


RESEARCH



On the notion of sequentially Cohen–Macaulay modules

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Abstract

In this survey paper, we first present the main properties of sequentially Cohen–Macaulay modules. Some basic examples are provided to help the reader with quickly getting acquainted with this topic. We then discuss two generalizations of the notion of sequential Cohen–Macaulayness which are inspired by a theorem of Jürgen Herzog and the third author.

Keywords: Sequentially Cohen–Macaulay module, Dimension filtration, Generic initial ideals, Local cohomology

Introduction

This survey note, which is dedicated to the work of Jürgen Herzog on the topic, cannot possibly be complete: the notion of sequentially Cohen–Macaulay module has been central in many papers in the literature from the late 90’s, starting maybe with [20]. On the other hand, in the late 90’s Herzog’s research activity was feverish as he counted very many visitors and collaborators. At the same time, the distribution of preprints in the form of postscript files over the internet expedited the dissemination of mathematical ideas. We thus must apologize in advance that our reference list is doomed to be incomplete.

It is in 1997 that Herzog, together with who would become his top co-author, Takayuki Hibi, published a paper on simplicial complexes [26], immediately followed by another series of papers of the two authors together with Annetta Aramova [5–7], where numerical problems about simplicial complexes were addressed through the study of Hilbert functions, Gröbner bases techniques and generic initial ideals, see also [3, 4, 19]. In 1999, another influential paper authored by Herzog and Hibi, is published, [27]: they introduce and study a new class of ideals, called componentwise linear. Componentwise linear Stanley–Reisner ideals I_{Δ} are characterized as those for which the pure i -th skeleton of the Alexander dual of Δ is Cohen–Macaulay for every i . Thanks to [20], this means that I_{Δ} is componentwise linear if and only its Alexander dual is sequentially Cohen–Macaulay, see also [30]. In this way, the authors were also able to generalize a well-known result of Eagon and Reiner, which says that the Stanley–Reisner ideal of a simplicial complex has a linear resolution if and only if its Alexander dual is Cohen–Macaulay. All the ideas behind the proofs of these facts led also to another fundamental result, which is the main theorem of [31] and provides a somewhat unexpected characterization of graded sequentially

Cohen–Macaulay modules over a polynomial ring R in terms of the Hilbert function of the local cohomology modules:

Theorem 1 *Let M be a finitely generated graded R -module, and let $M \cong F/U$ a free graded presentation of M . Then, F/U is sequentially Cohen–Macaulay if and only if $\text{Hilb}(H_m^i(F/U)) = \text{Hilb}(H_m^i(F/\text{Gin}(U)))$ for all $i \geq 0$.*

Here, $\text{Gin}(U)$ denotes the generic initial module of U with respect to the degree reverse lexicographic order.

This paper is organized as follows. In the first section, we introduce the definition of sequentially Cohen–Macaulay modules according to Stanley [37], and discuss some basic properties and examples. In Sect. 2, we present some of the main characterizations, or equivalent definitions, of sequential Cohen–Macaulayness, by recalling Schenzel’s results on the dimension filtration of a module, cf. Propositions 3, 5 and Theorem 2, and Peskine’s characterization in terms of deficiency modules, cf. Theorem 3. In Sect. 3, we recollect the definition of partial sequential Cohen–Macaulayness introduced by the third and the fourth author in [35], by requiring that only the queue of the dimension filtration is a Cohen–Macaulay filtration, see Definition 5 and some basic properties in the graded case in Lemmata 2 and 3. The next two results, Lemma 4 and Proposition 8, are dedicated to filling a gap in the proof of a fundamental lemma in [35], and we present the first of our generalizations of Theorem 1 in Theorem 4. We start Sect. 4 by recalling the definition of E-depth, as introduced by the first and second author in [12], see Definition 7. E-depth measures how much depth the deficiency modules of a finitely generated standard graded module M have altogether, and sequential Cohen–Macaulayness can be detected by E-depth as observed in Remark 10. Some interesting homological properties of E-depth, especially in connection with strictly filter regular elements, are shown in Proposition 9. In Definition 9, we introduce the other crucial notion for the following, what we might call partial generic initial ideal, making use of a special partial revlex order. Finally, by means of Proposition 11, we prove in Theorem 7 the main result of this section, which can be seen as another generalization of Theorem 1.

To Jürgen Herzog, a bright example of mathematician and teacher.

1 Sequentially Cohen–Macaulay modules

Throughout the paper, let (R, \mathfrak{m}, k) denote either a Noetherian local ring with maximal ideal \mathfrak{m} and residue field $k = R/\mathfrak{m}$, or a standard graded k -algebra $R = \bigoplus_{i \geq 0} R_i$, with $R_0 = k$ and $\mathfrak{m} = \bigoplus_{i > 0} R_i$. In the second case, every R -module we consider will be a graded R -module, and homomorphisms will be graded of degree zero.

One of the features that makes the Cohen–Macaulay property significant is its characterization in terms of the vanishing and non-vanishing of local cohomology: for a d -dimensional finitely generated module M with $t = \text{depth } M$, it holds that $H_m^i(M) = 0$ for all $i < t$ and $i > d$; also, $H_m^t(M) \neq 0$ and $H_m^d(M) \neq 0$. These results are originally due to Grothendieck, cf. [10], Theorem 3.5.8, Corollary 3.5.9, Corollary 3.5.11.a) and b). As a consequence, M is a Cohen–Macaulay module if and only if $H_m^i(M) = 0$ for all $i \neq d$.

Let $\dim(R) = n$. When R is Cohen–Macaulay and has a canonical module ω_R , by duality, the above conditions can be checked on the Matlis dual of the local cohomology modules, i.e., on the modules $\text{Ext}_R^{n-i}(M, \omega_R)$. With the above notation, we then have that $\text{Ext}_R^{n-t}(M, \omega_R)$ and $\text{Ext}_R^{n-d}(M, \omega_R)$ are nonzero, and furthermore that $\text{Ext}_R^i(M, \omega_R) = 0$ for all $i < n - d$ and all $i > n - t$. Moreover, M is Cohen–Macaulay if and only if $\text{Ext}_R^{n-i}(M, \omega_R) = 0$ for all $i \neq d$. Often in the literature, the modules $\text{Ext}_R^{n-i}(M, \omega_R)$, $i = 0, \dots, n$, are called the *deficiency modules of M* , since they also measure how far a module is from being Cohen–Macaulay.

We now introduce the class of sequentially Cohen–Macaulay modules, following [37], recalling their main properties, and in the next section, we discuss some of its equivalent definitions. Our main references here are [31] and [36].

Definition 1 A finitely generated R -module M is called *sequentially Cohen–Macaulay* if it admits a filtration of submodules $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ such that each quotient of the filtration M_i/M_{i-1} is Cohen–Macaulay and $\dim(M_i/M_{i-1}) < \dim(M_{i+1}/M_i)$ for all i . In this case, we say that the above filtration is a *sCM filtration* for M .

- Example 1*
- (1) It is clear from the definition that if $M \neq 0$ is a Cohen–Macaulay module, then it is sequentially Cohen–Macaulay. In fact, $M = M_1 \supsetneq M_0 = 0$.
 - (2) Any one-dimensional module M is sequentially Cohen–Macaulay; if M is not Cohen–Macaulay, one can take $M_1 = H_{\mathfrak{m}}^0(M)$ and $M_2 = M$.
 - (3) A domain R is sequentially Cohen–Macaulay if and only if it is Cohen–Macaulay. In fact, any nonzero submodule of R will have dimension $d = \dim(R)$, and the only possible sCM filtration for R is $0 = M_0 \subsetneq M_1 = R$.
 - (4) On the other hand, if M has a Cohen–Macaulay submodule M_1 such that M/M_1 is sequentially Cohen–Macaulay with a filtration $0 = M_1/M_1 \subsetneq N_1/M_1 \subsetneq \dots \subsetneq N_s/M_1 = M/M_1$, and $\dim M_1 < \dim N_1/M_1$, then $0 \subsetneq M_1 \subsetneq N_1 \subsetneq \dots \subsetneq N_s = M$ is a sCM filtration of M .

The notion of sequentially Cohen–Macaulay modules appears for the first time in the literature in [37] in the graded setting, in connection with the theory of Stanley–Reisner rings and simplicial complexes. Later on and independently, in [36], the notion of Cohen–Macaulay filtered modules has been introduced in the local case; in the same paper, it is proven that the two notions coincide. Since then, sequentially Cohen–Macaulay modules have been extensively studied especially in connection with shellability and graph theory, see for instance [1, 2, 22, 25, 40], and the definition has been extended in other directions, see [14], [33]. Sections 3 and 4 are devoted to two such extensions, due to the third and fourth author and the first two authors, respectively.

Example 2 Let M be a sequentially Cohen–Macaulay module with sCM filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ and let $d_i = \dim(M_i/M_{i-1})$ for all i .

- (1) Let N be another sequentially Cohen–Macaulay; then, $M \oplus N$ is sequentially Cohen–Macaulay. To see this, let $0 = N_0 \subsetneq N_1 \subsetneq \dots \subsetneq N_s = N$ be a sCM filtration of N and let $\delta_i = \dim(N_i/N_{i-1})$. For convenience, also let $d_0 = \delta_0 = -1$. Then, a filtration whose terms are of the form $M_i \oplus N_j$, where either $d_i = \max\{d_a \mid d_a \leq \delta_j\}$ or $\delta_j = \max\{\delta_a \mid \delta_a \leq d_i\}$, is a sCM filtration of $M_r \oplus N_s = M \oplus N$.

- (2) The completion \widehat{M} of M at the maximal ideal \mathfrak{m} is a sequentially Cohen–Macaulay \widehat{R} -module. To this end, let $M'_i = M_i \otimes_R \widehat{R}$, and by flatness of \widehat{R} , we have that $0 = M'_0 \subsetneq M'_1 \subsetneq \dots \subsetneq M'_r$ is a sCM filtration of $M \otimes_R \widehat{R} \cong \widehat{M}$.
- (3) It can be proven in general that being sequentially Cohen–Macaulay is preserved by faithfully flat base changes; with some extra effort, one can derive that M is a sequentially Cohen–Macaulay R -module if and only if $M[[x]]$ is sequentially Cohen–Macaulay as an $R[[x]]$ -module, cf [36, Theorem 6.2].

Remark 1 In the above example Part (1), the converse also holds, cf. [16, Proposition 4.5], [39, Proposition 3.2] or Corollary 4.

In Part (2) of the same example, the vice versa does not hold in general, see for instance [36, Example 6.1]. It is true though when R is a homomorphic image of a Cohen–Macaulay ring, see [23, Corollaries 2.8] and [23, Corollaries 2.9], where it is proved that in such a case M is a sequentially Cohen–Macaulay R -module if and only if \widehat{M} is a sequentially Cohen–Macaulay \widehat{R} -module.

Proposition 1 *Let M be a sequentially Cohen–Macaulay R -module with sCM filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$. Also let $d_i = \dim(M_i/M_{i-1})$, for all $i \in \{1, \dots, r\}$, then:*

- (1) *for all $j \in \mathbb{Z}$, we have that $H_{\mathfrak{m}}^j(M) \neq 0$ if and only if $j \in \{d_1, \dots, d_r\}$, and*

$$H_{\mathfrak{m}}^{d_i}(M) \cong H_{\mathfrak{m}}^{d_i}(M_i) \cong H_{\mathfrak{m}}^{d_i}(M_i/M_{i-1});$$

- (2) *for all $i \in \{1, \dots, r\}$ and $j < i$, the modules M_i and M_i/M_j are sequentially Cohen–Macaulay.*

Proof For Part (1), using the short exact sequences $0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_i/M_{i-1} \rightarrow 0$ one inductively shows that $\dim(M_i) = d_i$ for all $i \in \{1, \dots, r\}$, since $d_{i-1} < d_i$. The induced long exact sequences on local cohomology

$$\begin{aligned} \dots \rightarrow H_{\mathfrak{m}}^{j-1}(M_i/M_{i-1}) \rightarrow H_{\mathfrak{m}}^j(M_{i-1}) \rightarrow H_{\mathfrak{m}}^j(M_i) \rightarrow H_{\mathfrak{m}}^j(M_i/M_{i-1}) \\ \rightarrow H_{\mathfrak{m}}^{j+1}(M_{i-1}) \rightarrow \dots \end{aligned}$$

together with the fact that M_i/M_{i-1} is Cohen–Macaulay of dimension d_i , imply that

$$\begin{aligned} H_{\mathfrak{m}}^{d_i}(M_i) \cong H_{\mathfrak{m}}^{d_i}(M_i/M_{i-1}), \quad H_{\mathfrak{m}}^j(M_i) \cong H_{\mathfrak{m}}^j(M_{i-1}) \text{ if } j < d_i, \\ \text{and } H_{\mathfrak{m}}^j(M_i) = 0 \text{ otherwise.} \end{aligned}$$

In particular, $H_{\mathfrak{m}}^j(M) = 0$ for all $j > d_r$, and $H_{\mathfrak{m}}^j(M) = H_{\mathfrak{m}}^j(M_i)$ for all $j \leq d_i$. In conclusion, we have shown that

$$H_{\mathfrak{m}}^j(M) = \begin{cases} H_{\mathfrak{m}}^{d_i}(M_i) = H_{\mathfrak{m}}^{d_i}(M_i/M_{i-1}) & \text{if } j = d_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

Part (2) is clear once we notice that $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_i$ and $0 = M_j/M_j \subsetneq M_{j+1}/M_j \subsetneq \dots \subsetneq M_i/M_j$ are sCM filtrations. □

Example 3 Suppose that x is an M -regular element and that M/xM is sequentially Cohen–Macaulay; it is not true in general that M is sequentially Cohen–Macaulay, as observed after [38, Proposition 2.2]—the statement of [36, Theorem 4.7] is not correct. As a counterexample, one can take a two-dimensional not Cohen–Macaulay local domain R of depth 1, which is not sequentially Cohen–Macaulay by Example 1 (3). For instance, take

$R = \mathbb{Q}[[a^4, a^3b, ab^3, b^4]] \cong \mathbb{Q}[[z_1, z_2, z_3, z_4]]/(z_2z_3 - z_1z_4, z_3^2 - z_2z_4^2, z_1z_3^2 - z_2^2z_4, z_2^3 - z_1^2z_3)$.
 On the contrary, every $0 \neq x \in R$ is regular, $R/(x)$ is a one-dimensional R -module and, therefore, always sequentially Cohen–Macaulay by Example 1 (2).

