** *Let $f_\varepsilon(z) = z + (z^2 + \varepsilon^2)(1 + O(z) + O(\varepsilon^2))$ and (ε_ν, n_ν) be an α-sequence. Then*

$$\liminf J(f_{\varepsilon_\nu}) \supseteq J(f_0, L_\alpha) \text{ and } \limsup K(f_{\varepsilon_\nu}) \subseteq K(f_0, L_\alpha)$$

In particular, at $\varepsilon = 0$,

(1) the map $\varepsilon \to J(f_\varepsilon)$ is lower semicontinuous, but not continuous;
(2) the map $\varepsilon \to K(f_\varepsilon)$ is upper semicontinuous, but not continuous.

The goal of this paper is to make a step towards the generalization of Theorems 1.2 and 1.3 to the two-variables setting, by studying the perturbation of a class of maps tangent to the identity (i.e., with differential at a fixed point equal to the identity). More precisely, we consider an endomorphism of $\mathbb{C}^2$ of the form

$$F_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^2 \cdot \alpha_0(x, y) \\ y(1 + \rho x + O(x^2, xy, y^2)) \end{pmatrix},$$

where $\alpha_0(x, y) = 1 + O(x, y)$ and $\rho > 1$. For instance, F_0 may be the local expression of an endomorphism of $\mathbb{P}^2$ (e.g., if the two components of F_0 are polynomials of the same degree in

(x, y) with 0 as the only common root of their higher-degree homogeneous parts). We shall primarily be interested in this situation.

The map F_0 has a fixed point tangent to the identity at the origin, and two invariant lines $\{ x = 0 \}$ and $\{ y = 0 \}$. By the work of Hakim [12] (see Section 2), since $\rho > 1$ we know that $[1 : 0]$ is a *non-degenerate characteristic direction*, and that there exists an open set $\mathcal{B}$ of initial conditions, with the origin on the boundary, such that every point in $\mathcal{B}$ is attracted to the origin tangentially to the direction $[1 : 0]$. Moreover there exists, on an open subset $\widetilde{C}_0$ of $\mathcal{B}$, a (one dimensional) Fatou coordinate $\widetilde{\varphi^\iota}$, with values in $\mathbb{C}$, such that $\widetilde{\varphi^\iota} \circ F_0(p) = \widetilde{\varphi^\iota}(p) + 1$ (see Lemma 2.2).

A similar description holds for the inverse map. Indeed, after restricting ourselves to a neighbourhood U of the origin where F_0 is invertible, we can define the set $\mathcal{R}$ of point that are attracted to the origin tangentially to the direction $[1 : 0]$ by backward iteration. There is then a well defined map $\widetilde{\varphi^o} : -\widetilde{C}_0 \cap U \to \mathbb{C}$ such that $\widetilde{\varphi^o} \circ F_0(p) = \widetilde{\varphi^o}(p) + 1$ whenever the left hand side is defined. It is actually possible to construct two-dimensional Fatou coordinates (see [12]), but we shall not need them in this work. In the following, we shall need some specific coefficients of F_0. We thus write its expression as

$$F_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + x^2(1 + (q+1)x + ry + O(x^2, xy, y^2)) \\ y(1 + \rho x + O(x^2, xy, y^2)) \end{pmatrix}, \tag{1.1}$$

where ρ is real and greater than 1 and $q, r \in \mathbb{C}$.

Consider now a perturbation F_ε of F_0 of the form

$$\begin{aligned} F_\varepsilon \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + (x^2 + \varepsilon^2)\alpha_\varepsilon(x, y) \\ y(1 + \rho x + \beta_\varepsilon(x, y)) \end{pmatrix} \\ &= \begin{pmatrix} x + (x^2 + \varepsilon^2)(1 + (q+1)x + ry + O(x^2, xy, y^2) + O(\varepsilon^2)) \\ y(1 + \rho x + O(x^2, xy, y^2) + O(\varepsilon^2)) \end{pmatrix}. \end{aligned} \tag{1.2}$$

Our goal is to study the dependence of the large Julia set[1] $J^1(F_\varepsilon)$ on ε near the parameter $\varepsilon = 0$. Notice that the line $\{y = 0\}$ is invariant for all the maps in the family, and that on this line we have a classical (1-dimensional) parabolic implosion phenomenon.

Our main result is the following Theorem, which is a partial generalization of Theorem 1.2 to our setting. As in dimension 1, *α-sequences* play a crucial role. The set $\widetilde{C}_0$ introduced above will be precisely defined in Proposition 2.1, and the Fatou coordinates $\widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}$ in Lemma 2.2.

Theorem 1.4. *Let F_ε be a holomorphic family of endomorphisms of $\mathbb{C}^2$ as in* (1.2)*. Let $\mathcal{B}$ be the attracting basin for the origin for the map F_0 with respect to the characteristic direction* $[1 : 0]$*. Let α be a complex number and (n_ν, ε_ν) be an α-sequence. Then the family $F_{\varepsilon_\nu}^{n_\nu}$ is normal on a neighbourhood of $\mathcal{B} \cap \{ y = 0 \}$ in $\mathcal{B}$. In addition every limit L has the following properties:*

- *it is an open holomorphic map;*
- *it coincides with the 1-dimensional Lavaurs map L_α on the invariant line $\{y = 0\}$;*
- *it satisfies*

$$\widetilde{\varphi^o} \circ L(p) = \alpha + \widetilde{\varphi^\iota}(p) \tag{1.3}$$

whenever both sides are defined, where $\widetilde{\varphi^\iota} : \widetilde{C}_0 \to \mathbb{C}$ and $\widetilde{\varphi^o} : -\widetilde{C}_0 \to \mathbb{C}$ are suitably normalized (1-dimensional) Fatou coordinates for F_0.

The reason we do not get the normality of the sequence on all the basin $\mathcal{B}$ is that, a priori, the neighbourhood where we prove the normality does not necessarily contain a fundamental domain for the dynamics. If this were the case, the result would easily extend to the full basin.

Remark 1.5. Computer experiments suggest that given any α-sequence (ε_ν, n_ν) there is a neighbourhood of C_0 in $\widetilde{C}_0$ such that the sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a (unique) limit map L, without the need of extracting a subsequence.

[1] i.e., the complement of the Fatou set, which in general is larger than the Julia set defined as the support of the equilibrium measure for endomorphisms of $\mathbb{P}^2$, see [9].

As a consequence, we shall deduce an estimate of the discontinuity of the large Julia set in this context (notice that the discontinuity itself follows from an application of Theorem 1.2 to the invariant line $\{ y = 0 \}$). We say that, given $\mathcal{U} \subset \widetilde{C}_0$, a map $L : \mathcal{U} \to \mathbb{C}^2$ is a *Lavaurs map* if there exists an α-sequence (ε_ν, n_ν) such that $F_{\varepsilon_\nu}^{n_\nu} \to L$ on $\mathcal{U}$. We then have the following result (see Section 7 for the definition of the Lavaurs-Julia sets $J^1(F_0, L)$ in this setting).

Theorem 1.6. *Let F_ε be a holomorphic family of endomorphisms of $\mathbb{P}^2$ as in* (1.2) *and $L : \mathcal{U} \to \mathbb{C}^2$ be a Lavaurs map such that $F_{\varepsilon_\nu}^{n_\nu} \to L$ on $\mathcal{U}$ for some α-sequence (ε_ν, n_ν). Then*

$$\liminf J^1(F_{\varepsilon_\nu}) \supseteq J^1(F_0, L).$$

Finally, in the last section, we consider a family of *regular polynomials*, i.e., polynomial endomorphisms of $\mathbb{C}^2$ admitting an extension to $\mathbb{P}^2(\mathbb{C})$. For these maps, it is meaningful to define the *filled Julia set* K as the set of points with bounded orbit. In analogy with the one-dimensional theory, we deduce from Theorem 1.4 an estimate for the discontinuity of the filled Julia set at $\varepsilon = 0$ (see Section 8 for the definition of the set $K(F_0, L)$) and in particular deduce that $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$. Notice that, differently from the case of the large Julia set, this is not already a direct consequence of the 1-dimensional theory.

Theorem 1.7. *Let F_ε be a holomorphic family of regular polynomial maps of $\mathbb{C}^2$ as in* (1.2) *and $L : \mathcal{U} \to \mathbb{C}^2$ be a Lavaurs map such that $F_{\varepsilon_\nu}^{n_\nu} \to L$ on $\mathcal{U}$ for some α-sequence (ε_ν, n_ν). Then*

$$K(F_0, L) \supseteq \limsup K(F_{\varepsilon_\nu}).$$

Moreover, $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.

The paper is organized as follows. In Section 2 we recall the results by Hakim describing the local dynamics of the map (1.1) near the origin, and introduce the Fatou coordinates associated to the attracting and repelling basins. In Section 3 we define and study suitable perturbations of the Fatou coordinates, that allow to semiconjugate the iteration of F_ε to a translation by 1, up to a controlled error. In Sections 4 and 5 we carefully study the orbits of points under iteration by F_ε and prove some preliminary convergence result needed for the proof of Theorem 1.4, which is given in Section 6. In Section 7 and 8 we deduce from Theorem 1.4 the estimates of the discontinuity of the large Julia set and (for regular polynomials) of the filled Julia set at $\varepsilon = 0$.

2. Preliminaries and Fatou coordinates

Following the work of Hakim [12] (see also [13, 3]), we start giving a description of the local dynamics near the origin for F_0 by recalling some classical notions in this setting. Let Φ be a germ of transformation tangent to the identity at the origin of $\mathbb{C}^2$. We can locally write it near the origin as

$$\Phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + P(x,y) + \dots \\ y + Q(x,y) + \dots \end{pmatrix},$$

where P and Q are homogeneous polynomials of degree 2. In the following, we shall always assume that $P(x, y)$ is not identically zero. A *characteristic direction* is a direction $V = [x : y] \in \mathbb{P}^1(\mathbb{C})$ such that the complex line through the origin in the direction $[x : y]$ is invariant for (P, Q). The direction is *degenerate* if (P, Q) sends the line to the origin, *non degenerate* otherwise.

The homogeneous part of degree 2 of Φ can be seen as a self map of the tangent space at the origin. The characteristic directions correspond to the invariant lines through the origin for this map. When a direction is non degenerate, the homogeneous map induces a holomorphic map defined on (a neighbourhood of this direction in) $\mathbb{P}^1(\mathbb{C})$ and fixing this direction. The *director* (see [1, Definition 2.4]) of this direction is defined as $\rho - 1$, where ρ is the multiplier of the map at the fixed direction. Equivalently, one can take coordinates such that the characteristic direction is $V = [1 : u_0]$. Notice that the fact that $[1 : u_0]$ is a characteristic direction is equivalent to u_0

being a zero of $r(u) := Q(1,u) - uP(1,u)$. The director of the characteristic direction $[1 : u_0]$ is thus equal to

$$\frac{r'(u_0)}{P(1,u_0)}.$$

Given a germ Φ and a non degenerate characteristic direction V for Φ we can assume, without loss of generality, that $V = [1 : 0]$ and that the coefficient of x^2 in $P(x,y)$ is 1 (notice that Hakim has the opposite normalization, i.e., with the term $-x^2$). The following result by Hakim ([12, Proposition 2.6]) gives an explicit description of an invariant subdomain of the attracting basin $\mathcal{B}$. In all this work, we will restrict ourselves to points belonging to such an invariant domain.

Proposition 2.1 (Hakim)**.** *Let Φ be a germ of transformation of $\mathbb{C}^2$ tangent to the identity (normalized as above), such that $V = [1 : 0]$ is a nondegenerate characteristic direction with director δ whose real part is greater than some $0 < \alpha \in \mathbb{R}$. Then, if γ, s and R are small enough positive constants, every point of the set*

$$\widetilde{C}_0(\gamma, R, s) := \{ (x_0, y_0) \in \mathbb{C}^2 \colon \ |\operatorname{Im} x_0| \leq -\gamma \operatorname{Re} x_0, |x_0| \leq R, |y_0| \leq s\,|x_0| \}$$

is attracted to the origin in the direction V and $x(\Phi^n(x_0, y_0)) \sim -\frac{1}{n}$. Moreover we have

$$|x(\Phi^n(x_0,y_0))| \leq \frac{2}{n} \text{ and } |y(\Phi^n(x_0,y_0))|\,|x(\Phi^n(x_0,y_0))|^{-\alpha-1} \leq |y_0|\,|x_0|^{-\alpha-1}. \tag{2.1}$$

Notice that, for a γ_1 slightly smaller than γ, we have $F_0(\widetilde{C}_0(\gamma, R, s)) \subseteq \widetilde{C}_0(\gamma, R, s)$.

