ON THE CONTINUITY OF THE CONTINUOUS STEINER SYMMETRIZATION

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Dedicated to Roger Wets for his 85th birthday

Abstract. Starting from the Brock's construction of Continuous Steiner Symmetrization of sets, the problem of modifying continuously a given domain up to obtain a ball, preserving its measure and with decreasing first eigenvalue of the Laplace operator, is considered. For a large class of cases it is shown this is possible, while the general question remains still open.

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1. Introduction

The problem of *rounding* more and more a given set $\Omega \subset \mathbb{R}^d$, keeping fixed its measure and asymptotically reaching a ball of the same measure, enters in a number of problems and has been widely considered in the literature. More precisely, given a bounded open set $\Omega \subset \mathbb{R}^d$, the goal is to construct a family of domains (Ω_t), with $t \in [0,1]$, such that $\Omega_0 = \Omega$, $\Omega_1 = \Omega^*$ where Ω^* is a ball of the same measure as Ω, and $|\Omega_t| = |\Omega|$ for all $t \in [0,1]$, where by $|\cdot|$ we denote the Lebesgue measure.

In addition, we require that the mapping $t \mapsto \Omega_t$ be *continuous* with respect to some suitable topology, and that the family (Ω_t) satisfy some monotonicity property that will be specified later.

We notice that, without the last monotonicity requirement, a very simple construction would provide a solution. Take indeed a set Ω and a point x_0 far enough from Ω; denoting by $B(x_0, r)$ the ball of center x_0 and radius r and by ω_d the Lebesgue measure of the unit ball in $\mathbb{R}^d$, the family

$$\Omega_t = (1-t)^{1/d}\Omega \cup B(x_0, r_t) \qquad \text{with } r_t = \left(\frac{t|\Omega|}{\omega_d}\right)^{1/d}$$

satisfies the measure constraint $|\Omega_t| = |\Omega|$, is such that $\Omega_0 = \Omega$ and $\Omega_1 = \Omega^*$, and is continuous in several useful topologies. An example of such a family (Ω_t) is illustrated in Figure 1.

The additional monotonicity conditions that we impose consists in the requirement that a suitable shape functional F is monotone. For instance we could consider:

- $F(\Omega) = P(\Omega)$, the *perimeter* in the sense of De Giorgi, and we require $P(\Omega)$ is nonincreasing;
- $F(\Omega) = \mathcal{H}^{d-1}(\Omega)$, the *Hausdorff* $d-1$ *dimensional measure*, and we require $\mathcal{H}^{d-1}(\Omega)$ is nonincreasing;

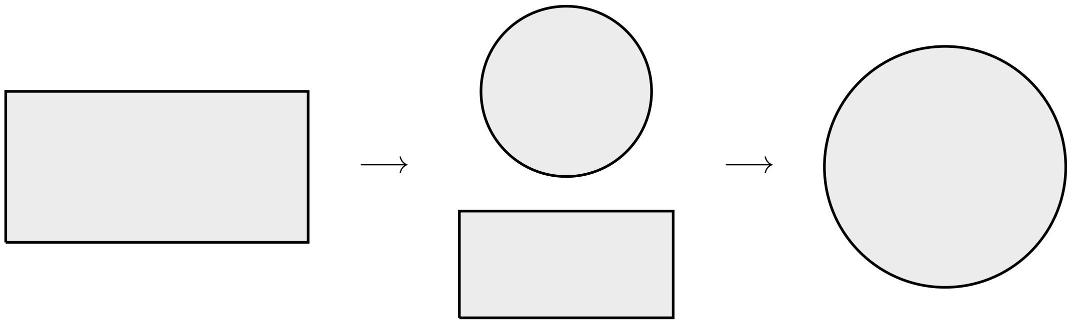

FIGURE 1. The sets Ω_0, $\Omega_{1/2}$, Ω_1 when Ω is the rectangle $]0,2[\times]0,1[$.