In the rest of this section, let $S = k[x_1, \dots, x_n]$, with the standard grading. Given a monomial ideal $I \subset S$, we let $G(I)$ denote the monomial minimal system of generators of I and $m(I) = \max\{i \mid x_i \text{ divides } u \text{ for some } u \in G(I)\}$.

An important class of sequentially Cohen–Macaulay modules is given by rings defined by weakly stable ideals.

Definition 2 A monomial ideal $I \subseteq S$ is said to be *weakly stable* if for all monomials $u \in I$ and all integers i, j with $1 \leq j < i \leq n$, there exists $t \in \mathbb{N}$ such that $x_j^t u/x_i^\ell \in I$, where ℓ is the largest integer such that x_i^ℓ divides u .

Observe that the condition of the previous definition is verified if and only if it is verified for all $u \in G(I)$. In the literature, weakly stable ideals are also called ideals of Borel type, quasi-stable ideals, or ideals of nested type, see [8, 11, 28]. It is easy too see from the definition that stable, strongly stable, and p -Borel ideals are weakly stable. In particular, no matter what the characteristic of the field is, Borel fixed ideals are weakly stable, see also [28, Theorem 4.2.10] and, thus, generic initial ideals are always weakly stable.

Observe that the saturation I^{sat} of a weakly stable ideal I is $I : \mathfrak{m}^\infty = I : x_n^\infty$, see [28, Proposition 4.2.9]; thus, I^{sat} is again weakly stable and x_n does not divide any $u \in G(I^{\text{sat}})$.

Proposition 2 Let I be a weakly stable ideal of $S = k[x_1, \dots, x_n]$. Then, S/I is sequentially Cohen–Macaulay.

Proof We prove the statement by induction on $m(I)$. If $m(I) = 1$, then $I = (x_1^r)$ for some positive integer r and S/I is Cohen–Macaulay. Assume now that S/J is sequentially Cohen–Macaulay for every weakly stable ideal J for which $m(J) < m(I)$.

Let $S' = k[x_1, \dots, x_{m(I)}]$ and $I' = I \cap S'$. Since $S/I \cong S'/I' \otimes_k k[x_{m(I)+1}, \dots, x_n]$ and a sCM filtration of S'/I' is easily extended into one of S/I , it is sufficient to prove that S'/I' is sequentially Cohen–Macaulay. Therefore, without loss of generality, we may assume that $m(I) = n$, and that S/I^{sat} is sequentially Cohen–Macaulay. Now, I^{sat}/I is a non-trivial Artinian module, and the first nonzero module of a sCM filtration of S/I^{sat} has positive dimension, see Proposition 1. We may thus conclude that S/I is sequentially Cohen–Macaulay, cf. Example 1 (4). \square

Remark 2 By [8, Proposition 3.2], see also [28, Proposition 4.2.9] and [11, Chap. 4], a monomial ideal I is weakly stable if and only if all its associated primes are generated by initial segments of variables, i.e., are of type (x_1, \dots, x_i) for some i . Since being sequentially Cohen–Macaulay is independent of coordinates changes on S , the above proposition shows that whenever the associated primes of a monomial ideal $I \subseteq S$ are totally ordered by inclusion, then S/I is sequentially Cohen–Macaulay.

Example 4 Let $I = (x_1^4, x_1^2x_2^2, x_1^3x_3, x_1^2x_2x_3, x_1^2x_2x_4) \subseteq S = k[x_1, x_2, x_3, x_4]$. It is easy to verify that I is a weakly stable ideal which is not strongly stable, by checking the definition or by computing its associated primes, by using the previous remark; thus, S/I is sequentially Cohen–Macaulay by Proposition 2. We can construct an explicit sCM filtration proceeding as in its proof. We have $m(I) = 4$, and $I^{\text{sat}} = I : x_4^\infty = (x_1^4, x_1^2x_2, x_1^3x_3)$.

This means that the first nonzero module of our filtration will be $(x_1^4, x_1^2x_2, x_1^3x_3)/I$. Now, we consider $I' = (x_1^4, x_1^2x_2, x_1^3x_3)$ as an ideal of $k[x_1, x_2, x_3]$ and compute its saturation $I'' = I : x_3^\infty = (x_1^3, x_1^2x_2)$. Hence, the second nonzero module of the filtration is $(x_1^3, x_1^2x_2)/I$. Proceeding in this way, we obtain the filtration $0 \subseteq (x_1^4, x_1^2x_2, x_1^3x_3)/I \subseteq (x_1^3, x_1^2x_2)/I \subseteq (x_1^2)/I \subseteq (1)/I = S/I$, and it is easily seen that it is a sCM filtration of S/I .

Example 5 For the case $M = R = S/I$, when $I = I_\Delta$ is the Stanley–Reisner ideal of a simplicial complex Δ , there is a beautiful characterization of sequential Cohen–Macaulayness due to Duval, [20, Theorem 3.3]. Given a simplicial complex Δ , let $\Delta(i)$ be the pure i -th skeleton of Δ , i.e., the pure subcomplex of Δ whose facets are the faces of Δ of dimension i . Then, S/I_Δ is sequentially Cohen–Macaulay if and only if $S/I_{\Delta(i)}$ is Cohen–Macaulay for all i . Another important result in this context is that I_Δ is componentwise linear if and only if the Stanley–Reisner ring of its Alexander dual Δ^* is sequentially Cohen–Macaulay, see [27, Theorem 2.1] and [28, Theorem 8.2.20]. Moreover, it is known that if Δ is (nonpure) shellable, then S/I_Δ is sequentially Cohen–Macaulay, see [28, Corollary 8.2.19].

Example 6 Another class of examples of sequentially Cohen–Macaulay modules is given by pretty clean modules, which have been introduced by Herzog and Popescu in [29] in order to characterize shellability of multicomplexes. A *pretty clean* module M is a module that admits a *pretty clean* filtration, i.e., a prime filtration of M by submodules $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_s = M$ such that each quotient M_i/M_{i-1} is isomorphic to S/\mathfrak{p}_i , for some prime ideals \mathfrak{p}_i , with the following property: if $\mathfrak{p}_i \subsetneq \mathfrak{p}_j$, then $i > j$. For example, if $M = S/I$ with I weakly stable, cf. [29, Proposition 5.2], or is such that $\text{Ass}(M)$ is a totally ordered set, cf. [29, Proposition 5.1] or Remark 2, then M is pretty clean. By [29, Theorem 4.1], pretty clean modules are sequentially Cohen–Macaulay, provided every prime \mathfrak{p}_i appearing in the pretty clean filtration is such that S/\mathfrak{p}_i is Cohen–Macaulay.

2 Characterizations of sequentially Cohen–Macaulay modules

The goal of this section is to present two characterizations of sequentially Cohen–Macaulay modules, due to Schenzel, see Theorem 2 and Peskine, see Theorem 3. As in the previous section, we let (R, \mathfrak{m}, k) be either a Noetherian local ring or a standard graded k -algebra with maximal homogeneous ideal \mathfrak{m} ; we let $\dim(R) = n$. In the second case, modules will be graded and homomorphisms homogeneous of degree 0.

2.1 Schenzel’s characterization

Our main reference here is [36]. By convention, the dimension of the zero module is set to be -1 . Let M be a finitely generated (graded) R -module of dimension d . Since R is Noetherian, for all $i = 0, \dots, d$, we can consider the largest (graded) submodule of M of dimension $\leq i$, and denote it by $\delta_i(M)$. By maximality, in this way, one obtains a filtration $\mathcal{M} : 0 \subseteq \delta_0(M) \subseteq \delta_1(M) \subseteq \dots \subseteq \delta_d(M) = M$, called the *dimension filtration* of M . Evidently, such a filtration is unique.

Given a set X of prime ideals of R , we denote by $X_i = \{\mathfrak{p} \in X : \dim R/\mathfrak{p} = i\}$. Similarly, we define $X_{\leq i}$ and $X_{> i}$.

Remark 3 Observe that M has a nonzero submodule of dimension i if and only if $\text{Ass}(M)_i \neq \emptyset$. In fact, if $\mathfrak{p} \in \text{Ass}(M)_i$, then R/\mathfrak{p} is a nonzero submodule of M of dimension i . Conversely, if N is a nonzero submodule of M of dimension i , then there must exist a

minimal prime \mathfrak{p} of N such that $\dim(R/\mathfrak{p}) = \dim(N) = i$. Since $N \subseteq M$, we must also have that $\mathfrak{p} \in \text{Ass}(M)$, as desired. In particular, we have that $\delta_i(M) = 0$ if and only if $\text{Ass}(M)_{\leq i} = \emptyset$.

Notice that $\delta_0(M) = H_m^0(M)$; the other modules in \mathcal{M} can be described similarly, with the help of the minimal primary decomposition of 0 as a submodule of M . In fact, let $\text{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and, for all $i = 0, \dots, d$, consider the set $\text{Ass}(M)_{\leq i}$. We set

$$\alpha_i = \begin{cases} \prod_{\mathfrak{p} \in \text{Ass}(M)_{\leq i}} \mathfrak{p} & \text{if } \text{Ass}(M)_{\leq i} \neq \emptyset, \\ R & \text{otherwise.} \end{cases}$$

Now, let us consider an irredundant primary decomposition $0 = \bigcap_{j=1}^m N_j$ inside M , where each M/N_j is a \mathfrak{p}_j -primary module.

Proposition 3 [36, Proposition 2.2] *Let M be a finitely generated R -module of dimension d . With the above notation, for all $i = 0, \dots, d$ we have that*

$$\delta_i(M) = H_{\alpha_i}^0(M) = \bigcap_{\{j \mid \mathfrak{p}_j \in \text{Ass}(M)_{> i}\}} N_j,$$

where we let the intersection over the empty set be equal to M .

Proof We start with the first equality and assume that $\delta_i(M) = 0$. By Remark 3, we have that $\text{Ass}(M)_{\leq i} = \emptyset$ and in this case $\alpha_i = R$. Thus, $H_{\alpha_i}^0(M) = H_R^0(M) = 0 = \delta_i(M)$.

Now, assume that $\delta_i(M) \neq 0$, and observe that every associated prime of $\delta_i(M)$ has dimension $\leq i$ and, therefore, $\text{Ass}(\delta_i(M)) \subseteq \text{Ass}(M)_{\leq i}$. Thus, there is a power of α_i which annihilates $\delta_i(M)$, and we get that $\delta_i(M) \subseteq H_{\alpha_i}^0(M)$. On the other hand, there is a power of α_i which annihilates $H_{\alpha_i}^0(M)$ and, thus, $\dim(H_{\alpha_i}^0(M)) \leq \dim(R/\alpha_i) \leq i$. From the maximality of $\delta_i(M)$, it follows that $H_{\alpha_i}^0(M) \subseteq \delta_i(M)$, as desired.

For the second equality, observe that $N_j :_M \mathfrak{p}_j^\infty = M$ and $N_j :_M x^\infty = N_j$ for any $x \notin \mathfrak{p}_j$, with $j \in \{1, \dots, m\}$, since M/N_j is \mathfrak{p}_j -primary. Also notice that

$$H_{\alpha_i}^0(M) = 0 :_M \alpha_i^\infty = \left(\bigcap_{j=1}^m N_j \right) :_M \alpha_i^\infty = \bigcap_{j=1}^m (N_j :_M \alpha_i^\infty).$$

Now, if $\mathfrak{p}_j \in \text{Ass}(M)_{\leq i}$, then $M \supseteq N_j :_M \alpha_i^\infty \supseteq N_j :_M \mathfrak{p}_j^\infty = M$, forcing equality everywhere. Next, assume that $\mathfrak{p}_j \notin \text{Ass}(M)_{\leq i}$. Then, $\mathfrak{p} \not\subseteq \mathfrak{p}_j$ for every $\mathfrak{p} \in \text{Ass}(M)_{\leq i}$ since, otherwise, we would have $\dim(R/\mathfrak{p}_j) \leq \dim(R/\mathfrak{p}) \leq i$ and, thus, $\mathfrak{p}_j \in \text{Ass}(M)_{\leq i}$. In this case, by choosing $x \in \alpha_i$ and $x \notin \mathfrak{p}_j$, we have that $N_j \subseteq N_j :_M \alpha_i^\infty \subseteq N_j :_M x^\infty = N_j$, and equalities hold. Summing up, we conclude that

$$H_{\alpha_i}^0(M) = \bigcap_{\{j \mid \mathfrak{p}_j \notin \text{Ass}(M)_{\leq i}\}} N_j = \bigcap_{\{j \mid \mathfrak{p}_j \in \text{Ass}(M)_{> i}\}} N_j.$$

□

Note that this is consistent with our convention that the intersection over the empty set is equal to M ; in fact, for $i = d$, we have that $\text{Ass}(M)_{\leq d} = \text{Ass}(M)$ and $\text{Ass}(M)_{> d} = \emptyset$. This agrees with the fact that $H_{\alpha_d}^0(M) = H_{\sqrt{0}}^0(M) = M$.

Proposition 4 [[36], Corollary 2.3] *Let M be a finitely generated R -module of dimension d ; then, for all $i = 0, \dots, d$,*

- (1) $\text{Ass}(\delta_i(M)) = \text{Ass}(M)_{\leq i}$;
- (2) $\text{Ass}(M/\delta_i(M)) = \text{Ass}(M)_{> i}$;
- (3) $\text{Ass}(\delta_i(M)/\delta_{i-1}(M)) = \text{Ass}(M)_i$.

Proof It is well-known that $\text{Ass}(M) = \text{Ass}(H_{\mathfrak{a}}^0(M)) \sqcup \text{Ass}(M/H_{\mathfrak{a}}^0(M))$, for any ideal \mathfrak{a} of R . Also notice that $\text{Ass}(H_{\mathfrak{a}}^0(M)) = \text{Ass}(M) \cap V(\mathfrak{a})$, for all \mathfrak{a} . Since $\delta_i(M) = H_{\mathfrak{a}_i}^0(M)$ by Proposition 3, the first equality descends from Remark 3 and the above observation.

For (2), consider the short exact sequence $0 \rightarrow \delta_i(M) \rightarrow M \rightarrow M/\delta_i(M) \rightarrow 0$. By Proposition 3, one has that $\text{Ass}(M) = \text{Ass}(\delta_i(M)) \sqcup \text{Ass}(M/\delta_i(M))$, which is equal to $\text{Ass}(M)_{\leq i} \sqcup \text{Ass}(M)_{> i}$ by (1), and the second equality follows.