Let us now consider F_0 as in (1.1). It is immediate to see that $[1 : 0]$ is a non-degenerate characteristic direction, with director equal to $\rho - 1$. This is the reason we made the assumption that $\rho > 1$. It will be even clearer later (Lemma 5.1) that this a crucial assumption.

An important feature of our setting is that the (local) inverse of a map tangent to the identity shares a lot of properties with the original map (this does not happen for instance in the semi-parabolic situation). In fact, it is immediate to see that the local inverse of an endomorphism tangent to the identity is still tangent to the identity, with the same characteristic directions and moreover the same Hakim directors (this follows since the homogeneous part of degree 2 of F^{-1} is the opposite of the homogeneous part of degree 2 of F, and thus induces the same map on $\mathbb{P}^1(\mathbb{C})$). In our situation, $(0,0)$ is still a double fixed point for the local inverse G_0, which has the following form (see for example the explicit description of the coefficients of the inverse of an endomorphism tangent to the identity given in [2]),

$$G_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - x^2(1 + (q-1)x + ry + O_2(x,y)) \\ y(1 - \rho x + O_2(x,y)) \end{pmatrix}$$

and the stated properties are readily verified.

In the following, we will fix a neighbourhood U of the origin where F_0 is invertible, and consider an invariant domain $\widetilde{C}_0$ as in Proposition 2.1 for F_0 such that $-\widetilde{C}_0$ satisfies the same property for G_0 and both $\widetilde{C}_0$ and $-\widetilde{C}_0$ are contained in U.

We now briefly recall how to construct a (one dimensional) Fatou coordinate $\widetilde{\varphi}^\iota$ on $\widetilde{C}_0$ semiconjugating F_0 to a translation by 1. We notice here that it is actually possible to construct a two-dimensional Fatou coordinate, on a subset of $\widetilde{C}_0$, with values in $\mathbb{C}^2$ and semiconjugating the system to the translation by $(1,0)$. Since we will not use it, we do not detail the construction here, but we refer the interested reader to [12].

The first step of the construction of $\widetilde{\varphi}^\iota$ is to consider the map

$$\widetilde{w}_0^\iota(x,y) := -\frac{1}{x} - q\log(-x). \tag{2.2}$$

Notice that, in the chart $\widetilde{w}_0^\iota$, the map F_0 already looks like a translation by 1. Indeed, by (1.1), we have

$$\begin{aligned}\widetilde{w}_0^\iota(F_0(x,y)) &= -\frac{1}{x(F_0(x,y))} - q\log(-x(F_0(x,y)))\\ &= -\frac{1}{x} - q\log(-x) + 1 + ry + O_2(x,y)\\ &= \widetilde{w}_0^\iota(x,y) + 1 + ry + O_2(x,y).\end{aligned} \tag{2.3}$$

In order to get an actual Fatou coordinate, we consider the functions

$$\widetilde{\varphi^\iota}_{0,n} := \widetilde{w}^\iota(F_0^n(x,y)) - n. \tag{2.4}$$

The following Lemma proves that the the sequence $\left(\widetilde{\varphi^\iota}_{0,n}\right)$ converges to an actual Fatou coordinate $\widetilde{\varphi^\iota}$ as $n \to \infty$.

Lemma 2.2. *The functions $\widetilde{\varphi^\iota}_{0,n}$ converge, locally uniformly on $\widetilde{C}_0$, to an analytic function $\widetilde{\varphi^\iota} : \widetilde{C}_0 \to \mathbb{C}$ satisfying*

$$\widetilde{\varphi^\iota}(F_0(p)) = \widetilde{\varphi^\iota}(p) + 1.$$

Proof. Set $A_0(x,y) := \widetilde{w}_0^\iota(F_0(x,y)) - \widetilde{w}_0^\iota(x,y) - 1 = \widetilde{\varphi^\iota}_{0,1}(x,y) - \widetilde{\varphi^\iota}_{0,0}(x,y)$ and notice that $A_0(F_0^n(x,y)) = \widetilde{\varphi^\iota}_{0,n+1}(x,y) - \widetilde{\varphi^\iota}_{0,n}(x,y)$. In order to ensure the convergence of the sequence $\left(\widetilde{\varphi^\iota}_{0,n}\right)$ we can prove that the series with general term $A_0(F_0^n(x,y))$'s converges normally on $\widetilde{C}_0$. It follows from (2.3) that

$$A_0(F_0^n(x,y)) = ry(F_0^n(x,y)) + O_2(x(F_0^n(x,y)), y(F_0^n(x,y))).$$

By Proposition 2.1, we have $|x(F_0^n(x,y))| \leq 2/n$ and $|y(F_0^n(x,y))| \leq 1/n^{\alpha+1}$, for some $\alpha > 0$. This implies that the series $\sum_{n=0}^{\infty} |A_0(F^n(x,y))|$ converges normally to

$$\widetilde{\varphi^\iota}(x,y) := \widetilde{\varphi^\iota}_0(x,y) + \sum_{n=0}^{\infty} A_0(F_0^n(x,y)).$$

The functional relation is also easily verified, since $|A_0(F^n(x,y))| \to 0$. □

In the repelling basin the situation is completely analogous. Setting $\widetilde{w}_0^o := -\frac{1}{x} - q\log(x)$ on $-\widetilde{C}_0$ and $\widetilde{\varphi^o}_{0,n} := \widetilde{w}_0^o(F_0^{-n}(x,y)) + n$, we have $\widetilde{\varphi^o}_{0,n} \to \widetilde{\varphi^o}$ locally uniformly on $-\widetilde{C}_0$, where $\widetilde{\varphi^o} : \widetilde{C}_0 \to \mathbb{C}$ satisfies the functional relation $\widetilde{\varphi^o} \circ F_0(p) = \widetilde{\varphi^o}(p) + 1$.

We notice that the Fatou coordinates are not unique. For instance, we can add any constant to them and still have a coordinate satisfying the desired functional relation. In the following (and in Theorem 1.4), we shall use as coordinate the one obtained in Lemma 2.2 above.

3. The perturbed Fatou coordinates

We consider now the perturbation (1.2) of the system F_0. The goal of this section is modify the Fatou coordinate $\widetilde{\varphi^\iota}$ built in Section 2 to an approximate coordinate for F_ε. More precisely, we are going to construct some coordinates $\widetilde{\varphi^\iota}_\varepsilon$ (with values in $\mathbb{C}$) that, on suitable subsets of $\widetilde{C}_0$:

1. almost conjugate F_ε to a translation by 1, in the sense that the error that we have in considering F_ε as a translation in this new chart will be bounded and explicitly estimated; and
2. tend to the one-dimensional Fatou coordinates $\widetilde{\varphi^\iota}$ for F_0 as $\varepsilon \to 0$.

We shall only be concerned with ε small and satisfying

$$\begin{cases}\operatorname{Re}\varepsilon > 0\\ |\operatorname{Im}\varepsilon| < c\left|\varepsilon^2\right|.\end{cases} \tag{3.1}$$

Notice that this means that ε is contained in the region, in a neighbouhood of the origin, of the points with positive real part and bounded by two circles of the same radius centered on the imaginary axis and tangent one to the other at the origin. Notice in particular that, by definition, every sequence ε_ν associated to an α sequence (ε_ν, n_ν) (see Definition 1.1) satisfies the above property.

First of all, we fix a small neighbourhood U of the origin, such that F_ε is invertible in U, for ε sufficiently small. In this section, we shall only be concerned with this local situation. Then, fix sufficiently small $\gamma < \gamma', R$ and s such that Proposition 2.1 holds on $\widetilde{C}_0(\gamma, R, s)$ and $\widetilde{C}_0(\gamma', R, s)$ for both F_0 and H_0, where $H_0(x,y) := -F_0^{-1}(-x,y)$. By taking γ and γ' sufficiently close, we can assume that $F_0(\widetilde{C}_0(\gamma', R, s))$ and $H_0(\widetilde{C}_0(\gamma', R, s))$ are contained in $\widetilde{C}_0$. Denote by $\widetilde{C}_0, \widetilde{C}_0' \subset U$ (dropping for simplicity the dependence on the parameters) these sets and by C_0, C_0' their projections on the x-plane. We shall assume that $R\rho \ll 1$, and so that $\widetilde{C}_0 \subset \widetilde{C}_0' \Subset U$.

We consider the classical 1-variable change of coordinates on x (and depending on ε) given by

$$u_\varepsilon(x) = \frac{1}{\varepsilon} \arctan\left(\frac{x}{\varepsilon}\right) = \frac{1}{2i\varepsilon} \log\left(\frac{i\varepsilon - x}{i\varepsilon + x}\right) \tag{3.2}$$

defined on $\mathbb{C} \setminus \{\pm i\varepsilon t \colon t \geq 1\}$, where log is the branch of the logarithm such that $u_\varepsilon(0) = 0$. The geometric idea behind this map is the following: for ε small as in (3.1), circular arcs connecting the two points $\pm i\varepsilon$ are sent to parallel (and almost vertical) lines. In particular, the image of the map u_ε is contained in the strip $\left\{-\frac{\pi}{2|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} w\right) < \frac{\pi}{2|\varepsilon|}\right\}$ and the image of the disc of radius ε centred at the origin is the strip $\left\{-\frac{\pi}{4|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} w\right) < \frac{\pi}{4|\varepsilon|}\right\}$. Notice that the inverse of this function on $\left\{-\frac{\pi}{2|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} w\right) < \frac{\pi}{2|\varepsilon|}\right\}$ is given by $w \mapsto \varepsilon \tan(\varepsilon w)$. We gather in the next Lemma the main properties of u_ε that we shall need in the sequel.

Lemma 3.1. *Let u_ε be given by (3.2). Then the following hold.*

(1) For every compact subset $\mathcal{C} \subset C_0$ there exist two positive constants $M^-(\mathcal{C})$ and $M^+(\mathcal{C})$ such that, for every $x \in \mathcal{C}$, we have

$$-\frac{\pi}{2|\varepsilon|} + M^- < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} u_\varepsilon(x)\right) < -\frac{\pi}{2|\varepsilon|} + M^+ \tag{3.3}$$

for every ε sufficiently small.

(2) If $-\frac{\pi}{2|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} u_\varepsilon(x)\right) < -\frac{\pi}{4|\varepsilon|}$, then $|x| \leq \frac{1}{\frac{\pi}{2|\varepsilon|} + \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} u_\varepsilon(x)\right)}$.

Proof. For the first assertion the main point is to notice that, by the compactness of $\mathcal{C}$, we have

$$u_\varepsilon(x) + \frac{\pi}{2\varepsilon} \to -\frac{1}{x}$$

uniformly on $\mathcal{C}$, as $\varepsilon \to 0$. From this we deduce the existence of constants M^-, M^+ such that (3.3) holds for every $x \in \mathcal{C}$.

For the second one, we exploit the inverse of u_ε on $\{-\frac{\pi}{2|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} w\right) < \frac{\pi}{2|\varepsilon|}\}$, which is given by $w \mapsto \varepsilon \tan(\varepsilon w)$. We have

$$\frac{\pi}{4} < |\operatorname{Re} w| < \frac{\pi}{2} \Rightarrow |\tan w| \leq \tan |\operatorname{Re} w| < \frac{1}{\frac{\pi}{2} - |\operatorname{Re} w|}$$

and the assertion follows putting $w = \varepsilon u_\varepsilon(x)$. □

We define now, by means of the functions u_ε, different regions in the dynamical plane. In order to do this, we have to define some constants (independent on ε) that we shall repeatedly use in the sequel.