- $F(\Omega)=T(\Omega)$, the *torsional rigidity* defined below, and we require $T(\Omega)$ is nondecreasing;
- $F(\Omega)=\lambda(\Omega)$, the *first eigenvalue* of the Dirichlet Laplacian defined below, and we require $\lambda(\Omega)$ is nonincreasing;
- $F(\Omega)=h(\Omega)$, the *Cheeger constant*, and we require $h(\Omega)$ is nonincreasing.

In this paper we focus the attention mostly on the first eigenvalue $\lambda(\Omega)$ and on the torsional rigidity $T(\Omega)$.

More precisely, $\lambda(\Omega)$ is the first eigenvalue of the Laplace operator $-\Delta$ with Dirichlet conditions on $\partial\Omega$, that is the minimal value λ such that the PDE

$$\begin{cases}-\Delta u=\lambda u & \text{in } \Omega\,,\\ u\in H^1_0(\Omega),\end{cases}$$

has a nonzero solution. Equivalently, by the min-max principle (see for instance [11]) $\lambda(\Omega)$ can be defined through the minimization of the Rayleigh quotient, as

$$\lambda(\Omega)=\min\left\{\Big[\int_\Omega|\nabla u|^2\,dx\Big]\Big[\int_\Omega u^2\,dx\Big]^{-1}\ :\ u\in H^1_0(\Omega),\ u\ne0\right\}.$$

An important bound for $\lambda(\Omega)$ is the *Faber-Krahn inequality* (see for instance [11], [12])

$$\lambda(\Omega^*)\le\lambda(\Omega)\,,$$

which can be stated in a scaling free form as

$$|\Omega|^{2/d}\lambda(\Omega)\ge|B|^{2/d}\lambda(B),$$

where B is any ball in $\mathbb{R}^d$.

The torsional rigidity $T(\Omega)$ is defined as $\int_\Omega u_\Omega\,dx$, where u_Ω is the unique solution of the PDE

$$\begin{cases}-\Delta u=1 & \text{in } \Omega\,,\\ u\in H^1_0(\Omega)\,,\end{cases}$$

or equivalently through the maximization problem

$$T(\Omega)=\max\left\{\Big[\int_\Omega u\,dx\Big]^2\Big[\int_\Omega|\nabla u|^2\,dx\Big]^{-1}\ :\ u\in H^1_0(\Omega),\ u\ne0\right\}\,,$$

where the maximum is reached by u_Ω itself. Also for $T(\Omega)$ an important inequality holds, the *Saint-Venant inequality*

$$T(\Omega)\le T(\Omega^*)\,,$$

which can be stated in a scaling free form as

$$|\Omega|^{-(d+2)/d}T(\Omega) \le |B|^{-(d+2)/d}T(B)$$

where B is any ball in $\mathbb{R}^d$.

The monotonicity properties we require to the family (Ω_t) are then:

- the mapping $t \mapsto \lambda(\Omega_t)$ is nonincreasing;
- the mapping $t \mapsto T(\Omega_t)$ is nondecreasing.

Concerning the continuity of the map $t \mapsto \Omega_t$ our requirement is that the solutions u_t of the PDEs

$$\begin{cases} -\Delta u_t = f & \text{in } \Omega_t\,, \\ u_t \in H^1_0(\Omega_t)\,, \end{cases}$$

vary continuously in t with respect to the strong $H^1(\mathbb{R}^d)$ convergence, for every right-hand side $f \in L^2(\mathbb{R}^d)$. This is the γ-convergence, that we describe more precisely in Section 2.

When instead of a continuous family (Ω_t) we consider the discrete case of a sequence (Ω_n) such that

(i) $\Omega_0 = \Omega$, $|\Omega_n| = |\Omega|$ for every n, $\Omega_n \to \Omega^*$ in the γ-convergence,
(ii) $\lambda(\Omega_{n+1}) \le \lambda(\Omega_n)$ and $T(\Omega_{n+1}) \ge T(\Omega_n)$ for every n,

we have the problem that was first considered by Steiner, who proposed to use successive symmetrizations through different hyperplanes. More precisely, given a domain $\Omega \subset \mathbb{R}^d$ and a direction ν, the *Steiner symmetrization* of Ω with respect to ν is defined as

$$\Omega^*_\nu = \left\{ x \in \mathbb{R}^d \;:\; |x\cdot\nu| < \frac{\varphi\big(\pi(x)\big)}{2} \right\}.$$