Finally, consider the short exact sequence $0 \rightarrow \delta_{i-1}(M) \rightarrow \delta_i(M) \rightarrow \delta_i(M)/\delta_{i-1}(M) \rightarrow 0$ and observe that $\text{Ass}(\delta_i(M)/\delta_{i-1}(M)) \subseteq \text{Ass}(M/\delta_{i-1}(M))$. We also have $\text{Ass}(\delta_i(M)/\delta_{i-1}(M)) \subseteq \text{Ass}(\delta_i(M))$ since $\delta_{i-1}(M) = H_{\mathfrak{a}_{i-1}}^0(M) = H_{\mathfrak{a}_{i-1}}^0(\delta_i(M))$. Thus, by (1) and (2), we have necessarily that $\text{Ass}(\delta_i(M)/\delta_{i-1}(M)) \subseteq \text{Ass}(M)_i$. On the other hand, by (1), if $\mathfrak{p} \in \text{Ass}(M)_i$, then $\mathfrak{p} \notin \text{Ass}(M_{i-1})$, and therefore $\mathfrak{p} \in \text{Ass}(\delta_i(M)/\delta_{i-1}(M))$. \square

As a corollary of Propositions 3 and 4, one can obtain another characterization of the dimension filtration, cf. [29, Proposition 1.1].

Corollary 1 *Let M be a d -dimensional R -module. A filtration of M by submodules $0 \subseteq M_0 \subseteq \dots \subseteq M_d = M$ is the dimension filtration of M if and only if $\text{Ass}(M_i/M_{i-1}) = \text{Ass}(M)_i$ for all i .*

The following definition was introduced in [36, Definition 4.1].

Definition 3 Let M be a d -dimensional finitely generated R -module, and $\mathcal{M} : 0 \subseteq \delta_0(M) \subseteq \dots \subseteq \delta_d(M) = M$ be its dimension filtration. Then, M is called *Cohen–Macaulay filtered* if for all $i \in \{0, \dots, d\}$ the module $\delta_i(M)/\delta_{i-1}(M)$ is either zero or Cohen–Macaulay.

Notice that from the definition, it immediately follows that if $\delta_i(M)/\delta_{i-1}(M) \neq 0$, then it has dimension i .

A Cohen–Macaulay filtered module M is sequentially Cohen–Macaulay. In fact, we may let i_1 denote the smallest integer such that $\delta_{i_1}(M) \neq 0$ and set $M_1 = \delta_{i_1}(M)$. Then, if i_j is the smallest integer such that $\delta_{i_{j-1}}(M) \subsetneq \delta_{i_j}(M)$, we let $M_j = \delta_{i_j}(M)$. The resulting filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ is clearly a sCM filtration of M .

A filtration $0 = C_{-1} \subseteq C_0 \subseteq C_1 \subseteq \dots \subseteq C_d = M$ such that each of its quotients C_i/C_{i-1} is either zero or Cohen–Macaulay of dimension i is called a *CM filtration* of M . To see that the notions of sequentially Cohen–Macaulay and Cohen–Macaulay filtered modules coincide, we need the following result, which shows that if a CM filtration of a module exists, then it is unique, and it is equal to its dimension filtration.

Proposition 5 [[36], Proposition 4.3] *Let M be a d -dimensional finitely generated R -module. If M has a CM filtration $0 = C_{-1} \subseteq C_0 \subseteq C_1 \subseteq \dots \subseteq C_d = M$, then $C_i = \delta_i(M)$ for every $i \in \{0, \dots, d\}$.*

Proof We proceed by induction on d . If $d = 0$, then $C_0 = M = \delta_0(M)$ and this case is complete. Assume henceforth that $d > 0$, so that by induction, we have that $C_i = \delta_i(C_{d-1})$

for all $i < d$. Since for $i < d$ we have that $\delta_i(M) = \delta_i(\delta_{d-1}(M))$, it suffices to show that $\delta_{d-1}(M) = C_{d-1}$.

Observe that $\text{Ass}(C_0) \subseteq \text{Ass}(M)_{\leq 0}$ and, for all $\mathfrak{p} \in \text{Ass}(C_i/C_{i-1})$, we have $\dim A/\mathfrak{p} = i$ if $C_i \neq C_{i-1}$. With this information, we can inductively show that $\dim C_i \leq i$ and, accordingly $C_i \subseteq \delta_i(M)$ for all i ; in particular, $C_{d-1} \subseteq \delta_{d-1}(M)$. If $\text{Ass}(M)_{\leq d-1} = \emptyset$, then $\delta_{d-1}(M) = 0$ by Remark 3, and the desired equality is trivial. Otherwise, we let $\mathfrak{a} = \mathfrak{a}_{d-1} = \prod_{\mathfrak{p} \in \text{Ass}(M)_{\leq d-1}} \mathfrak{p}$ and we claim that $H_{\mathfrak{a}}^0(M/C_{d-1}) = 0$. This is obvious if $M/C_{d-1} = 0$. Thus, assume $C_{d-1} \neq M$ and observe that \mathfrak{a} contains a regular element of M/C_{d-1} . To see the latter, assume that $\mathfrak{a} \subseteq \bigcup_{\mathfrak{p} \in \text{Ass}(M/C_{d-1})} \mathfrak{p}$; then, by prime avoidance, we can find $\mathfrak{p} \in \text{Ass}(M/C_{d-1})$ such that $\mathfrak{a} \subseteq \mathfrak{p}$ and, therefore, \mathfrak{p} contains a prime $\mathfrak{p}' \in \text{Ass}(M)_{\leq d-1}$. This is a contradiction, since it would imply $d = \dim(M/C_{d-1}) = \dim(R/\mathfrak{p}) \leq \dim(R/\mathfrak{p}') \leq d - 1$. Finally, consider the short exact sequence $0 \rightarrow C_{d-1} \rightarrow M \rightarrow M/C_{d-1} \rightarrow 0$. By Proposition 3, we have that $\delta_{d-1}(M) = H_{\mathfrak{a}}^0(M) = H_{\mathfrak{a}}^0(C_{d-1}) \subseteq C_{d-1}$. Thus, $C_{d-1} = \delta_{d-1}(M)$, and we are done. \square

Theorem 2 (Schenzel) *Let M be a finitely generated R -module; then M is sequentially Cohen–Macaulay if and only if M is Cohen–Macaulay filtered. Moreover, if M is sequentially Cohen–Macaulay, then its sCM filtration is unique.*

Proof We only need to show that the “only if” part. Let $\mathcal{M} : 0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ be any sCM filtration of M , with $d = \dim(M)$, $d_0 = -1$ and $d_i = \dim(M_i/M_{i-1})$. For $j \in \{-1, 0, \dots, d\}$, we also let $i(j)$ be the largest integer i such that $d_i \leq j$, and we set $C_j = M_{i(j)}$. In this way, we have constructed a filtration $\mathcal{C} : 0 = C_{-1} \subseteq C_0 \subseteq \dots \subseteq C_d = M$, which is a CM filtration of M . By Proposition 5, we may conclude that $C_i = \delta_i(M)$, and that \mathcal{C} is the unique dimension filtration of M . Thus, M is Cohen–Macaulay filtered.

Observe that one can reconstruct \mathcal{M} from \mathcal{C} ; it follows that the modules M_i only depend on M as well, and \mathcal{M} is also unique. \square

Notice that by the above theorem and Proposition 4, if M is sequentially Cohen–Macaulay with sCM filtration \mathcal{M} , then $\text{Ass}(M) = \sqcup_i \text{Ass}(M_i/M_{i-1})$. The comparison between sCM filtrations and dimension filtrations has also the following useful consequences.

Corollary 2 *Let M be a d -dimensional finitely generated R -module. The following are equivalent:*

- (1) M is sequentially Cohen–Macaulay;
- (2) The modules $\delta_i(M)$ and $M/\delta_i(M)$ are sequentially Cohen–Macaulay for all $i \in \{0, \dots, d\}$;
- (3) There exists $i \in \{0, \dots, d\}$ such that $\delta_i(M)$ and $M/\delta_i(M)$ are sequentially Cohen–Macaulay.

In particular, M is sequentially Cohen–Macaulay if and only if $M/H_{\mathfrak{m}}^0(M)$ is sequentially Cohen–Macaulay.

Proof Observe that given any $i \in \{0, \dots, d\}$, we have that $\delta_j(M) = \delta_j(\delta_i(M))$ for every $j \leq i$, and $\delta_j(M/\delta_i(M)) = \delta_j(M)/\delta_i(M)$ for every $j \geq i$. It follows that $0 \subseteq \delta_0(M) \subseteq \dots \subseteq \delta_i(M)$ and $0 = \delta_i(M)/\delta_i(M) \subseteq \delta_{i+1}(M)/\delta_i(M) \subseteq \dots \subseteq \delta_d(M)/\delta_i(M)$ are the dimension filtrations of $\delta_i(M)$ and $M/\delta_i(M)$, respectively. This yields all the assertions at once. \square

Recall now that a finitely generated R -module M is *unmixed* if $\dim(R/\mathfrak{p}) = \dim(M)$ for all $\mathfrak{p} \in \text{Ass}(M)$.

Corollary 3 *Let M be a finitely generated unmixed R -module. Then, M is sequentially Cohen–Macaulay if and only if M is Cohen–Macaulay.*

Proof One direction is clear. Let $d = \dim(M)$; in view of Remark 3, the fact that M is unmixed guarantees that $\delta_i(M) = 0$ for all $i < d$. By Theorem 2, M is Cohen–Macaulay filtered, and thus $\delta_d(M)/\delta_{d-1}(M) = M$ is Cohen–Macaulay. \square

2.2 Peskine’s characterization

We will always assume that R has a canonical module ω_R . In our setup, this assumption is not too restrictive: it is always the case when R is standard graded, and it is true for instance if R is complete local, cf. Remark 1.

Proposition 6 *Let R be an n -dimensional Cohen–Macaulay ring with canonical module ω_R and M be sequentially Cohen–Macaulay, with sCM filtration $0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$; also let $d_i = \dim M_i/M_{i-1}$, for $i = 1, \dots, r$. Then,*

- (1) *for all $i = 1, \dots, r$, one has that $\text{Ext}_R^{n-d_i}(M, \omega_R) \simeq \text{Ext}_R^{n-d_i}(M_i/M_{i-1}, \omega_R)$ is Cohen–Macaulay and has dimension d_i ;*
- (2) *$\text{Ext}_R^{n-j}(M, \omega_R) = 0$ whenever $j \notin \{d_1, \dots, d_r\}$;*
- (3) *$\text{Ext}_R^{n-d_i}(\text{Ext}_R^{n-d_i}(M, \omega_R), \omega_R) \simeq M_i/M_{i-1}$ for $i = 1, \dots, r$.*

Proof In the graded case, (1) and (2) follow immediately from Proposition 1, graded local duality [10, Theorem 3.6.19] and [10, Theorem 3.3.10 (c) (i)]. In the local case, local duality yields (1) and (2) for the completion \widehat{M} of M as an \widehat{R} -module, see Example 2 (2). Since $\omega_{\widehat{R}} \cong \widehat{\omega}_R \cong \omega_R \otimes_R \widehat{R}$ and $\text{Ext}_{\widehat{R}}^i(\widehat{N}, \omega_{\widehat{R}}) \cong \text{Ext}_R^i(N, \omega_R) \otimes_R \widehat{R}$ for any finitely generated R -module N , we conclude by faithful flatness of \widehat{R} that (1) and (2) hold also in the local case.

Finally, by [10, Theorem 3.3.10 (c) (iii)], we have that $N \cong \text{Ext}_R^{n-d}(\text{Ext}_R^{n-d}(N, \omega_R), \omega_R)$ for any Cohen–Macaulay module N of dimension d and, thus, the last statement follows immediately from (1). \square

Remark 4 Let M be as in the above proposition.

- (1) By Parts (2) and (3) of the previous proposition, we have an isomorphism

$$M_1 \cong \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R),$$

where $t = \text{depth } M$ and M_1 is t -dimensional and Cohen–Macaulay. We will show an extension of this fact in Lemma 1.

- (2) Notice that if M is sequentially Cohen–Macaulay with $\text{depth}(M) = 0$, then in particular, $M_1 = H_m^0(M)$. In fact, using [10, Theorem 3.3.10 (c) (iii)] and the short exact sequence $0 \rightarrow H_m^0(M) \rightarrow M \rightarrow M/H_m^0(M) \rightarrow 0$, we get that

$$M_1 \cong \text{Ext}_R^n(\text{Ext}_R^n(M, \omega_R), \omega_R) \cong \text{Ext}_R^n(\text{Ext}_R^n(H_m^0(M), \omega_R), \omega_R) \cong H_m^0(M).$$

We recall now the following crucial lemma, see [31, Lemma 1.5].

Lemma 1 *Let R be a Cohen–Macaulay n -dimensional ring with canonical module ω_R and M be a finitely generated R -module.*

Let also $\text{depth}(M) = t$ and assume that $\text{Ext}_R^{n-t}(M, \omega_R)$ is Cohen–Macaulay of dimension t . Then, there is a natural monomorphism $\alpha : \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R) \rightarrow M$ such that

$$\text{Ext}_R^{n-t}(\alpha) : \text{Ext}_R^{n-t}(M, \omega_R) \rightarrow \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R)\omega_R)$$

is an isomorphism.

Proof We only give a proof in the graded case; the local one is handled similarly. Without loss of generality, we may assume that R is a polynomial ring, as we show next. We write $R = S/I$, where S is a standard graded polynomial ring of dimension m over a field k and $I \subseteq S$ is homogeneous. Let \mathfrak{m} and \mathfrak{n} denote the graded maximal ideals of R and S , respectively. By graded Local Duality [10, Theorem 3.6.19], we have that $\text{Ext}_S^{m-i}(M, \omega_S) \cong \text{Hom}_S(H_{\mathfrak{n}}^i(M), E_S(k))$ and $\text{Ext}_R^{n-i}(M, \omega_R) \cong \text{Hom}_R(H_{\mathfrak{m}}^i(M), E_R(k))$. By Base Independence, $H_{\mathfrak{n}}^i(M) \cong H_{\mathfrak{m}}^i(M)$ for all i . Thus, by Hom $-\otimes$ adjointness and [10, Lemma 3.1.6], we obtain

$$\text{Hom}_S(H_{\mathfrak{n}}^i(M), E_S(k)) \cong \text{Hom}_R(H_{\mathfrak{m}}^i(M), \text{Hom}_S(R, E_S(k))) \cong \text{Hom}_R(H_{\mathfrak{m}}^i(M), E_R(k)).$$

In particular, we have shown that $\text{Ext}_R^{n-t}(M, \omega_R) \cong \text{Ext}_S^{m-t}(M, \omega_S)$ and that we may assume that R is a polynomial ring.

