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Hilbert, Completeness and Geometry

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Abstract. This paper aims to show how the mathematical content of Hilbert's Axiom of Completeness consists in an attempt to solve the more general problem of the relationship between intuition and formalization. Hilbert found the accordance between these two sides of mathematical knowledge at a logical level, clarifying the necessary and sufficient conditions for a good formalization of geometry. We will tackle the problem of what is, for Hilbert, the definition of geometry. The solution of this problem will bring out how Hilbert's conception of mathematics is not as innovative as his conception of the axiomatic method. The role that the demonstrative tools play in Hilbert's foundational reflections will also drive us to deal with the problem of the purity of methods, explicitly addressed by Hilbert. In this respect Hilbert's position is very innovative and deeply linked to his modern conception of the axiomatic method. In the end we will show that the role played by the Axiom of Completeness for geometry is the same as the Axiom of Induction for arithmetic and of Church-Turing thesis for computability theory. We end this paper arguing that set theory is the right context in which applying the axiomatic method to mathematics and we postpone to a sequel of this work the attempt to offer a solution similar to Hilbert's for the completeness of set theory.[1]

Keywords. Hilbert, Axiom of Completeness, Geometry, Axiomatic Method.

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1 The Axiom of Completeness

In 1899, after a series of lectures on geometry held at the University of Göttingen[2], Hilbert published a book not only fundamental for the subsequent development of geometry, but also for the way of thinking about and doing mathematics of the century that would shortly thereafter start: the "Foundations of Geometry" (*Grundlagen der Geometrie*)[3].

One of the most innovative aspects of this work is the way of thinking about and using the axiomatic method, which is no longer treated as a hypothetical-deductive method capable of proving theorems from true, self-evident, axioms. On the contrary Hilbert transformed it into a versatile method, useful to investigate the foundations of a science and building independence proofs among its axioms.

The system of axioms that Hilbert sets up in the *Grundlagen der Geometrie* is divided into five groups. In order: connection, order, parallels, congruence and continuity. We have two axioms of continuity: Archimedes's axiom and the Axiom of Completeness.

We now want to analyze the latter and try to understand what led Hilbert to formulate this axiom and why it occupies such an important role in the whole construction of the foundation of geometry.

> To the preceeding five groups of axioms, we may add the following one, which, although not of a purely geometrical nature, merits particular attention from a theoretical point of view[4].

Moreover Hilbert argues that the Axiom of Completeness "*forms the cornerstone of the entire system of axioms*"[5].

In the first German edition of 1899 there is no trace of the Axiom of Completeness. It appears from the second, in 1903, to the sixth, in 1923, in the following form:

> V.2 (Axiom der Vollständigkeit) Die Elemente (Punkte, Geraden, Ebenen) der Geometrie bilden ein System von Dingen, welches bei Aufrechterhaltung sämtlicher genannten Axiome keiner Erweiterung mehr fähig

[2]See (Toepell 1986a) and (Hallet and Majer 2004), for a precise exposition of the origins of the *Grundlagen der Geometrie* and of the development of Hilbert's reflections on geometry in this early period.

[3]When referring and quoting it we will use (Hilbert 1899) to indicate the first German edition and (Hilbert 1900F) for the first French edition. Otherwise (Hilbert 1950) refers to the first English edition, translated from the second German edition (Hilbert 1903G), while (Hilbert 1971) indicates the second English edition, translated from the tenth German edition (Hilbert 1968). However, when quoting from (Hilbert 1971), we will point the German edition where the quote first appeared. Moreover, when quoting (Hilbert 1950), we will indicate if the quote can be found also in (Hilbert 1899).

[4](Hilbert 1950, p. 15).

[5](Hilbert 1971, p. 28); original emphasis. From the seventh German edition onward.

> ist, d.h.: zu dem System der Punkte, Geraden, Ebenen ist es nicht möglich, ein anderes System von Dingen hinzuzufügen, so dass in dem durch Zusammensetzung entstehenden System sämliche aufgeführten Axiome I-IV, V 1 erfüllt sind[6].

The axiom, however, appeared in print for the first time in the French edition, in 1900, in the following form.

> Au système de points, droites et plans, il est impossible d'adjoindre d'autres êtres de maniére que le système ainsi généralisé forme une nouvelle géométrie oé les axiomes des cinq groupes I-V soient tous vérifiés; en d'autres termes: les elèments de la Géométrie forment un système d'êtres qui, si l'on conserve tous les axiomes, n'est susceptible d'aucune extension[7].

There is also an axiom of completeness for the axiomatization of real numbers in *Über den Zahlbegriff*, published in 1900.

> IV.2 (Axiom of Completeness) It is not possible to add to the system of numbers another system of things so that the axioms I, II, III, and IV 1 are also all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms[8].

Furthermore, from the seventh edition onward the Completeness Axiom is replaced by a Linear Completeness Axiom, which in the context of the other axioms implies the Axiom of Completeness in the apparently more general form.

> V.2 (Axiom of Line Completeness) It is not possible to extend the system of points on a line with its order and congruence relations in such a way that the relations holding among the original elements as well as the fundamental properties of the line order and congruence following from Axioms I-III and from V.1 are preserved[9].

The literal translation of the Axiom of Completeness is the following.

> V.2 (Axiom of Completeness) The elements (points, straight lines, planes) of geometry form a system of things that, compatibly with the other

[6] (Hilbert 1903G, p. 16).

[7] (Hilbert 1900F, p. 25).

[8] (Hilbert 1900a, p. 1094) in (Ewald 1996). In German: *IV.2 (Axiom der Vollständigkeit) Es ist nicht möglich dem Systeme der Zahlen ein anderes System von Dingen hinzuzufügen, so dass auch in dem durch Zusammensetzung entstehenden Systeme bei Erhaltung der Beziehungen zwischen den Zahelen die Axiome I, II, III, IV.1 sämtlich erfüllt sind; oder kurz: die Zahlen bilden ein System von Dingen, wleches bei Aufrechterhaltung sämtlicher Beziehungen und sämtlicher aufgeführten Axiome keiner Erweiterung mehr fähig ist.*

[9] In (Hilbert 1971, p. 26).

> axioms, can not be extended; i.e. it is not possible to add to the system of points, straight lines, planes another system of things in such a way that in the resulting system all the axioms I-IV, V.1 are satisfied.

In order to set about analyzing the content of this axiom, there are some terms that need to be clarified: Axiome, Dingen, Geometrie. The clarification of the concepts related to these terms will be used to explain Hilbert's axiomatic approach to the foundations of geometry and the central role that the Axiom of Completeness plays in this respect. We do not want to trace the history of these terms, but to investigate the role that these concept played in that historical context.

1.1 Axiome

It is easy to imagine that the concept of axiom in Hilbert's thought mirrors his use of the axiomatic method.

> The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual do- mains of knowledge[10].

This *deepening of the foundations* amounts in an analysis of the basic principles of a theory that are formalized by means of axioms. The goal of the axiomatic method is to answers questions about why certain theorems can be proved with those principles and others can not[11]. But how it is possible to link axiomatic analysis and mathematical practice? In other words, where does the meaning of the axioms come from? In this first period of reflections on foundational issues[12], Hilbert seems to offer a point of view so far from a formalist conception of mathematics that it may be almost seen at odds with a modern approach.

> These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition[13].

This quote is partly the result of an immature reflection on the sources of knowledge in geometry[14], but it also springs from a notion of intuition that is

[10] (Hilbert 1918, p. 1109) in (Ewald 1996).

[11] In a letter to Frege, dated December 29th, 1899 (in (Frege 1980, pp. 38-39)) Hilbert wrote: *I wanted to make possible to understand and answer such questions as why the sum of the angles in a triangle is equal to two right angles ad how this fact is connected with the parallel axiom.*

[12] In the second period of Hilbert's foundational studies, whose beginning can be placed in the early twenties, with the beginning of the proof theory, we can see an evolution of the concept of axiom. Yet even in this second period, the label "formalist" does not match with his concept of mathematical practice. See (Venturi) in this respect.