Here $\pi(x) = x - \nu(x\cdot\nu)$ is the projection of a point $x \in \mathbb{R}^d$ on the hyperplane orthogonal to ν and, for each y in this hyperplane,

$$\varphi(y) = \mathcal{H}^1\big(\Omega \cap \pi^{-1}(y)\big)$$

is the length of the y-section of Ω, where by $\mathcal{H}^1$ we denote the 1-dimensional Hausdorff measure.

Note that the set Ω^*_ν has the same volume of Ω and is symmetric with respect to the hyperplane orthogonal to ν. In addition, it is well-known (see for instance [1]) that the Steiner symmetrization decreases the first eigenvalue and increases the torsional rigidity, that is

$$\lambda(\Omega^*_\nu) \le \lambda(\Omega) \qquad \text{and} \qquad T(\Omega^*_\nu) \ge T(\Omega)\,.$$

By repeating this symmetrization procedure for a dense sequence of directions ν, one obtains a sequence Ω_n of sets, all with the same measure, which γ-converge as $n \to \infty$ to the ball Ω^*.

The question is now to pass from the discrete Steiner symmetrization to a continuous one. Since successive Steiner symmetrizations allow to pass from a generic set to a ball, it is enough to construct a continuous family Ω_t of sets which transforms a set Ω into its Steiner symmetrization Ω^*_ν for a fixed direction ν. An explicit construction of a family Ω_t was proposed by Brock in [4] (see also [5]) and was called *Continuous Steiner Symmetrization*. We shortly recall the Brock's construction in Section 3.

Unfortunately, the Brock's construction provides the γ-continuity of the family Ω_t only in very particular situations, as for instance when the initial domain Ω is convex, while in general discontinuities may occur, due to irregularities of the domains Ω_t. On the other hand, the γ-continuity would be very useful in several situations, as for instance in the study of some Blaschke-Santaló diagrams, as illustrated in [8].

In the present paper we show that a modification of Brock's construction could be enough to provide the required γ-continuity of the family Ω_t, at least for a larger class of domains Ω. In [8] a similar construction was made for polyhedral domains Ω. Even if the arguments are not complete, we believe it could help to better understand the difficulties behind the Continuous Steiner Symmetrization.

In the last section we consider a possible alternative approach based on the De Giorgi theory of minimizing movements.

2. The γ-convergence

In this section we recall the definition and the main properties of γ-convergence; for all details, proofs, and generalization to the class of capacitary measures, we refer the interested reader to [6]. For simplicity, we make the assumption that all the domains we consider are included in a given bounded open subset D of $\mathbb{R}^d$, which is satisfied for the domains we consider later. In the following, for every domain Ω, a function in $H^1_0(\Omega)$ is considered extended by zero on $\mathbb{R}^d \setminus \Omega$.

Definition 2.1. *A sequence (Ω_n) of domains ia said to γ-converge to a domain Ω if for every $f \in L^2(\mathbb{R}^d)$ the solutions $u_{n,f}$ of the PDEs*

$$\begin{cases} -\Delta u = f & \text{in } \Omega_n \\ u \in H^1_0(\Omega_n) \end{cases}$$

converge weakly in $H^1(\mathbb{R}^d)$ to the solution u_f of the PDE

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u \in H^1_0(\Omega)\,. \end{cases}$$

We summarize here below the main properties of the γ-convergence. We refer to [6] for all the details, properties, and proofs.