First of all, fix some $1 < \rho' < \rho$. Then, fix some $1 < \rho'' < 5/4$ such that

$$\left| \frac{4\pi(\rho'' - 1)}{\tan(4\pi(\rho'' - 1))} \right| > \frac{1}{\rho'}.$$

This is possible since $\rho' > 1$. In particular, ρ'' may be very close to 1. Finally, set

$$K := 2\pi(\rho'' - 1) \text{ and } \tau := \left|\tan\left(-\frac{\pi}{2} + \frac{K}{2}\right)\right|. \tag{3.4}$$

Without loss of generality, we can take ρ'' small enough to ensure that $K \leq \pi/4$. Moreover, we shall assume that γ' and s are small enough such that

$$\begin{cases} \rho' < \rho \frac{1-\gamma'}{\sqrt{1+\gamma'^2}}, \\ 4\tau s < 1. \end{cases} \tag{3.5}$$

Denote by D_ε the subset of $\mathbb{C}$ given by

$$x \in D_\varepsilon \Leftrightarrow -\frac{\pi}{2|\varepsilon|} + \frac{K}{|\varepsilon|} < \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} u_\varepsilon(x)\right) < \frac{\pi}{2|\varepsilon|} - \frac{K}{2|\varepsilon|}. \tag{3.6}$$

Notice the asymmetry in the definition of D_ε. This will be explained in Lemma 6.2.

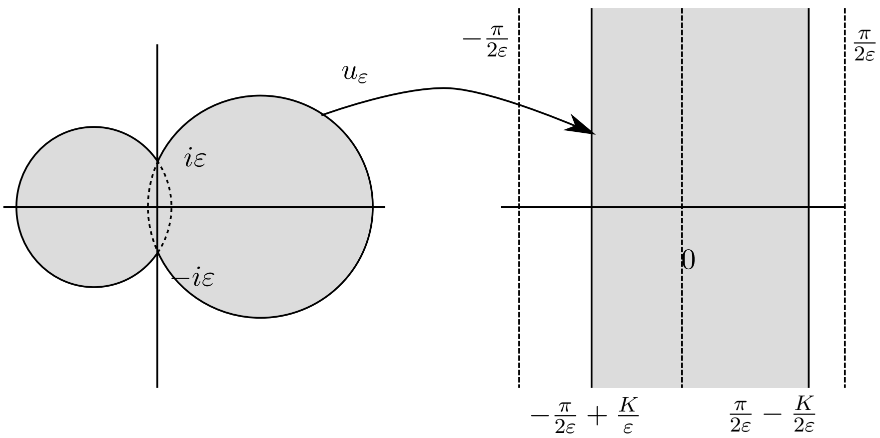

FIGURE 1. The region D_ε, for a real ε

Let us now move to $\mathbb{C}^2$. Let $\widetilde{D}_\varepsilon$ be the product $D_\varepsilon \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|} \subset \mathbb{C}^2$ (the constant $e^{4\pi\rho\tau}$ will be explained in Proposition 4.7), where $\mathbb{D}_r \subset \mathbb{C}$ denotes the open disc of radius r. By definition, since $K \leq \pi/4$, we have

$$\mathbb{D}_{|\varepsilon|} \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|} \subset \widetilde{D}_\varepsilon \subset \mathbb{D}_{\tau|\varepsilon|} \times \mathbb{D}_{2e^{4\pi\rho\tau}|\varepsilon|}. \tag{3.7}$$

Notice in particular that the ratios τ and $2e^{4\pi\rho\tau}$ are independent of ε.

Set $C_\varepsilon := \frac{\varepsilon}{|\varepsilon|} C_0 \setminus D_\varepsilon$ and $\widetilde{C}_\varepsilon := \left(\frac{\varepsilon}{|\varepsilon|}, 1\right) \cdot \widetilde{C}_0 \setminus \widetilde{D}_\varepsilon$ the rotations of C_0 and $\widetilde{C}_0$ of $\frac{\varepsilon}{|\varepsilon|}$ around the y plane. Notice that $\widetilde{C}_\varepsilon \to \widetilde{C}_0$ and $\widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon \to \widetilde{C}_0$ as $\varepsilon \to 0$. Morevover, we have $\widetilde{C}_\varepsilon \subset \widetilde{C}'_0$ for ε sufficiently small (and satisfying (3.1)) The following Lemma will be very useful in the sequel.

Lemma 3.2. *For ε sufficiently small, we have $F_\varepsilon(\widetilde{C}_\varepsilon) \subset \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$.*

Proof. By the choice of $\widetilde{C}_0$ and $\widetilde{C}'_0$, we have $F_0(\widetilde{C}'_0) \subset \widetilde{C}_0$. Moreover, $F_\varepsilon = F_0 + O(\varepsilon^2)$ and F_ε uniformly converges to F_0 on compact subsets of $\widetilde{C}'_0$. The assertion then follows from the the first inclusion in (3.7). □

The first step in the construction of the almost Fatou coordinates consists in considering the functions $\widetilde{u}_\varepsilon$ given by

$$\widetilde{u}_\varepsilon(x, y) := u_\varepsilon(x).$$

The following lemma gives the fundamental estimate on $\widetilde{u}_\varepsilon$: in this chart, the map $F_\varepsilon(x, y)$ approximately acts as a translation by 1 on the first coordinate. Here and in the following, it will be useful to consider the expression

$$\gamma_\varepsilon(x, y) := \frac{\alpha_\varepsilon(x, y)}{1 + x\alpha_\varepsilon(x, y)},$$

where $\alpha_\varepsilon(x,y)$ is as in (1.2). It is immediate to see that $\gamma_\varepsilon(x,y) = 1+qx+ry+O_2(x,y)+O(\varepsilon^2)$.

Lemma 3.3. *Take $p=(x,y)\in \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$. Then*

$$\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) = 1+qx+ry+O_2(x,y)+O(\varepsilon^2).$$

In particular, when γ, R, s and $\varepsilon(\gamma,R,s)$ are small enough, for $p=(x,y)\in \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$ we have

$$|\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) - 1| < \rho'' - 1 \text{ and } \left| \frac{\varepsilon}{|\varepsilon|}\left(\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p)\right) - 1\right| < \rho''-1.$$

Proof. Since $x(F_\varepsilon(x,y)) = x + (x^2+\varepsilon^2)\alpha_\varepsilon(x,y)$, it follows that

$$\frac{i\varepsilon - x(F_\varepsilon(x,y))}{i\varepsilon + x(F_\varepsilon(x,y))} = \frac{(i\varepsilon - x)\left(1+(x+i\varepsilon)\alpha_\varepsilon(x,y)\right)}{(i\varepsilon+x)\left(1+(x-i\varepsilon)\alpha_\varepsilon(x,y)\right)}$$

and so

$$\frac{i\varepsilon + x}{i\varepsilon - x}\frac{i\varepsilon - x(F_\varepsilon(x,y))}{i\varepsilon + x(F_\varepsilon(x,y))} = \frac{1+i\varepsilon\gamma_\varepsilon(x,y)}{1-i\varepsilon\gamma_\varepsilon(x,y)}.$$

The desired difference is then equal to

$$\begin{aligned}
\widetilde{u}_\varepsilon(F_\varepsilon(p)) - \widetilde{u}_\varepsilon(p) &= \frac{1}{2i\varepsilon}\log\frac{1+i\varepsilon\gamma_\varepsilon(x,y)}{1-i\varepsilon\gamma_\varepsilon(x,y)}\\
&= \frac{1}{i\varepsilon}\left[i\varepsilon\gamma_\varepsilon(x,y) + \frac{1}{3}\left(i\varepsilon\gamma_\varepsilon(x,y)\right)^3 + O(\varepsilon^4)\right]\\
&= \gamma_\varepsilon(x,y) + O(\varepsilon^2)\\
&= 1+qx+ry+O_2(x,y)+O(\varepsilon^2)
\end{aligned}$$

and the assertion is proved. ☐

The next step is to slightly modify our coordinate $\widetilde{u}_\varepsilon$ to a coordinate $\widetilde{w}^\iota_\varepsilon$ satisfying the following two properties:

1. $\widetilde{w}^\iota_\varepsilon \to \widetilde{w}^\iota_0$ (with $\widetilde{w}^\iota_0$ as in (2.2)) as $\varepsilon \to 0$, and
2. $\widetilde{w}^\iota_\varepsilon(F^n_\varepsilon(p)) - n \to \widetilde{\varphi}^\iota$ when $\varepsilon\to 0$ and $n\to\infty$ satisfying some relation to be determined later.

We also look for functions $\widetilde{w}^o_\varepsilon$ satisfying analogous properties on $-\widetilde{C}_0$. Recall that the functions $\widetilde{w}^\iota_0(x,y)$ and $\widetilde{w}^o_0(x,y)$ almost semiconjugate the (first coordinate of the) system F_0 to a translation by 1 (by (2.3)).

We set

$$\widetilde{w}_\varepsilon(x,y) := \widetilde{u}_\varepsilon(x,y) - \frac{q}{2}\log(\varepsilon^2+x^2) = \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon+x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2).$$

and consider their incoming and outgoing normalizations $\widetilde{w}^\iota_\varepsilon$ and $\widetilde{w}^o_\varepsilon$ given by

$$\begin{aligned}
\widetilde{w}^\iota_\varepsilon(x,y) &:= \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon+x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2) + \frac{\pi}{2\varepsilon},\\
\widetilde{w}^o_\varepsilon(x,y) &:= \frac{1}{2i\varepsilon}\log\left(\frac{i\varepsilon - x}{i\varepsilon+x}\right) - \frac{q}{2}\log(\varepsilon^2+x^2) - \frac{\pi}{2\varepsilon}.
\end{aligned}$$

It is immediate to check that the first request is satisfied, i.e., that $\widetilde{w}^\iota_\varepsilon(x,y)\to\widetilde{w}^\iota_0$ on $\widetilde{C}_0$ (and $\widetilde{w}^o_\varepsilon(x,y)\to\widetilde{w}^o_0$ on $-\widetilde{C}_0$) as $\varepsilon\to 0$. In the next proposition we estimate the distance between the reading of F_ε in this new chart $\widetilde{w}_\varepsilon$ and the translation by 1. We want to prove, in particular, that now the error has no linear terms in the x variable. Indeed, notice that also for the system F_0 we had to remove this term (see Lemma 2.2) to ensure the convergence of the series with general term $A_0(F^n_0(p))$, by the harmonic behaviour of $x(F^n_0(p))$. For convenience of notation, we denote this error by

$$A_\varepsilon(x,y) := \widetilde{w}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{w_\varepsilon}(x,y) - 1$$

We then have the following estimate.

Proposition 3.4. $A_\varepsilon(x,y) = ry + O_2(x,y) + O(\varepsilon^2)$.

Notice that, differently from [6], here the error is still linear in y. The reason is that we do not add any correction term in y in the expression of $\widetilde{w}_\varepsilon$. On the other hand, by our assumptions we do not have any linear dependence in ε.