[13] (Hilbert 1950, p. 2). Also in (Hilbert 1899).

[14] For a detailed study of the origins and the early influences on Hilbert's conception of geometry see (Toepell 1986a), (Toepell 1986b) and (Toepell 2000).

not the empirical intuition of space, as in the Euclidean formulation. Although recognizing the intuition of space as the starting point of any geometrical reflection, Hilbert maintains that it is not the ultimate source of meaning and truth of geometrical propositions. A different notion of intuition leads Hilbert to argue that the analysis of the foundations of geometry consists of "a rigorous axiomatic investigation of their [of the geometrical signs] conceptual content"[15]. As a matter of fact Hilbert is explicit in recognizing that the axioms of geometry have different degrees of intuitiveness.

> *A general remark on the character of our axioms I-V* might be pertinent here. The axioms I-III [incidence, order, congruence] state very simple, one could even say, original facts; their validity in nature can easily be demonstrated through experiment. Against this, however, the validity of IV and V [parallels and continuity in the form of the Archimedean Axiom] is not so immediately clear. The experimental confirmation of these demands a greater number of experiments.[16].

Accordingly, Hilbert's notion of axiom, even if it is deeply linked with intuition, does not have the evident character that it had classically. We cannot find in Hilbert the substantial coincident between intuition and evidence, that in Euclid's conception of geometry was based on the notion of spatial intuition. In this modern formulation, axioms draw their meaning from a kind of intuition that we can define *contextual*. It is an intuition encoding the *modus operandi* that is obtained working in a field of research, in this case geometry.

We can find an antecedent of this kind of intuition in Klein's words:

> Mechanical experiences, such as we have in the manipulation of solid bodies, contribute to forming our ordinary metric intuition, while optical experiences with light-rays and shadows are responsible for the developement of a 'projective' intuition[17].

However a different conception of the axiomatic method and of a formalistic treatment of mathematics[18] will lead Klein to a different approach to geometry. Indeed Klein's geometrical enquires and the Erlangen's Programme will always presuppose an uncritical treatment of the intuitive data on the nature of space, contrary to the basic principle that aims Hilbert's axiomatic method. Indeed, while Klein will try to analyze and classify the different kind of spaces, Hilbert will deal with intuitions prior to the concept of space. We will come back later to this point, while explaining the different stages that Hilbert saw in the development of a science.

[15](Hilbert 1900, p. 1101) in (Ewald 1996).
[16](Hilbert *1898-1899, p. 380) in (Hallet and Majer 2004).
[17]In (Klein 1897, p. 593).
[18]On this subject see (Torretti 1984).

In the preface to the *Grundlagen der Geometrie*, Hilbert is explicit in pointing out the requirements that a system of axioms must meet to be considered a good presentation of a theory.

> The following investigation is a new attempt to choose for geometry a *simple* and *complete* [*vollständiges*] set of *independent* axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance [*Bedeutung*] of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms. [19].

Here we see clearly that the meaning of the axioms is related to the technical tools they provide, as they are used in proving geometric theorems. This meaning is therefore intrinsic to the context of the theory.

Hilbert thus requires that a formal system be simple, complete and independent. We will consider later the meaning of completeness; however it is now useful to note that a certain idea of completeness is related to the requirement that the system of axioms should be able to prove *all* important geometrical theorems. Moreover, as shown by mathematical practice, the more the ideas are simple, the more they are deep and fundamental[20]. Finally, the demand for independence is for Hilbert a necessary condition for a good application of the axiomatic method. Indeed, for Hilbert the independence of a system of axioms is an index of the depth of the principles expressed by the axioms[21].

We still have to explain what accounts for the truth of the axioms. The answer to this question is clearly shown in a letter to Frege[22] in the form of the well-known equation that Hilbert saw between coherence, truth and existence.

Once shown that the criterion of existence is identified with that of consistency, we still need to clarify what in Hilbert's view exists and how.

[19] (Hilbert 1950, p. 1). Also in (Hilbert 1899).

[20] We will not discuss here the problem of simplicity, although it is partially linked to that of purity of the methods we will address later. In the *Mathematische Notizbücher* (Hilbert *1891) Hilbert writes: *The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite evident what variant conditions (and aids) have been used in the two proofs. Given two routes, it is not right to take either of these two or to look for a third; it is necessary to investigate the area lying between the two routes.* As can be seen from this quote, the problem of simplicity is linked to what would be the development of Hilbert's proof theory; but this would lead us too far from the historical period we are examining. On this subject see (Thiele 2003).

[21] Notice however that the system of axioms proposed by Hilbert was not entirely independent. A truly independent system of axioms for geometry, but not categorical, will be proposed in 1904 by Oscar Veblen in (Veblen 1904).

[22] Letter from Hilbert to Frege December 29th, 1899; in (Frege 1980).

1.2 Dingen

For Hilbert, the existence of mathematical entities is intimately linked to the axioms of a specific formal system. Hilbert considers the axioms as *implicit definitions* of mathematical objects.

> The axioms so set up are at the same time the definitions of those elementary ideas[23].

The idea behind this position is a clear distinction between formal theory and intuitive theory. The latter refers to any mathematical field of research that features only one subject of enquiry and homogeneous methods.

Hilbert is explicit in saying that the axiomatic method leads to a more general conceptual level.

> According to this point of view, the method of the axiomatic construction of a theory presents itself as the procedure of the mapping [*Abbildung*] of a domain of knowledge onto a framework of concepts, which is carried out in such a way that to the objects of the domain of knowledge there now correspond the concepts, and to statements about the objects there correspond the logical relations between the concepts[24].

It is important here to stress that for Hilbert the mathematical objects defined by the axioms of the *Grundlagen der Geometrie* are not strictly speaking geometrical objects but conceptual entities that can be interpreted as geometrical objects. The intended interpretation is of course that of geometry, but this does not narrow the range of possible interpretations that can be give to formulas that constitute the formal system.

We then can see three distinct levels of things: 1) empirical entities 2) formal objects 3) elementary ideas. This distinction mirrors the evolutive steps of a theory that we will see in the next paragraph.

This distinction explains the Kantian exergue that Hilbert places at the beginning of the *Grundlagen der Geometrie*: *All human knowledge begins with intuitions, thence passes to concepts and ends with ideas*[25].

One of the main problem of a formal treatment of a theory is to explain why the axiomatic system so constructed should be a good formalization of the intended intuitive theory. This is the content of an objection raised by Frege.

> Your system of definitions is like a system of equations with several unknowns, where there remains a doubt whether the equations are

[23] (Hilbert 1900, p. 1104) in (Ewald 1996).
[24] (Hilbert *1921-1922, p. 3). Translation in (Hallet 2008).
[25] (Hilbert 1950). Also in (Hilbert 1899).

> soluble and, especially, whether the unknown quantities are uniquely determined. If they were uniquely determined, it would be better to give the solutions, i.e. to explain each of the expressions 'point', 'line', 'between' individually through something that was already known. Given your definitions, I do not know how to decide the question whether my pocket watch is a point. The very first axiom deals with two points; thus if I wanted to know whether it held for my watch, I should first have to know of some other object that is was a point. But even if I knew this, e.g. of my penholder, I still could not decide whether my watch and my penholder determined a line, because I would not know what a line was[26].

The objection is justified on the basis of Frege's studies on the foundations of geometry. Indeed, he acknowledged that the axioms were self-evident propositions and that geometrical objects were abstractions of empirical objects. Frege's critic, however, is easily rebutted by Hilbert[27]. In fact he argues that that was exactly the strength of his method: to establish a formal system able to define an abstract concept, which would respond only to the requirements imposed by the axioms.