- The weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to u_f is equivalent to the strong $H^1(\mathbb{R}^d)$ convergence. Indeed, integrating by parts we obtain

$$\int |\nabla u_{n,f}|^2 dx = \int u_{n,f} f\,dx \to \int u_f f\,dx = \int |\nabla u_f|^2 dx.$$

- In the definition above it is not difficult to show that it is equivalent to require the weak $H^1(\mathbb{R}^d)$ convergence of $u_{n,f}$ to u_f for every $f \in L^2(\mathbb{R}^d)$ or for every $f \in H^{-1}(\mathbb{R}^d)$. Indeed, if $f \in H^{-1}(\mathbb{R}^d)$ it is enough to approximate f by a sequence $f_k \in L^2(\mathbb{R}^d)$, in the H^{-1} norm, to obtain for every test function ϕ

$$\begin{aligned}\left|\int \nabla u_{n,f}\nabla\phi\,dx-\int\nabla u_f\nabla\phi\,dx\right| &= \left|\langle f,\phi\rangle_{H_0^1(\Omega_n)}-\langle f,\phi\rangle_{H_0^1(\Omega)}\right|\\ &\le \left|\langle f_k,\phi\rangle_{H_0^1(\Omega_n)}-\langle f_k,\phi\rangle_{H_0^1(\Omega)}\right|+\varepsilon_k\|\phi\|\\ &= \left|\int\nabla u_{n,f_k}\nabla\phi\,dx-\int\nabla u_{f_k}\nabla\phi\,dx\right|+\varepsilon_k\|\phi\|.\end{aligned}$$

where $\varepsilon_k\to 0$. Passing to the limit first as $n\to\infty$ and then as $k\to\infty$ gives what claimed.

- The γ-convergence can be defined in a similar way for *quasi-open* sets $\Omega\subset D$ or more generally for *capacitary measures* μ confined into D (that is $\mu=+\infty$ outside D). Quasi-open sets are sets of positivity $\{u>0\}$ of functions $u\in H^1(\mathbb{R}^d)$, while capacitary measures are regular nonnegative Borel measures μ on D, possibly $+\infty$ valued, such that $\mu(E)=0$ for every Borel set $E\subset D$ with $\operatorname{cap}(E)=0$. For all details on quasi-open sets and capacitary measures we refer the interested reader to the book [6]. Here we only recall that for a capacitary measure μ the corresponding PDE is formally written as

$$\begin{cases}-\Delta u+\mu u=f & \text{in } D\\ u\in H_0^1(D)\cap L^2_\mu(D)\end{cases}$$

and has to be intended in the weak sense, that is, $u\in H_0^1(D)\cap L^2_\mu(D)$ and

$$\int_D \nabla u\nabla\phi\,dx+\int_D u\phi\,d\mu=\langle f,\phi\rangle$$

for all $\phi\in H_0^1(D)\cap L^2_\mu(D)$. We notice that open sets or more generally quasi-open sets can be seen as capacitary measures: for a given domain Ω the capacitary measure representing it is the measure ∞_{Ω^c} defined as

$$\infty_{\Omega^c}(E)=\begin{cases}0 & \text{if } \operatorname{cap}(E\cap\Omega)=0\\ +\infty & \text{otherwise.}\end{cases}$$

- In Definition 2.1 it is possible to show (see Remark 4.3.10 of [6]) that requiring the convergence of the solutions u_n to u for every right-hand side f is equivalent to require the convergence $u_n\to u$ only for $f\equiv 1$ and in the $L^2(D)$ sense. In particular, calling u_μ the unique solution of the PDE $-\Delta u+\mu u=1$ in $H_0^1(D)\cap L^2_\mu(D)$, the quantity

$$d_\gamma(\mu_1,\mu_2)=\|u_{\mu_1}-u_{\mu_2}\|_{L^2(D)} \tag{2.1}$$

is a distance on the space $\mathcal{M}$ of capacitary measures, which is equivalent to γ-convergence, and so $\mathcal{M}$ endowed with the distance d_γ above is a compact metric space. Since the solutions u_μ are all equi-bounded (for instance they are all below by the solution w of the Dirichlet problem $-\Delta w=1$ on $H_0^1(D)$, which is a bounded function) the L^2 norm in (2.1) can be replaced by any L^p norm, with $1\le p<+\infty$. In particular, if $p=1$ and $\Omega_1\subset\Omega_2$ we have

$$\|u_{\Omega_1}-u_{\Omega_2}\|_{L^1}=\int u_{\Omega_2}dx-\int u_{\Omega_1}dx=T(\Omega_2)-T(\Omega_1),$$

and the γ-convergence is then reduced to the convergence of the corresponding torsional rigidities.