Proof. The computation is analogous to the one in [6]. By the definition of $\widetilde{w}_\varepsilon$ and the analogous property of $\widetilde{u}_\varepsilon$ (Lemma 3.3) we have

$$\begin{aligned}\widetilde{w}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{w_\varepsilon}(x,y) = {} & \widetilde{u}_\varepsilon(F_\varepsilon(x,y)) - \widetilde{u}_\varepsilon(x,y) \\ & - \frac{q}{2}\log(\varepsilon^2 + x(F_\varepsilon(x,y))^2) + \frac{q}{2}\log(\varepsilon^2 + x^2) \\ = {} & 1 + qx + ry + O_2(x,y) + O(\varepsilon^2) \\ & - \frac{q}{2}\log\frac{\varepsilon^2 + x(F_\varepsilon(x,y))^2}{\varepsilon^2 + x^2}.\end{aligned}$$

It is thus sufficient to prove that

$$\frac{\varepsilon^2 + x(F_\varepsilon(x,y))^2}{\varepsilon^2 + x^2} = 1 + 2x + O_2(x,y) + O(\varepsilon^2).$$

But

$$\begin{aligned}\varepsilon^2 + x(F_\varepsilon(x,y))^2 &= \varepsilon^2 + x^2 + (x^2+\varepsilon^2)^2\alpha_\varepsilon^2(x,y) + 2x(x^2+\varepsilon^2)\alpha_\varepsilon(x,y) \\ &= (x^2+\varepsilon^2)(1 + 2x\alpha_\varepsilon(x,y) + O(x^2,\varepsilon^2)) \\ &= (x^2+\varepsilon^2)(1 + 2x + O_2(x,y) + O(\varepsilon^2))\end{aligned}$$

and the assertion follows. □

Let us finally introduce the *incoming almost Fatou coordinate*, by means of the $\widetilde{w}^\iota_\varepsilon$, as it was done for the map F_0 in (2.4). Set

$$\widetilde{\varphi^\iota}_{\varepsilon,n}(p) := \widetilde{w}^\iota_\varepsilon(F^n_\varepsilon(p)) - n = \widetilde{w}^\iota_\varepsilon(p) + \sum_{j=0}^{n-1} A_\varepsilon(F^j_\varepsilon(p)). \tag{3.8}$$

We shall be particularly interested in the following relation between the parameter ε and the number of iterations.

Definition 3.5. A sequence $(\varepsilon_\nu, m_\nu) \subset (\mathbb{C}\times\mathbb{N})^{\mathbb{N}}$ such that $\varepsilon_\nu \to 0$ will be said *of bounded type* if $\frac{\pi}{2\varepsilon_\nu} - m_\nu$ is bounded in ν.

Notice that, given an α-sequence (ε_ν, n_ν), the sequence $(\varepsilon_\nu, n_\nu/2)$ is of bounded type.

The following result in particular proves that the coordinates $\widetilde{w}^\iota_\varepsilon$ satisfy the second request. This convergence will be crucial in order to prove Theorem 1.4. Here $\widetilde{\varphi}^\iota$ denotes the Fatou coordinate on $\widetilde{C}_0$ given by Lemma 2.2.

Proposition 3.6. *Let $(\varepsilon_\nu, m_\nu)_{\nu\in\mathbb{N}}$ be a sequence of bounded type. Then*

$$\widetilde{\varphi^\iota}_{\varepsilon_\nu,m_\nu} \to \widetilde{\varphi}^\iota$$

locally uniformly on $\widetilde{C}_0$.

We can also define the outgoing almost Fatou coordinates on $-\widetilde{C}_0$ as

$$\widetilde{\varphi^o}_{\varepsilon,n}(p) := \widetilde{w}^o(F_\varepsilon^{-n}(p)) + n$$

(recall that by assumption $-\widetilde{C}_0$ is contained in a neighbourhood U of the origin where F_ε is invertible, for ε sufficiently small). The following convergence is then an immediate consequence of Proposition 3.6 applied to the inverse system.

Corollary 3.7. *Let (ε_ν, m_ν) be a sequence of bounded type. Then*

$$\widetilde{\varphi^o}_{\varepsilon_\nu,m_\nu} \to \widetilde{\varphi^o}$$

locally uniformly on $-\widetilde{C}_0$.

To prove Proposition 3.6, we need to estimate the series of the errors in (3.8). In particular, we need to bound the modulus of the two coordinates of the orbit $F_\varepsilon^j(p)$, for $p \in \widetilde{\mathbb{C}}_0$ and j up to (approximately) $\pi/(2|\varepsilon|)$. This is the content of the next section. The proof of Proposition 3.6 will then be given in Section 5.

In our study, we will need to carefully compare the behaviour of F_ε in $\widetilde{C}_0$ and the one of F_ε^{-1} on $-\widetilde{C}_0$. Notice that F_ε^{-1} is given by

$$F_\varepsilon^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - (x^2+\varepsilon^2)(1+(q-1)x+ry+ +O(\varepsilon^2)+O_2(x,y)) \\ y(1-\rho x + O(\varepsilon^2)+O_2(x,y)) \end{pmatrix}$$

In order to compare the behaviour of the orbits for F_ε^{-1} with the ones for F_ε, it will be useful to consider the change of coordinate $(x,y) \mapsto (-x,y)$ and thus study the maps

$$\begin{aligned} H_\varepsilon\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x + (x^2+\varepsilon^2)(1+(-q+1)x+ry+ +O(\varepsilon^2)+O_2(x,y)) \\ y(1+\rho x + O(\varepsilon^2)+O_2(x,y)) \end{pmatrix} \\ &= \begin{pmatrix} x + (x^2+\varepsilon^2)\alpha_\varepsilon^H(x,y) \\ y(1+\rho x + \beta_\varepsilon^H(x,y)) \end{pmatrix} \end{aligned} \tag{3.9}$$

In this way, we can study both F_ε and H_ε in the same region of space. Notice that the main difference between F_ε and H_ε is that the coefficient q has changed sign.

4. The estimates for the points in the orbit

In this section we are going to study the orbit of a point $p \in \widetilde{C}_0$ under the iteration of F_ε. In particular, since the main application we have in mind is the study of $F_{\varepsilon_\nu}^{n_\nu}$ when (ε_ν, n_ν) is an α-sequence, we shall be primarily interested in the study of orbit up to an order of $\pi/|\varepsilon|$ iterations.

Recall that the set $\widetilde{C}_0$ is given by Proposition 2.1 and in particular consists of points that converge to the origin under F_0 tangentially to the (negative) real axis of the complex direction $[1:0]$. We shall still assume (by taking $R \ll 1$ small enough) that $\widetilde{C}_0$ is contained in a small neighbourhood U of the origin where F_0 and F_ε are invertible, for ε sufficiently small.

By Lemma 3.1, for every compact $\mathcal{C} \subset \widetilde{C}_0$ there exist two constants $M^-(\mathcal{C})$ and $M^+(\mathcal{C})$ such that

$$-\frac{\pi}{2|\varepsilon|} + M^-(\mathcal{C}) \le \operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(p)\right) \le -\frac{\pi}{2|\varepsilon|} + M^+(\mathcal{C}) \quad \forall p \in \mathcal{C}, \forall \varepsilon \le \varepsilon_0. \tag{4.1}$$

Without loss of generality, we will assume that M^- and M^+ are integers and $\gg 1$ (since $R \ll 1$).

We shall divide the estimates of the coordinates of $F_\varepsilon^j(p)$ according to its position with respect to the set $\widetilde{D}_\varepsilon$, i.e., according to the position of $x(F_\varepsilon^j(p))$ with respect to D_ε as in (3.6). The following notation will be consistently used through all our study.

Definition 4.1. Given $p \in \widetilde{C}_0$ and ε such that $p \in \widetilde{C}_\varepsilon$, we define the *entry time* $n_p(\varepsilon)$ and the *exit time* $n'_p(\varepsilon)$ by

$$\begin{aligned} n_p(\varepsilon) &:= \min\{\, j \in \mathbb{N} \colon F_\varepsilon^j(p) \in \widetilde{D}_\varepsilon \,\} \\ n'_p(\varepsilon) &:= \min\{\, j \in \mathbb{N} \colon F_\varepsilon^j \notin \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon \,\} \end{aligned} \tag{4.2}$$

The next Proposition gives the bounds on $n_p(\varepsilon)$ that we shall need in the sequel.

Proposition 4.2. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^-, M^+ be as in* (4.1)*. Then, for every $p = (x,y) \in \mathcal{C}$ and ε sufficiently small,*

$$\frac{K}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} \le n_p(\varepsilon) \le \frac{K}{(2-\rho'')|\varepsilon|} - \frac{M^-}{2-\rho''}.$$

In particular, $F_\varepsilon^j(p) \in \widetilde{C}_\varepsilon$ for $0 \le j < \frac{K}{\rho''|\varepsilon|} - \frac{M^+}{\rho''}$.

Proof. Notice that, since $F_\varepsilon(\widetilde{C}_\varepsilon) \subset \widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$ (by Lemma 3.2), we only have to study the first coordinate of the orbit. Since $\widetilde{C}_\varepsilon \to \widetilde{C}_0$, we have that $\mathcal{C} \subset \widetilde{C}_\varepsilon$ for ε sufficiently small. From Lemma 3.3 it follows that

$$2 - \rho'' < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon(p))\right) - \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(p)\right) < \rho''.$$

Thus, we deduce that

$$-\frac{\pi}{2|\varepsilon|} + M^- + (2-\rho'')j < \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(q))\right) < -\frac{\pi}{2|\varepsilon|} + M^+ + \rho'' j \tag{4.3}$$

and the assertion follows from the definition of D_ε (see (3.6)). $\square$

4.1. **Up to $n_p(\varepsilon)$.** Given p in some compact subset $\mathcal{C} \in \widetilde{C}_0$, here we study the modulus of the two coordinates of the points in the orbit of p for F_ε until they fall in $\widetilde{D}_\varepsilon$, i.e., for a number of iteration up to $n_p(\varepsilon)$. We start estimating the first coordinate. Here we shall make use of the definition of K (see (3.4)).

Lemma 4.3. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in (4.1). Then*

$$\left|x(F_\varepsilon^j(p))\right| \le \frac{2}{j + M^-}$$

for every $p \in \mathcal{C}$, for ε small enough and $j \le n_p(\varepsilon)$.

Proof. The statement follows from Lemma 3.1 (2) and the (first) inequality in (4.3). Indeed, we have (recall that $3/4 < 2 - \rho'' < 1$)

$$\begin{aligned}\left|x(F_\varepsilon^j(p))\right| &< \frac{1}{\frac{\pi}{2|\varepsilon|} + \mathrm{Re}(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p)))}\\ &\le \frac{1}{\frac{\pi}{2|\varepsilon|} - \frac{\pi}{2|\varepsilon|} + (2-\rho'')j + M^-} \le \frac{1}{2-\rho''}\frac{1}{j+M^-} \le \frac{2}{j+M^-}.\end{aligned}$$

and the inequality is proved. $\square$

We now come to the second coordinate. Estimating this is the main difference between our setting and the semiparabolic one. Notice that, by (1.2), in order to bound the terms $\left|y(F_\varepsilon^j(p))\right|$, we will need to get an estimate from below of the first coordinate. This will be done by means of the following lemma.

Lemma 4.4. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in (4.1). Let $p, q \in \mathcal{C}$ and set $q_j := \varepsilon\left(\tan\left(\varepsilon(\widetilde{u}_\varepsilon(q) + j)\right)\right)$ and $\widetilde{q}_j := \varepsilon\left(\tan\left(\varepsilon(\widetilde{u}_\varepsilon(q) + |\varepsilon|\, j/\varepsilon)\right)\right)$. Then, for some positive constants C depending on $\mathcal{C}$ and C_ε depending on $\mathcal{C}$ and ε, and going to zero as $\mathrm{Re}\,\varepsilon \to 0$,*

$$\left|x(F_\varepsilon^j(p)) - q_j\right| < C\frac{1 + \log(M^- + j)}{(M^- + j)^2} \tag{4.4}$$

and

$$\left|x(F_\varepsilon^j(p)) - \widetilde{q}_j\right| < C\frac{1 + \log(M^- + j)}{(M^- + j)^2} + C_\varepsilon \frac{1}{M^+ + j} \tag{4.5}$$

for every $0 \le j \le n_p(\varepsilon)$.

Notice in particular that the two estimates reduce to the same for ε real.

Proof. The idea is to first estimate the distance between the two sequences $\widetilde{u}_\varepsilon(F_\varepsilon^j(p))$ and $\widetilde{u}_\varepsilon(q) + j$ (and between $\widetilde{u}_\varepsilon(F_\varepsilon^j(p))$ and $\widetilde{u}_\varepsilon(q) + |\varepsilon|\, j/\varepsilon$) and then to see how this distance is transformed by the application of the inverse of u_ε. Notice that, since $j \le n_p(\varepsilon)$, by definition of $\widetilde{D}_\varepsilon$ (see (3.6)) we have $\mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p))\right) < -\frac{\pi}{4|\varepsilon|}$ for the points in the orbit under consideration (since $K \le \pi/4$).