> This is apparently where the cardinal point of the misunderstanding lies. I do not want to assume anything as known in advance; I regard my explanation in sec. 1 as the definition of the concepts point, line, plane - if one ads again all the axioms of groups I to V as characteristic marks. If one is looking for another definitions of a 'point', e.g. through paraphrase in terms of extensionless, etc., then I must indeed oppose such attempts in the most decisive way; one is looking for something one can never find because there is nothing there[28].

The problem with Hilbert's reply is that it just points out a distinction of levels but does not give an explanation to the problem implicit in Frege's objection. We will call it Frege's problem and we formulate it as follows: why is the axiomatic system presented by Hilbert in the *Grundlagen der Geometrie* to be considered an axiomatization of geometry? In other words, if the axioms formalize the fundamental ideas of a theory and they are what allow the most important geometrical facts to be proved, what are the criteria that allow to identify the class of theorems we are interested in axiomatizing as theorems of geometry? And finally: in Hilbert's view, what is the definition of geometry once the axiomatic method has cut off the link between formalization and spatial intuition?

[26] Letter from Frege to Hilbert January 6th, 1900; in (Frege 1980, p. 45).

[27] Or at least this is what Hilbert would have answered, because he chose not to replay. Anyway next quote is from the letter just before the one just quoted; and we can assume that if Hilbert did not wrote Frege back is because he had already made his point.

[28] Letter from Hilbert to Frege December 29th, 1899; in (Frege 1980, p. 39).

2 Completeness

In order to understand the meaning of the Axiom of Completeness we promised to explain the meaning of the terms involved. However, the main thesis of this paper is that it is not possible to understand what Hilbert's conception of geometry was without explaining the role that the Axiom of Completeness has in the process of its axiomatization.

If we undertake the difficult task of clarifying the ideas of an author far from us in time, and in the progress of the discipline he contributed to, some methodological precautions are necessary. First of all we must avoid the use of contemporary conceptual results in anachronistic contexts. As a matter of fact, understanding the genesis of concepts means going back to the time when those ideas were not clear, not completely understood. For this reason an historical analysis of this kind, even when it is precise and competent, risks obscuring not only the intentions of those who went through that experience, but the scope and extent of the ideas that are investigated. So, the analysis we would like to pursue here aims to contextualize the choices made by Hilbert as regards foundations of geometry, without altering the originality of those ideas. We therefore propose to go to the root of the problems that Hilbert addressed, trying to understand the mathematical choices and also to unravel the philosophical ideas that moved them.

We assume as our methodological stance that concepts do not proceed in a straight line of reasoning, but they get more and more clear once they are used in solving problems. In this way, ideas and conceptions at first vague are modeled on solutions given to problems. These concepts then become indispensable tools for the discipline that uses them, so that they cannot be disregarded if we want to understand a certain matter completely. In the exact sciences the historical process is easily mystified in two forms: firstly, a retrospective look tends to discover a linear progression of knowledge, and secondly the narrative of a discipline often proceeds in the opposite direction to the one that led to its formation.

In the case under discussion here it is interesting to see how this idea of completeness, which is still vague in Hilbert's discussions, and for this reason so fruitful, contained both the synthesis and the difficulty of concepts that a few decades after played a crucial role in studies of logic and beyond.

2.1 Completeness of the axioms

Coming back to the concept of geometry, in the lectures on projective geometry in 1891, Hilbert divides geometry in three parts:

> The divisions of geometry.

1. Intuitive geometry.
2. Axioms of geometry.
   (investigates which axioms are used in the established facts in intuitive geometry and confronts these systematically with geometries in which some of these axioms are dropped)
3. Analytical geometry.
   (in which from the outset a number is ascribed to the points in a line and thus reduces geometry to analysis)[29].

There is here an important distinction: the one between geometry and geometries. It is also possible to find this distinction in the *Grundlagen der Geometrie*, but for orthographic reasons it can be found only in the French version of 1900, where in the statement of the Axiom of Completeness we can find the distinction between *Géométrie* and *géométrie* The presence of new additions and comments indicates that Hilbert followed closely the editing of this translation[30]. From now on, with Geometry we mean the intuitive theory that is the object of formalization in the *Grundlagen der Geometrie*.

Hilbert's emphasis on analytic geometry stems from its importance in geometrical investigations at that time, as, for example, in Klein's representations of geometries as groups of transformations over manifolds. However, Hilbert's goal is not analyze the nature of space, as Klein did, but to make an axiomatic inquire of our geometrical intuitions. These intuitions are prior to the concept of space and hence they cannot presuppose anything about it. Indeed few years later Hilbert sharpens his reflections on the general concept of a mathematical theory and he says that

> Usually, in the story of a mathematical theory we can easily and clearly distinguish three stages of development: naïve, formal and critical[31].

Then, for geometry, Hilbert's task is to analyze critically the continuity assumption hidden in the intuition of space.

In (Hilbert 1903), Hilbert too contributed to the clarification of the nature of the space, assuming continuity since the beginning. However, since a foundation and not just a classification was sought in the *Grundlagen der Geometrie*, Hilbert sees his work as a contribution to the *kritische* stage of the development of geometry. Thus, following the basic principle of the axiomatic method of deepening the foundations, Hilbert tries to elucidates the more fundamental principles of Geometry.

[29] (Hilbert *1891, p. 3).

[30] In the volume (Hallet and Majer 2004) there is a careful account of the editorial vicisitudes of the French translation.

[31] In (Hilbert 1903a, p. 383) in (Hilbert 1970b). In German: *In der Geschichte einer mathematischen Theorie lassen sich meist 3 Entwicklungsperioden leicht und deutlich unterscheiden: Die naive, die formale und die kritische.* My translation.

Here is outlined one of the most difficult tasks of Hilbert's axiomatization of Geometry: to find a system of axioms able to formalize all the means, also analytical, used in geometrical proofs. Linked to these problems, there are considerations on the purity of method, but we will face them later. Here it is sufficient to say that Hilbert is not concerned with problems of uniformity of methods of proofs[32].

In the same lectures on projective geometry we can find the following sentence, which still suffers from a conception that shortly thereafter would be radically changed.

> Geometry is the theory about the properties of space[33].

However, in Hilbert's lectures for the summer semester, in 1984, entitled *Die Grundlagen der Geometrie* there is no longer an explicit definition of geometry, but rather of geometrical facts. It is also worth noting that in the 1899 *Grundlagen der Geometrie* we do not find a definition of space.

> Among the phenomena, or facts of experience that we take into account observing nature, there is a particular group, namely the group of those facts which determine the external form of things. Geometry concerns itself with these facts[34].

Here there is a subtle, but basic, shift in addressing the problem of a foundation for geometry. Hilbert is not trying to define what Geometry is by means of the axioms, on the contrary he just tries to find a simple, independent and consistent system of axioms that allows a formalization of all geometrical facts. The completeness of the axioms to which Hilbert refers at the beginning of the *Grundlagen der Geometrie* has therefore to be understood in the sense of maximizing the class of geometrical facts that can be proved thanks to the proposed system of axioms.

In 1894, Hilbert was explicit in describing the goals he wanted to achieve by means of his foundational studies.

> Our colleague's problem is this: what are the *necessary* and *sufficient*[35] conditions, independent of each other, which one must posit for a system of things, so that every property of these things corresponds to a geometrical fact and vice versa, so that by means of such a sys-

[32] This is a concern typical of a classical conception of the axiomatic method that dates back to Aristotele: "[...] we cannot in demonstrating pass from one genus to another. We cannot, for instance, prove geometrical truths by arithmetic" (Posterior Analytics: 75a29-75b12). For an historical survey of this subject see (Detlefsen 2008).

[33] (Hilbert *1891, p. 5).

[34] (Hilbert *1894, p. 7).