- The first eigenvalue $\lambda(\Omega)$ (as well as all the other eigenvalues $\lambda_k(\Omega)$) and the torsional rigidity $T(\Omega)$ are continuous with respect to the γ-convergence.
- The Lebesgue measure $|\Omega|$, or more generally integral functionals as $\int_\Omega f(x)\,dx$ with $f \geq 0$ and measurable, are lower semicontinuous with respect to the γ-convergence on the domains Ω.
- As stated above, the space $\mathcal{M}$ of capacitary measures, endowed with the γ-convergence, is a compact metric space. On the contrary, the family of open sets (or also quasi-open sets) is not compact in $\mathcal{M}$; it is actually a dense subset of $\mathcal{M}$. The first example of a sequence of open sets Ω_n which γ-converges to a capacitary measure which is not a domain (actually to the Lebesgue measure) was given in [9].
- Several subclasses of $\mathcal{M}$ are dense with respect to the γ-convergence (see Proposition 4.3.7 and Remark 4.3.8 of [6]). For instance:
  - the class of measures $a(x)\,dx$ with $a \geq 0$ and smooth;
  - the class of smooth domains $\Omega \subset D$.
  - the class of polyhedral domains $\Omega \subset D$;
  - the class of measures of the form $a(x)\,d\mathcal{H}^{d-1}$ with $a \geq 0$ and smooth, where $\mathcal{H}^{d-1}$ is the $d-1$ dimensional Hausdorff measure;
  - the class of measures of the form $\mathcal{H}^{d-1}\lfloor S$, where $S \subset D$ is a smooth $d-1$ manifold.

3. The Brock's construction

We summarize rapidly here the construction by Brock (see [4], [5]) of the continuous Steiner symmetrization, together with the properties important for our purpose. The first construction is for the unidimensional case; here taking the variable t in $[0,+\infty]$ or in $[0,1]$ does not make any real difference.

- If I is the interval $]a,b[$, then the continuous Steiner symmetrization I^t is the interval $]a^t,b^t[$, where

$$a^t = \big(a-b+e^{-t}(a+b)\big)/2, \qquad b^t = \big(b-a+e^{-t}(a+b)\big)/2.$$

- If A is an open subset of $\mathbb{R}$ we consider the properties:
  (i) $A(0)=A$;
  (ii) if I is an interval with $I \subset A(s)$, then $I^t \subset A(s+t)$ for every $t \geq 0$.
  We define then the continuous Steiner symmetrization A^t as

$$A^t = \bigcap \big\{A(t)\ :\ A(t) \text{ satisfies (i) and (ii)}\big\}.$$

  In [4] Brock proves that if A is open then A^t are open sets; in addition the monotonicity property

$$A \subset B \Longrightarrow A^t \subset B^t \text{ for every } t$$

  holds.
- Finally, if $A \subset \mathbb{R}$ is only measurable, we have

$$A = \bigcap_n A_n \setminus N$$

with A_n open sets and N Lebesgue negligible. We then define the continuous Steiner symmetrization A^t of A as

$$A^t = \bigcap_n A_n^t.$$

This definition is unique up to a nullset, and we still call continuous Steiner symmetrization a family A^t such that $|A^t \triangle (\bigcap_n A_n^t)| = 0$.

We can now pass to define the continuous Steiner symmetrization for subsets of $\mathbb{R}^d$, with respect to a hyperplane that, with no loss of generality, we can suppose to be R^{d-1}. For a general set A we define the projection of A on $\mathbb{R}^{d-1}$ as

$$A' = \big\{x' \in \mathbb{R}^{d-1} \ : \ (x', y) \in A \text{ for some } y \in \mathbb{R}\big\},$$

and for $x' \in A'$ the intersection of A with $(x', \mathbb{R})$ as

$$A(x') = \big\{y \in \mathbb{R} \ : \ (x', y) \in A\big\}.$$

Note that $A(x')$ is a one-dimensional set. When A is an open subset of $\mathbb{R}^d$ we define its continuous Steiner symmetrization A^t by

$$A^t = \big\{x = (x', y) \ : \ x' \in A', \ y \in (A(x'))^t\big\}. \tag{3.1}$$

If $A \subset \mathbb{R}^d$ is only measurable, we define its continuous Steiner symmetrization by the same formula as (3.1), but up to Lebesgue negligible sets.