Let now $F_{\bullet} : 0 \rightarrow F_{n-t} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be the minimal graded free resolution of M ; applying the functor $\text{Hom}_R(-, \omega_R)$, we obtain the dual complex

$$F_{\bullet}^* : 0 \rightarrow F_{n-t}^* \rightarrow \dots \rightarrow F_1^* \rightarrow F_0^* \rightarrow 0,$$

with $H_0(F_{\bullet}^*) = \text{Ext}_R^{n-t}(M, \omega_R)$. Observe that F_{\bullet}^* is a complex of free modules, since $\omega_R \cong R(-n)$. If we also let G_{\bullet} denote the minimal graded free resolution of $\text{Ext}_R^{n-t}(M, \omega_R)$, then there is a map of complexes $\phi_{\bullet} : F_{\bullet}^* \rightarrow G_{\bullet}$ which lifts the identity map of $\text{Ext}_R^{n-t}(M, \omega_R)$. Since the last one is Cohen–Macaulay of dimension t , the length of G_{\bullet} is the same as that of F_{\bullet}^* , namely $n - t$. Applying the functor $\text{Hom}_R(-, \omega_R)$ to ϕ_{\bullet} , we obtain a map of complexes $\phi_{\bullet}^* : G_{\bullet}^* \rightarrow F_{\bullet}^{**}$ which, in turn, gives a map on the zeroth cohomology:

$$\alpha = H_0(\phi_{\bullet}^*) : \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R) = H_0(G_{\bullet}^*) \rightarrow H_0(F_{\bullet}^{**}) \cong H_0(F_{\bullet}) = M.$$

Applying again the functor $\text{Hom}_R(-, \omega_R)$, this time to ϕ_{\bullet}^* , we obtain a map of complexes $\phi_{\bullet}^{***} : F_{\bullet}^{***} \rightarrow G_{\bullet}^{**}$ and, thus, a map

$$\begin{aligned} H_0(\phi_{\bullet}^{***}) : H_0(F_{\bullet}^{***}) &\cong H^0(F_{\bullet}^*) \cong \text{Ext}_R^{n-t}(M, \omega_R) \\ &\rightarrow \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R), \omega_R) = H_0(G_{\bullet}^{**}), \end{aligned}$$

which coincides with the map $\text{Ext}_R^{n-t}(\alpha)$. The canonical isomorphism $\psi : G_{\bullet}^{**} \rightarrow G_{\bullet}$, together with the fact that $\text{Ext}_R^{n-t}(M, \omega_R)$ is Cohen–Macaulay of dimension t , gives an isomorphism $H_0(\psi) : \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R), \omega_R) \rightarrow \text{Ext}_R^{n-t}(M, \omega_R)$, and one can verify that

$$H_0(\psi) \circ H_0(\phi_{\bullet}^{***}) = H_0(\phi_{\bullet}) = \text{id}_{\text{Ext}_R^{n-t}(M, \omega_R)}.$$

Note that $H_0(\psi)$ is the inverse of the isomorphism of [10, Theorem 3.3.10 (c) (iii)], and this implies that $H_0(\phi_{\bullet}^{***}) = \text{Ext}_R^{n-t}(\alpha)$ is the natural isomorphism of the same theorem.

To conclude the proof, it remains to be shown that α is a monomorphism.

Let $N = \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R)$. We show that α is injective once we localize at every associated prime of N and then we are done, since, if $\ker(\alpha) \neq 0$, its associated primes would be contained in those of N .

Let $\mathfrak{p} \in \text{Ass}(N)$; since N is Cohen–Macaulay of dimension t , we have that $\dim(R/\mathfrak{p}) = t$ and $\dim(R_{\mathfrak{p}}) = n - t$. By replacing R with $R_{\mathfrak{p}}$, M with $M_{\mathfrak{p}}$, and ω_R with $(\omega_R)_{\mathfrak{p}} \cong \omega_{R_{\mathfrak{p}}}$, the proof will be complete once we show that α is injective in the case $t = 0$. To this end observe that as in Remark 4, the short exact sequence $0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow M/H_{\mathfrak{m}}^0(M) \rightarrow 0$ and [10, Theorem 3.3.10 (c) (iii)] yield

$$\text{Ext}_R^n(\text{Ext}_R^n(M, \omega_R), \omega_R) \cong \text{Ext}_R^n(\text{Ext}_R^n(H_{\mathfrak{m}}^0(M), \omega_R), \omega_R) \cong H_{\mathfrak{m}}^0(M),$$

and α composed with this isomorphism becomes just the inclusion of $H_{\mathfrak{m}}^0(M)$ inside M . \square

The equivalence between the first two conditions in the next theorem was announced in [37] without a proof, but citing a spectral sequence argument due to Peskine. Here, we give another proof of this fact, see also [36, Theorem 5.5] where the equivalence with the third condition is proved.

Theorem 3 (Peskine) *Let R be a Cohen–Macaulay ring of dimension n with canonical module ω_R , and M be a finitely generated d -dimensional R -module. Then, the following are equivalent:*

- (1) M is sequentially Cohen–Macaulay;
- (2) $\text{Ext}_R^{n-i}(M, \omega_R)$ is either 0 or Cohen–Macaulay of dimension i for all $i \in \{0, \dots, d\}$;
- (3) $\text{Ext}_R^{n-i}(M, \omega_R)$ is either 0 or Cohen–Macaulay of dimension i for all $i \in \{1, \dots, d - 1\}$.

Proof The implication (1) \Rightarrow (2) follows at once by Proposition 6, and clearly (2) implies (3).

Now assume (3); we proceed by induction on $d - t$, where $t = \text{depth}(M)$. If $t = d$, then M is Cohen–Macaulay, and hence sequentially Cohen–Macaulay. Assume that $t < d$. If $t = 0$, then $0 \neq \text{Ext}_R^n(M, \omega_R)$ has finite length, and hence it is Cohen–Macaulay. Either way, thanks to our assumption we have that $0 \neq \text{Ext}_R^{n-t}(M, \omega_R)$ is t -dimensional and Cohen–Macaulay. By Lemma 1, there is an injective homomorphism $\alpha : \text{Ext}_R^{n-t}(\text{Ext}_R^{n-t}(M, \omega_R), \omega_R) \rightarrow M$. Since the first module is t -dimensional Cohen–Macaulay by [10, Theorem 3.3.10 (c)], then so is its image, say M_1 , which is a submodule of M . It follows that $\text{depth}(M_1) = \dim(M_1) = t = \text{depth}(M) < d = \dim(M)$.

Consider the short exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0$ and the induced sequence in cohomology obtained by applying the functor $\text{Hom}_R(-, \omega_R)$. We then have isomorphisms $\text{Ext}_R^j(M/M_1, \omega_R) \cong \text{Ext}_R^j(M, \omega_R)$ for all $j \neq n - t, n - t + 1$ and the exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ext}_R^{n-t}(M/M_1, \omega_R) &\rightarrow \text{Ext}_R^{n-t}(M, \omega_R) \xrightarrow{\beta} \text{Ext}_R^{n-t}(M_1, \omega_R) \\ &\rightarrow \text{Ext}_R^{n-t+1}(M/M_1, \omega_R) \rightarrow 0. \end{aligned}$$

By Lemma 1 we know that the map $\text{Ext}_R^{n-t}(\alpha)$ is an isomorphism, and therefore β is an isomorphism as well. It follows that $\text{Ext}_R^j(M, \omega_R) \cong \text{Ext}_R^j(M/M_1, \omega_R)$ for every $j \neq n - t$, while $\text{Ext}_R^{n-t}(M/M_1, \omega_R) = 0$.

This shows in particular that $\text{depth}(M/M_1) > t$; since $\dim(M/M_1) = d$, we may apply induction and obtain that M/M_1 is sequentially Cohen–Macaulay. Let $0 = M_1/M_1 \subsetneq$

$M_2/M_1 \subsetneq \dots \subsetneq M_r/M_1 = M/M_1$ be a sCM filtration. Since M_1 is Cohen–Macaulay of dimension t and $\text{depth}(M_2/M_1) = \text{depth}(M/M_1) > t$ by Example 1 (4), we deduce that $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_r = M$ is a sCM filtration and, thus M is sequentially Cohen–Macaulay. \square

Corollary 4 *Let R be Cohen–Macaulay with canonical module ω_R , and M, N be finitely generated R -modules. Then, M and N are sequentially Cohen–Macaulay if and only if $M \oplus N$ is sequentially Cohen–Macaulay.*

Proof Let $n = \dim R$. We have already showed in Example 2 (1) that, if M and N are sequentially Cohen–Macaulay, then so is $M \oplus N$. This also follows immediately from Theorem 3, since if $\text{Ext}_R^{n-i}(M, \omega_R)$ and $\text{Ext}_R^{n-i}(N, \omega_R)$ is either zero or Cohen–Macaulay of dimension i , then so is $\text{Ext}_R^{n-i}(M, \omega_R) \oplus \text{Ext}_R^{n-i}(N, \omega_R) \cong \text{Ext}_R^{n-i}(M \oplus N, \omega_R)$. For the converse, it suffices to observe that if the direct sum of two modules is zero or Cohen–Macaulay of a given dimension, then so is each of its summand. \square

Remark 5 We remark that sequential Cohen–Macaulayness behaves well with respect to localization; see for instance [16, Proposition 4.7] or [15, Proposition 2.6]. For any sequentially Cohen–Macaulay R -module M and $\mathfrak{p} \in \text{Supp}(M)$, one has that $M_{\mathfrak{p}}$ is a sequentially Cohen–Macaulay $R_{\mathfrak{p}}$ -module and, in fact, one can recover its dimension filtration from that of M . Let $s = \dim(R/\mathfrak{p})$ and consider the quotients $\delta_i(M)/\delta_{i-1}(M)$ of the dimension filtration of M . If not zero, they are i -dimensional Cohen–Macaulay, and their localization is either zero or Cohen–Macaulay of dimension $i - s$. Now let $N_i = (\delta_{i+s}(M))_{\mathfrak{p}}$ for all $i \geq 0$ such that $i + s \leq d = \dim(M)$, i.e., for $i = 0, \dots, d - s$, and observe that $0 \subseteq N_0 \subseteq \dots \subseteq N_{d-s} = M_{\mathfrak{p}}$ is a CM filtration of $M_{\mathfrak{p}}$, with quotients $N_i/N_{i-1} \simeq (\delta_{i+s}(M)/\delta_{i+s-1}(M))_{\mathfrak{p}}$. If R is Cohen–Macaulay, the fact that $M_{\mathfrak{p}}$ is sequentially Cohen–Macaulay for all $\mathfrak{p} \in \text{Supp}(M)$ is also a consequence of Theorem 3. See [39] for other results about localization and sequentially Cohen–Macaulay modules.

We now recall some definitions needed to state the next result, and that we will use frequently in the next sections. Given an R -module N , we let $\text{Ass}^\circ(N) = \text{Ass}(N) \setminus \{\mathfrak{m}\}$.

Definition 4 Let $M \neq 0$ be a finitely generated graded R -module.

- (1) A homogeneous element $0 \neq y \in \mathfrak{m}$ is *filter regular* for M if $y \notin \bigcup_{\mathfrak{p} \in \text{Ass}^\circ(M)} \mathfrak{p}$.
A sequence of homogeneous elements $y_1, \dots, y_t \in \mathfrak{m}$ is a *filter regular sequence* for M if y_{i+1} is a filter regular element for $M/(y_1, \dots, y_i)M$ for all $i \in \{0, \dots, t - 1\}$.
- (2) A homogeneous element $0 \neq y \in \mathfrak{m}$ is *strictly filter regular* for M if $y \notin \bigcup_{\mathfrak{p} \in \text{Ass}^\circ(X(M))} \mathfrak{p}$, where $X(M) = \bigoplus_{i \in \mathbb{N}} \text{Ext}_R^i(M, R)$.
A sequence of homogeneous elements $y_1, \dots, y_t \in \mathfrak{m}$ is a *strictly filter regular sequence* for M if y_{i+1} is a strictly filter regular element for $M/(y_1, \dots, y_i)M$ for all $i \in \{0, \dots, t - 1\}$.

Remark 6 Regular sequences are clearly filter regular sequences. Moreover, strictly filter regular sequences are filter regular by the graded version of [9, Corollary 11.3.3]. When the field k is infinite, by Prime Avoidance any sequence of general forms is strictly filter regular.

We conclude this section with the following result, which clarifies how sequential Cohen–Macaulayness behaves with respect to quotients, cf. Example 3; it will also be useful later on.

Proposition 7 *Let R be a Cohen–Macaulay ring of dimension n with canonical module ω_R ; let M be a d -dimensional finitely generated R -module, and $x \in R$ a strictly filter regular element for M . Then,*

- (1) *If M is sequentially Cohen–Macaulay, then M/xM is sequentially Cohen–Macaulay.*
- (2) *The converse holds if x is regular for all nonzero $\text{Ext}_R^{n-i}(M, \omega_R)$ with $i > 0$.*

Proof Assume that M is sequentially Cohen–Macaulay. Since x is strictly filter regular, it is filter regular by Remark 2.17, $L = 0 :_M x$ has finite length and $\text{Ext}_R^{n-i}(\overline{M}, \omega_R) \cong \text{Ext}_R^{n-i}(M, \omega_R)$ for all $i > 0$, where $\overline{M} = M/L$. Then, the long exact sequence obtained by applying the functor $\text{Hom}_R(-, \omega_R)$ to the short exact sequence $0 \rightarrow \overline{M} \xrightarrow{x} M \rightarrow M/xM \rightarrow 0$ gives a long exact sequence

$$\begin{aligned} \dots &\xrightarrow{x} \text{Ext}_R^{n-i}(M, \omega_R) \rightarrow \text{Ext}_R^{n-(i-1)}(M/xM, \omega_R) \rightarrow \text{Ext}_R^{n-(i-1)}(M, \omega_R) \rightarrow \dots \\ \dots &\xrightarrow{x} \text{Ext}_R^{n-1}(M, \omega_R) \rightarrow \text{Ext}_R^n(M/xM, \omega_R) \rightarrow \text{Ext}_R^n(M, \omega_R) \rightarrow 0. \end{aligned}$$

By Theorem 3, we have that each nonzero $\text{Ext}_R^{n-i}(M, \omega_R)$ is Cohen–Macaulay of dimension i , and in this case x is $\text{Ext}_R^{n-i}(M, \omega_R)$ -regular when $i > 0$. For $i > 1$, we then have short exact sequences

$$0 \rightarrow \text{Ext}_R^{n-i}(M, \omega_R) \xrightarrow{x} \text{Ext}_R^{n-i}(M, \omega_R) \rightarrow \text{Ext}_R^{n-(i-1)}(M/xM, \omega_R) \rightarrow 0,$$

and it follows that $\text{Ext}_R^{n-(i-1)}(M/xM, \omega_R) \cong \text{Ext}_R^{n-i}(M, \omega_R) \otimes_R R/(x)$ is Cohen–Macaulay of dimension $i - 1$ for all $i > 1$. We conclude that M/xM is sequentially Cohen–Macaulay using the implication (3) \Rightarrow (1) of Theorem 3.