[35] My emphasis.

> tem of things a complete description and ordering of all geometrical facts is possible[36].

Hilbert's statement of intent is clear: find necessary and sufficient conditions to describe every geometrical fact. Then the problem of defining geometry disappears, since it is implicitly and extensionally defined by geometrical facts. This is precisely the purpose of an analysis conducted with the axiomatic method. As a matter of fact, in 1902, Hilbert says:

> I understand under the axiomatical exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems being connected with this truth, but to determine the position of this theorem within the system of known truths in such a way that it can be clearly said which conditions are necessary and suffcient for giving a foundation of this truth[37].

Thanks to this precise statement, we can make some general consideration on the axiomatic method. First of all, this method is primarily designed to formalize an already developed field of knowledge. Therefore it is a method that can be applied when a science has already reached a sufficient level of maturity, such that it can be divided from other branches of knowledge. Then it is possible to develop an intuition internal to the theory capable of identifying the class of facts that have to be axiomatized, together with the basic principles that allow their proofs. Moreover, it should be noted that Hilbert says explicitly that the goal of the axiomatic method is a clear understanding of geometrical proofs, thanks to the analysis of the *meaning* of the axioms[38], and not the discovery of new theorems.

Besides, Hilbert does not consider the axiomatic method primarily as a source of mathematical rigor, capable of giving an epistemological foundation for mathematical knowledge[39], but rather as a tool which allows us to answer why some proofs are possible and some others are not.

One of Hilbert's greatest achievements in the field of the foundational studies has been to recognize not only the distinction of levels between theory and metatheory, but also to understand that the metatheory was analyzable with mathematical tools. However, Hilbert considered meta-mathematical investigation as a deepening of knowledge about mathematics, and not as a genuine source of new results; contrary to his subsequent work and what the development of twentieth-century logic would show[40].

[36] (Hilbert *1894, p. 8).

[37] (Hilbert 1902-1903, p. 50).

[38] Recall the quote from the introduction of the *Grundlagen der Geometrie* (p. 1), where Hilbert declares that the aim of the book is "*to bring out as clearly as possible the significance [*Bedeutung*] of the different groups of axioms*".

[39] See (Ogawa 2004) in this respect.

[40] Following this line of reasoning it is perhaps reasonable to find an explanation for Hilbert's mild

In 1908, Hilbert still express opinions similar to those of 1902.

> In the case of modern mathematical investigations, ...I remember the investiga tions into the foundations of geometry, of arithmetic, and of set theory-they are concerned not so much with proving a particular fact or establishing the correctness of a particular proposition, but rather much more with carrying through the proof of a proposition with restriction to particular means or with demonstrating the impossibility of such a proof[41].

If the main point in axiomatizing Geometry is the axiomatization of all geometrical facts, what distinguishes them from other facts, whether empirical or mathematical? Hilbert answers this question clearly, but he is not clear on what motivates his choice; and it is on this terrain that Frege's problem regains strength.

2.2 Axiom der Vollständigkeit

Hilbert's aim is to find necessary and sufficient conditions to prove all relevant geometrical fact. So it is possible to define Geometry as the field of knowledge whose true propositions are the theorems that can be proved by means of the axioms presented in the *Grundlagen der Geometrie*.

As we saw in the last paragraph Hilbert's critical investigation of our geometrical intuitions should also take care of the continuity principles that are deeply linked with our intuition of space. This partially explains Hilbert's attention to analytical geometry. Judson Webb, in (Webb 1980), suggests that Hilbert's goal was to free Geometry from analytical considerations, in order to restore its dignity and autonomy. However, more than historical or methodological observations, there is also another reason that led Hilbert to deal with analytic geometry and in particular with analysis.

Hilbert talks explicitly of the "introduction of the number [*Einführung der Zahl*]", within Geometry, and its goal seems to be the arithmetization[42] of Geometry in the axiomatic context. Moreover, following his concept of axioms, as revealing their meaning in the demonstrative use, Hilbert's aim was to formalize analytical tools by means of geometrical axioms.

As a matter of fact, logic and analysis always play an important role in Hilbert's foundational work. In 1922, Hilbert expresses this view in these terms:

reaction to Gödel's incompleteness theorems. However, the quotes above are from the first period of Hilbert's interest on foundational issues i.e. before the twenties; while Gödel's theorems where proved in 1930.

[41] (Hilbert 1909, p. 72). Translation in (Ogawa 2004, p. 100).

[42] By arithmetic Hilbert means analysis and in this sense we use the expression "arithmetization of geometry".

> This circumstance corresponds to a conviction I have long maintained, namely, that a simultaneous construction of arithmetic and formal logic is necessary because of the close connection and inseparability of arithmetical and logical truth[43].

The foundational view proposed here by Hilbert is radically different from the standard one that tries to ground all mathematical knowledge on a single concept. This is what Frege and Russell tried to do with logic; or how a set theoretical, functional or categorical foundation of mathematics is interpreted in modern times. Rather Hilbert was convinced that the tools offered by logic and arithmetic were essential for a proper development of any branch of mathematics. In other words, Hilbert does not seem to have any ontological or epistemological commitments in using numbers and logic; rather it is a methodological concern[44].

> In all exact sciences we gain accurate results only if we introduce the concept of number[45].

However, according to Hilbert these tools must be investigated in a critical manner.

> But if science is not to fall into a bare formalism, in a later stage of its development it has to come back and reflect on itself, and at least verify the basis upon which it has come to introduce the concept of number[46].

In order to introduce the concept of number in Geometry, Hilbert defines a calculus of segments and then he uses the axiomatic method to show which algebraic properties of the calculus follow from the validity of geometrical propositions.

Here the axiomatic method is used with the aim of understanding the demonstrative role of the axioms of Geometry. The idea is to generate a coordinate system internal to Geometry, showing that some fundamental theorems implie

[43](Hilbert 1922, pp. 1131-1132) in (Ewald 1996).

[44]This is why it is not easy to attribute any philosophical position to Hilbert, although the problems he addresses have obvious philosophical implications.

[45](Hilbert *1894). In German: *In allen exakten Wissenschaften gewinnt man erst dann präzise Resultate wenn die Zahl eingefuhrt ist.* , in (Hallet and Majer 2004, p. 194).

[46](Hilbert *1898). In German: *Aber wenn die Wissenschaft nicht einem unfruchtbaren Formalismus anheimfallen soll, so wird sie auf einem spateren Stadium der Entwickelung sich wieder auf sich selbst besinnen mussen und mindestens die Grundlagen prufen, auf denen sie zur Einfuhrung der Zahl gekommen ist*, in (Hallet and Majer 2004, p. 194). However, even in this mixture of geometry and analysis we need to be guided by intuition. In (Hilbert *1905, pp. 87-88), Hilbert says: *[O]ne should always be guided by intuition when laying things down axiomatically, and one always has intuition before oneself as a goal [*Zielpunkt*].* Translation in (Hallet 2008).

certain properties of numbers that are used as coordinates. In this way, the system of real numbers is not imposed from outside, as in analytic geometry, but arises from geometrical argumentation.

For example, the validity of Pappus's theorem (called Pascal's theorem by Hilbert) is used to show that the multiplication that it is possible to define on the coordinate system must necessarily be commutative. Thanks to axioms I-V1 Hilbert shows that the coordinate system thus defined forms an Archimedean field. However, since this Archimedean field can be countable, it is clear to Hilbert that the geometry that satisfies all axioms I-V1 can not be immediately identified with analytic geometry.