We stress that, for a bounded quasi-open set A, the previous construction only provides a measurable set defined up to a set of zero Lebesgue measure. In order to obtain that the symmetrized sets be still quasi-open and defined quasi-everywhere, it is convenient, for a bounded quasi-open set A, to define (by an abuse of notation) the symmetrized set A^t in the following way: consider a decreasing sequence of bounded open sets (A_n) with $\mathrm{cap}(A_n \setminus A) \to 0$ and $A \subset A_n$. For any $t \in [0, 1]$ the set A_n^t is well defined, and by monotonicity we may define $A_n^t \supset A_{n+1}^t$. Then (A_n^t) is γ-convergent and we define

$$A^t = \gamma - \lim_{n\to\infty} A_n^t.$$

In this way, the set A^t is quasi-open. More details on this issue can be found in [6]; in particular, the proofs that the construction above is independent of the sequence A_n and that the Lebesgue measure is preserved, are still missing.

The continuous Steiner symmetrization can be defined for any positive measurable function u by symmetrizing its level sets:

$$\forall s > 0 \qquad \{u^t > s\} := \{u > s\}^t.$$

The main properties of the Brock's construction are summarized here below, where $\lambda_k(\Omega)$ denotes the k-th eigenvalue of the Dirichlet Laplacian in Ω.

Proposition 3.1. *For every bounded quasi-open set $\Omega \subset \mathbb{R}^d$ and every positive integer k the mapping $t \mapsto \lambda_k(\Omega^t)$, is lower semicontinuous on the left and upper semicontinuous on the right.*

When the starting set Ω is convex, or more generally when the one-dimensional sections $\Omega(x')$ above are intervals, the γ-continuity actually occurs. However, this is not always the case, as the example of Figure 2 shows. Up to the moment when the internal fracture appears the γ-continuity is verified; on the other hand, the

Brock's construction removes the fracture instantaneously, and the γ-continuity is lost.

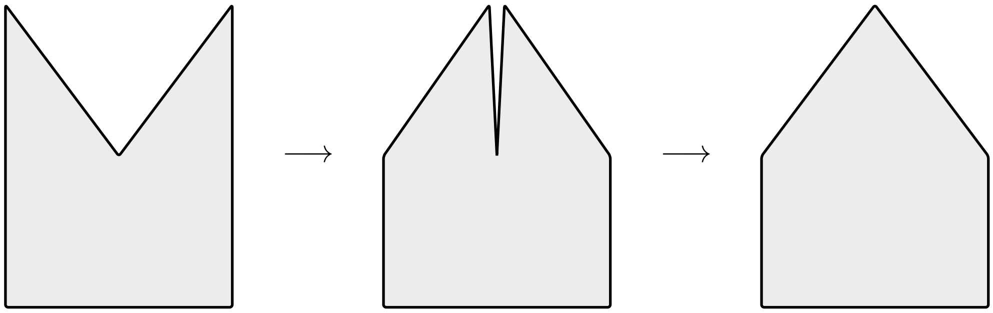

FIGURE 2. A set Ω such that $t \mapsto \lambda(\Omega_t)$ is discontinuous.

Since the torsional rigidity $T(\Omega_t)$ is increasing along the family (Ω_t), it has only countably many discontinuity points. Let t_0 be one of these points and assume that at t_0 we have two domains Ω^-, Ω^+ such that $\Omega^- \subset \Omega^+$ and

$$\begin{cases} T(\Omega_t) \to T(\Omega^-) & \text{as } t \to t_0 \text{ from the left} \\ T(\Omega_t) \to T(\Omega^+) & \text{as } t \to t_0 \text{ from the right} \end{cases} \tag{3.2}$$

In other words Ω^- is the domain with fractures, while Ω^+ is the domain where the fractures have been removed.