For the converse, the fact that x is regular for all nonzero $\text{Ext}_R^{n-i}(M, \omega_R)$ with $i > 0$ shows that the above long exact sequence of Ext modules breaks into short exact sequences

$$0 \rightarrow \text{Ext}_R^{n-i}(M, \omega_R) \xrightarrow{x} \text{Ext}_R^{n-i}(M, \omega_R) \rightarrow \text{Ext}_R^{n-(i-1)}(M/xM, \omega_R) \rightarrow 0.$$

for all $i > 1$. By Theorem 3, we have that $\text{Ext}_R^{n-(i-1)}(M/xM, \omega_R)$ is either zero or Cohen–Macaulay of dimension $i - 1$, and thus $\text{Ext}_R^{n-i}(M, \omega_R)$ is either zero or Cohen–Macaulay of dimension i for all $i > 1$. Since x is assumed to be regular on $\text{Ext}_R^{n-1}(M, \omega_R)$, and $\dim(\text{Ext}_R^{n-1}(M, \omega_R)) \leq 1$, we have that $\text{Ext}_R^{n-1}(M, \omega_R)$ is either zero, or Cohen–Macaulay of dimension 1. It follows again from the implication (3) \Rightarrow (1) of Theorem 3 that M is sequentially Cohen–Macaulay. □

Remark 7 There are many other interesting results about sequentially Cohen–Macaulay modules and their characterizations which do not find space in this note. For instance, in [14, Theorem 5.1], it is proven that a module is sequentially Cohen–Macaulay if and only if each module of its dimension filtration is pseudo Cohen–Macaulay. In [18, Theorem 1.1], the sequential Cohen–Macaulayness of M is characterized in terms of the existence of one good system of parameters of M which has the property of parametric decomposition; see also Theorems 3.9 and 4.2 in [14] for other characterizations which involve good systems of parameters and dd -sequences.

Moreover, in [15], it is investigated how the sequential Cohen–Macaulay property behaves in relation to taking associated graded rings and Rees algebras, see also [38] for more results of this type.

3 Partially sequentially Cohen–Macaulay modules

We are going to study next the notion of partially sequentially Cohen–Macaulay module, as introduced in [35], which naturally generalizes that of sequentially Cohen–Macaulay modules. Thanks to Schenzel’s Theorem 2, the definition can be given in terms of the dimension filtration of the module. Throughout this section, we let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over an infinite field k with homogeneous maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$. Recall that, in this case, R has a graded canonical module $\omega_R \cong R(-n)$. We consider finitely generated graded R -modules M ; when $M = 0$, we set $\text{depth}(M) = +\infty$ and $\text{dim}(M) = -1$, as usual. We let $d = \text{dim}(M)$.

Definition 5 Let $i \in \{0, \dots, d\}$ and let $\{\delta_j(M)\}_j$ be the dimension filtration of M ; M is called *i -partially sequentially Cohen–Macaulay*, *i -sCM* for short, if $\delta_j(M)/\delta_{j-1}(M)$ is either zero or Cohen–Macaulay for all $i \leq j \leq d$.

By definition and Corollary 2, a module M is 0-sCM if and only if M sequentially Cohen–Macaulay if and only if M is 1-sCM.

Example 7 (1) Let M be a sequentially Cohen–Macaulay module. The simplest way of constructing an i -sCM module which is not sequentially Cohen–Macaulay is perhaps taking a non-sequentially Cohen–Macaulay module N of dimension strictly smaller than i , and consider their direct sum $M \oplus N$, cf. Example 2 (1).

(2) Let $M = R/I$, where $I = (x_1) \cap (x_2, x_3) \cap (x_1^2, x_4, x_5) \subset R = k[x_1, x_2, x_3, x_4, x_5]$. With the help of Proposition 3, we can construct the dimension filtration $0 = \delta_{-1} = \delta_0 = \delta_1 \subseteq \delta_2 = ((x_1) \cap (x_2, x_3))/I \subseteq \delta_3 = (x_1)/I \subseteq \delta_4 = R/I$ of M . Then, δ_4/δ_3 and δ_3/δ_2 are Cohen–Macaulay of dimension 4 and 3, respectively, but $0 \neq \delta_2/\delta_1$ is not Cohen–Macaulay. Hence, M is an example of a 3-sCM which is not 2-sCM.

Remark 8 Observe that M is i -sCM if and only if $M/\delta_{i-1}(M)$ is sequentially Cohen–Macaulay. This follows at once recalling that the dimension filtration $\{\gamma_j\}_j$ of $M/\delta_{i-1}(M)$ is such that $\gamma_j = \delta_j(M)/\delta_{i-1}(M)$ for $j \geq i$ and $\gamma_j = 0$ otherwise. Notice that, since $\gamma_j = 0$ for all $j \leq i - 1$, if M is i -sCM then $H_{\mathfrak{m}}^j(M/\delta_{i-1}(M)) = 0$ for all $j \leq i - 1$, by Proposition 1.

Given a graded free presentation of $M \cong F/U$, we denote by $\{e_1, \dots, e_r\}$ a graded basis of F . We consider R together with the pure reverse lexicographic ordering $>$ such that $x_1 > \dots > x_n$; recall that $>$ is not a monomial order on R , but by definition it agrees with the reverse lexicographic order that refines it on monomials of the same degree.

We extend $>$ to F in the following way: given monomials ue_i and ve_j of F , set

$$ue_i > ve_j \text{ if } (\deg(ue_i) > \deg(ve_j)), \text{ or } (\deg(ue_i) = \deg(ve_j) \text{ and } u > v) \\ \text{or } (\deg(ue_i) = \deg(ve_j), u = v \text{ and } i < j).$$

We shall consider this order until the end of the section, and denote by $\text{Gin}(U)$ the generic initial module of U with respect to $>$. Since the action of $\text{GL}_n(k)$ on R as change of coordinates can be extended in an obvious way to F , $\text{Gin}(U)$ simply results to be the initial submodule in $_{>}(gU)$ where g is a general change of coordinates.

We prove next some preliminary facts which are needed later on. Given a graded submodule $V \subseteq F$ with $\text{dim}(F/V) = d$, for all $j \in \{-1, \dots, d\}$, we denote by $V^{(j)}$ the

R -module such that $V^{(j)}/V = \delta_j(F/V)$. Several results contained in the next two lemmata can be found in [22], where they are proved in the ideal case.

Lemma 2 *With the above notation,*

- (1) $\text{Gin}(U^{(j)}) \subseteq \text{Gin}(U)^{(j)}$;
- (2) $U^{(j)} = (U^{(j)})^{(j)}$;
- (3) if V is a graded submodule of F such that $U \subseteq V$, then $U^{(j)} \subseteq V^{(j)}$;
- (4) $\text{Gin}(U^{(j)})^{(j)} = \text{Gin}(U)^{(j)}$.

Proof (1) Notice that $\text{Gin}(U^{(j)})/\text{Gin}(U)$ and $U^{(j)}/U$ have the same Hilbert series, hence the same dimension, which is less than or equal to j . Since $\text{Gin}(U)^{(j)}/\text{Gin}(U) = \delta_j(F/\text{Gin}(U))$, we have the desired inclusion.

(2) Since $U \subseteq U^{(j)}$, one inclusion is clear. Now, consider the short exact sequence

$$0 \longrightarrow U^{(j)}/U \longrightarrow (U^{(j)})^{(j)}/U \longrightarrow (U^{(j)})^{(j)}/U^{(j)} \longrightarrow 0.$$

Since the dimensions of $U^{(j)}/U$ and $(U^{(j)})^{(j)}/U^{(j)}$ are less than or equal to j , it follows that also $\dim((U^{(j)})^{(j)}/U) \leq j$ and, hence, $(U^{(j)})^{(j)} \subseteq U^{(j)}$.

(3) Since $U \subseteq U^{(j)} \cap V$, we have that $\dim((U^{(j)} + V)/V) \leq \dim(U^{(j)}/U) \leq j$, which implies $U^{(j)} \subseteq U^{(j)} + V \subseteq V^{(j)}$.

(4) Since $U \subseteq U^{(j)}$, it immediately follows from (1) that $\text{Gin}(U) \subseteq \text{Gin}(U^{(j)}) \subseteq \text{Gin}(U)^{(j)}$ and, accordingly, $\text{Gin}(U)^{(j)} \subseteq \text{Gin}(U^{(j)})^{(j)}$. On the other hand, by Parts (1) and (2), the latter is contained in $(\text{Gin}(U)^{(j)})^{(j)} = \text{Gin}(U)^{(j)}$. □

We denote the Hilbert series of a graded R -module N by $\text{Hilb}(N) = \text{Hilb}(N, z)$. We also let $h^i(N) = \text{Hilb}(H_m^i(N))$.

Lemma 3 *Let $M \cong F/U$, with dimension filtration $\{\delta_j\}_j$; then, the following holds:*

- (1) M is i -sCM if and only if $M/H_m^0(M)$ is i -sCM.
- (2) If M is i -sCM, then $H_m^i(M) \cong H_m^i(\delta_j) \cong H_m^i(\delta_j/\delta_{j-1})$ for all $j \geq i$.
- (3) If M is i -sCM, then $(z - 1)^j h^i(M) = (1 - z)^j \text{Hilb}(\delta_j/\delta_{j-1})$ for all $j \geq i$.
- (4) Let $\{\gamma_j\}_j$ be the dimension filtration of $F/\text{Gin}(U)$; if M is i -sCM, then $\text{Hilb}(\delta_j/\delta_{j-1}) = \text{Hilb}(\gamma_j/\gamma_{j-1})$ for all $j \geq i$.

Proof Since $H_m^0(R) = \delta_0$, the dimension filtration of $M/H_m^0(R)$ is $\{\delta_j/\delta_0\}_j$, which shows the first part. The long exact sequence in cohomology induced by $0 \rightarrow \delta_{j-1} \rightarrow \delta_j \rightarrow \delta_j/\delta_{j-1} \rightarrow 0$ and the Cohen–Macaulayness of δ_j/δ_{j-1} for all $j \geq i$ easily imply (2).

Let us fix $j \geq i$ and prove (3). If $j = 0$ the assertion is clear; thus we may assume $j > 0$ and by way of Part (2) that δ_j/δ_{j-1} is j -dimensional Cohen–Macaulay. Let x be a (δ_j/δ_{j-1}) -regular element of degree one; then, the short exact sequence given by multiplication by x induces a short exact sequence in cohomology

$$0 \longrightarrow H_m^{j-1} \left(\frac{\delta_j/\delta_{j-1}}{x(\delta_j/\delta_{j-1})} \right) \longrightarrow H_m^j(\delta_j/\delta_{j-1})(-1) \longrightarrow H_m^j(\delta_j/\delta_{j-1}) \longrightarrow 0$$

together with (2) imply that $h^{j-1}((\delta_j/\delta_{j-1})/(x(\delta_j/\delta_{j-1}))) = (z - 1) h^j(\delta_j/\delta_{j-1}) = (z - 1) h^j(M)$.

Thus, one can easily prove that $h^0((\delta_j/\delta_{j-1})/(x(\delta_j/\delta_{j-1}))) = (z - 1)^j h^j(M)$, where x is a (δ_j/δ_{j-1}) -maximal regular sequence. Since $\dim((\delta_j/\delta_{j-1})/(x(\delta_j/\delta_{j-1}))) = 0$, we also

have $h^0((\delta_j/\delta_{j-1})/(\mathbf{x}\delta_j/\delta_{j-1})) = \text{Hilb}((\delta_j/\delta_{j-1})/(\mathbf{x}\delta_j/\delta_{j-1})) = (1 - z)^j \text{Hilb}(\delta_j/\delta_{j-1})$, and the proof of (3) is complete.

Finally, to prove (4), we show that $\text{Hilb}(U^{(j)}/U^{(j-1)}) = \text{Hilb}(\text{Gin}(U)^{(j)}/(\text{Gin}(U)^{(j-1)}))$ holds for all $j \geq i$. Actually, we prove more, i.e., that $\text{Hilb}(U^{(j)}) = \text{Hilb}(\text{Gin}(U)^{(j)})$ for all $j \geq i$; since $\text{Gin}(U^{(j)}) \subseteq \text{Gin}(U)^{(j)}$ by Lemma 2 (1) for all j , the last equality is equivalent to proving that $\text{Gin}(U^{(j)}) = \text{Gin}(U)^{(j)}$ for all $j \geq i$, and this is what we do.

Consider now, for all j , the short exact sequences $0 \rightarrow U^{(j)}/U^{(j-1)} \rightarrow F/U^{(j-1)} \rightarrow F/U^{(j)} \rightarrow 0$; we see inductively that $\text{depth}(F/U^{(j)}) \geq j + 1$ for all $j \geq i$. For $j = d$ and if $U^{(j)}/U^{(j-1)} = 0$, this is obvious; otherwise, since M is i -sCM, $U^{(j)}/U^{(j-1)}$ is j -dimensional Cohen–Macaulay for all $j \geq i$, and by [10, Proposition 1.2.9], we get that $\text{depth } F/U^{(j-1)} \geq \min\{j, j + 1\} = j$.

For all graded submodules $V \subseteq F$, it is well-known that $\text{depth}(F/\text{Gin}(V)) = \text{depth}(F/V)$ and that $F/\text{Gin}(V)$ is sequentially Cohen–Macaulay. Thus, $j + 1 \leq \text{depth}(F/\text{Gin}(U^{(j)}))$ by what we proved above, and Proposition 1 together with Theorem 2 imply that the latter is also equal to the smallest integer t such that $\text{Gin}(U^{(j)}) \subsetneq \text{Gin}(U^{(j)})^{(t)}$. Therefore, we have shown that $\text{Gin}(U^{(j)}) = \text{Gin}(U^{(j)})^{(j)}$ for all $j \geq i$. Now, the conclusion follows from Lemma 2 (4). \square

Definition 6 Let M be a finitely generated graded R -module. We let $\delta(M) = 0$ if $M = 0$, and we let $\delta(M)$ be the largest graded R -submodule of M of dimension at most $\dim(M) - 1$ otherwise. Given $j \geq 0$, we also define the module $\delta^j(M)$ inductively by letting

$$\delta^0(M) = M, \quad \delta^1(M) = \delta(M), \quad \text{and} \quad \delta^j(M) = \delta(\delta^{j-1}(M)).$$

Since $\delta_i(M)$ is the largest submodule of M of dimension at most i , it is easy to see that for all $i \in \{0, \dots, d\}$ there exists $j = j(i) \geq 0$ such that $\delta_i(M) = \delta^j(M)$. In particular, $\delta_{d-1}(M) = \delta(M) = \delta^1(M)$.