We first prove that

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(q) - j\right| \le C_1\left(1+\log(M^-+j)\right). \tag{4.6}$$

Notice that this is an improvement with respect to the estimate obtained in Lemma 3.3, but that we shall need both that estimate and the bound from above obtained in Lemma 4.3 in order to get this one.

By the definition of M^-, we have that $|x(p)|$ and $|x(q)|$ are bounded above by $2/M^-$. Recalling that $|y| \le s\,|x|$ for every $(x,y) \in \widetilde{C}_\varepsilon$, Lemma 3.3 gives

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(p) - j\right| \le c_1 \sum_{i<j} \left|x(F_\varepsilon^i(p))\right| + c_2 \sum_{i<j}\left(\left|x(F_\varepsilon^i(p))\right|^2 + |\varepsilon|^2\right).$$

Since by Lemma 4.3 we have $\left|x(F_\varepsilon^j(p))\right| \le 2/(j+M^-)$ and the maximal number of iterations $n_p(\varepsilon)$ is bounded by a constant times $1/|\varepsilon|$, this gives

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(p) - j\right| \le C_2\left(1+\log(M^-+j)\right)$$

for some positive C_2, and the estimate (4.6) follows since the two sequences $\left(\widetilde{u}_\varepsilon(p)+j\right)_j$ and $\left(\widetilde{u}_\varepsilon(q)+j\right)_j$ obviously stay at constant distance.

We then consider the sequence $\widetilde{q}_j$. Using (4.6), it is immediate to see that

$$\left|\widetilde{u}_\varepsilon(F_\varepsilon^j(p)) - \widetilde{u}_\varepsilon(q) - |\varepsilon|\,j/\varepsilon\right| \le C_1\left(1+\log(M^-+j)\right) + |\arg(\varepsilon)|\,j, \tag{4.7}$$

since the distance between the two sequences $\widetilde{u}_\varepsilon(q)+j$ and $\widetilde{u}_\varepsilon(q)+|\varepsilon|\,j/\varepsilon$. is bounded by the last term.

We now need to estimate how the errors in (4.6) and (4.7) are transformed when passing to the dynamical space, and in particular recover the quadratic denominator in (4.4). By (4.7) we have

$$\begin{aligned}\mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(F_\varepsilon^j(p))\right) &\ge -\frac{\pi}{2\,|\varepsilon|} + M^- + j - C_1\left(1+\log(M^-+j)\right) - |\arg\varepsilon|\,j\\ &> -\frac{\pi}{2\,|\varepsilon|} + C_3(M^-+j)\end{aligned}$$

for ε sufficiently small (as in (3.1)), $j \le n_p(\varepsilon)$ and some $C_3 > 0$. So, given $L > 0$, it is enough to bound from above the modulus of the derivative of the inverse of u_ε on the strip $\left\{-\frac{\pi}{2|\varepsilon|} + L \le \mathrm{Re}\left(\frac{\varepsilon}{|\varepsilon|}w\right) < -\frac{\pi}{4|\varepsilon|}\right\}$ by (a constant times) $1/|L|^2$. This can be done with a straightforward computation. Recall that $u_\varepsilon(z) = \frac{1}{\varepsilon}\arctan\left(\frac{z}{\varepsilon}\right)$, so that its inverse is given by $\varepsilon\tan(\varepsilon w)$. The derivative of this inverse at a point $-\pi/2\varepsilon + w$ is thus given by $\psi_\varepsilon(w) = \varepsilon^2\left(\cos(\varepsilon w)\right)^{-2}$. On the strip in consideration, ψ_ε takes its maximum at $w = -\frac{\pi}{2\varepsilon} + L$, where we have $\psi_\varepsilon(-\frac{\pi}{2\varepsilon}+L) = \varepsilon^2/\sin^2(\varepsilon L)$. The estimate then follows since $x \le 2\sin(x)$ on $[0,\pi/4]$. □

Proposition 4.5. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset, M^-, M^+ be as in (4.1) and C, C_ε as in Lemma 4.4. Then*

$$\left(\frac{1}{\rho'} - C_\varepsilon\right)\frac{1}{M^+ + j} - C\frac{1+\log(M^-+j)}{(M^-+j)^2} \le \left|x(F_\varepsilon^j(p))\right| \le \frac{2}{j+M^-}$$

for every $p \in \mathcal{C}$, for ε small enough and $j \le n_p(\varepsilon)$.

Proof. The second inequality is the content of Lemma 4.3. Let us then prove the lower bound. By Lemma 4.4, it is enough to get the bound

$$\frac{1}{\rho'(M^+ + j)} \le |\widetilde{q}_j|$$

where $\widetilde{q}_j := \varepsilon\tan\left(\varepsilon(\mathrm{Re}(\widetilde{u}_\varepsilon(p)) + |\varepsilon|\,j/\varepsilon)\right)$ as in Lemma 4.4. Notice that we arranged the points $\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(\widetilde{q}_j)$ to be on the real axis. Since we have $\mathrm{Re}\,\frac{\varepsilon}{|\varepsilon|}\widetilde{u}_\varepsilon(q_0) < -\frac{\pi}{2|\varepsilon|} + M^+$ (and thus $\mathrm{Re}\,\frac{\varepsilon}{|\varepsilon|}u_\varepsilon\left(\widetilde{q}_j\right) \le$

$-\frac{\pi}{2|\varepsilon|} + M^+ + j)$, it follows that

$$|\widetilde{q}_j| \geq |\varepsilon| \left| \tan\left(\varepsilon \left(-\frac{\pi}{2|\varepsilon|} + (M^+ + j)\frac{|\varepsilon|}{\varepsilon} \right) \right) \right| = \frac{|\varepsilon|}{\tan\left(M^+ |\varepsilon| + j|\varepsilon|\right)}.$$

We thus have to prove that, for ε sufficiently small and $j \leq n_p(\varepsilon)$,

$$\frac{|\varepsilon| M^+ + |\varepsilon| j}{\tan\left(M^+ |\varepsilon| + j|\varepsilon|\right)} > \frac{1}{\rho'}.$$

The left hand side is decreasing in j, so we can evaluate it at $j = n_p(\varepsilon)$, which is less or equal than $\frac{K}{(2-\rho'')|\varepsilon|}$ by Proposition 4.2. We thus need to prove that, for ε sufficiently small,

$$\frac{|\varepsilon| M^+ + \frac{K}{2-\rho''}}{\left|\tan\left(M^+ |\varepsilon| + \frac{K}{2-\rho''}\right)\right|} > \frac{1}{\rho'}.$$

This follows since $|\varepsilon| M^+ + \frac{K}{2-\rho''} < 2K$ for $|\varepsilon| \ll 1$ and, by assumption, K satisfies $\left|\frac{2K}{\tan(2K)}\right| > \frac{1}{\rho'}$. This concludes the proof. □

We can now give the estimate for the second coordinate.

Proposition 4.6. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^+ be as in (4.1). There exists a positive constant c_1, depending on $\mathcal{C}$, such that for $p \in \mathcal{C}$ and $J \leq n_p(\varepsilon)$,*

$$\left|y(F_\varepsilon^J(p))\right| \leq c_1 |y(p)| \prod_{l=M^+}^{M^+ + J - 1} \left(1 - \frac{\widetilde{\rho}}{l}\right)$$

for some $1 < \widetilde{\rho} < \frac{\rho}{\rho'} \frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$.

Notice that $1 < \frac{\rho}{\rho'} \frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$ by the assumption (3.5).

Proof. We shall make use of both estimates obtained in Proposition 4.5. Since the part of orbit which we are considering is in $\widetilde{C}_\varepsilon$ (at least) up to $J-1$, we have $\left|y(F_\varepsilon^j(p))\right| \leq s \left|x(F_\varepsilon^j(p))\right|$ and $\left|x(F_\varepsilon^j(p))\right| > |\varepsilon|$, for $j \leq J-1$. So, by the expression of $y(F_\varepsilon(p))$ in (1.2), we get

$$\begin{aligned}
\left|y(F_\varepsilon^J(p))\right| &\leq |y(p)| \prod_{j=0}^{J-1} \left|1 + \rho x(F_\varepsilon^j(p)) + O(x^2(F_\varepsilon^j(p))\right| \\
&\leq |y(p)| \prod_{j=0}^{J-1} \left(\left|1 + \rho x(F_\varepsilon^j(p))\right| + \widetilde{c}_1 \left|x^2(F_\varepsilon^j(p))\right|\right)
\end{aligned}$$

for some positive $\widetilde{c}_1$. For ε sufficiently small, we have $\widetilde{C}_\varepsilon \subset \widetilde{C}'_0 = \widetilde{C}_0(\gamma', R, s)$ (see Proposition 2.1). This implies that $\left|\mathrm{Im}\left(x(F_\varepsilon^j(p))\right)\right| < \gamma' \left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right|$ for every $j < n_p(\varepsilon)$. Thus

$$\begin{aligned}
\left|1 + \rho x(F_\varepsilon^j(p))\right| &\leq 1 - \rho \left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right| + \rho \left|\mathrm{Im}\left(x(F_\varepsilon^j(p))\right)\right| \\
&\leq 1 - \rho(1-\gamma') \left|\mathrm{Re}\left(x(F_\varepsilon^j(p))\right)\right| \\
&\leq 1 - \rho \frac{1-\gamma'}{\sqrt{1+\gamma'^2}} \left|x(F_\varepsilon^j(p))\right|
\end{aligned}$$

and thus, by the estimates on $x(F_\varepsilon^j(p))$ in Proposition 4.5 we deduce that (for ε sufficiently small)

$$
\begin{aligned}
\left|y(F_\varepsilon^J(p))\right| &\leq |y(p)| \prod_{j=0}^{J-1} \left(1 - \rho \frac{1-\gamma'}{\sqrt{1+\gamma'^2}} \left(\frac{1}{\rho'} - C_\varepsilon\right) \frac{1}{M^+ + j} + \widetilde{c}_1' \frac{1+\log(M^- + j)}{(M^- + j)^2}\right) \\
&\leq c_1 |y(p)| \prod_{j=0}^{J-1} \left(1 - \widetilde{\rho} \frac{1}{M^+ + j}\right)
\end{aligned}
$$

where $\widetilde{\rho}$ is some constant such that $1 < \widetilde{\rho} < \frac{\rho}{\rho'} \frac{1-\gamma'}{\sqrt{1+\gamma'^2}}$, and the assertion follows. $\square$

4.2. **From $n_p(\varepsilon)$ to $n'_p(\varepsilon)$.** Notice that $\widetilde{D}_\varepsilon$ needs not to be F_ε-invariant. In this section we estimate the second coordinate for points in an orbit entering $\widetilde{D}_\varepsilon$ (and in particular explain the constant $e^{4\pi\rho\tau}$ in the definition of $\widetilde{D}_\varepsilon$). Our goal is prove a lower bound on $n'_p(\varepsilon)$ (and moreover to prove that the orbit cannot come back to $\widetilde{C}_\varepsilon$). This will in particular give an estimate for the coordinates of the point in the orbit for j up to the lower bound of $n'_p(\varepsilon)$ (since in $\widetilde{D}_\varepsilon$ both $|x|$ and $|y|$ are bounded by (a constant times) $|\varepsilon|$).