Indeed, the domain of the latter is uncountable, because it makes use of all real numbers. So, Hilbert's major concern is to define axiomatically a bijection between the points of a straight line and the real numbers. The solution of this problem is precisely the mathematical content of the Axiom of Completeness

> If in a geometry only the validity of the Archimedean Axiom is assumed, then it is possible to extend the set of points, lines, and planes by "*irrational*" elements so that in the resulting geometry on every line a point corresponds, without exception, to every set of three real numbers that satisfy the equation. By suitable interpretations it is possible to infer at the same time that *all* Axioms I-V are valid in the extended geometry. Thus extended geometry (by the adjunction if irrational elements) is none other than the ordinary space Cartesian geometry in which the completeness axiom V.2 also holds[47]

In this quotation it is possible to see how the Axiom of Completeness is used to fill that gap between Hilbertian plane geometry and analytic geometry. The way to achieve this is by adding irrational elements to the coordinate system presented in the *Grundlagen der Geometrie*. As a matter of fact, the axiomatization of the real numbers is simultaneous with the introduction of the Axiom of Completeness for geometry[48].

The irrational elements are also called ideal elements, by Hilbert. However, he immediately makes it clear that the ideal character of these elements is only relative the specific presentation of the system[49].

> That to every real number there corresponds a point of the straight line does not follow from our axioms. We can achieve this, however, by the introduction of ideal (irrational) points (Cantor's Axiom). It

[47] (Hilbert 1950, pp. 35-36).

[48] Remember that the Axiom of Completeness first appears in (Hilbert 1900a) and then in the first French edition of *Grundlagen der Geometrie*

[49] In (Hilbert *1919, p. 149), Hilbert says, *The terminology of ideal elements thus properly speaking only has its justification from the point of view of the system we start out from. In the new system we do not at all distinguish between actual and ideal elements.*

> can be shown that these ideal points satisfy all the axioms I-V [...]. Their use is purely a matter of method: *first with their help is it possible to develop analytic geometry to its fullest extent* [50].

The reference to irrational elements echoes the problem of the purity of methods, which is explicitly mentioned by Hilbert. However Hilbert's solution is not to restrict the demonstrative tools, allowing just those conforming to the essential properties of the object of the theory. Indeed, the same idea of an extra-logical property of mathematical objects is contrary to the conception of axiomatic method, as Hilbert made clear also in correspondence with Frege.

> In fact, the geometric investigation carried out here seeks in general to cast light on the question of which axioms, assumptions or auxiliary means are necessary in the proof of a given elementary geometrical truth, and it is left up to discretionary judgement [*Ermessen*] in each individual case which method of proof is to be preferred, depending on the standpoint adopted[51].

Since its aim is to show the possibility or the impossibility of a proof, the axiomatic method is the highest expression of the search for the purity of methods. In an interlineated addition to the 1898/1899 lessons Hilbert writes: "Thus, solution of a problem impossible or impossible with certain means. With this is connected the demand for the purity of methods[52]". Hilbert considers the application of the axiomatic method as a precondition for any consideration on the purity of methods. Indeed, thanks to that it is possible to clear necessary conditions for the proof of a mathematical theorem. So, the choice of the demonstrative methods becomes a subjective question, since it does not depend on the nature of the problem.

> This basic principle, according to which one ought to elucidate the possibility of proofs, is very closely connected with the demand for the 'purity of method' of proof methods stressed by many modern mathematicians[53]. At root, this demand is nothing other than a subjective interpretation of the basic principle followed here.[54].

[50] (Hilbert *1899, pp. 166-167).

[51] (Hilbert 1950, pp. 82).

[52] See (Hilbert *1898-1899, p. 284) in (Hallet and Majer 2004).

[53] Remember that Hilbert's proofs were not easily accepted by the mathematical community of the late nineteenth century. Therefore, instead of restricting the methods of proof, Hilbert put forward an analysis of proofs that does not rest on external considerations on the nature of mathematical entities, but that aim to show if a demonstrative tool is necessary in a particular proof. Moreover, given the link between methods of proof and axioms, the justification of the means used in a proof is brought back to the justification of the axioms and to the inference rules. In 1925, in (Hilbert 1925), Hilbert writes: *If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success.*

[54] (Hilbert 1950, p. 82).

As a matter of fact Hilbert used analytic geometry to the full in the application of the axiomatic method to Geometry; for example in the proof that it is possible to develop a non-Deserguean geometry. This choice shows also that Hilbert's goal was not a foundation of analytic geometry in the contemporary sense, short of running into an obvious circularity in his reasoning.

In this context we can also explain how the axiomatic method can be used to improve our mathematical knowledge. Remember that Hilbert says: "I understand under the axiomatical exploration of a mathematical truth [or theorem] an investigation which does not aim at finding new or more general theorems"[55].

> This basic principle [to enquire the main possibility of a proof] seems to me to contain a general and natural prescription. In fact, whenever in our mathematical work we encounter a problem or conjecture a theorem, our drive for knowledge [*Erkenntnistrieb*] is only then satisfied when we have succeeded in giving the complete solution of the problem and the rigorous proof of the theorem, or when we recognise clearly the grounds for the impossibility of success and thereby the necessity of the failure[56].

Therefore, we can clearly see in Hilbert's thought a dichotomy between the subjective side of the demonstrative tools and the objective side of the the logical relations between concepts. However, the objectivity of mathematics is not needed to ground the mathematical discourse; indeed, this is done by means of a consistency proof. The emphasis given to the objectivity of mathematics is just a matter of justification of the methods of proof, hence of the axioms. We need to stress here the difference between giving a foundation or a justification. As a matter of fact, if we try to interpret Hilbert's foundational efforts as a modern foundation for a mathematical theory, we see that we ran into an apparent circularity of the argumentation, because analytic geometry is used in order to show the necessity of the axioms that should give a foundation for analytic geometry. Then, this seems to support the autonomy of Hilbert's foundation of mathematics[57]. But, this point of view is incorrect, since a foundation is sought where there is no foundation in the traditional sense. Hilbert does not try to find an epistemological explanation for mathematical arguments, or an ontological classification of mathematical entities, on a mathematical ground. On the contrary he tries to justify the possibility to give a formal treatment of an intuitive theory. Even if Hilbert avoids any extra-logical commitments about objects and methods of proof, however the way he constructs the formal theory for Geometry is not autonomous from extra-mathematical considerations; we can say philosophical. Indeed Hilbert justifies the formalization of a theory appealing to

[55] (Hilbert 1902-1903, p. 50).
[56] (Hilbert 1950, p. 82).
[57] See for example (Franks 2009).

intuition, logical reasoning and the concept of number. These concepts seem to be for Hilbert the starting points for any mathematical knowledge and construction. Appealing to these notions he is able to say that the axioms presented in the *Grundlagen der Geometrie* formalize precisely analytic geometry, in its intuitive character. This choice is indeed philosophical, because it implies a precise definition of mathematics: the science of calculation, carried out by logical means. This conception is quite astonishing if we think of the development of mathematics in the last century. However it explains the role of arithmetic in Hilbert's conception of mathematics, throughout all his work; where arithmetic is here to be understood in the widest sense, including also transfinite cardinal arithmetic.

Recalling that Hilbert's goal was to find necessary and sufficient conditions for proving the more relevant geometrical facts, we can affirm that the axioms of groups I-IV, together with Axiom of Archimedes, are necessary conditions for the development of analytic geometry, and the Axiom of Completeness plays the role of a sufficient condition to adapt the formal presentation given in the *Grundlagen der Geometrie* to the intuitive idea of a geometrical theory that makes use of the whole class of real numbers. Already in 1872 Cantor felt the need for an axiom to make compatible these two sides of geometry.

> In order to complete the connection [...] with the geometry of the straight line, one must only add an axiom which simply says that conversely every numerical quantity also has a determined point on the straight line, whose coordinate is equal to that quantity [...] I call this proposition an *axiom* because by its nature it cannot be universally proved. A certain objectivity is then subsequently gained thereby for the quantities although they are quite independent of this[58].

So we can distinguish two different kinds of axioms: the ones that are *necessary* for the development of a theory and the *sufficient* ones used to match intuition and formalization.