In the following remark, we collect two known facts which are useful in the following.

Remark 9 Recall that in our setting $R \cong \omega_R(n)$. It is a well-known fact that for a d -dimensional graded R -module M it holds that $\dim \text{Ext}_R^{n-i}(M, \omega_R) \leq i$ for all i .

- (1) Given a graded submodule $N \subseteq M$ such that $\dim(N) < \dim(M) = d$, we have that $N = \delta(M)$ if and only if M/N is unmixed of dimension d , i.e., $\dim(R/\mathfrak{p}) = \dim(M/N) = d$ for all $\mathfrak{p} \in \text{Ass}(M/N)$. This is a straightforward application of Proposition 3.
- (2) A d -dimensional finitely generated graded R -module M is unmixed if and only if $\dim(\text{Ext}_R^{n-j}(M, R)) < j$ for all $j \in \{0, \dots, d - 1\}$. In fact, if $\mathfrak{p} \in \text{Ass}(M)$ were a prime of height $n - j$ for some $j < d$, we would have that $\text{Ext}_R^{n-j}(M, R)_{\mathfrak{p}} \cong \text{Ext}_{R_{\mathfrak{p}}}^{n-j}(M_{\mathfrak{p}}, R_{\mathfrak{p}}) \neq 0$, since the latter is, up to shift, the Matlis dual of $H_{\mathfrak{p}R_{\mathfrak{p}}}^0(M_{\mathfrak{p}})$ which is not zero because $\text{depth}(M_{\mathfrak{p}}) = 0$ by [10, Proposition 1.2.13]. It follows that $\mathfrak{p} \in \text{Supp}(\text{Ext}_R^{n-j}(M, R))$ and, thus, $\dim(\text{Ext}_R^{n-j}(M, R)) \geq j$, contradiction. The converse is analogous, observing that if $\text{Ext}_R^{n-j}(M, R)$ has dimension at least j and, hence, necessarily equal to j , then it must have a prime of height $n - j$ in its support.

The following lemma can be regarded as an enhanced graded version of [17, Proposition 4.16].

Lemma 4 *Let M be a finitely generated graded R -module. For a sufficiently general homogeneous element $x \in \mathfrak{m}$ and for all $j \geq 0$ there is a short exact sequence $0 \rightarrow \delta^j(\delta(M)/x\delta(M)) \rightarrow \delta^{j+1}(M/xM) \rightarrow L_j \rightarrow 0$, where L_j is a module of finite length.*

Proof Let $d = \dim(M)$. The case $d \leq 1$ is trivial; therefore, we will assume that $d \geq 2$ and proceed by induction on $j \geq 0$.

First assume that $j = 0$; let $\overline{M} = M/\delta(M)$, and observe that \overline{M} is unmixed of positive depth and dimension d . In particular, $\text{Ext}_R^n(\overline{M}, R) = 0$ and, by Remark 9, we have $\dim(\text{Ext}_R^{n-\ell}(\overline{M}, R)) < \ell$ for all $0 < \ell < d$.

For x sufficiently general, we have that x is \overline{M} -regular and, thus, $\delta(M) \cap xM = x(\delta(M) :_M x) = x\delta(M)$; moreover, either $\dim(\delta(M)/x\delta(M)) \leq 0$ or $\dim(\delta(M)/x\delta(M)) = \dim(\delta(M)) - 1 < d - 1 = \dim(\overline{M}/x\overline{M}) = \dim(M/xM)$. If we let $T = \delta(M) + xM$, then $\delta(M)/x\delta(M) \cong T/xM \subseteq \delta(M/xM)$, and therefore, we have an exact sequence $0 \rightarrow \delta(M)/x\delta(M) \xrightarrow{\varphi} \delta(M/xM)$. Moreover, we have that $\text{coker}(\varphi) \cong \delta((M/xM)/(T/xM)) \cong \delta(M/T)$. Since $M/T \cong \overline{M}/x\overline{M}$, we then have a short exact sequence

$$0 \longrightarrow \delta(M)/x\delta(M) \xrightarrow{\varphi} \delta(M/xM) \longrightarrow \delta(\overline{M}/x\overline{M}) \longrightarrow 0.$$

Since x is sufficiently general, by Remark 6, we may assume that x is also strictly filter regular for \overline{M} . We then have that either $\dim(\text{Ext}_R^{n-\ell}(\overline{M}, R) \otimes_R R/(x)) = \dim(\text{Ext}_R^{n-\ell}(\overline{M}, R)) \leq 0$ or $\dim(\text{Ext}_R^{n-\ell}(\overline{M}, R) \otimes_R R/(x)) = \dim(\text{Ext}_R^{n-\ell}(\overline{M}, R)) - 1 < \ell - 1$ for all $0 < \ell < d$. Since x is strictly filter regular for \overline{M} , from the short exact sequences

$$\begin{aligned} 0 &\longrightarrow \text{Ext}_R^{n-\ell}(\overline{M}, R) \otimes_R R/(x) \longrightarrow \text{Ext}_R^{n-(\ell-1)}(\overline{M}/x\overline{M}, R) \\ &\longrightarrow 0 :_{\text{Ext}_R^{n-(\ell-1)}(\overline{M}, R)} x \longrightarrow 0 \end{aligned}$$

it also follows that $\dim(\text{Ext}_R^{n-(\ell-1)}(\overline{M}/x\overline{M}, R)) = \dim(\text{Ext}_R^{n-\ell}(\overline{M}, R) \otimes_R R/(x)) < \ell - 1$ for all $0 \leq \ell - 1 < d - 1$. Thus, by Remark 9, it follows that $\overline{M}/x\overline{M}$ is unmixed of dimension $d - 1$ and that $L_0 = \delta(\overline{M}/x\overline{M})$ is necessarily $\delta_0(\overline{M}/x\overline{M}) = H_{\mathfrak{m}}^0(\overline{M}/x\overline{M})$, which has finite length.

Now suppose that the statement of the lemma is proved for $j - 1$, so that we have a short exact sequence $0 \rightarrow \delta^{j-1}(\delta(M)/x\delta(M)) \rightarrow \delta^j(M/xM) \rightarrow L_{j-1} \rightarrow 0$, with L_{j-1} of finite length.

It is clear from the definition of δ that there is an exact sequence $0 \rightarrow \delta^j(\delta(M)/x\delta(M)) \xrightarrow{\varphi_j} U \rightarrow L_j \rightarrow 0$, where we let $U = \delta^{j+1}(M/xM)$ and $L_j = \text{coker}(\varphi_j)$. If U has finite length we are done, so let us assume that $\dim(U) > 0$. In this case, we necessarily have that $h = \dim(\delta^j(M/xM)) > 0$, and since L_{j-1} has finite length we conclude that $\dim(\delta(M)/x\delta(M)) = h$. Again because L_{j-1} has finite length, we can find $p \gg 0$ such that $\mathfrak{m}^p U \subseteq \delta^{j-1}(\delta(M)/x\delta(M))$. Moreover, $\dim(\mathfrak{m}^p U) \leq \dim(U) < h$, and therefore $\mathfrak{m}^p U$ is contained in $\delta^j(\delta(M)/x\delta(M))$. This shows that $\mathfrak{m}^p L_j = 0$, and thus L_j has finite length. \square

In [35], it is claimed that if $x \in R$ is M -regular, it is possible to prove that M is i -sCM if and only if M/xM is $(i - 1)$ -sCM following the same lines of the proof [36, Theorem 4.7], which is not utterly correct, as we already pointed out in Example 3. The claim is indeed false: if x is M -regular and M/xM is $(i - 1)$ -sCM, then M is not necessarily i -sCM, as the following example shows.

Example 8 Let R be a 2-dimensional domain which is not Cohen–Macaulay, cf. Example 3. For all $0 \neq x \in R$, we have that $R/(x)$ is 0-sCM, but R is not even 2-sCM.

In the following proposition, we show that the claimed result holds true under some additional assumption.

Proposition 8 *Let i be a positive integer, and M be a finitely generated graded R -module with $\text{depth}(M) > 0$. Also assume that $\text{depth}(\text{Ext}_R^{n-\ell}(M, R)) > 0$ for all $\ell \geq i - 1$. For a sufficiently general $x \in R$, if M/xM is $(i - 1)$ -sCM, then M is i -sCM.*

Proof Since a module is 0-sCM if and only if it is 1-sCM, cf. Definition 5 and Corollary 2, when $i = 1, 2$ the statement follows immediately from Proposition 7; therefore, we may let $3 \leq i \leq d = \dim(M)$. Let j be the smallest integer such that $\delta_{i-1}(M) = \delta^j(M)$, and let $N = M/\delta^j(M)$. Observe that $\text{depth}(N) > 0$ by Proposition 4. By Remark 8, it is enough to prove that N is sequentially Cohen–Macaulay, and this is what we do.

Observe that since $\dim(\delta^j(M)) \leq i - 1$, we have that $\text{Ext}_R^{n-\ell}(M, R) \cong \text{Ext}_R^{n-\ell}(N, R)$ for all $\ell \geq i$, and there is an injection $0 \rightarrow \text{Ext}_R^{n-(i-1)}(N, R) \rightarrow \text{Ext}_R^{n-(i-1)}(M, R)$. In particular, from our assumptions, we get that

$$\text{depth}(\text{Ext}_R^{n-\ell}(N, R)) > 0 \text{ for all } \ell \geq i - 1.$$

By a repeated application of Lemma 4, there exists a short exact sequence

$$0 \rightarrow \delta^j(M)/x\delta^j(M) \xrightarrow{\varphi} \delta^j(M/xM) \rightarrow L \rightarrow 0,$$

where L is a module of finite length. Now, either $\dim(\delta^j(M/xM)) = \dim(\delta^j(M)/x\delta^j(M)) = \dim(\delta^j(M)) - 1$, or $\dim(\delta^j(M/xM)) \leq 0$, and in both cases, we have that $\dim(\delta^j(M/xM)) \leq i - 2$. By minimality of j , we have that $\dim(\delta^{j-1}(M)) > i - 1$ and applying iteratively Lemma 4 as we did above, we also obtain $\dim(\delta^{j-1}(M/xM)) = \dim(\delta^{j-1}(M)) - 1 > i - 2$; thus, we may conclude that

$$\delta^j(M/xM) = \delta_{i-2}(M/xM).$$

Since $L = \text{coker}(\varphi)$ and x is regular for $N = M/\delta^j(M)$, we have $(M/xM)/(\delta^j(M)/x\delta^j(M)) \cong M/(\delta^j(M) + xM) \cong N/xN$, and the above yields a short exact sequence $0 \rightarrow L \rightarrow N/xN \rightarrow (M/xM)/\delta_{i-2}(M/xM) \rightarrow 0$.

Since L has finite length, the associated long exact sequence of Ext-modules yields that

$$\text{Ext}_R^{n-\ell}(N/xN, R) \cong \text{Ext}_R^{n-\ell}((M/xM)/\delta_{i-2}(M/xM), R) \text{ for all } \ell \neq 0.$$

In particular, being M/xM a $(i - 1)$ -sCM module by assumption, Remark 8 and Peskine Theorem 3 imply that $\text{Ext}_R^{n-\ell}(N/xN, R)$ is either zero or Cohen–Macaulay of dimension ℓ for all $\ell \neq 0$. Remark 8 together with local duality also imply that

$$\text{Ext}_R^{n-(\ell-1)}(N/xN, R) = 0 \text{ for all } 1 \leq \ell - 1 \leq i - 2.$$

We may assume that x , which is N -regular, is also $\text{Ext}_R^{n-\ell}(N, R)$ -regular for all $\ell \geq i - 1$, for we proved above that all these modules have positive depth. For $\ell \geq i$, we thus have short exact sequences $0 \rightarrow \text{Ext}_R^{n-\ell}(N, R) \xrightarrow{x} \text{Ext}_R^{n-\ell}(N, R) \rightarrow \text{Ext}_R^{n-(\ell-1)}(N/xN, R) \rightarrow 0$, from which it follows that $\text{Ext}_R^{n-\ell}(N, R)$ is either zero or Cohen–Macaulay of dimension ℓ for all $\ell \geq i$.

From the above, we also have that for all $2 \leq \ell \leq i - 1$ the maps $\text{Ext}_R^{n-\ell}(N, R) \xrightarrow{x} \text{Ext}_R^{n-\ell}(N, R)$ are isomorphisms, and by graded Nakayama’s Lemma that $\text{Ext}_R^{n-\ell}(N, R) = 0$ for all $2 \leq \ell \leq i - 1$. Finally, we also have an injection $0 \rightarrow \text{Ext}_R^{n-1}(N, R) \xrightarrow{x} \text{Ext}_R^{n-1}(N, R)$, which implies that $\text{Ext}_R^{n-1}(N, R)$ is Cohen–Macaulay of dimension one by Remark 9.

Applying Peskine’s Theorem 2.9, we have thus showed that N is sequentially Cohen–Macaulay, that is, M is i -sCM. □

The next theorem provides a characterization of partially sequentially Cohen–Macaulay modules. It was proved for the first time in [35, Theorem 3.5] in the ideal case. Here, we generalize the result to finitely generated modules and fix the gap in the original proof thanks to Proposition 8. We let $R_{[n-1]} = k[x_1, \dots, x_{n-1}] \cong R/x_nR$ and denote by $N_{[n-1]}$ the $R_{[n-1]}$ -module $N/x_nN \cong N \otimes_R R/x_nR$ by restriction of scalars. We let $x \in R$ be a general linear form which, without loss of generality, we may write as $l = a_1x_1 + \dots + a_{n-1}x_{n-1} - x_n$ and consider the map $g_n: R \rightarrow R_{[n-1]}$, defined by $x_i \mapsto x_i$ for $i = 1, \dots, n - 1$ and $x_n \mapsto a_1x_1 + \dots + a_{n-1}x_{n-1}$. Then, the surjective homomorphism $F/U \rightarrow F_{[n-1]}/g_n(U)$ has kernel $(U + xF)/U$ and induces the isomorphism

$$\frac{F}{U + xF} \cong \frac{F_{[n-1]}}{g_n(U)}. \tag{1}$$

Moreover, the image of $\text{Gin}(U)$ in $F_{[n-1]}$ via the mapping $x_n \mapsto 0$ is $\text{Gin}(U)_{[n-1]}$. With this notation, the module version of [24, Corollary 2.15] states that

$$\text{Gin}(g_n(U)) = \text{Gin}(U)_{[n-1]}. \tag{2}$$

Theorem 4 *Let M be a finitely generated graded R -module of dimension d , and let $M \cong F/U$ be a free graded presentation of M . The following conditions are equivalent:*

- (1) F/U is i -sCM;
- (2) $h^i(F/U) = h^i(F/\text{Gin}(U))$ for all $i \leq j \leq d$.