Remark 3.2. In the one-dimensional case the existence of a γ-continuous family (Ω_t) cannot be obtained in general, since starting by Ω_0 made of two segments and ending by Ω_1 made of a single segment will necessarily produce a discontinuity of $T(\Omega_t)$ at some point t_0, independently of the construction of the family (Ω_t).

In the case $d \geq 2$ on the contrary, we can fill the discontinuity between Ω^- and Ω^+ by constructing a γ-continuous family (Ω_t), with Ω_t increasing with respect to the set inclusion, and $\Omega_0 = \Omega^-$, $\Omega_1 = \Omega^+$.

Theorem 3.3. *Let $d \geq 2$ and let $\Omega_0 \subset \Omega_1$ be two bounded open sets. Then there exists a γ-continuous family Ω_t of open sets ($t \in [0,1]$) such that*

$$\Omega_s \subset \Omega_t \qquad \textit{for every } s < t. \tag{3.3}$$

Proof. Let us denote by C a large cube containing Ω_1 and by $\Gamma(t)$ a Peano curve from $[0,1]$ onto C, that is a continuous mapping $\Gamma : [0,1] \to \mathbb{R}^d$ such that $\Gamma([0,1]) = C$; we also choose $\Gamma(0) \in \Omega_0$. We define

$$\Omega_t = \big(\Omega_1 \setminus \Gamma([0, 1-t])\big) \cup \Omega_0 \qquad \text{for every } t \in [0,1].$$

Note that Ω_t are open subsets of $\mathbb{R}^d$ and that for $t = 0$ we obtain Ω_0, while for $t = 1$ we obtain Ω_1. The family Ω_t above clearly satisfies the monotonicity property (3.3).

In order to show that the family Ω_t is γ-continuous, it is enough to prove that

$$\operatorname{cap}(\Omega_{t_n} \triangle \Omega_t) \to 0 \qquad \text{whenever } t_n \to t.$$

This comes from the fact that the mapping $\Gamma(t)$ is uniformly continuous, so that

$$|\Gamma(t) - \Gamma(t_n)| \leq \omega(|t - t_n|)$$

for a suitable modulus of continuity ω. Therefore Ω_t and Ω_{t_n} differ by a set which has a diameter less than $2\omega(|t-t_n|)$, hence of capacity which vanishes as $t_n \to t$. □

Remark 3.4. Since the proof of Theorem 3.3 is only based on capacitary arguments, the same statement is valid in the more general case when Ω_0 and Ω_1 are quasi-open sets.

Remark 3.5. When working with polyhedral domains (i.e. whose boundary is made of a finite number of subsets of hyperplanes) we are in the situation above. In fact, if Ω is a polyhedral domain, the Brock's construction provides a family Ω_t made of polyhedral domains, and we have a finite number of discontinuity points $t_1, t_2, \dots t_N$. In addition, for every discontinuity point t_k, the fracture S is a $d-1$ dimensional polyhedral set, $\Omega^- = \Omega_{t_k}$ while $\Omega^+ = \Omega_{t_k} \setminus S$, and then Theorem 3.3 applies.

In several situations (see for instance [8]), thanks to the γ-density of polyhedral domains in the class of all domains, Remark 3.5 is sufficient to achieve the required goals. However, the question of existence of γ-continuous paths (Ω_t), with monotone $\lambda(\Omega_t)$ and $T(\Omega_t)$, between a general domain Ω_0 and the ball B with the same Lebesgue measure, remains.

Similar questions arise if, instead of the quantities $\lambda(\Omega_t)$ and $T(\Omega_t)$, one considers for instance the perimeter $P(\Omega_t)$, requiring the continuity of the map $t \mapsto P(\Omega_t)$ and its decreasing monotonicity.

The procedure of *removing fractures* mentioned after (3.2) needs to be more rigorous. This can be made through the following result.