In the lectures that precede the first edition of the *Grundlagen der Geometrie* Hilbert proposed that continuity be formalized, in ways similar to Cantor's[59] and Dedekind's[60], which were able, together with the other axioms, to guarantee the existence of a bijection between the point lying on a straight line and the real numbers. However, Hilbert soon realized that there was need of less continuity for developing Geometry. Thus, following the general principle of the axiomatic method of pointing out the necessary conditions, Hilbert chose the Axiom of Archimedes. Indeed Hilbert's aim was to explain how and why geo-

[58] In (Cantor 1872, p. 128).

[59] (Cantor continuity axiom): every descending (with respect to the relation of inclusion) sequence of non empty real intervals has no-empty intersection.

[60] (Dedekind continuity axiom): given any partition of the real line in two classes $A \leq B$ (i.e. $\forall a \in A$ and $\forall b \in B$, we have $a \leq b$) there is a real number c such that $a \leq c \leq b$, for every $a \in A$ and $b \in B$.

metrical proofs were possible, considering knowledge as knowledge of causes. In a letter to Frege, on December 29th 1899 (in (Frege 1980, pp. 38-39)), Hilbert wrote: "*It was of necessity that I had to set up my axiomatic system:* I wanted to make it possible to understand *those geometrical proposition that I regard as the most important results of geometrical inquiries*" (my emphasis).

> By the above treatment the requirement of continuity has been decomposed into two essentially different parts, namely into Archimedes' Axiom, whose role is to prepare the requirement of continuity, and the Completeness Axiom which *forms the cornerstone of the entire system of axioms*. The subsequent investigations rest essentially only on Archimedes' Axiom and the completeness axiom is in general not assumed[61].

Following this line of reasoning, the Axioms of Completeness can be seen as the first, historically documented, instance of Skolem's paradox; of course Hilbert was not driven by considerations on the nature of logic, but the Axiom of Completeness can be seen as a way of solving the problem of the existence of a theory for analytic geometry that cannot prove that real numbers are uncountable. As a matter of fact Hilbert seems to argue in favor of an intuitive connection with real numbers. Writing against the genetic method that tries to define real numbers, starting with natural numbers, Hilbert says:

> The totality of real numbers, i.e. the continuum [...] is not the totality of all possible series of decimal fractions, or of all possible laws according to which elements of a fundamental sequence may proceed. It is rather a system of things whose mutual relations are governed by the axioms set up and for which all propositions, and only those, are true which can be derived from the axioms by a finite number of logical processes[62].

In other words, this matching of intuition and formalization, which tries to harmonize the intuitions behind the system of real numbers and the real line, is the intuitive content of the fifth group of axioms of the *Grundlagen der Geometrie*.

In conclusion, Hilbert's analysis of the notion of continuity led him to formalize the Axiom of Completeness as a sufficient condition for analytic geometry, in the form of a *maximality* principle.

There are some presuppositions that need to be made explicit in Hilbert's ideas. First of all, the scope of the axiomatization needs to be known right from

[61] (Hilbert 1971, p. 28). From the seventh German edition onward

[62] In (Hilbert 1900, p. 1105) in (Ewald 1996).

the beginning. Moreover, Hilbert chose to include analytical tools in the formalization of Geometry. This choice seems surprising if we consider that at that time the development of Geometry led to the introduction of very remote concepts, not only from classical geometry, but also from considering calculation as the most important tool in Geometry[63]. The answer to this problem may be Hilbert's conviction that "In all exact sciences we gain accurate results only if we introduce the concept of number[64]".

All this shows how important logic and arithmetic are for Hilbert. So, together with the fact that formalization needs to take care of demonstrative methods used in a certain field of knowledge, it explains how Hilbert's ideas developed to the proof theory.

3 Idea of completeness and contemporary axiomatics

Having explained what Hilbert means by completeness and what he was aiming for in placing it at the center of his axiomatic presentation of Geometry, we would like here to study how this idea developed after Hilbert.

We would like to say here that we do not want to explain how the notion of completeness became what we now call semantic completeness, syntactic completeness and categoricity[65]. On the contrary, we would like to see if the idea of a maximal axiom that tries to match intuition and formalization has been used in other contexts.

In the analysis of the foundations of Geometry, Hilbert faced the problem of finding a link between the subjective perception of mathematical reality and the objective character of mathematical truth. However, this link was not fully justified, because he never even try to address the problem of explaining the concept of Geometry. Hilbert's solution is satisfactory as far as the Axiom of Completeness, translated into a modern terminology -with second order logic-, implies the categoricity of the model. However, since it is possible to develop arithmetic in the system of the *Grundlagen der Geometrie*, by the first Gödel's incompleteness theorem, this system is deductively incomplete, with respect to first order logic. But for what concern the sense of completeness we used to explain the Axiom of Completeness, we can say that Hilbert did managed to build a complete system of axioms, i.e. capable to prove all relevant theorems of Euclidean geometry and to formalize all methods of proof used in it. Anyway at that time not only Gödel's results were lacking, but also a good formalization of logic, able

[63] See (Hintikka 1997) for a detailed analysis of the importance of combinatorial aspects in Hilbert's thought.

[64] (Hilbert *1894).

[65] See (Awoday and Reck 2002) in this respect.

to represet the logical tools used in formalizing Geometry.

There is a substantial link between the problem of matching intuition and formalization and the problem of a mathematical treatment of logic. Indeed whenever there is a need to formalize concepts that have intuitive roots, we have to reflect on whether reasoning on these concepts is really possibile; and at the border between subjectivity of judgements and objectivity of truths there is logic.

In this respect the Axiom of Completeness is used to delimit the scope of axiomatization and it witnesses an extra-logical relation with the subject matter of Geometry.

Hilbert's work can be seen as an instance of a more general procedure aiming to establish some necessary conditions for the development of a theory and to find a maximal principle as a sufficient condition for the formalization.

Another example, besides the case of geometry, is the formalization of the concept of computability. In this case the need for a principle capable for completing the theory is really important, since what is formalized is a meta-mathe matical concept. In this context, the analogue of the Axiom of Completeness is Church-Turing thesis. It says that the class of functions defined by the λ-caluculus (equivalently of general recursive functions and of functions computa ble by a Turing machine[66]) is the class of all the functions that are intuitively computable. Then, since all these functions are intuitively computable, Church-Turing thesis is a sufficient condition that characterizes the class of computable functions. There seems to be an important difference between the Axiom of Completeness and Church-Turing thesis, since one is an axiom, but the other a thesis. However the difference is only apparent, because as far as their use in proofs in concerned both serve as a justification of the use of the other axioms. Indeed Hilbert says explicitly that the Axiom of Completeness is not used in his geometrical investigations; exactly as the Church-Turing thesis is not used in proving theorem of recursion theory, but just invoked to justify that all and only those functions are intuitively computable. Again we can see that Church-Turing thesis bridges the gap between the formalization of a concept and the our intuitive idea.

Another example of this kind is the formalization of the concept of natural number by means of the Peano-Dedekind axioms[67]. In this case the presentation is completed by the axiom of induction as a second order principle: given a

[66] Indeed all these classes are provably the same.

[67] Besides the scheme for induction we have:

1. $\forall x(x \neq 0 \rightarrow \exists y(S(y) = x))$,
2. $\neg\exists x(S(x) = 0)$,
3. $\forall x, y(x \neq y \rightarrow S(x) \neq S(y))$.

non empty set M, an element $0 \in M$ and an injective function $S : M \to M$

$$\forall P \subseteq M\big(0 \in P \wedge \forall x(x \in P \to S(x) \in P) \to P = M\big).$$

This axiom says that every subset of M satisfying the axioms and closed under the successor function, must necessarily be the structure of natural numbers. In other words is not possible to extend the system of natural numbers with new objects and to get a new system of things that satisfies the Dedekind-Peano axioms, minus induction.