Proof (1) \Rightarrow (2) is a direct consequence of Lemma 3 (3) and (4), since also $F/\text{Gin}(U)$ is i -sCM.

We prove the converse by induction on d . If $d = 0$, F/U is Cohen–Macaulay and sequentially Cohen–Macaulay. Therefore, without loss of generality we may assume that F/U and $F/\text{Gin}(U)$ have positive dimension and, by Lemma 3 (1), also positive depth. Since $F/\text{Gin}(U)$ is sequentially Cohen–Macaulay, Peskine’s Theorem 3 implies that there exists a linear form $l \in R$ which is $F/\text{Gin}(U)$ -regular and also regular for all non-zero $\text{Ext}_R^{n-j}(F/\text{Gin}(U), \omega_R)$ with $j > 0$. Starting with the exact sequence $0 \rightarrow F/\text{Gin}(U)(-1) \xrightarrow{l} F/\text{Gin}(U) \rightarrow F/(\text{Gin}(U) + lF) \rightarrow 0$, by the above and local duality, we obtain the short exact sequences

$$\begin{aligned} 0 \longrightarrow H_m^{j-1}(F/(\text{Gin}(U) + lF)) \longrightarrow H_m^j(F/\text{Gin}(U))(-1) \\ \xrightarrow{l} H_m^j(F/\text{Gin}(U)) \longrightarrow 0, \end{aligned}$$

from which it follows that $h^{j-1}(F/(\text{Gin}(U) + lF)) = (z - 1)h^j(F/\text{Gin}(U))$ for all j .

Consider now a sufficiently general linear form $x \in R$. For all j , there are exact sequences

$$0 \longrightarrow B^{(j)} \longrightarrow H_m^{j-1}(F/(U + xF)) \longrightarrow H_m^j(F/U)(-1) \longrightarrow H_m^j(F/U) \longrightarrow C^{(j)} \longrightarrow 0$$

for some R -modules $B^{(j)}$ and $C^{(j)}$, and these imply that $h^{j-1}(F/(U + xF)) = (z - 1)h^j(F/U) + \text{Hilb}(B^{(j)}) + \text{Hilb}(C^{(j)})$ for all j . By (1) and (2), we obtain

$$\begin{aligned} (z - 1)h^j(F/U) &\leq (z - 1)h^j(F/U) + \text{Hilb}(B^{(j)}) + \text{Hilb}(C^{(j)}) \\ &= h^{j-1}(F/(U + xF)) = h^{j-1}(F_{[n-1]}/g_n(U)) \\ &\leq h^{j-1}(F_{[n-1]}/\text{Gin}(g_n(U))) = h^{j-1}(F_{[n-1]}/\text{Gin}(U)_{[n-1]}) \\ &= h^{j-1}(F/(\text{Gin}(U) + lF)) = (z - 1)h^j(F/\text{Gin}(U)). \end{aligned}$$

Thus, from our hypothesis, it follows that the above inequalities are equalities for all $j \geq i$, that means that $\text{Hilb}(B^{(j)}) = \text{Hilb}(C^{(j)}) = 0$, i.e., $B^{(j)} = C^{(j)} = 0$ for $j \geq i$. Moreover, since

$C^{(i-1)} = B^{(i)} = 0$, it follows that x is regular for all nonzero $\text{Ext}_R^{n-j}(F/U, \omega_R)$ with $j \geq i - 1$, which thus have positive depth.

From the above equalities we also get that $h^j(F_{[n-1]}/g_n(U)) = h^j(F_{[n-1]}/\text{Gin}(g_n(U)))$ for all $j \geq i - 1$; by induction, this implies that $F/(U + xF) \cong F_{[n-1]}/g_n(U)$ is $(i - 1)$ -sCM.

The conclusion follows now by a straightforward application of Proposition 8. \square

As a corollary, we immediately obtain Theorem 1.

Theorem 5 *Let M be a finitely generated graded R -module, and let $M \cong F/U$ a free graded presentation of M . Then, F/U is sequentially Cohen–Macaulay if and only if $h^i(F/U) = h^i(F/\text{Gin}(U))$ for all $j \geq 0$.*

One can wonder whether the equality $h^i(F/U) = h^i(F/\text{Gin}(U))$ is enough to imply that F/U is i -sCM; however, this is not the case.

Example 9 (1) Consider a graded Cohen–Macaulay $k[x_1, \dots, x_n]$ -module M_1 of dimension i and a graded non-sequentially Cohen–Macaulay $k[x_{i+2}, \dots, x_n]$ -module N . Let $M_2 = N \otimes_k k[x_1, \dots, x_{i+1}]$ and take $M = M_1 \oplus M_2$. Then, x_1, \dots, x_{i+1} is a strictly filter-regular sequence for M_2 , and it follows from Proposition 7 (1) that M_2 is not sequentially Cohen–Macaulay. By Corollary 4 also M is not sequentially Cohen–Macaulay. On the other hand, since $\text{depth}(M_2) > i$, we have that $H_m^i(M) \cong H_m^i(M_1)$. If we write $M_1 \cong F_1/U_1$ and $M_2 \cong F_2/U_2$, where F_1 and F_2 are graded free R -modules and U_1, U_2 are graded submodules, then $M_1 \oplus M_2 \cong F/U$ where $F = F_1 \oplus F_2$ and $U = U_1 \oplus U_2$, and it follows that $\text{Gin}(U) = \text{Gin}(U_1) \oplus \text{Gin}(U_2)$. Since M_1 is Cohen–Macaulay, hence sequentially Cohen–Macaulay, and $\text{depth}(F_2/\text{Gin}(U_2)) = \text{depth}(M_2) > i$, we therefore conclude by Theorem 5 that $h^i(F/U) = h^i(F_1/U_1) = h^i(F_1/\text{Gin}(U_1)) = h^i(F/\text{Gin}(U))$.

(2) The following is another explicit example of such instance in the ideal case. Consider the polynomial ring $R = k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$ and the monomial ideal $I = (x_1^3, x_1^2x_2x_4, x_1x_5, x_1x_6, x_2x_5, x_2x_6, x_2^2x_7^2, x_3x_5, x_3x_6, x_3x_7, x_4x_5, x_4x_6, x_4x_7, x_7^3)$. Then, one can check that $\text{depth}(R/I) = 0$ and $\dim(R/I) = 3$; moreover, $h^j(R/I) = h^j(R/\text{Gin}(I))$ for $j = 0, 3$, and $h^j(R/I) \neq h^j(R/\text{Gin}(I))$ for $j = 1, 2$. By Theorem 4, this means that R/I is 3-sCM but not 2-sCM and, *a fortiori*, not 1-sCM.

On the other hand, if I^{lex} is the lexicographic ideal associated with I , then the equality $h^i(R/I) = h^i(R/I^{\text{lex}})$ ensures the i -partial sequential Cohen–Macaulayness of R/I . Notice that this is a stronger condition, though, since $h^i(R/I) \leq h^i(R/\text{Gin}(I)) \leq h^i(R/I^{\text{lex}})$, coefficientwise, see [34, Theorems 2.4 and 5.4]. Actually, in [35, Theorem 4.4], the following result is proved.

Theorem 6 *Let i be a positive integer and I a homogeneous ideal of R ; then, the following conditions are equivalent:*

- (1) $h^i(R/I) = h^i(R/I^{\text{lex}})$;
- (2) $h^i(R/\text{Gin}(I)) = h^i(R/I^{\text{lex}})$;
- (3) $h^j(R/I) = h^j(R/I^{\text{lex}})$ for all $j \geq i$.

If any of the above holds, then I is i -sCM.

We conclude this section by observing that the conditions in the previous theorem are still equivalent if we replace $\text{Gin}(I)$ with $\text{Gin}_0(I)$, the zero-generic initial ideal of I introduced in [13]. We also remark that the equivalence between conditions (2) and (3) is not true if we replace the Hilbert series of local cohomology modules with graded Betti numbers, see [32, Theorem 3.1].

4 E-depth

As in the previous section, k will denote an infinite field, $R = k[x_1, \dots, x_n]$ a standard graded polynomial ring and $\mathfrak{m} = (x_1, \dots, x_n)$ its homogeneous maximal ideal. As before, when $M = 0$, we let $\dim(M) = -1$ and $\text{depth}(M) = \infty$. We start with the main definition of this section.

Definition 7 Given a nonzero finitely generated graded \mathbb{Z} -module M and a nonnegative integer r , we say that M satisfies condition (E_r) if $\text{depth}(\text{Ext}_R^i(M, R)) \geq \min\{r, n - i\}$ for all $i \in \mathbb{Z}$. We let

$$\text{E-depth}(M) = \min \{n, \sup\{r \in \mathbb{N} \mid M \text{ satisfies } (E_r)\}\}.$$

Remark 10 Observe that a nonzero module M is sequentially Cohen–Macaulay if and only if it satisfies condition (E_r) for all $r \geq 0$, see Theorem 3. It is therefore clear that a sequentially Cohen–Macaulay R -module has E-depth equal to n . The converse is also true, since if $\text{E-depth}(M) = n$, then $\text{depth}(\text{Ext}_R^i(M, R)) \geq n - i$; as $\dim(\text{Ext}_R^i(M, R)) \leq n - i$ always holds, cf. Remark 9, this implies the claim.

Lemma 5 Let $M \neq 0$ be a finitely generated \mathbb{Z} -graded R -module with positive depth and E-depth, and let ℓ be a linear form which is a strictly filter regular for M ; then, the graded short exact sequence $0 \rightarrow M(-1) \xrightarrow{\cdot\ell} M \rightarrow M/\ell M \rightarrow 0$ induces graded short exact sequences

$$0 \rightarrow \text{Ext}_R^i(M, R) \xrightarrow{\cdot\ell} \text{Ext}_R^i(M, R)(1) \rightarrow \text{Ext}_R^{i+1}(M/\ell M, R) \rightarrow 0, \text{ for all } i < n,$$

and

$$0 \rightarrow H_{\mathfrak{m}}^{i-1}(M/\ell M) \rightarrow H_{\mathfrak{m}}^i(M)(-1) \xrightarrow{\cdot\ell} H_{\mathfrak{m}}^i(M) \rightarrow 0, \text{ for all } i > 0.$$

Proof Consider the induced long exact sequence of Ext modules

$$\dots \longrightarrow \text{Ext}_R^i(M, R) \xrightarrow{\cdot\ell} \text{Ext}_R^i(M(-1), R) \longrightarrow \text{Ext}_R^{i+1}(M/\ell M, R) \longrightarrow \dots,$$

and observe that $\text{Ext}_R^i(M(-1), R) \cong \text{Ext}_R^i(M, R)(1)$. Our assumption that $\text{E-depth}(M) > 0$ guarantees that $\text{depth}(\text{Ext}_R^i(M, R)) > 0$ for all $i < n$. Since ℓ is strictly filter regular for M , the multiplication by ℓ is injective on $\text{Ext}_R^i(M, R)$ for all $i < n$, and the long exact sequence breaks into short exact sequences, as claimed, and the graded short exact sequences of local cohomology modules are obtained by graded Local Duality. \square

Proposition 9 Let $M \neq 0$ be a finitely generated \mathbb{Z} -graded R -module.

- (1) $\text{E-depth}(M) = \text{E-depth}(M/H_{\mathfrak{m}}^0(M))$.
- (2) Assume that $\text{E-depth}(M) > 0$ and ℓ is homogeneous strictly filter regular for M ; then, either $\text{E-depth}(M/(H_{\mathfrak{m}}^0(M) + \ell M)) = \text{E-depth}(M) = n$, or $\text{E-depth}(M/(H_{\mathfrak{m}}^0(M) + \ell M)) = \text{E-depth}(M) - 1$.

Proof We first prove (1); clearly, we may assume that $H_m^0(M) \neq 0$. Applying the functor $\text{Hom}_R(-, R)$ to the short exact sequence $0 \rightarrow H_m^0(M) \rightarrow M \rightarrow M/H_m^0(M) \rightarrow 0$ we get a long exact sequence of Ext modules, from which we obtain that $\text{Ext}_R^{n-i}(M, R) \cong \text{Ext}_R^{n-i}(M/H_m^0(M), R)$ for all $i \neq 0$ because $H_m^0(M)$ has finite length; the first statement is now clear, since $\text{depth}(\text{Ext}_R^n(H_m^0(M), R)) = 0$.

For the proof of (2), let $N = M/H_m^0(M)$; by (1), M and N have same E-depth and $\text{Ext}_R^i(M, R) \cong \text{Ext}_R^i(N, R)$ for all $i \neq n$. First assume that M is sequentially Cohen–Macaulay, i.e., $\text{E-depth}(M) = n$. Then, N is sequentially Cohen–Macaulay by Remark 10 or by Corollary 2, and since ℓ is filter regular for M by Remark 2.17, ℓ is N -regular. It follows from Proposition 7 that $M/(H_m^0(M) + \ell M)$ is sequentially Cohen–Macaulay and, therefore, has E-depth equal to n .

Now assume that $0 < \text{E-depth}(N) = \text{E-depth}(M) = r < n$; from an application of Lemma 5, we obtain short exact sequences $0 \rightarrow \text{Ext}_R^i(N, R) \xrightarrow{\ell} \text{Ext}_R^i(N, R)(1) \rightarrow \text{Ext}_R^{i+1}(N/\ell N, R) \rightarrow 0$ and, therefore, $\text{Ext}_R^{i+1}(N/\ell N, R) \cong \text{Ext}_R^i(M, R)(1)/\ell \text{Ext}_R^i(M, R)$ for all $i < n$. It follows that $\text{depth}(\text{Ext}_R^{i+1}(N/\ell N, R)) = \text{depth}(\text{Ext}_R^i(M, R)) - 1$ for all $i < n$, which clearly implies $\text{E-depth}(N/\ell N) = \text{E-depth}(M) - 1$, as desired. \square

We now introduce a special grading on a polynomial ring which refines the standard grading, and that can be further refined to the monomial \mathbb{Z}^n -grading.