Proposition 4.7. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset. Then, for every $p \in \mathcal{C}$, and $n_p(\varepsilon) < j \leq n'_p(\varepsilon)$, we have*

$$
\left|y(F_\varepsilon^j(p))\right| \leq e^{4\pi\rho\tau} \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \leq e^{4\pi\rho\tau} |\varepsilon|
$$

Proof. Recall that $\tau = \tan\left(-\frac{\pi}{2} + \frac{K}{2}\right)$ and that by the assumption (3.5) we have $4s\tau < 1$. Since the part of orbit under consideration is contained in $\widetilde{D}_\varepsilon$ (and thus $\left|x(F_\varepsilon^j(p))\right| \leq \tau |\varepsilon|$, by (3.7)), we have

$$
\begin{aligned}
\left|y(F_\varepsilon^j(p))\right| &\leq \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \prod_{i=n_p(\varepsilon)}^{j-1} (1 + 2\rho\tau|\varepsilon|) \\
&\leq \left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \prod_{i=n_p(\varepsilon)}^{\lfloor \frac{\pi - K/2}{(2-\rho'')|\varepsilon|} \rfloor} (1 + 2\rho\tau|\varepsilon|).
\end{aligned}
$$

The product is bounded by $(1+2\rho\tau|\varepsilon|)^{2\pi/|\varepsilon|} \leq e^{4\pi\rho\tau}$ as $\varepsilon \to 0$. Moreover, we have $\left|y(F_\varepsilon^{n_p(\varepsilon)}(p))\right| \leq \left|y(F_\varepsilon^{n_p(\varepsilon)-1}(p))\right| \left|1 + \rho x(F_\varepsilon^{n_p(\varepsilon)-1})\right| \leq 4s\tau|\varepsilon| < |\varepsilon|$. This gives the assertion. $\square$

We can now give the estimate on $n'_p(\varepsilon)$.

Proposition 4.8. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^-, M^+ be as in (4.1). Then, for every $p \in \mathcal{C}$,*

$$
\frac{\pi - K/2}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} \leq n'_p(\varepsilon) \leq \frac{\pi - K/2}{(2-\rho'')|\varepsilon|} - \frac{M^-}{2-\rho''}.
$$

Moreover, we have $\left|y(F_\varepsilon^j(p))\right| \leq e^{4\pi\rho\tau}|\varepsilon|$ for $n_p(\varepsilon) \leq j < n'_p(\varepsilon)$ and

$$
\operatorname{Re}\left(\frac{\varepsilon}{|\varepsilon|} \widetilde{u}_\varepsilon \left(F_\varepsilon^{n'_p(\varepsilon)}\right)\right) \geq \frac{\pi - K}{2|\varepsilon|}.
$$

In particular, once entered in $\widetilde{D}_\varepsilon$, the orbit cannot come back to $\widetilde{C}_\varepsilon$.

Proof. By Proposition 4.7, the modulus of the second coordinate of the points of the orbit is bounded by $e^{4\rho\pi\tau}|\varepsilon|$ for $n_p(\varepsilon) < j \leq n'_p(\varepsilon)$. Since for $j \leq n_p(\varepsilon)$ it is bounded by $s\left|x(F_\varepsilon^j(p))\right|$, the assertion follows from Equation (4.3). $\square$

4.3. **After $n'_p(\varepsilon)$.** In order to study the behaviour of F_ε after $\widetilde{D}_\varepsilon$, we shall make use of the family H_ε introduced in (3.9). The following proposition is an immediate consequence of the analogous results for F_ε (first assertion of Lemma 4.4). We denote by $n_p^H(\varepsilon)$ the entry time for H (see Definition 4.1).

Lemma 4.9. *Let $\mathcal{C} \subset \widetilde{C}_0$ be a compact subset and M^- be as in* (4.1)*. Let p, q be contained in some compact subset $\mathcal{C} \subset \widetilde{C}_0$. Then, for ε sufficiently small,*

$$\left|x(F_\varepsilon^j(p)) - x(H_\varepsilon^j(q))\right| < C\frac{1+\log(M^-+j)}{(M^-+j)^2}$$

for every $0 \le j \le \min(n_p(\varepsilon), n_q^H(\varepsilon))$, for some positive constant C.

We will get the estimates on the second coordinate in this part of the orbit directly in Section 6, when proving Theorem 1.4, by applying Proposition 4.6 to both F_ε and H_ε.

5. A preliminary convergence: proof of Proposition 3.6

In this section we prove Proposition 3.6. Namely, given a sequence (ε_ν, m_ν) of bounded type (see Definition 3.5), we prove that $\widetilde{\varphi^\iota}_{\varepsilon_\nu, m_\nu} \to \widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}_{\varepsilon_\nu, m_\nu} \to \widetilde{\varphi^o}$, locally uniformly on $\widetilde{C}_0$ and $-\widetilde{C}_0$, where $\widetilde{\varphi^\iota}$ and $\widetilde{\varphi^o}$ are the Fatou coordinates for F_0 given by Lemma 2.2. Recall that by assumption these two sets are contained in a neighbourhood U of the origin where F_ε is invertible, for ε sufficiently small, ans thus in particular where $\widetilde{\varphi^o}$ is well defined. We shall need the following elementary Lemma.

Lemma 5.1. *Let $a \in \mathbb{R}$, be strictly greater than 1. Then, for every $j_0 \ge l_0 \ge 1$ such that $0 < 1 - \frac{a}{l} < 1$ for every $l \ge l_0$, the series*

$$\sum_{j=j_0}^{\infty} \prod_{l=l_0}^{j} \left(1 - \frac{a}{l}\right)$$

converges.

Notice that the Lemma is false when $a = 1$, since the series reduces to an harmonic one. In our applications a will essentially be ρ, which we assume by hyphotesis to be greater than 1.

Proof. As in [16, Lemma 4], let us set $P_j := \prod_{l=l_0}^{j}(1 - \frac{a}{l})$ and notice that the P_j's admit an explicit expression as

$$P_j = c\,\frac{\Gamma(j+1-a)}{\Gamma(j+1)}$$

for some constant $c = c(l_0)$, where Γ is the Euler Gamma function. Since $\Gamma(j+1-a) \sim \frac{1}{j^a} j!$ as $j \to \infty$, we deduce that $P_j \sim c\frac{1}{j^a}$, and so $\sum_j P_j$ converges. $\square$

We can now prove Proposition 3.6. The proof follows the main ideas of the one of [6, Theorem 2.6]. The major issue (and the main difference with respect to [6]) will be to take into account the errors due the $O(y)$-terms in the estimates. This will be done by means of the following Lemma, which relies on Propositions 4.6 and 4.7.

Lemma 5.2. *Let $p \in \widetilde{C}_0$ and $n_p(\varepsilon)$ be as in* (4.2)*. Let $\overline{n}(\varepsilon)$ be such that $n_p(\varepsilon) \le \overline{n}(\varepsilon) \le \frac{3\pi}{5|\varepsilon|}$. Then the following hold:*

1. *the function $\varepsilon \mapsto \sum_{j=1}^{\overline{n}(\varepsilon)} \left(\left| y\left(F_\varepsilon^j(p)\right)\right| + \left| y\left(F_0^j(p)\right)\right| \right)$ is bounded, locally uniformly on p, for ε sufficiently small;*
2. *$\lim_{\varepsilon \to 0} \sum_{j=n_p(\varepsilon)+1}^{\overline{n}(\varepsilon)} \left| y(F_\varepsilon^j(p)) \right| = 0$, locally uniformly on p.*

Notice that, by Proposition 4.8, $n'_p(\varepsilon) \geq \frac{\pi - K/2}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} \geq \frac{7\pi}{8}\frac{5}{4|\varepsilon|} - \frac{M^+}{\rho''} \geq \frac{3\pi}{5|\varepsilon|}$ for ε sufficiently small. So, in particular, the orbit up to time $\overline{n}(\varepsilon)$ is contained in $\widetilde{C}_\varepsilon \cup \widetilde{D}_\varepsilon$. On the other hand, we have $n_p(\varepsilon) + \frac{M^-}{2-\rho''} \leq \frac{K}{(2-\rho'')|\varepsilon|} \leq \frac{\pi/4}{(2-5/4)|\varepsilon|} - \frac{M^-}{2-\rho''} \leq \frac{\pi}{3|\varepsilon|}$. So, in particular, the assumption of Lemma 5.2 is satisfied when $(\varepsilon_\nu, \overline{n}(\varepsilon_\nu))$ is of bounded type.

Proof. We start with the first point. The convergence of the second part of the series is immediate from Proposition 2.1, by the harmonic behaviour of $x(F_0^j(p))$ and the estimate (2.1). Let us thus consider the first part. Here we split this series in a first part, with the indices up to $n_p(\varepsilon)$ and in the remaining part starting from $n_p(\varepsilon)+1$. The sum is thus given by

$$\sum_{j=1}^{n_p(\varepsilon)} \left|y\left(F_\varepsilon^j(p)\right)\right| + \sum_{j=n_p(\varepsilon)+1}^{\overline{n}(\varepsilon)} \left|y\left(F_\varepsilon^j(p)\right)\right|$$

and, by Propositions 4.6 and 4.7, this is bounded by (a constant times)

$$\sum_{j=1}^{n_p(\varepsilon)} \prod_{l=M^+}^{M^++j-1} \left(1 - \frac{\widetilde{\rho}}{l}\right) + \left(\prod_{j=M^+}^{n_p(\varepsilon)-1+M^+} \left(1 - \frac{\widetilde{\rho}}{j}\right)\right) \cdot \sum_{j=n_p(\varepsilon)}^{\overline{n}(\varepsilon)} e^{4\pi\rho\tau}$$

where M^+ is as in (4.1) and $\widetilde{\rho}$ is (as in Proposition 4.6) a constant greater than 1. By the lower estimates on $n_p(\varepsilon)$ in Proposition 4.2 and the asymptotic behaviour proved in Lemma 5.1, the last expression is bounded by

$$\sum_{j=1}^{\infty} \prod_{l=M^+}^{j-1+M^+} \left(1 - \frac{\widetilde{\rho}}{l}\right) + \frac{3\pi}{5|\varepsilon|} \cdot e^{4\pi\rho\tau} \cdot \left(\frac{1}{\frac{K}{\rho''|\varepsilon|} - \frac{M^+}{\rho''} - 1 + M^+}\right)^{\widetilde{\rho}}.$$

The first term is bounded, again by Lemma 5.1, and the second one (which, up to a constant, is in particular a majorant for the sum in the second point in the statement) goes to zero as $\varepsilon \to 0$ (since $\widetilde{\rho} > 1$). This proves both statements. □

Proof of Proposition 3.6. First of all, recall that by Lemma 2.2 the sequence

$$\widetilde{\varphi}^\iota{}_{0,m_\nu} = \widetilde{w}_0^\iota + \sum_{j=0}^{m_\nu - 1} A_0(F_0^j(p))$$

converges to a (1-dimensional) Fatou coordinate $\widetilde{\varphi}^\iota$ (for this we just need that $m_\nu \to \infty$). It is then enough to show that the difference $\widetilde{\varphi}^\iota{}_{\varepsilon_\nu,m_\nu} - \widetilde{\varphi}^\iota{}_{0,m_\nu}$ goes to zero as $\nu \to \infty$. Here we shall make use of the hypothesis that the sequence (ε_ν, m_ν) is of bounded type. The difference is equal to

$$\widetilde{\varphi}^\iota{}_{\varepsilon_\nu,m_\nu}(p) - \widetilde{\varphi}^\iota{}_{0,m_\nu}(p) = \widetilde{w}_{\varepsilon_\nu}^\iota(p) - \widetilde{w}_0^\iota(p) + \sum_{j=0}^{m_\nu-1} \left(A_{\varepsilon_\nu}(F_{\varepsilon_\nu}^j(p)) - A_0(F_0^j(p))\right)$$

and we see that the first difference goes to zero as $\nu \to \infty$. We thus only have to estimate the second part, whose modulus is bounded by

$$\begin{aligned}\sum_I + \sum_{II} := &\sum_{j=0}^{m_\nu-1} \left|A_0(F_{\varepsilon_\nu}^j(p)) - A_0(F_0^j(p))\right| \\ &+ \sum_{j=0}^{m_\nu-1} \left|A_{\varepsilon_\nu}(F_{\varepsilon_\nu}^j(p)) - A_0(F_{\varepsilon_\nu}^j(p))\right|.\end{aligned}$$

Let us consider the first sum. First of all, we prove that the majorant

$$\sum_{j=1}^{m_\nu-1} \left(\left|A_0(F_{\varepsilon_\nu}^j(p))\right| + \left|A_0(F_0^j(p))\right|\right)$$

converges. This follows from the fact that $A_0(p) = O(x^2, y)$ by Proposition 3.4, the estimates on $\left|x(F_0^j(p))\right|$ and $\left|x(F_{\varepsilon_\nu}^j(p))\right|$ in Propositions 2.1 and 4.5 and from Lemma 5.2 (1). Indeed, with M^+ as in (4.1), we have (for some positive constant K_0),

$$
\begin{aligned}
\sum_I &\leq \sum_{j=1}^{m_\nu-1} \left(\left|A_0(F_{\varepsilon_\nu}^j(p))\right| + \left|A_0(F_0^j(p))\right|\right) \\
&\leq K_0 \sum_{j=1}^{m_\nu-1} \left(\left|x(F_{\varepsilon_\nu}^j(p))\right|^2 + \left|x(F_0^j(p))\right|^2\right) + K_0 \sum_{j=1}^{m_\nu-1} \left(\left|y(F_{\varepsilon_\nu}^j(p))\right| + \left|y(F_0^j(p))\right|\right) \\
&\leq K_0 \sum_{j=1}^{m_\nu-1} \left(\frac{8}{(j+M^+)^2} + |\varepsilon_\nu|^2\right) + K_0 \sum_{j=1}^{m_\nu-1} \left(\left|y(F_{\varepsilon_\nu}^j(p))\right| + \left|y(F_0^j(p))\right|\right) \leq B
\end{aligned}
$$

where in the last passage we used the assumption that the sequence (ε_ν, m_ν) is of bounded type to estimate the sum of the $|\varepsilon_\nu|^2$'s and in order to apply Lemma 5.2 (1) for the second sum.