Proposition 3.6. *Let Ω_0 be a given quasi open set and let $m \geq |\Omega_0|$. Then there exists a quasi open set $\hat{\Omega}$ solving the shape optimization problem*

$$\min\big\{\lambda(\Omega)\ :\ \Omega_0 \subset \Omega,\ |\Omega| \leq m\big\}.$$

Proof. The proof can be obtained directly by applying the existence result of [7]. □

In an analogous way we can obtain a solution for the shape optimization problem

$$\max\big\{T(\Omega)\ :\ \Omega_0 \subset \Omega,\ |\Omega| \leq m\big\}.$$

In particular, the case $m = |\Omega_0|$ is interesting; this allows to obtain, for every given Ω_0, an optimal domain $\hat{\Omega}$ containing Ω_0 and with the same measure as Ω_0, which solves simultaneously the two shape optimization problems

$$\begin{cases}\min\big\{\lambda(\Omega)\ :\ \Omega_0 \subset \Omega,\ |\Omega| = |\Omega_0|\big\},\\ \max\big\{T(\Omega)\ :\ \Omega_0 \subset \Omega,\ |\Omega| = |\Omega_0|\big\}.\end{cases}$$

Indeed, if Ω_1 is an optimal domain for the eigenvalue optimization problem and Ω_2 an optimal domain for the torsion optimization problem, it is enough to take $\hat{\Omega} = \Omega_1 \cup \Omega_2$.

In other words, if Ω_0 is a Lipschitz domain, we have $\hat{\Omega} = \Omega_0$ while, in the case the set Ω_0 presents some internal fractures, the set $\hat{\Omega}$ removes them.

4. The minimizing movement approach

An alternative approach to the Brock's construction of the family Ω_t through the Continuous Steiner Simmetrization could be the use of the De Giorgi minimizing movement theory, introduced in [10] (see for instance [2], [3] for a detailed presentation and further developments).

In our framework of shape functionals, the metric space X could be the one of all measurable subsets Ω of the Euclidean space $\mathbb{R}^d$ with a prescribed Lebesgue measure, say $|\Omega| = 1$, endowed with the L^1 distance

$$d(\Omega_1, \Omega_2) = |\Omega_1 \triangle \Omega_2|.$$

Given a shape functional F defined on X one can consider the so-called *implicit Euler scheme* of time step ε and initial condition Ω_0, which provides a discrete family $\Omega_{n,\varepsilon}$ constructed recursively in the following way:

$$\Omega_{0,\varepsilon} = \Omega_0, \qquad \Omega_{n+1,\varepsilon} \in \operatorname{argmin}_{\Omega \in X} \Big\{ F(\Omega) + \frac{1}{2\varepsilon} |\Omega \triangle \Omega_{n,\varepsilon}|^2 \Big\}.$$

We may then set $\Omega_{t,\varepsilon} = \Omega_{[t/\varepsilon],\varepsilon}$, where $[\cdot]$ stands for the integer part function, and say that Ω_t is a family of sets constructed by the minimizing movement procedure associated to the shape functional F if for every $t \in [0, T]$ we have

$$|\Omega_t \triangle \Omega_{[t/\varepsilon],\varepsilon}| \to 0 \quad \text{as } \varepsilon \to 0.$$

If the limit above occurs only for a sequence (ε_n) (independent of t), we say that Ω_t is a generalized minimizing movement.

It is easy to see that the discrete sequence $\Omega_{n,\varepsilon}$ is such that $F(\Omega_{n,\varepsilon})$ decreases. It would be interesting to show, at least in the particular cases when the shape functional $F(\Omega)$ is the first eigenvalue $\lambda(\Omega)$, the opposite $-T(\Omega)$ of the torsional rigidity, or the perimeter $P(\Omega)$, or some convex combination of them, that the map $t \mapsto F(\Omega_t)$ is continuous and decreasing.

We do not know if the map $t \mapsto F(\Omega_t)$ above is continuous and decreasing, and the cases in which, as $t \to \infty$, the limit domain is a ball. Some results in this direction, in the case $F(\Omega) = P(\Omega)$ can be found in [14], while some partial results in the case of spectral functionals can be found in [13].

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