Definition 8 Let r be a positive integer, A be a \mathbb{Z} -graded ring and $S = A[y_1, \dots, y_r]$ a polynomial ring over A . For $i \in \{0, \dots, r\}$ let $\eta_i \in \mathbb{Z}^{r+1}$ be the vector whose $(i + 1)$ -st entry is 1, and all other entries equal 0. We consider S as a $\mathbb{Z} \times \mathbb{Z}^r$ -graded ring by letting

$$\text{deg}_S(a) = \text{deg}_A(a) \cdot \eta_0, \text{ for all } a \in A \text{ and } \text{deg}_S(y_i) = \eta_i, \text{ for } i \in \{1, \dots, r\}.$$

By means of the previous definition, we may consider $R = k[x_1, \dots, x_n]$ as a $\mathbb{Z} \times \mathbb{Z}^r$ -graded ring for any $0 \leq r \leq n - 1$ by letting $\text{deg}_R(x_i) = \eta_0$ for all $1 \leq i \leq n - r$ and $\text{deg}_R(x_i) = \eta_i$ for all $n - r + 1 \leq i \leq n$. Observe that an element $f \in R$ is graded with respect to such grading if and only if f can be written as $f = \bar{f} \cdot u$, where $\bar{f} \in k[x_1, \dots, x_{n-r}]$ is homogeneous with respect to the standard grading, and u is a monomial in $k[x_{n-r+1}, \dots, x_n]$. In particular, when $r = 0$ this is just the standard grading on R , while for $r = n - 1$, it coincides with the monomial \mathbb{Z}^n -grading.

Remark 11 We can extend in a natural way such a grading to free R -modules F with a basis by assigning degrees to the elements of the given basis. Accordingly, any R -module M is $\mathbb{Z} \times \mathbb{Z}^r$ -graded if and only if $M \cong F/U$, where F is a free $\mathbb{Z} \times \mathbb{Z}^r$ -graded R -module and U is a $\mathbb{Z} \times \mathbb{Z}^r$ -graded submodule of F .

We see next that the grading just introduced is very relevant to the purpose of estimating the E-depth of a module; the following can be seen as a refinement of Proposition 2.

Proposition 10 Let $R = k[x_1, \dots, x_n]$, and M be a finitely generated $\mathbb{Z} \times \mathbb{Z}^r$ -graded R -module such that x_n, \dots, x_{n-r+1} is a filter regular sequence for M . Then, x_n, \dots, x_{n-r+1} is a strictly filter regular sequence for M , and $\text{E-depth}(M) \geq r$.

Proof Write M as $M = F/U$, where F is a finitely generated $\mathbb{Z} \times \mathbb{Z}^r$ -graded free R -module, and U is a $\mathbb{Z} \times \mathbb{Z}^r$ -graded submodule of F . Since U is $\mathbb{Z} \times \mathbb{Z}^r$ -graded, x_n is $\mathbb{Z} \times \mathbb{Z}^r$ -homogeneous and filter regular, $U^{\text{sat}} = U :_F x_n^\infty$ is also $\mathbb{Z} \times \mathbb{Z}^r$ graded and,

accordingly $M/H_m^0(M) \cong F/U^{\text{sat}}$ is too. If $F/U^{\text{sat}} = 0$, then M has dimension zero and, thus, is sequentially Cohen–Macaulay; then $\text{E-depth}(M) = n \geq t$, and every sequence of non-zero elements of (x_1, \dots, x_n) is a strictly filter regular sequence for M . In particular, x_n, \dots, x_{n-r+1} is a strictly filter regular sequence for M .

Now suppose that $F/U^{\text{sat}} \neq 0$. By assumption, x_n is filter regular for M , and thus regular for F/U^{sat} . Since the latter is $\mathbb{Z} \times \mathbb{Z}^r$ -graded, we have that $F/U^{\text{sat}} \cong \bar{F}/\bar{U} \otimes_k k[x_n]$ for some $\mathbb{Z} \times \mathbb{Z}^{r-1}$ -graded $\bar{R} = k[x_1, \dots, x_{n-1}]$ -module \bar{F} , and some $\mathbb{Z} \times \mathbb{Z}^{r-1}$ -graded submodule \bar{U} of \bar{F} such that \bar{F}/\bar{U} can be identified with the hyperplane section $F/U^{\text{sat}} \otimes_R R/(x_n)$. Thus, for $i < n$, we have

$$\text{Ext}_R^i(M, R) \cong \text{Ext}_R^i(M/H_m^0(M), R) \cong \text{Ext}_R^i(F/U^{\text{sat}}, R) \cong \text{Ext}_R^i(\bar{F}/\bar{U}, \bar{R}) \otimes_k k[x_n].$$

It follows that x_n is a nonzero divisor on $\text{Ext}_R^i(M, R)$ for all $i < n$, and thus $\text{E-depth}(M) > 0$. Since $\text{Ext}_R^n(F/U, R)$ has finite length it also follows that x_n is a strictly filter regular element for M . Now we can consider \bar{F}/\bar{U} , and an iteration of this argument will imply the desired conclusion. □

We now introduce a weight order on R . Given integers $0 \leq r \leq n$, consider the following $r \times n$ matrix

$$\Omega_{r,n} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & -1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & -1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

and let ω_i be its i -th row; then, this induces a “partial” revlex order rev_r on R by declaring that a monomial $x^{\underline{a}} = x_1^{a_1} \dots x_n^{a_n}$ is greater than a monomial $x^{\underline{b}} = x_1^{b_1} \dots x_n^{b_n}$ if and only if there exists $1 \leq j \leq r$ such that $\omega_i \cdot \underline{a} = \omega_i \cdot \underline{b}$ for all $i \leq j$ and $\omega_{j+1} \cdot \underline{a} > \omega_{j+1} \cdot \underline{b}$.

Remark 12 Observe that rev_n coincides with the usual revlex order on $R = k[x_1, \dots, x_n]$.

Give a graded free R -module F with basis $\{e_1, \dots, e_s\}$, we can write any $f \in F$ uniquely as a finite sum $f = \sum u_j e_j$, where u_j are monomials, and we assume the sum has minimal support. We let the initial form $\text{in}_{\text{rev}_r}(f)$ of f written as above be the sum of those $u_j e_j$ for which u_j is maximal with respect to the order rev_r introduced above.

Definition 9 Let F be a finitely generated graded free R -module, and U be a graded submodule; let also $M = F/U$. We say that *the r -partial general initial submodule of U satisfies a given property \mathcal{P}* if there exists a non-empty Zariski open set \mathcal{L} of r -uples of linear forms such that $F/\text{in}_{\text{rev}_r}(g_\ell(U))$ satisfies \mathcal{P} for any $\ell = (\ell_{n-r+1}, \dots, \ell_n) \in \mathcal{L}$, where g_ℓ is the automorphism on F induced by the change of coordinates of R which sends $\ell_i \mapsto x_i$ and fixes the other variables.

With some abuse of notation, we denote any partial initial submodule $\text{in}_{\text{rev}_r}(g_\ell(U))$ which satisfies \mathcal{P} the r -partial general submodule of U , and denote it by $\text{gin}_r(U)$.

Proposition 11 *Let F be a finitely generated graded free R -module, and $U \subseteq F$ be a graded submodule. For every $0 \leq r \leq n$, we have that $\text{E-depth}(F/\text{gin}_r(U)) \geq r$.*

Proof For a sufficiently general change of coordinates, we have that x_n, \dots, x_{n-r+1} form a filter regular sequence for $F/(g_\ell(U))$. Since $\text{in}_{\text{rev}_r}(U) :_F x_n = \text{in}_{\text{rev}_r}(U :_F x_n)$, see [21, 15.7], we have that x_n, \dots, x_{n-r+1} also form a filter regular sequence for $F/\text{in}_{\text{rev}_r}(g_\ell(U))$, and hence for $F/\text{gin}_r(U)$. By construction, the module $F/\text{gin}_r(U)$ is $\mathbb{Z} \times \mathbb{Z}^r$ -graded, and the claim now follows from Proposition 10. \square

Theorem 7 *Let F be a finitely generated graded free R -module, and $U \subseteq F$ be a graded submodule. For every $0 \leq r \leq n$, we have that $\text{E-depth}(F/U) \geq r$ if and only if $h^i(F/U) = h^i(F/\text{gin}_r(U))$ for all $i \in \mathbb{N}$.*

Proof After performing a sufficiently general change of coordinates, we may assume that $V = \text{in}_{\text{rev}_r}(U)$ has the same properties as $\text{gin}_{\text{rev}_r}(U)$, and that x_n, \dots, x_{n-r+1} form a strictly filter regular sequence for F/U . By using again that $\text{in}_{\text{rev}_r}(U :_F x_n) = \text{in}_{\text{rev}_r}(U) :_F x_n$, and because strictly filter regular sequences are filter regular by Remark 6, we have that x_n, \dots, x_{n-r+1} form a filter regular sequence for F/V . Since V is $\mathbb{Z} \times \mathbb{Z}^r$ -graded, it follows from Proposition 10 that x_n, \dots, x_{n-r+1} is a strictly filter regular sequence for F/V . By [21, 15.7], we also have that

$$V^{\text{sat}} = \text{in}_{\text{rev}_r}(U) :_F x_n^\infty = \text{in}_{\text{rev}_r}(U :_F x_n^\infty) = \text{in}_{\text{rev}_r}(U^{\text{sat}}),$$

and

$$V^{\text{sat}} + x_n F = \text{in}_{\text{rev}_r}(U^{\text{sat}}) + x_n F = \text{in}_{\text{rev}_r}(U^{\text{sat}} + x_n F).$$

Viewing $F/(U^{\text{sat}} + x_n F)$ as a quotient of a free $\bar{S} = k[x_1, \dots, x_{n-1}]$ -module \bar{F} by a graded submodule \bar{U} , we see that $V^{\text{sat}} + x_n F$ can be identified with a submodule $\bar{V} \subseteq \bar{F}$, with $\bar{V} = \text{in}_{\text{rev}_{r-1}}(\bar{U})$ and \bar{V} has the same properties of $\text{gin}_{\text{rev}_{t-1}}(\bar{U})$, see [12, Lemma 3.4] and the proof of [12, Theorem 3.6] for more details.

First assume that $\text{E-depth}(F/U) \geq r$, and we prove equality for the Hilbert series of $H_m^i(F/U)$ and $H_m^i(F/V)$ for all $i \in \mathbb{N}$ by induction on $r \geq 0$. The base case is trivial since $\text{in}_{\text{rev}_0}(U) = U$. By Proposition 9, we have that $\text{E-depth}(F/U^{\text{sat}}) \geq r > 0$ and, accordingly, $\text{E-depth}(F/(U^{\text{sat}} + x_n F)) \geq r - 1$. By induction, we have that $H_m^i(F/(U^{\text{sat}} + x_n F)) = H_m^i(F/(V^{\text{sat}} + x_n F))$. Since x_n is strictly filter regular for both F/U^{sat} and F/V^{sat} , and these have positive E-depth, by Lemma 5, we have graded short exact sequences for all $i > 0$:

$$0 \longrightarrow H_m^{i-1}(F/(U^{\text{sat}} + x_n F)) \longrightarrow H_m^i(F/U^{\text{sat}})(-1) \xrightarrow{\cdot x_n} H_m^i(F/U^{\text{sat}}) \longrightarrow 0, \tag{3}$$

and

$$0 \longrightarrow H_m^{i-1}(F/(V^{\text{sat}} + x_n F)) \longrightarrow H_m^i(F/V^{\text{sat}})(-1) \xrightarrow{\cdot x_n} H_m^i(F/V^{\text{sat}}) \longrightarrow 0. \tag{4}$$

Since the first modules in both sequences have the same Hilbert series, a straightforward computation shows that $h^i(F/U^{\text{sat}}) = h^i(F/V^{\text{sat}})$. Since $H_m^i(F/U^{\text{sat}}) \cong H_m^i(F/U)$ and $H_m^i(F/V^{\text{sat}}) \cong H_m^i(F/V)$ for all $i > 0$, the equality between Hilbert series is proved for $i > 0$. Finally, since $\text{Hilb}(F/U) = \text{Hilb}(F/V)$ and $\text{Hilb}(F/U^{\text{sat}}) = \text{Hilb}(F/V^{\text{sat}})$, we have that

$$h^0(F/U) = \text{Hilb}(U^{\text{sat}}/U) = \text{Hilb}(V^{\text{sat}}/V) = h^0(F/V).$$

Conversely, assume that the local cohomology modules of F/U and F/V have the same Hilbert series. If $r = 0$ there is nothing to show, otherwise it is enough to prove that $\text{E-depth}(F/U^{\text{sat}}) \geq r$.

Since x_n is a filter regular element for F/V , it is strictly filter regular for F/V , being the latter $\mathbb{Z} \times \mathbb{Z}^r$ -graded, and hence it is filter regular for F/V^{sat} . Thus, the sequence (4) is exact for $i > 0$. On the other hand, suppose by way of contradiction that the sequence (3) is not exact for some $i \in \mathbb{Z}$, and let i be the smallest such integer, so that we still have an exact sequence

$$0 \longrightarrow H_m^{i-1}(F/(U^{\text{sat}} + x_n F)) \longrightarrow H_m^i(F/U^{\text{sat}})(-1) \xrightarrow{\cdot x_n} H_m^i(F/U^{\text{sat}}).$$

Counting dimensions in such a sequence, and comparing them to those obtained from (4) we obtain that $h^{i-1}(F/(U^{\text{sat}} + x_n F))_j > h^{i-1}(F/(V^{\text{sat}} + x_n F))_j$. However, by upper semi-continuity, we know that the reverse inequality always holds, which gives a contradiction. Thus, the sequence (3) is exact for all $i \in \mathbb{N}$. By induction we have that $\text{E-depth}(F/(U^{\text{sat}} + x_n F)) \geq r - 1$. If $\text{E-depth}(F/(U^{\text{sat}} + x_n F)) = n$, by Proposition 9, we see that $\text{E-depth}(F/U) = n \geq r$, as desired. Otherwise, again by Proposition 9, we have that $\text{E-depth}(F/U) = \text{E-depth}(F/(U^{\text{sat}} + x_n F)) + 1 \geq r$. \square

Recalling Remark 12, Theorem 7 can be viewed as another extension of Theorem 1.

Funding

The first author was partially supported by a grant from the Simons Foundation (41000748, G.C.). The second author was partially supported by the PRIN 2020 project 2020355B8Y "Squarefree Gröbner degenerations, special varieties and related topics". The third author was partially supported by the PRIN project 2017SSNZAW004 "Moduli Theory and Birational Classification" of Italian MIUR.

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Received: 15 January 2022 Accepted: 20 April 2022

Published online: 05 July 2022

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