As in the case of the Axiom of Completeness we are here dealing with a method which, by using second order principles, fixes the structure intended to formalize an intuitive concept uniquely. As for Geometry, by means of these axioms we give a definition of natural number. It is interesting to note that in both situations the result is achieved through the identification of a property which formalizes the demonstrative power of a concept: continuity in the first case, induction in the second.

Therefore, it is interesting to ask whether this axiomatic notion is still relevant and how the progress of logic served to clarify this relationship between intuition and formalism.

4 The case of set theory

In the axiomatic context set theory plays a prominent role. This theory, in fact, was given a satisfactory axiomatization capable of formalizing almost all mathematics, and maintained and improved the ability to analyze up to a minimum the demonstrative tools used in mathematical practice, thanks to versatile methods for building independence proofs.

In the last century the development of mathematics and the invention of category theory have undermine the widespread idea that set theory could be the foundation of mathematics[68]. However if we confine to Hilbert's idea of foundation of a science as outlined in these pages -to apply the axiomatic method in order to find necessary and sufficient conditions- we can say that set theory provide fine tools to analyze the main possibility of proof of a theorem[69] and a unifying language where it is possible to pose any mathematical problem. Hence it is a good framework for applying Hilbert's axiomatic method to mathematics.

Indeed, set theory deals mainly with problems independent of ZFC, the classical first order formalization of the theory. However, the analysis of these prob-

[68] On this topic see MacLane's and Mathias's articles in (Judah, Just and Woodin 1992).

[69] This is also the aim of what is now called Reverse Mathematics, although its main focus are systems that leis in between RCA_0 and second order arithmetic. For this reason in Reverse Mathematics the axiomatic method is applied to theorems about countable structures. So, even if its analysis is finer, its scope is much smaller then that of set theory. See (Marcone 2009), and (Simpson 2009) for a presentation of aims and methods of Reverse Mathematics.

lems does not end with their independence proofs, but seeks to identify which principles are needed for their proofs; as Hilbert did in the case of Geometry. As a consequence these principles often cannot hide their combinatorial origin.

Secondly, as the analysis of the concept of axiom has shown, for Hilbert the idea of completeness is related to the idea of exhaustiveness of the methods of proof. However, the incompleteness phenomenon arising from Gödel's theorems makes it always possible to extend these methods, although in such a way that it is possible to compare them by means of their consistency strength[70].

A further source of difficulty is the fact that set theory uses arguments that, even if formalized in first order logic, are substantially of higher order. For example, the axioms expressing the existence of large cardinals, while affirming the existence of sets with certain first order properties, imply the existence of a model for set theory, or of class-size objects. For this reason a reflection about the methods used in set theory should also take into account a meta-theoretical discussion of logic, not necessarily first order logic[71].

Moreover, if we try to formulate an axiom that makes set theory complete with respect to the intuitive idea of set, one collides with some conceptual difficulties. These are due to the fact that the very concept of set is a mental operation of reducing to unity a plurality of things. Therefore the "set of" operation cannot be limited to a fixed domain, without asking if this latter is itself a set. The history of the axiomatization of the concept of set is in fact a continuing attempt to impose the least restrictive limitations, in order to avoid an inconsistent system; as Russell's paradox showed for Frege's system.

However, even facing these inherent difficulties, the need for an axiom similar to Hilbert's Axiom of Completeness was historically felt quite early in the development of set theory.

In 1921 Fraenkel expressed this idea as follows:

> Zermelo's axiom system do not ensure any character of "categorical" uniqueness. For this reason there should be an "Axiom of Narrowness" similar, but opposite, to Hilbert's Axiom of Completeness, in order to impose the domain to be the smallest possible, compatibly with the other axioms. In this way we can eliminate those classes, existing in Zermelo's system, that are unnecessary for a mathematical purpose[72].

[70] We say that a theory T has consistency strength stronger than a theory S if in first order Peano arithmetic (i.e. the induction axiom is a scheme) it is possible to prove $con(T) \rightarrow con(S)$, where $con(T)$ is the sentence expressing the consistency of T. Surprisingly, and luckily, the theories that are the object of study are linearly ordered, with respect to consistency strength. This order is induced by the one existing among the axioms that postulate the existence of large cardinals. For an overview of this subject see (Kanamori 1994).

[71] For an historical presentation of this problem, see (Moore 1980).

[72] (Fraenkel 1921). In German: *Das Zermelosche Axiomensystem sichert dem Bereich keinen "kat-*

Once noticing that set theory is a good framework for applying the axiomatic method, as Hilbert conceived it, to mathematics, it would be interesting to inquire about the possibility of defining a notion of completeness able to capture some intrinsic aspect of set theory, within an axiomatic framework. In other words, in which way it makes sense to try to reconcile the idea of a complete theory with the phenomenon of incompleteness?

We defer the attempt to answer these questions to another work.

egoriscehn" Eindeutigkeitscharakter. Dazu ist ein weiteres, dem Hilbertschen Vollständigkeitsaxiom umgekehrt analoges "Beschränktheitsaxiom" erforderlich das dem Bereich den kleinsten mit den Axiomen verträglichen Umfang auferlegt. Hierdurch werden verschiedene, für mathematische Zwecke unnötige Klassen von Mengen ansageschieden , die im Zermeloschen System Platz haben. My translation.

Bibliography

Awodey, S. and E. H. Reck (2002). Completeness and categoricity. Part I: nineteenth-century axiomatics to twentieth-century metalogic, *History and Philosophy of Logic* 23, 2002.

Cantor, G. (1872). Über die Ausdehnung eines Satzes aus der Theorie der trignometrischen Reihen, in *Mathematische Annalen*, 5, pp. 123-132.

Corry. L. (2006). "The origin of Hilbert's axiomatic method", in Renn (2006).

Detlefsen, M. (2008). The purity of methods, in (Mancosu 2008).

Dreben, B. and Kanamori, A. (1997). "Hilbert and set theory", *Synthese* 110, 1997.

Ehrlich, P (1997). "From completeness to archimedean completeness: an essay in the foundations of euclidean geometry", *Synthese* 110, 1997.

Ewald, W. ed. (1996). *From Kant to Hilbert: a source book in the foundations of mathematics* Volume II, Clarendon Press, Oxford.

Ewald, W. and W. Sieg, (eds.) (2008). *David Hilbert's lectures on the foundations of logic and arithmetic, 1917-1933*, Springer, Berlin.

Ewald, W. Hallet, M. and W. Sieg, (eds.) *David Hilbert's lectures on the foundations of logic and arithmetic, 1894-1917*, Springer, Berlin. Forthcoming.

Fraenkel, A. (1921). Über die Zermelosche Begründung der Mengenlehre, abstract, in *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 30(2).

Franks, C. (2009). *The autonomy of mathematical knowledge: Hilbert's program revisited*, Cambridge University Press, Cambridge.

Frege, G. (1980). *Philosophical and mathematical correspondence*, Basil Blackwell, Oxford.

Hallet, M. (2008). Reflections on the purity of method in Hilbert's *Grundlagen der Geometrie*", in (Mancosu 2008).

Hallet, M. and U. Majer, (eds.) (2004). *David Hilbert's lectures on the foundations of geometry, 1891-1902*, Springer, Berlin.

Hendricks, V. F. and al. (eds.) (2007). *Interactions. Mathematics, physics and philosophy, 1860-1930*, Springer, New York.

Hilbert, D. *Mathematische Notizbücher*, 3 notebooks, Niedersächsische Staats- und Universitätsbibliothek Göttingen, Handschriftenabteilung, Cod. Ms. D.

Hilbert 600:1-3.

Hilbert, D. (*1891). Projektive Geometrie, lectures notes for a course held during the year 1891, in Könisberg. In (Hallet and Majer 2004) pp. 21-55.