We now prove that $\sum_I$ goes to zero, as $\nu \to \infty$. Given any small η, we look for a sufficiently large J such that the sum

$$
\sum_{j=J}^{m_\nu-1} \left|A_0(F_{\varepsilon_\nu}^j(p)) - A_0(F_0^j(p))\right|
$$

is less than η for $|\varepsilon_\nu|$ smaller than some ε_0. The convergence to 0 of $\sum_I$ will then follow from the fact that $A_0(F_{\varepsilon_\nu}^j(p)) - A_0(F_0^j(p)) \to 0$ as $\nu \to \infty$, for every fixed j. As above, this sum is bounded by

$$
\sum_{j=J}^{m_\nu-1} \left(\frac{8}{(j+M^+)^2} + |\varepsilon_\nu|^2\right) + \sum_{j=J}^{m_\nu-1} \left|y(F_{\varepsilon_\nu}^j(p))\right| + \sum_{j=J}^{m_\nu-1} \left|y(F_0^j(p))\right|. \tag{5.1}
$$

For J sufficiently large, the first sum is smaller than $\eta/3$ (uniformly in ε), since (ε_ν, m_ν) is of bounded type. The same is true for the third one, by the harmonic behaviour of $x(F_0^j(p))$ and the estimate (2.1). We are thus left with the second sum of (5.1). We split it as in Lemma 5.2:

$$
\sum_{j=J}^{m_\nu-1} \left|y(F_{\varepsilon_\nu}^j(p))\right| \leq \sum_{j=J}^{n_p(\varepsilon_\nu)} \left|y(F_{\varepsilon_\nu}^j(p))\right| + \sum_{j=n_p(\varepsilon_\nu)+1}^{m_\nu-1} \left|y(F_{\varepsilon_\nu}^j(p))\right|. \tag{5.2}
$$

Lemma 5.2 (2) implies that the second sum of the right hand side goes to zero as $\varepsilon_\nu \to 0$. We are thus left with the first sum in the right hand side of (5.2). We estimate it by applying twice Proposition 4.6 and Lemma 5.1:

$$
\begin{aligned}
\sum_{j=J}^{n_p(\varepsilon_\nu)} \left|y(F_{\varepsilon_\nu}^j(p))\right| &\leq c_1 \sum_{j=J}^{n_p(\varepsilon_\nu)} \left|y(F_{\varepsilon_\nu}^J(p))\right| \prod_{l=J+M^+}^{j-1+M^+} \left(1 - \frac{\widetilde{\rho}}{l}\right) \\
&\leq c_1 \left|y(F_{\varepsilon_\nu}^J(p))\right| \sum_{j=J}^{\infty} \prod_{l=J+M^+}^{j-1+M^+} \left(1 - \frac{\widetilde{\rho}}{l}\right) \\
&\leq C_1 \left|y(F_{\varepsilon_\nu}^J(p))\right| \\
&\leq C_2 \left|y(p)\right| \prod_{l=M^+}^{J-1+M^+} \left(1 - \frac{\widetilde{\rho}}{l}\right).
\end{aligned}
$$

We can then take J large enough (and independent from ε) so that the last term is smaller than $\frac{\eta}{6}$. Notice in particular the independence of J from ε (for ε sufficiently small).

So, until now we have proved that $\sum_I$ goes to zero as $\nu \to \infty$. It is immediate to check that the same holds for $\sum_{II}$. Indeed,

$$\sum_{II} \leq \sum_{j=0}^{m_\nu - 1} \left| A_{\varepsilon_\nu}(F^j_{\varepsilon_\nu}(p)) - A_0(F^j_{\varepsilon_\nu}(p)) \right| \leq \sum_{j=0}^{m_\nu - 1} K_1 \left|\varepsilon_\nu\right|^2$$

for some positive constant K_1. The assertion then follows since (ε_ν, m_ν) is of bounded type. $\square$

6. The convergence to the Lavaurs map

In this section we prove Theorem 1.4. We shall exploit the 1-dimensional Theorem 1.2, i.e., the convergence of the restriction of $F^{n_\nu}_{\varepsilon_\nu}$ on $C_0 = \widetilde{C}_0 \cap \{ y = 0 \}$ to the 1-dimensional Lavaurs map L_α.

Lemma 6.1. *Let $p_0 \in \widetilde{C}_0 \cap \{ y = 0 \}$ and (ε_ν, n_ν) be an α-sequence. Assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0 \cap \{ y = 0 \}$. Then for every δ there exists η such that (after possibly shrinking $\widetilde{C}_0$)*

$$\widetilde{\varphi^o}\left(-\widetilde{C}_0 \cap F^{n_\nu}_{\varepsilon_\nu}\left(\widetilde{C}_0 \cap \left(\widetilde{\varphi^\iota}\right)^{-1}\left(\mathbb{D}(\widetilde{\varphi^\iota}(p_0), \eta)\right)\right)\right) \subset \mathbb{D}(\widetilde{\varphi^o}(q_0), \delta)$$

for every ν sufficiently large.

The need of shrinking $\widetilde{C}_0$ is just due to the fact that Proposition 3.6 and Corollary 3.7 give the convergence on compact subsets of $\widetilde{C}_0$ (and $-\widetilde{C}_0$).

Proof. Let m^o_ν and m^ι_ν be sequences of bounded type such that $m^\iota_\nu + m^o_\nu = n_\nu$. By definition of $\widetilde{\varphi^\iota}_{\varepsilon,n}$ and $\widetilde{\varphi^o}_{\varepsilon,n}$ we have

$$\begin{aligned}
\widetilde{\varphi^o}_{\varepsilon_\nu, m^o_\nu} \circ F^{n_\nu}_{\varepsilon_\nu}(p) &= \widetilde{w}_{\varepsilon_\nu}\left(F^{-m^o}_{\varepsilon_\nu}(F^{n_\nu}_{\varepsilon_\nu}(p))\right) - \frac{\pi}{2\varepsilon_\nu} + m^o_\nu \\
&= \widetilde{w}_{\varepsilon_\nu}\left(F^{m^\iota_{\varepsilon_\nu}}_{\varepsilon_\nu}(p)\right) - \frac{\pi}{2\varepsilon_\nu} - m^\iota_\nu + n_\nu \\
&= \widetilde{\varphi^\iota}_{\varepsilon_\nu, m^o_\nu}(p) + n_\nu - \frac{\pi}{\varepsilon_\nu}
\end{aligned} \tag{6.1}$$

whenever $F^{n_\nu}_{\varepsilon_\nu}(p) \in -\widetilde{C}_0$. The assertion follows from Proposition 3.6 and Corollary 3.7. $\square$

Lemma 6.2. *Let $p_0 \in \widetilde{C}_0 \cap \{ y = 0 \}$ and (ε_ν, n_ν) be an α-sequence. Assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0 \cap \{ y = 0 \}$. Then, for every polydisc Δ_{q_0} centered at q_0 and contained in $-\widetilde{C}_0$ there exists a polydisc Δ_{p_0} centered at p_0 and contained in $\widetilde{C}_0$ such that $F^{n_\nu}_{\varepsilon_\nu}(\Delta_{p_0}) \subset \Delta_{q_0}$ for ν sufficiently large.*

Proof. Set $\Delta_{q_0} = \mathbb{D}^1_{q_0} \times \mathbb{D}^2_{q_0}$ and analogously $\Delta_{p_0} = \mathbb{D}^1_{p_0} \times \mathbb{D}^2_{p_0}$. By Lemma 6.1 it is enough to prove that, if Δ_{p_0} is sufficiently small, for every ν sufficiently large we have

$$\max_{\mathbb{D}^1_{p_0} \times \partial \mathbb{D}^2_{p_0}} \left| y(F^{m^\iota_\nu}_{\varepsilon_\nu}) \right| \leq \frac{1}{2} \min_{\mathbb{D}^1_{q_0} \times \partial \mathbb{D}^2_{q_0}} \left| y(F^{-m^o_\nu}_{\varepsilon_\nu}) \right|.$$

We shall use the estimates collected in Section 4. First of all, notice that, by Proposition 4.7, it is enough to prove that

$$\max_{p \in \mathbb{D}^1_{p_0} \times \partial \mathbb{D}^2_{p_0}} \left| y(F^{n_p(\varepsilon_\nu)}_{\varepsilon_\nu}) \right| \leq c \min_{q \in -\mathbb{D}^1_{q_0} \times \partial \mathbb{D}^2_{q_0}} \left| y(H^{n_\nu - n'_p(\varepsilon_\nu)}_{\varepsilon_\nu}) \right|$$

for some constant c, where H_ε is as in (3.9). Geometrically, we want to ensure that the vertical expansion in the third part of the orbit (i.e., after $n'_p(\varepsilon)$) is balanced by a suitable contraction during the first part (i.e., up to $n_p(\varepsilon)$).

This means proving that, for every $p \in \mathbb{D}^1_{p_0} \times \partial \mathbb{D}^2_{p_0}$ and $q \in -\mathbb{D}^1_{q_0} \times \partial \mathbb{D}^2_{q_0}$,

$$
(6.2)\qquad
\begin{aligned}
&\left|\prod_{j=0}^{n_p(\varepsilon)} \left(1+\rho x(F^j_{\varepsilon_\nu}(p)) + \beta_{\varepsilon_\nu}(x(F^j_{\varepsilon_\nu}(p)), y(F^j_{\varepsilon_\nu}(p)))\right)\right|\\
&\le c'\left|\prod_{j=0}^{n_\nu - n'_p(\varepsilon_\nu)} \left(1+\rho x(H^j_{\varepsilon_\nu}(q)) + \beta^H_{\varepsilon_\nu}(x(H^j_{\varepsilon_\nu}(q)), y(H^j_{\varepsilon_\nu}(q)))\right)\right|
\end{aligned}
$$

for some positive c'. First of all, we claim that there exists a constant K_1 (independent from ν) such that $K_1 + n_p(\varepsilon_\nu) \ge n_\nu - n'_p(\varepsilon_\nu)$, i.e., the number of points in the orbit for F_ε before entering in $\widetilde{D}_{\varepsilon_\nu}$ (and thus in the contracting part) is at least the same (up to the constant) of the number of points in the expanding part. Indeed, recalling the definition (3.4) of K, we have $\frac{K}{\rho''|\varepsilon_\nu|} \ge \frac{\pi}{|\varepsilon_\nu|} - \frac{\pi - K/2}{\rho''|\varepsilon_\nu|}$. So, by Propositions 4.2 and 4.8 we have, with M^+ as in (4.1),

$$
\begin{aligned}
1+|\alpha| + \frac{M^+}{\rho''} + n_p(\varepsilon_\nu) &\ge 1+|\alpha| + \frac{M^+}{\rho''} + \frac{K}{\rho''|\varepsilon_\nu|} - \frac{M^+}{\rho''}\\
&\ge n_\nu - \frac{\pi}{|\varepsilon_\nu|} + \frac{\pi}{|\varepsilon_\nu|} - \frac{\pi - K/2}{\rho''|\varepsilon_\nu|} \ge n_\nu - n'_p(\varepsilon_\nu)
\end{aligned}
$$

for ν sufficiently large, and the desired inequality is proved. The inequality (6.2) now follows from Lemma 4.9 (and Proposition 4.5), and the assertion follows. □

We can now prove Theorem 1.4.