Hilbert, D. (*1894). Die Grundlagen der Geometrie, lectures notes for a course held during the year 1894, in Könisberg. In (Hallet and Majer 2004), pp. 72-123.

Hilbert, D. (*1898). Feriencursus: Über den Begriff des Unendlichen, lectures notes for a course held during the years 1898, in Göttingen. In (Hallet and Majer 2004), pp. 160-178.

Hilbert, D. (*1898-1899). Grundlagen der Euklidischen Geometrie, lectures notes for a course held during the years 1898-1899, in Göttingen. In (Hallet and Majer 2004), pp. 221-286.

Hilbert, D. (*1899). Elemente der Euklidischen Geometrie, *Ausarbeitung* form (Hilbert *1898-1899). In (Hallet and Majer 2004), pp. 302-395.

Hilbert, D. (1899). *Grundlagen der Geometrie* (*Festschrift*). In (Hallet and Majer 2004), pp. 436-525.

Hilbert, D. (1900F). *Les principes fondamentaux de la géométrie*, Gauthier-Villars, Paris, 1900.

Hilbert, D. (1900). Mathematishen Probleme, 1900. Translated as Mathematical problems in (Ewald 1996), pp. 1096-1105.

Hilbert, D. (1900a). Über den Zahlbegriff. Translated as On the concept of number in (Ewald 1996), pp. 1089-1095.

Hilbert, D. (1902-190303). Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck, in *Proceedings of the London Mathematical Society*, 35, pp. 50-68.

Hilbert, D. (1903G). *Grundlagen der Geometrie*, Zweite, durch Zusätze vermehrte und mit fünf Anhängen versehene Auflage. Teubner, Leipzig.

Hilbert, D. (1903). Über die Grundlagen der Geometrie", *Mathematiche Annalen*, 56, pp. 381-422. Translated in English in (Hilbert 1971), p. 150-190.

Hilbert, D. (1903a). Über die Theorie der algebraischen Invarianten", Mathematical paper read at the international Mathematical Congress Chicago. Printed in (Hilbert 1970b), pp. 376-383.

Hilbert, D. (1905). "Über die Grundlagen der Logik und der Arithmetik", 1905.

Translated as "On the foundations of logic and arithmetic", in (Van Heijenoort 1967).

Hilbert, D. (*1905). Logische Principien des mathematischen Denkens", lecture notes for a course held during the year 1905, in Göttingen. In (Ewald, Hallet and Sieg).

Hilbert, D. (1909). Wesen und Ziele einer Analysis der unendlichvielen unabhängigen Variablen. In (Hilbert 1970c).

Hilbert, D. (1918). Axiomatishes Denken. Translated as "Axiomatic thought", in (Ewald 1996), pp. 1105-1115.

Hilbert, D. (*1919). Natur und mathematisches Erkennen, lecture notes for a course held during the year 1919, in Göttingen. Published for the first time by David Rowe, in 1992.

Hilbert, D. (*1921-1922). Grundlagen der Mathematik, lecture notes for a course held during the academic year 1921-1922, in Göttingen. In (Ewald and Sieg 2008).

Hilbert, D. (1922). Neubegründung der Mathematik. Erste Mittelung. Translated as The new grounding of mathematics. First report, in (Ewald 1996), pp. 1117-1148.

Hilbert, D. (1925). Über das Unendliche. Translated as On the infinite, in (Ewald 1996).

Hilbert, D. (1950). *The Foundations of Geometry*, The Open Court Publishing Company, La Salle. Translated from the 2^{nd} German edition.

Hilbert, D. (1968). *Grundlagen der Geometrie*. Mit Supplementen von Paul Bernays. 10. Auflage. Teubner, Stuttgart.

Hilbert, D. (1970b). *Gesammelte Abhandlungen II*, Springer.

Hilbert, D. (1970c). *Gesammelte Abhandlungen III*, Springer.

Hilbert, D. (1971). *Foundations of geometry*, (Second English ed.), La Salle: Open Court. Translated by the 10^{th} German edition.

Hintikka, J. (ed.) (1995). *From Dedekind to Gödel*, Synthese Library vol. 251.

Hintikka, J. (1997). Hilbert vindicated, *Synthese* 110.

Judah, H. Just, W. and W. H. Woodin (eds.) (1992). *Set theory and the continuum*, Springer, New York.

Kanamori, A. (1994). *The higher Infinite. Large cardinals in set theory from their beginnings*, Springer, Berlin.

Klein, F. (1897). Gutachen, betreffend den dritten Band der Theorie der Transformationsgruppen von S. Lie anlässlich der ersten Verteilung des Lobatschewsky Preises, *Mathematichen Annalen*, 50, pp. 583-600.

Gabriele Lolli: "Hilbert e la logica", in *Le Matematiche*, LV(1), 2000.

Mancosu, P. (ed.) (2008). *The philosophy of mathematical practice*, Oxford University Press, Oxford.

Majer, U. (1997). "Husserl and Hilbert on completeness. A neglected chapter in early twentieth century foundations of mathematics", *Synthese* 110, 1997.

Majer, U. (2007). "Hilbert's axiomatic approach to the foundations of science - a failed research program?", in (Hendricks 2007).

Marcone, A. (2009). Equivalenze tra teoremi: il programma di ricerca della reverse mathematics, *La Matematica nella Società e nella Cultura, Rivista dell'Unione Matematica Italiana* 2, pp. 101-126.

Moore, G. H. (1980). Beyond first-order logic: the historical interplay between mathematical logic and axiomatic set theory, *History and Philosophy of Logic* 1.

Ogawa, Y. (2004). The Pursuit of Rigor: Hilbert's axiomatic method and the objectivity of mathematics, *Annals of the Japan Association for Philosophy of Science* 12(2).

Ortiz-Hill, C. (1995). "Husserl and Hilbert on completeness", in (Hintikka 1995).

Peckhaus, V. (2003). "The pragmatism of Hilbert's program", *Synthese* 137, 2003.

Peckhaus, V. (2005). "Pro and contra Hilbert : Zermelo's set theories", *Philosophia Scientiae* 9(2), 2005.

Renn, J. and al. (eds.) (2006). *The genesis of general relativity, Vol. 4 Theories of gravitation in the twilight of classical physics: the promise of mathematics and the dream of a unified theory*, Springer, New York, 2002.

Shore, R. (2010). Reverse mathematics: the playground of logic, *The Bulletin of Symbolic Logic*, 16, pp. 378-402.

Simpson, S. C. (2009). *Subsystems of second order arithmetic*, Cambridge University Press, Cambridge.

Thiele, R. (2003). Hilbert's Twenty-fourth problem, in *America Mathematical Monthly*, January.

Toepell, M. (1986a). *Über die Entstehung von David Hilberts "Grundlagen der Geometrie"*, Dissertation, Göttingen. Vandenhoeck & Ruprech.

Toepell, M. (1986b). On the origins of David Hilbert's *Grundlagen der Geometrie*, in *Archive for History of Exact Sciences*, 35(4).

Toepell, M. (2000). The origin and the further development of Hilbert's *Grundlagen der Geometrie*, in *Le Matematiche*, LV(1).

Torretti, R. (1984). *Philosophy of geometry from Riemann to Poincaré*, Springer, New York.

Van Heijenoort, J: *From Frege to Gödel: a source book in mathematical logic, 1879–1931*, Harvard University Press, Cambridge, 1967.

Veblen, O. (1904). A system of axioms for geometry, in *Transaction of the American mathematical society*, 5.

Venturi, G. The concept of axiom in Hilbert's thought, preprint.

Webb, J. C. (1980). *Mechanicism, mentalism and metamathematics*, Kluwer, Boston.

Judson C. Webb: "Hilbert's formalism and arithmetization of mathematics", *Synthese* 110, 1997.