Proof of Theorem 1.4. First of all, we can assume that p_0 belongs to $C_0 = \{\, y = 0 \,\} \cap \widetilde{C}_0$. Indeed, there exists some N_0 such that $F_0^{N_0}(p_0) \in \widetilde{C}_0$. So, we can prove the Theorem for the $(\alpha - N_0)$-sequence $(\varepsilon_\nu, n_\nu - N_0)$ and the base point $F_0^{N_0}(p_0)$ and the assertion then follows since $F_{\varepsilon_\nu}^{N_0} \to F_0^{N_0}$. For the same reason, we can assume that $q_0 := L_\alpha(p_0)$ belongs to $-\widetilde{C}_0$.

By Lemma 6.2, there exists a polydisc Δ_{p_0} centered at p_0 such that the sequence $F_{\varepsilon_\nu}^{n_\nu}$ is bounded on Δ_{p_0}. In particular, up to a subsequence, this sequence converges to a limit map L, defined in Δ_{p_0} with values in $-\widetilde{C}_0$. Notice that the limit must be open, since the same arguments apply to the inverse system. The relation (1.3) then follows from (6.1) and the assertion follows. □

In the following, given a subset $\mathcal{U} \subset \widetilde{C}_0$, we denote by $\mathcal{T}_\alpha(\mathcal{U})$ the set

$$
\mathcal{T}_\alpha(\mathcal{U}) := \{\, L : \mathcal{U} \to \mathbb{C}^2 \colon \exists (\varepsilon_\nu, n_\nu)\ \alpha - \text{sequence such that } F_{\varepsilon_\nu}^{n_\nu} \to L \text{ on } \mathcal{U} \,\}
$$

We denote by $\mathcal{T}_\alpha$ the union of all the $\mathcal{T}_\alpha(\mathcal{U})$'s, where $\mathcal{U} \subset \widetilde{C}_0$, and call the elements of $\mathcal{T}_\alpha$ *Lavaurs maps*. Theorem 1.4 can then be restated as follows: every compact subset $\mathcal{C}_0 \subset C_0$ has a neighbouhhood $\mathcal{U}_{\mathcal{C}_0} \subset \widetilde{C}_0$ such that every $\mathcal{T}_\alpha(\mathcal{U}_{\mathcal{C}_0})$ is not empty.

7. The discontinuity of the large Julia set

In this section we shall prove Theorem 1.6. By means of the Lavaurs maps L, we first define a 2-dimensional analogous of the Julia-Lavaurs set $J^1(F_0, L)$, and use this set to estimate the discontinuity of the Julia set at $\varepsilon = 0$.

Definition 7.1. Let $\mathcal{U} \subset \widetilde{C}_0$ and $L \in \mathcal{T}_\alpha(\mathcal{U})$. The Julia-Lavaurs set $J^1(F_0, L)$ is the set

$$
J^1(F_0, L) := \overline{\{\, z \in \mathbb{P}^2 | \exists m \in \mathbb{N} \colon L^m(z) \in J^1(F_0) \,\}}.
$$

The condition $L^m(z) \in J^1(F_0)$ means that we require $L^i(z)$ to be defined, for $i = 0, \dots m$. In particular, we have $z, \dots, L^{m-1}(z) \in \mathcal{U}$.

From the definition it follows that $J^1(F_0) \subseteq J^1(F_0, L)$, for every $L \in \mathcal{T}_\alpha$. The following result gives the key estimate for the lower-semicontinuity of the large Julia sets at $\varepsilon = 0$. The proof is analogous to the 1-dimensional case, exploiting the fact that the maps L are open.

Theorem 7.2. *Let $L \in \mathcal{T}_\alpha$ be defined on $\mathcal{U} \subset \widetilde{C}_0$ and (n_ν, ε_ν) be an α-sequence such that $F_{\varepsilon_\nu}^{n_\nu} \to L$ on $\mathcal{U}$. Then*

$$\liminf J^1(F_{\varepsilon_\nu}) \supseteq J^1(F_0, L).$$

Proof. The key ingredients are the lower semicontinuity of $J^1(F_\varepsilon)$ and Theorem 1.4. By definition, the set of all z's admitting an m such that $L^m(z) \in J^1(F_0)$ is dense in $J^1(F_0, L)$. Thus, given z_0 and m satisfying the previous condition, we only need to find a sequence of points $z_\nu \in J^1(F_{\varepsilon_\nu})$ such that $z_\nu \to z_0$, for some sequence $\varepsilon_\nu \to 0$.

Set $p_0 := L^m(z_0)$. By the lower semicontinuity of $\varepsilon \mapsto J^1(F_\varepsilon)$ we can find a sequence of points $p_\nu \in J^1(F_{\varepsilon_\nu})$ such that $p_\nu \to p_0$. By Theorem 1.4 we have $F_{\varepsilon_\nu}^{mn_\nu} \to L^m$ uniformly near z_0, and this (since L is open) gives a sequence z_ν converging to z_0 such that $F_{\varepsilon_\nu}^{mn_\nu}(z_\nu) = p_\nu \in J^1(F_{\varepsilon_\nu})$. This implies that $z_\nu \in J^1(F_{\varepsilon_\nu})$, and the assertion follows. □

Notice the function $\varepsilon \mapsto J^1(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$ since, by means of just the one-dimensional Lavaurs Theorem 1.2, we can create points in $\widetilde{C}_0 \cap \{ y = 0 \}$ (which is contained in the Fatou set) satisfying $L_\alpha(p) \in J^1(F_0)$. Indeed, the following property holds:

$$\forall p \in \widetilde{C}_0 \cap \{ y = 0 \} \text{ there exists } \alpha \text{ such that } p \in J^1((F_0)_{|y=0}, L_\alpha). \tag{7.1}$$

where L_α is the 1-dimensional Lavaurs map on the invariant line $\{ y = 0 \}$ associated to α. Indeed, since $\partial\mathcal{B} \subseteq J^1(F_0)$ and $\mathcal{B}$ intersects the repelling basin $\mathcal{R}$, we can find $q \in J^1(F_0) \cap \{ y = 0 \}$ in the image of the Fatou parametrization ψ^o for $(F_0)_{|\{ y=0 \}}$. The assertion follows considering α such that $L_\alpha(p) = q$.

In our context, given any $p \in C_0$ and $q \in -C_0$ as above, by means of Theorem 1.4 we can consider a neighbourhood of p where a sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a Lavaurs map L (necessarily coinciding with L_α on the line $\{ y = 0 \}$). Since L is open, we have that $L^{-1}(J^1(F_0))$ is contained in the liminf of the Julia sets $J^1(f_{\varepsilon_\nu})$. This gives a two-dimensional estimate of the discontinuity.

8. The discontinuity of the filled Julia set

For regular polynomial endomorphism of $\mathbb{C}^2$ it is meaningful to consider the filled Julia set, defined in the following way.

Definition 8.1. Given a regular polynomial endomorphism F of $\mathbb{C}^2$, the *filled Julia set* $K(F)$ is the set of points whose orbit is bounded.

Equivalently, given any sufficiently large ball B_R, such that $F^{-1}(B_R) \Subset B_R$, the filled Julia set is equal to

$$K(F) := \bigcap_{n \geq 0} F^{-n}(B_R).$$

In this section we shall prove that, if the family (1.2) is induced by regular polynomials, then the set-valued function $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.

Recall that the function $\varepsilon \mapsto K(F_\varepsilon)$ is always upper semicontinuous (see [10]). Here the key definition will be the following analogous of the filled Lavaurs-Julia set in dimension 1 ([14]).

Definition 8.2. Given $\mathcal{U} \subset \widetilde{C}_0$ and $L \in \mathcal{T}_\alpha(\mathcal{U})$, the filled Lavaurs-Julia set $K(F_0, L)$ is the complement of the points p such that there exists $m \geq 0$ such that $L^m(p)$ is defined and is not in $K(F_0)$.

Notice in particular that $K(F_0, L) \subseteq K(F_0)$ and coincides with $K(F_0)$ outside $\mathcal{U}$. Moreover, notice that $K(F_0, L)$ is closed.

Theorem 8.3. *Let $L \in \mathcal{T}_\alpha$ be defined on some $\mathcal{U} \subset \widetilde{C}_0$. and let (ε_ν, n_ν) be an α-sequence such that $F_{\varepsilon_\nu}^{n_\nu} \to L$ on $\mathcal{U}$. Then*

$$K(F_0, L) \supseteq \limsup K(F_{\varepsilon_\nu}).$$

Proof. Since the set-valued function $\varepsilon \mapsto K_\varepsilon$ is upper-semicontinuous, there exists a large closed ball B such that, for $\nu \geq \nu_0$, we have $\cup_\nu K(F_{\varepsilon_\nu}) \subset B$. Without loss of generality we can assume that $\nu_0 = 1$. Let us consider the space

$$P := \{ \{ 0 \} \cup \bigcup_\nu \{\varepsilon_\nu\} \} \times B$$

and its subset X given by

$$X := \{ (0, z) \colon x \in K(F_0, L) \} \cup \bigcup_\nu \{ (\varepsilon_\nu, z) \colon z \in K(F_{\varepsilon_\nu}) \}.$$

By [10, Proposition 2.1] and the fact that P is compact, it is enough to prove that X is closed in P. This follows from Theorem 1.4. Indeed, let z be in the complement of $K(F_0, L)$. Since this set is closed, a small ball B_z around z is outside $K(F_0, L)$, too. By definition, this means that, for some m, we have $L^m(B_z) \subset K(F_0)^c$. Theorem 1.4 implies that, up to shrinking the ball B_z, we have $F_{\varepsilon_\nu}^{n_\nu}(B_z) \subset K(F_0)^c$ for ν sufficiently large. The upper semicontinuity of $\varepsilon \mapsto K(F_\varepsilon)$ then implies that $F_{\varepsilon_\nu}^{n_\nu}(B_z) \subset K(F_{\varepsilon_\nu})^c$, for ν large enough. So, $B_z \subset K(F_{\varepsilon_\nu})^c$ and this gives the assertion. $\square$

Corollary 8.4. *Let F_ε be a holomorphic family of regular polynomials of $\mathbb{C}^2$ as in* (1.2)*. Then the set-valued function $\varepsilon \mapsto K(F_\varepsilon)$ is discontinuous at $\varepsilon = 0$.*

Proof. The argument is the same used to prove the discontinuity of $J^1(F_\varepsilon)$ in Section 7. If the function $\varepsilon \mapsto K(F_\varepsilon)$ were continuous, Theorem 8.3 and the fact that $K(F_0, L) \subseteq K(F_0)$ for every Lavaurs map L would imply that all the $K(F_0, L)$'s were equal to $K(F_0)$. Since $\widetilde{C}_0 \subseteq K(F_0)$, it is enough to find $p \in \widetilde{C}_0$ and α such that $p \notin K(F_0, L)$. To do this, it is enough to take any point q in $\{ y = 0 \}$ not contained in $K(F_0)$ (recall that $K(F_0)$ is compact) and then consider a point $p \in \widetilde{C}_0 \cap \{ y = 0 \}$ and α such that $L_\alpha(p) = q$. The existence of such points is a consequence of the property (7.1). Then, consider a neighbourhood $\mathcal{U}$ of p such that some sequence $F_{\varepsilon_\nu}^{n_\nu}$ converges to a Lavaurs map L on $\mathcal{U}$. The assertion follows since L is open and coincides with L_α on the intersection with the invariant line $\{ y = 0 \}$. $\square$

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